Nonlinear Differ. Equ. Appl. (2018) 25:6

1021-9722/18/010001-31
published online January 29, 2018
https://doi.org/10.1007/s00030-018-0495-9

Nonlinear Differential Equations and Applications NoDEA

Homogenization of Cahn–Hilliard-type equations via evolutionary Γ-convergence

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Abstract. In these notes we discuss two approaches to evolutionary Γ-convergence of gradient systems in Hilbert spaces. The formulation of the gradient system is based on two functionals, namely the energy functional and the dissipation potential, which allows us to employ Γ-convergence methods. In the first approach we consider families of uniformly convex energy functionals such that the limit passage of the time-dependent problems can be based on the theory of evolutionary variational inequalities as developed by Daneri and Savaré 2010. The second approach uses the equivalent formulation of the gradient system via the energy-dissipation principle and follows the ideas of Sandier and Serfaty 2004.
We apply both approaches to rigorously derive homogenization limits for Cahn–Hilliard-type equations. Using the method of weak and strong two-scale convergence via periodic unfolding, we show that the energy and dissipation functionals Γ-converge. In conclusion, we will give specific examples for the applicability of each of the two approaches.

Mathematics Subject Classification. 35B27, 35K55, 35K30, 35B30, 49J40, 49J45.

Keywords. Evolutionary Γ-convergence, Gradient systems, Homogenization, Cahn–Hilliard equation, Evolutionary variational inequality, Energy-dissipation principle, Two-scale convergence.

The research of S.R. was supported by *Deutsche Forschungsgesellschaft* within *Collaborative Research Center 910: Control of self-organizing nonlinear systems: Theoretical methods and concepts of application* via the project A5 *Pattern formation in systems with multiple scales*. The research of M.L. was supported by Research Center Matheon under the ECMath Project SE2 *Electrothermal modeling of large-area OLEDs*.

1. Introduction

Multiscale problems arise in various applications in mechanics, physics, chemistry, and in the natural sciences in general, e.g. classical and stochastic homogenization [3,24,38], dimension reduction [13,29,30], atomistic-to-continuous passages [23], sharp-interface limits [40]. Therefore, the development of new tools for the treatment of such problems is an important and challenging field. In particular, tools that are based on variational methods are of great interest since they usually reflect the physical principle behind the problem, and in this way they can provide more insight into the problem.

In this text, we are interested in evolutionary problems that have a gradient structure, i.e. the evolution of the system is written in terms of an entropy or energy functional $\mathcal{E}$ defined on a state space X and a dissipation potential $\mathcal{R}$ in the form of an abstract balance between viscous and potential restoring forces:

$$0 = \mathrm{D}\mathcal{R}\big(\dot{u}(t)\big) + \mathrm{D}\mathcal{E}\big(u(t)\big). \tag{1.1}$$

Here, we consider "classical" gradient systems $(X, \mathcal{E}, \mathcal{R})$ meaning that the dissipation potential $\mathcal{R}$ is a quadratic functional.

The multiscale nature of the problems under consideration is given by a small parameter $\varepsilon > 0$, which characterizes the ratio between the microscopic and macroscopic length scales. Hence, we consider a family of gradient systems $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ and address the central question of characterizing the conditions on the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ that guarantee the convergence of solutions u_ε of the multiscale problems associated with $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ to solutions of an effective problem in the limit $\varepsilon \to 0$. In particular, as the evolution is entirely driven by functionals we aim for methods based on Γ-convergence and, following [36], call this approach evolutionary Γ-convergence, *E-convergence* for short.

Here, we present two distinct approaches: The first approach is based on the uniform Λ-*convexity* of the driving functionals $\mathcal{E}_\varepsilon$ with respect to the potentials $\mathcal{R}_\varepsilon$, see Sect. 2.2 for the definition. In this case we can reformulate the evolution of the system in terms of an *Evolutionary Variational Estimate (EVE)*, see (2.12). We refer to [4,17,18] for an extensive survey on the topic of Λ-convex gradient systems.

The second approach to E-convergence is based on the equivalent formulation of (1.1) via the *Energy Dissipation Principle (EDP)*, which reads

$$\mathcal{E}_\varepsilon\big(u_\varepsilon(T)\big) + \int_0^T \mathcal{R}_\varepsilon(\dot{u}_\varepsilon) + \mathcal{R}^*_\varepsilon\big(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon)\big)\,\mathrm{d}t \le \mathcal{E}_\varepsilon\big(u_\varepsilon(0)\big). \tag{1.2}$$

In contrast to the first approach based on (EVE), the (EDP) formulation does not rely on any convexity assumptions of the energy functional and follows from the Legendre–Fenchel equivalences and the chain rule. An important point is that in the later application to homogenization problems the lower liminf estimate for the dissipation potentials with respect to weak convergence in X is not satisfied. Therefore, we generalize the abstract E-convergence results via (EDP) in [36] to fit in our setting. Let us remark, that this approach is related to the well-known Sandier–Serfaty principle [47], which is also based on (EDP). However, there the conditions are formulated in a very general manner. In contrast, we give explicit conditions on the energy and dissipation potentials to prove E-convergence. Moreover, we do not need to impose two separate estimates for the primal and dual dissipation potentials.

Having established the two approaches for E-convergence in the abstract case, we apply both methods to rigorously prove a homogenization result for the multiscale Cahn–Hilliard-type equation

$$\partial_t u_\varepsilon = \operatorname{div}\big[M_\varepsilon(x)\nabla\big(\partial_u W_\varepsilon(x,u_\varepsilon) - \operatorname{div}(A_\varepsilon(x)\nabla u_\varepsilon)\big)\big]. \tag{1.3}$$

The multiple scales are given by the rapidly oscillating coefficient functions $M_\varepsilon(x) = \mathbb{M}(x, x/\varepsilon)$ and $A_\varepsilon(x) = \mathbb{A}(x, x/\varepsilon)$ as well as the potential $W_\varepsilon(x,u) = \mathbb{W}(x, x/\varepsilon, u)$. We show that limits of (subsequences of) solutions to (1.3) solve the limiting equation

$$\partial_t u = \operatorname{div}\big[M_{\text{eff}}(x)\nabla\big(\partial_u W_{\text{eff}}(x,u) - \operatorname{div}(A_{\text{eff}}(x)\nabla u)\big)\big], \tag{1.4}$$

where the effective coefficient functions $M_{\text{eff}}, A_{\text{eff}}$ are given via the classical unit cell problem and $W_{\text{eff}}(x,u)$ is the usual average of $\mathbb{W}$ over the microscopic cells for fixed u. We refer to [11,51] for a physical application of this model. Therein, the dewetting process of thin films on heterogeneous substrates is modeled via the Cahn–Hilliard equation with nonlinear mobility and spatially periodic oscillating potential.

It is well-known that (1.3) has the gradient structure $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, where X is isomorphic to the dual of H^1-functions with fixed average, $\mathcal{E}_\varepsilon$ is the classical Allen–Cahn energy functional, and $\mathcal{R}_\varepsilon$ is an H^{-1}-norm-like dissipation potential, namely

$$\begin{aligned} \mathcal{E}_\varepsilon(u) &= \int_\Omega \frac{1}{2}\nabla u \cdot A_\varepsilon(x)\nabla u + W_\varepsilon(x,u)\,\mathrm{d}x \quad \text{and} \\ \mathcal{R}_\varepsilon(\dot u) &= \int_\Omega \frac{1}{2}\nabla \xi_{\dot u} \cdot M_\varepsilon(x)\nabla \xi_{\dot u}\,\mathrm{d}x, \quad \text{where} \quad -\operatorname{div}(M_\varepsilon(x)\nabla\xi_{\dot u}) = \dot u. \end{aligned} \tag{1.5}$$

Then, the PDE (1.3) is (formally) equivalent to the force-balance formulation

$$0 = \mathrm{D}\mathcal{R}_\varepsilon\big(\dot u_\varepsilon(t)\big) + \mathrm{D}\mathcal{E}_\varepsilon\big(u_\varepsilon(t)\big). \tag{1.6}$$

Using two-scale convergence techniques, we prove that under suitable assumptions on the potential W_ε the energy functionals $\mathcal{E}_\varepsilon$ Γ-converge to an effective energy functional $\mathcal{E}_0$ with respect to the weak topology on $\mathrm{H}^1(\Omega)$. With the same arguments we can show that the dual dissipation potentials Γ-converge to an effective potential in the *weak topology of* X^* and thus, by a duality principle for Γ-convergence we obtain the Γ-convergence of the primal dissipation potentials in the *strong topology of* X .

In order to apply the abstract E-convergence results based on (EVE), we assume that the potential W_ε is uniformly λ-convex on $\mathbb{R}$. In that case, we can deduce the uniform Λ-convexity of $\mathcal{E}_\varepsilon$ with Λ related to λ. In particular, in this case the first approach yields the desired homogenized equation (1.4).

In the second approach, based on the (EDP) formulation, we can drop the convexity assumption on W_ε. However, we need to verify closedness properties of the subdifferential of $\mathcal{E}_\varepsilon$. In the concrete case of the Cahn–Hilliard equation in (1.3) this follows e.g. from suitable uniform growth estimates for $\partial_u W_\varepsilon$ or uniform λ-convexity of W_ε. However, we need to additionally impose the *well-preparedness of the initial conditions*, i.e. $u_\varepsilon(0) \to u(0)$ in X and $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0))$, whereas this condition was not needed in (EVE). In particular, this means that $u_\varepsilon(0)$ is a recovery sequence for $\mathcal{E}_\varepsilon$.

We remark that both approaches allow us to consider the classical logarithmic- and double-well potential. However, we show that there are certain examples of potentials that highlight the distinction between the approaches.

Finally, let us shortly review the literature on E-convergence and homogenization results related to the Cahn–Hilliard equation. An effective macroscopic Cahn–Hilliard equation in a porous media setting is derived in [48] via the method of asymptotic expansion. To our knowledge no rigorous homogenization results concerning the Cahn–Hilliard equation exist so far. In [47], energy-based methods, which we term energy-dissipation principle, are developed to derive evolutionary Γ-convergence results for gradient flows in an abstract setting. Based on this, the sharp interface limit of the Cahn–Hilliard equation is investigated in [28] using the classical Modica–Mortola energy functional. In [49], the abstract scheme for energies defined on spaces with Hilbert space structure in [47] is generalized to metric spaces. In [7, 19], the convergence of the one dimensional Cahn–Hilliard equation to a Stefan problem is proved for nonconvex potentials relying once more on [47]. In [42, 43], sharp interface limits are rigorously derived by exploiting the gradient structure of the Cahn–Hilliard equation, Γ-convergence, and the Rayleigh principle. Finally, let us mention that the concept of evolutionary Γ-convergence was used in [35] for Hamiltonian systems. In particular, a homogenization result for the wave equation was obtained. In [39] E-convergence of rate-independent systems, which can be seen as generalized gradient systems, was discussed using an energetic formulation which corresponds to the (EDP) formulation.

This paper is structured as follows. In Sect. 2, we introduce abstract gradient systems $(X, \mathcal{E}, \mathcal{R})$ consisting of a separable Hilbert space X, an energy functional $\mathcal{E}$, and a quadratic dissipation potential $\mathcal{R}$. We discuss the notion of evolutionary Γ-convergence in Sect. 2.1 and state the two abstract results on the (EVE) and (EDP) formulation in Sects. 2.2 and 2.3, respectively. Section 3 is devoted to the homogenization of the Cahn–Hilliard-type equation (1.3). We collect the assumptions on the data in Sect. 3.1, explain the gradient structure in Sect. 3.2, and derive the Γ-convergence of the energy and dissipation functionals in Sect. 3.3. Finally, we apply the abstract results of Sect. 2.1 to the concrete setting in Sect. 3.4 and 3.5, respectively. In Sect. 3.6, we present exemplary potentials W_ε, that fit into our theory. Finally, we conclude the paper in Sect. 4 by discussing the benefits and differences of the two approaches via (EVE) and (EDP), respectively. Moreover, we compare our E-convergence results with that of [47].

2. Abstract gradient systems

A gradient system is a triple $(X, \mathcal{E}, \mathcal{R})$ consisting of a separable Hilbert space X, a proper and lower semicontinuous driving functional $\mathcal{E} : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, and a quadratic dissipation potential $\mathcal{R} : X \to [0, \infty)$. The latter means that $\mathcal{R}$ is of the form $\mathcal{R}(v) = \frac{1}{2}\langle Gv, v\rangle$ with $\langle\cdot,\cdot\rangle$ denoting the dual pairing between X and its dual X^* (which we do not identify to distinguish between velocities and forces) and $G \in \mathrm{Lin}(X, X^*)$ is symmetric and positive definite. In particular, we assume that $\mathcal{R}$ satisfies

$$\exists\, \alpha, \beta > 0 : \quad \tfrac{\alpha}{2}\|v\|_X^2 \le \mathcal{R}(v) \le \tfrac{\beta}{2}\|v\|_X^2 \quad \text{for all } v \in X. \tag{2.1}$$

The gradient-flow equation associated with $\mathcal{E}$ and $\mathcal{R}$ is now given in terms of the force balance, also called Biot's equation, which reads

$$0 \in \mathrm{D}\mathcal{R}\big(\dot u(t)\big) + \partial_X \mathcal{E}\big(u(t)\big), \quad u(0) = u_0, \tag{2.2}$$

where $\partial_X\mathcal{E}(u) \subset X^*$ denotes a suitable notion of a set-valued subdifferential of $\mathcal{E}$. Let us remark that the right notion of subdifferential, e.g. convex, Fréchet, or strong/weak limiting subdifferential, is dictated by the concrete problem. On the one hand, it has to be "big" enough such that all relevant limits are contained. On the other hand it has to be "small" enough to satisfy a chain rule condition (see below). We refer to [45] for a discussion of sufficient conditions on $\mathcal{E}$, $\partial_X\mathcal{E}$ and the data u_0 that guarantee the existence of solutions of (2.2), see also Remark 2.1. In the following we always assume that solutions $u \in \mathrm{H}^1(0, T; X)$ of the force-balance formulation in (2.2) exist.

With the primal dissipation potential $\mathcal{R}$ we can associate the dual dissipation potential $\mathcal{R}^* : X^* \to [0, \infty)$, which is given via the Legendre transform, i.e.

$$\mathcal{R}^*(\xi) := \sup\big\{\langle\xi, v\rangle - \mathcal{R}(v) \,\big|\, v \in X\big\}.$$

In particular, we have that $\mathcal{R}^*(\xi) := \frac{1}{2}\langle\xi, G^{-1}\xi\rangle$ and the estimates $\frac{\alpha^*}{2}\|\xi\|_{X^*}^2 \le \mathcal{R}^*(\xi) \le \frac{\beta^*}{2}\|\xi\|_{X^*}^2$ are satisfied for all $\xi \in X^*$, where $\alpha^* = 1/\beta$ and $\beta^* = 1/\alpha$.

For the driving functional $\mathcal{E}$ we assume that there exists a reflexive Banach space $Z \subset X$ such that the embedding is compact and

$$\exists\, c, C > 0,\ q \ge 1 :\ \mathcal{E}(u) \ge c\|u\|_Z^q - C \ \text{ for all } u \in Z. \tag{2.3}$$

As usual, we extend $\mathcal{E}$ to the bigger space X by setting $\mathcal{E}(u) = +\infty$ for $u \in X \setminus Z$.

Finally, we make the crucial assumption that $\partial_X\mathcal{E}$ satisfies a chain rule condition: If $u \in \mathrm{H}^1(0, T; X)$, $\xi \in \mathrm{L}^2(0, T; X^*)$ is such that $\xi(t) \in \partial_X\mathcal{E}(u(t))$ for a.a. $t \in [0, T]$, and $t \mapsto \mathcal{E}(u(t))$ is bounded, then it is also absolutely continuous on $[0, T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\big(u(t)\big) = \langle\xi(t), \dot u(t)\rangle \quad \text{for a.e. } t \in [0, T]. \tag{2.4}$$

Remark 2.1. *Our setting can be cast in the framework of* [45] *by considering the Hilbert space* X *with the norm* $\|v\|_G^2 = \langle Gv, v\rangle$ *and the corresponding subdifferential* $\partial_G\mathcal{E} = G^{-1}\partial_X\mathcal{E} \subset X$*, meaning that* $v \in \partial_G\mathcal{E}(u)$ *if and only if* $Gv \in \partial_X\mathcal{E}(u)$*.*

If $u_0 \in \mathrm{dom}(\mathcal{E})$*, the coercivity and the chain rule conditions in* (2.3) *and* (2.4) *are satisfied, then solutions* $u \in \mathrm{H}^1(0,T;X)$ *of* (2.2) *exist according to [45, Thm. 3] with* $\partial_X \mathcal{E}$ *being the strong-weak limiting subdifferential.*

2.1. Evolutionary Γ-convergence for abstract gradient systems

For a parameter $\varepsilon \in [0,1]$ we consider a family of gradient systems $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, where X, $\mathcal{E}_\varepsilon$, and $\mathcal{R}_\varepsilon$ are as above for each ε. Following [36, Def. 2.10] we define the notion of evolutionary Γ-convergence with or without well-prepared initial conditions – *E-convergence* respective *well-prepared E-convergence* for short.

Definition 2.2. (E-convergence) *For* $\varepsilon > 0$*, let* $u_\varepsilon : [0,T] \to X$ *be a solution of* $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ *in the sense of* (2.2) *and assume that* $u_\varepsilon(0) \to u_0$ *in* X*. We say that* $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ E-converges *to* $(X, \mathcal{E}_0, \mathcal{R}_0)$ *if there exists a solution* $u : [0,T] \to X$ *of* $(X, \mathcal{E}_0, \mathcal{R}_0)$ *with* $u(0) = u_0$ *and a subsequence* $\varepsilon_k \to 0$ *such that* $u_{\varepsilon_k}(t) \to u(t)$ *in* X *and* $\mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}(t)) \to \mathcal{E}_0(u(t))$ *for all* $t \in (0,T]$.

If we need to impose additionally $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u_0) < \infty$*, we say that* $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ E-converges with well-prepared initial conditions *to* $(X, \mathcal{E}_0, \mathcal{R}_0)$.

In the upcoming subsections we discuss two abstract E-convergence results: In Theorem 2.5 we impose a uniform Λ-convexity condition on $\mathcal{E}_\varepsilon$ to show the E-convergence of $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ using an equivalent formulation based on evolutionary variational inequalities and without well-preparedness of the initial conditions. Secondly, we prove the same result in Theorem 2.6 assuming well-preparedness and a closedness property of the subdifferentials instead of the Λ-convexity condition by passing to the limit in the energy-dissipation formulation of (2.2). Both approaches are based on the Γ-convergence of the functionals whose definition we recall here.

Definition 2.3. (Γ- and Mosco convergence) *On a reflexive Banach space* X *we say that the functionals* $\mathcal{E}_\varepsilon$ Γ-converge *to* $\mathcal{E}_0$ *in the* weak (resp. strong) *topology on* X*, and write* $\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}_0$ *(resp.* $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$*), if the following two estimates are satisfied*

(i) liminf estimate

$$\forall\, u_\varepsilon \rightharpoonup u \ (\text{resp. } u_\varepsilon \to u): \quad \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u);$$

(ii) limsup estimate (existence of recovery sequences)

$$\forall\, \widehat{u}\ \exists\, \widehat{u}_\varepsilon \rightharpoonup \widehat{u} \ (\text{resp. } \widehat{u}_\varepsilon \to \widehat{u}): \quad \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \leq \mathcal{E}_0(\widehat{u}).$$

We say that $\mathcal{E}_\varepsilon$ converges in the sense of Mosco *to* $\mathcal{E}_0$*, written* $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$*, if (i) holds with respect to the weak convergence in* X *and (ii) is satisfied with respect to the strong convergence, i.e. strongly converging recovery sequences exist.*

Let the systems $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ satisfy the assumptions (2.1) and (2.3) uniformly with respect to ε, i.e. there exist constants $\alpha, \beta, C, c > 0$, a reflexive Banach space $Z \subset X$ compactly, and $q \geq 1$, all independent of ε, such that

$$\forall\, \varepsilon \in [0,1]: \quad \begin{cases} \forall\, v \in X: & \frac{\alpha}{2}\|v\|_X^2 \leq \mathcal{R}_\varepsilon(v) \leq \frac{\beta}{2}\|v\|_X^2; \\ \forall\, u \in X: & \mathcal{E}_\varepsilon(u) \geq c\|u\|_Z^q - C. \end{cases} \tag{2.5}$$

Moreover, we assume in the following that the driving functionals $\mathcal{E}_\varepsilon$ and the dissipation potentials $\mathcal{R}_\varepsilon$ Γ-converge in the strong sense on X, respectively, namely

$$\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0 \text{ in } X \quad \text{and} \quad \mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0 \text{ in } X. \tag{2.6}$$

Finally, in the uniform Λ-convex case in Sect. 2.2 we will additionally assume that the dissipation potentials $\mathcal{R}_\varepsilon$ *converge continuously* along strongly converging sequences in X, denoted $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{C}} \mathcal{R}_0$, i.e.

$$\forall\, u_\varepsilon \to u \text{ in } X: \quad \lim_{\varepsilon\to 0} \mathcal{R}_\varepsilon(u_\varepsilon) = \mathcal{R}_0(u). \tag{2.7}$$

Since Z is compactly embedded in X and the family $\mathcal{E}_\varepsilon$ is equi-coercive on Z, the weak Γ-convergence on Z is equivalent to Mosco convergence on X, see e.g. [36, Prop. 2.5]. Moreover, the strong Γ-convergence on X of the dissipation potentials $\mathcal{R}_\varepsilon$ is equivalent to the weak Γ-convergence of $\mathcal{R}^*_\varepsilon$ on X^* due to the continuity properties of the Legendre transform.

Proposition 2.4. *[6, pp. 271] Let $\mathcal{R}^*_\varepsilon$ denote the Legendre transform of $\mathcal{R}_\varepsilon$, then*

$$\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0 \textit{ in } X \quad \Longleftrightarrow \quad \mathcal{R}^*_\varepsilon \xrightharpoonup{\Gamma} \mathcal{R}^*_0 \textit{ in } X^*. \tag{2.8}$$

2.2. A convergence result based on variational inequalities

In this section we state the first abstract Γ-convergence result for the gradient systems $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ in the case that $\mathcal{E}_\varepsilon$ is uniformly Λ-convex with respect to the dissipation potential $\mathcal{R}_\varepsilon$, i.e. we assume that there exists a constant $\Lambda \in \mathbb{R}$, independent of ε, such that

$$u \mapsto \mathcal{E}_\varepsilon(u) - \Lambda\mathcal{R}_\varepsilon(u) \quad \text{is convex.} \tag{2.9}$$

If the driving functional $\mathcal{E}_\varepsilon$ is Λ-convex with respect to $\mathcal{R}_\varepsilon$ in the sense of (2.9) we obtain the equivalent formulation of the (differential) gradient-flow equation in (2.2) as an evolutionary variational estimate (EVE). We recall that the *Fréchet subdifferential* $\partial_{\mathrm{F}}\mathcal{E}_\varepsilon : X \rightrightarrows X^*$ is defined via

$$\partial_{\mathrm{F}}\mathcal{E}_\varepsilon(u) := \left\{ \xi \in X^* \;\Big|\; \liminf_{w\to u} \frac{\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u) - \langle \xi, w-u\rangle}{\|w-u\|_X} \geq 0 \right\} \tag{2.10}$$

and is in general multi-valued. In particular, in the Λ-convex case we have that $\xi \in \partial_{\mathrm{F}}\mathcal{E}_\varepsilon(u)$ for $u \in X$ if and only if

$$\text{for all } w \in X: \quad \mathcal{E}_\varepsilon(w) \geq \mathcal{E}_\varepsilon(u) + \langle \xi, w-u\rangle + \Lambda\mathcal{R}_\varepsilon(w-u). \tag{2.11}$$

Moreover, if $\mathcal{E}_\varepsilon$ is Λ-convex $\partial_{\mathrm{F}}\mathcal{E}_\varepsilon$ satisfies the chain rule condition (see e.g. [10, Lem. 3.3]) as well as the strong-weak closedness condition, cf. Proposition 2.7.

Using this convexity estimate and the gradient-flow equation in (2.2) for $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ we arrive at the *Evolutionary Variational Estimate* (EVE)

$$\forall\, t > 0,\, w \in X: \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_\varepsilon(u(t)-w) + \Lambda\mathcal{R}_\varepsilon(u(t)-w) \leq \mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t)), \tag{2.12}$$

which corresponds to the Hilbert space version of Bénilan's weak formulation [8] in the case $\Lambda = 0$, see also [4, Ch. 4] and [18]. Multiplying the estimate in

(2.12) with $\mathrm{e}^{\Lambda t}$ and integrating over an interval $[r,s]$, for $s > r \geq 0$, gives the equivalent *Integrated Evolutionary Variational Estimate* (IEVE)

$$\mathrm{e}^{\Lambda(s-r)}\mathcal{R}_\varepsilon(u_\varepsilon(s)-w) - \mathcal{R}_\varepsilon(u_\varepsilon(r)-w) \leq \mathsf{M}_\Lambda(s-r)\Big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u_\varepsilon(s))\Big) \quad (2.13)$$

for all $w \in X$ with $\mathsf{M}_\Lambda(\tau) = (\mathrm{e}^{\Lambda\tau}-1)/\Lambda$ for $\Lambda \neq 0$ and $\mathsf{M}_0(\tau) = \tau$, see also [17, Prop. 3.1]. Note, that this formulation is only written in terms of functionals and no derivatives appear.

We state the main result of this subsection on the evolutionary Γ-convergence of the gradient system $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ that can be found in [37]. Therein the result is formulated by assuming $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$ in X, however in the proof only $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ in X is used. Nevertheless this discrepancy is not relevant for our application to the Cahn–Hilliard equation. Note that the following theorem is a variant of [18, Thm. 2.17], see also [36].

Theorem 2.5. [37, Thm. 3.2] *Let $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ satisfy the equi-coercivity conditions in (2.5) and assume that $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ and $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{C}} \mathcal{R}_0$ in X. Assume moreover that the convexity property in (2.9) is satisfied and that the initial conditions are such that $u_\varepsilon(0) \to u(0)$ in X with $u(0) \in \overline{\mathrm{dom}(\mathcal{E}_0)}^X$. Then, $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ E-converges to $(X, \mathcal{E}_0, \mathcal{R}_0)$ and the limit $t \mapsto u(t)$ satisfies for all $t > 0$, $w \in X$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_0(u(t)-w) + \Lambda\mathcal{R}_0(u(t)-w) \leq \mathcal{E}_0(w) - \mathcal{E}_0(u(t)). \quad (2.14)$$

Moreover, for each $t \in (0,T]$ the energies converge, i.e. $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u(t))$.

2.3. A convergence result for the energy-dissipation principle

In this section, we establish the second approach for E-convergence based on the energy-dissipation principle in (1.2). Indeed, the latter gives an equivalent formulation of (2.2) if the chain rule (2.4) is satisfied. The crucial point is that for general convex potentials $\Psi : X \to [0,\infty]$ the Legendre–Fenchel equivalences hold, namely

$$v \in X,\, \xi \in X^* : \quad \xi \in \partial\Psi(v) \;\Leftrightarrow\; v \in \partial\Psi^*(\xi) \;\Leftrightarrow\; \Psi(v) + \Psi^*(\xi) \leq \langle \xi, v\rangle.$$

Hence, assuming that $u_\varepsilon \in \mathrm{H}^1(0,T;X)$ is a solution of the differential formulation (2.2) with respect to $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ we have $\mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}^*_\varepsilon(\xi_\varepsilon) \leq \langle \xi_\varepsilon, \dot u_\varepsilon\rangle$ a.e. in $[0,T]$, where $\xi_\varepsilon \in \mathrm{L}^2(0,T;X^*)$ satisfies $\xi_\varepsilon(t) \in \partial_X\mathcal{E}_\varepsilon(u_\varepsilon(t))$ for a.a. $t \in [0,T]$. Using the chain rule (2.4) we obtain the energy-dissipation principle (EDP) after integrating over $[0,T]$

$$\mathcal{E}_\varepsilon\big(u_\varepsilon(T)\big) + \int_0^T \mathcal{R}_\varepsilon\big(\dot u_\varepsilon(s)\big) + \mathcal{R}^*_\varepsilon\big(\xi_\varepsilon(s)\big)\,\mathrm{d}s \leq \mathcal{E}_\varepsilon\big(u_\varepsilon(0)\big),$$
$$\xi_\varepsilon(t) \in \partial_X\mathcal{E}_\varepsilon\big(u_\varepsilon(t)\big). \quad (2.15)$$

Conversely, if (2.15) is satisfied we easily check that u_ε also solves the differential formulation (2.2) (see e.g. [36, Thm. 3.2]). Moreover, note that estimate (2.15) is in fact an equality. Indeed, by the elementary estimate

$\mathcal{R}_\varepsilon(v)+\mathcal{R}^*_\varepsilon(\xi)\geq\langle\xi,v\rangle$ and the chain rule (2.4), we obtain

$$\begin{aligned}&\text{if } \widehat{u}\in \mathrm{H}^1(0,T;X),\ \widehat{\xi}\in \mathrm{L}^2(0,T;X^*),\ \widehat{\xi}(t)\in\partial_X\mathcal{E}_\varepsilon(\widehat{u}(t)) \text{ for a.a. } t\in[0,T],\\ &\text{then } \mathcal{E}_\varepsilon\big(\widehat{u}(t)\big)+\textstyle\int_s^t\mathcal{R}_\varepsilon\big(\dot{\widehat{u}}\big)+\mathcal{R}^*_\varepsilon\big(\widehat{\xi}\big)\,\mathrm{d}r\geq\mathcal{E}_\varepsilon(\widehat{u}(s)) \text{ for all } 0\leq s<t\leq T.\end{aligned}\tag{2.16}$$

The following result, being a slight variation of [36, Thm. 3.3 & 3.6], based on (2.15) is in the spirit of Sandier & Serfaty's approach [47,49] (see Sect. 4 for a comparison). Note that in contrast to the subsequent section, we do not require any convexity properties of $\mathcal{E}_\varepsilon$ and the continuous convergence of $\mathcal{R}_\varepsilon$ to $\mathcal{R}_0$ can be relaxed to strong Γ-convergence. However, we have to impose additionally well-preparedness of the initial conditions and a closedness condition on the subdifferential of $\mathcal{E}_\varepsilon$ to be able to identify the limit formulation. The latter is formulated such that it fits into our general setting and can weakened in more concrete situations, see e.g. Proposition 2.7. The novelty of the following proof is to use time-discretizations for the solutions and Jensen's inequality in order to derive $\liminf_{\varepsilon\to0}\int_0^T\mathcal{R}_\varepsilon(\dot{u}_\varepsilon)\,\mathrm{d}t\geq\int_0^T\mathcal{R}_0(\dot{u})\,\mathrm{d}t$ although $\mathcal{R}_\varepsilon\xrightarrow{\Gamma}\mathcal{R}_0$ *strongly* and $\dot{u}_\varepsilon\rightharpoonup\dot{u}$ *weakly* in X, only.

Theorem 2.6. *Let $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ satisfy the assumptions (2.5) and (2.6) on equi-coercivity and Γ-convergence. Moreover, we assume that the initial conditions are well-prepared, i.e.*

$$u_\varepsilon(0)\to u(0) \ \text{in } X \quad\text{and}\quad \mathcal{E}_\varepsilon(u_\varepsilon(0))\to\mathcal{E}_0(u(0))<\infty,\tag{2.17}$$

and that the subdifferential $\partial_X\mathcal{E}_\varepsilon$ is closed in the sense

$$\left.\begin{array}{c}\widehat{u}_\varepsilon\overset{*}{\rightharpoonup}\widehat{u}\ \text{in } \mathrm{L}^\infty(0,T;Z),\ \widehat{u}_\varepsilon\rightharpoonup\widehat{u}\ \text{in } \mathrm{H}^1(0,T;X),\\ \widehat{\xi}_\varepsilon\rightharpoonup\widehat{\xi}\ \text{in } \mathrm{L}^2(0,T;X^*),\\ \widehat{\xi}_\varepsilon(t)\in\partial_X\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon(t))\ \text{f.a.a. } t\in[0,T]\end{array}\right\}\Rightarrow\begin{array}{c}\text{f.a.a. } t\in[0,T]:\\ \widehat{\xi}(t)\in\partial_X\mathcal{E}_0(\widehat{u}(t)).\end{array}\tag{2.18}$$

Then, we have the well-prepared E-convergence of $(X,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ to $(X,\mathcal{E}_0,\mathcal{R}_0)$. In particular, the limit $t\mapsto u(t)$ satisfies

$$\mathcal{E}_0(u(T))+\int_0^T\mathcal{R}_0\big(\dot{u}(t)\big)+\mathcal{R}^*_0\big(\xi(t)\big)\,\mathrm{d}t\leq\mathcal{E}_0(u(0)),\quad \xi(t)\in\partial_X\mathcal{E}_0\big(u(t)\big).\tag{2.19}$$

Moreover, for each $t\in[0,T]$ the energies converge, i.e. $\mathcal{E}_\varepsilon(u_\varepsilon(t))\to\mathcal{E}_0(u(t))$.

Proof. Step 1. Uniform bounds. Using the well-preparedness of the initial conditions (2.17), we find a constant $C>0$ such that $\mathcal{E}_\varepsilon(u_\varepsilon(0))\leq C$. Since the energy-dissipation estimate (2.15) is satisfied we immediately get $\int_0^T\mathcal{R}_\varepsilon(\dot{u}_\varepsilon)+\mathcal{R}^*_\varepsilon(\xi_\varepsilon)\,\mathrm{d}t\leq C$ such that by the uniform coercivity of $\mathcal{R}_\varepsilon$ and $\mathcal{R}^*_\varepsilon$ we obtain uniform bounds for $\|\dot{u}_\varepsilon\|_{\mathrm{L}^2(0,T;X)}$ and $\|\xi_\varepsilon\|_{\mathrm{L}^2(0,T;X^*)}$.

Moreover, the upper bound (2.16) holds for the time-reversed curve $\widehat{u}_\varepsilon(t)=u_\varepsilon(T-t)$. Due to the invariance of the dissipation potentials with respect to this transformation we obtain for $t=T$

$$\mathcal{E}_\varepsilon(u_\varepsilon(0))+\int_0^T\mathcal{R}_\varepsilon(\dot{u}_\varepsilon)+\mathcal{R}^*_\varepsilon(\xi_\varepsilon)\,\mathrm{d}r\geq\mathcal{E}_\varepsilon(u_\varepsilon(T-s)).$$

Thus, the coercivity (2.5), the well-preparedness (2.17) and the uniform bound for the total dissipation imply $\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_Z \le C$. In particular, we have shown the uniform a priori bounds

$$\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;Z)} + \|u_\varepsilon\|_{\mathrm{H}^1(0,T;X)} + \|\xi_\varepsilon\|_{\mathrm{L}^2(0,T;X^*)} \le C. \tag{2.20}$$

Step 2. Convergent subsequence. Due to (2.20) we can extract a converging subsequence (not relabeled) giving

$$\begin{aligned} &u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } \mathrm{L}^\infty(0,T;Z), \quad u_\varepsilon \rightharpoonup u \text{ in } \mathrm{H}^1(0,T;X), \text{ and} \\ &\xi_\varepsilon \rightharpoonup \xi \text{ in } \mathrm{L}^2(0,T;X^*). \end{aligned} \tag{2.21}$$

Moreover, by Arzelà–Ascoli's theorem and the compact embedding $Z \subset X$, we have

$$\forall\, t \in [0,T]: \quad u_\varepsilon(t) \rightharpoonup u(t) \text{ in } Z \text{ and } u_\varepsilon(t) \to u(t) \text{ in } X. \tag{2.22}$$

Step 3. Passing to the limit. We show that the limit u satisfies (2.19). The right-hand side in (2.15) converges because of the well-preparedness of the initial data. Moreover, from $u_\varepsilon(T) \to u(T)$ in X and $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ in X (cf. (2.6)), we obtain $\mathcal{E}_0(u(T)) \le \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(T))$. Thus, it remains to prove a lower estimate for the total dissipation, namely

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \ge \int_0^T \mathcal{R}_0(\dot u) + \mathcal{R}_0^*(\xi)\,\mathrm{d}t. \tag{2.23}$$

For this, let $0 = t_0^N < t_1^N < \cdots < t_N^N = T$ denote an equidistant partition of the interval $[0,T]$ with time step $\tau_N = T/N$, $N \in \mathbb{N}$. Then, Jensen's inequality yields

$$\begin{aligned} \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t &= \sum_{k=1}^N \int_{t_{k-1}^N}^{t_k^N} \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \\ &\ge \sum_{k=1}^N \tau_N \left\{ \mathcal{R}_\varepsilon\Big(\tfrac{1}{\tau_N}\textstyle\int_{t_{k-1}}^{t_k} \dot u_\varepsilon\,\mathrm{d}t\Big) + \mathcal{R}_\varepsilon^*\Big(\tfrac{1}{\tau_N}\textstyle\int_{t_{k-1}}^{t_k} \xi_\varepsilon\,\mathrm{d}t\Big)\right\}. \end{aligned} \tag{2.24}$$

For $k = 1,\dots,N$ we introduce $V_k^{N,\varepsilon} := (u_\varepsilon(t_k^N) - u_\varepsilon(t_{k-1}^N))/\tau_N \in X$ and $\Xi_k^{N,\varepsilon} := \frac{1}{\tau_N}\int_{t_{k-1}^N}^{t_k^N} \xi_\varepsilon\,\mathrm{d}s \in X^*$. Using $u_\varepsilon(t_k^N) \to u(t_k^N)$ in X and $\xi_\varepsilon \rightharpoonup \xi$ in $\mathrm{L}^2(0,T;X^*)$ we obtain

$$V_k^{N,\varepsilon} \to V_k^N := \frac{u(t_k^N) - u(t_{k-1}^N)}{\tau_N} \text{ in } X, \quad \Xi_k^{N,\varepsilon} \rightharpoonup \Xi_k^N := \frac{1}{\tau_N}\int_{t_{k-1}^N}^{t_k^N} \xi\,\mathrm{d}s \text{ in } X^*.$$

Hence, $\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0$ in X and $\mathcal{R}_\varepsilon^* \xrightharpoonup{\Gamma} \mathcal{R}_0^*$ in X^* (cf. (2.6) and (2.8)) yield the lower estimate

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \ge \sum_{k=1}^N \tau_N \left\{\mathcal{R}_0(V_k^N) + \mathcal{R}_0^*(\Xi_k^N)\right\}. \tag{2.25}$$

Next, we aim to pass to the limit $N \to \infty$. Let $u_N \in \mathrm{H}^1(0,T;X)$ denote the piecewise affine interpolant such that $u_N(t_k^N) = u(t_k^N)$ and $\dot u_N(t) = V_k^N$ for $t \in (t_{k-1}^N, t_k^N]$. Moreover, we denote by $\xi_N \in \mathrm{L}^2(0,T;X^*)$ the piecewise constant interpolant satisfying $\xi_N(t) = \Xi_k^N$ for $t \in (t_{k-1}^N, t_k^N]$. We easily check that $u_N \rightharpoonup u$ in $\mathrm{H}^1(0,T;X)$ and $\xi_N \rightharpoonup \xi$ in $\mathrm{L}^2(0,T;X^*)$ such that by Ioffe's lower semicontinuity result [27], we are able to pass to the limit $N \to \infty$ in (2.25) and finally arrive at

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \geq \int_0^T \mathcal{R}_0(\dot u) + \mathcal{R}_0^*(\xi)\,\mathrm{d}t.$$

By the closedness of the subdifferentials (2.18), we immediately have $\xi(t) \in \partial_X \mathcal{E}_0(u(t))$ for a.a. $t \in [0,T]$. Thus, we have shown that u solves the limiting energy-dissipation formulation (2.19).

Step 4. Convergence of the energies. Recalling the derivation of (2.15) resp. (2.19) via the chain rule, we indeed have equality in (2.19) on each time interval. Since we have the convergence of the initial energies $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0))$ by (2.17), the lim inf-estimate derived in Step 3 must actually attain a limit. Hence, we have for all $t \in [0,T]$

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u(t)) \text{ and } \int_0^t \mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \to \int_0^t \mathcal{R}_0(\dot u) + \mathcal{R}_0^*(\xi)\,\mathrm{d}t.$$

Thus, we have established the well-prepared E-convergence of $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$. ☐

Note, that the usual strong-weak closedness of the graph of the subdifferential $\partial_X \mathcal{E}_\varepsilon$ in the sense of

$$\left.\begin{array}{c} u_\varepsilon \to u \text{ in } X,\ \mathcal{E}_\varepsilon(u_\varepsilon) \to e_0, \\ \xi_\varepsilon \in \partial_X \mathcal{E}_\varepsilon(u_\varepsilon),\ \xi_\varepsilon \rightharpoonup \xi \text{ in } X^* \end{array}\right\} \Rightarrow e_0 = \mathcal{E}_0(u) \text{ and } \xi \in \partial_X \mathcal{E}_0(u) \qquad (2.26)$$

is in general not sufficient to conclude $\xi(t) \in \partial_X \mathcal{E}_0(u(t))$ for a.e. $t \in [0,T]$ since we only have weak convergence of ξ_ε in $\mathrm{L}^2(0,T;X^*)$. Hence, we need the stronger assumption (2.18) in Theorem 2.6. However, if we additionally assume that $\partial_X \mathcal{E}_0(u) \subset X^*$ is convex (e.g. if $\partial_X \mathcal{E}_0$ is the Fréchet-subdifferential or actually single-valued) it is indeed sufficient to impose (2.26).

Proposition 2.7. *Assume that for each $u \in X$ the subdifferential $\partial_X \mathcal{E}_0(u)$ is convex. Then, the strong-weak closedness of the graph of $\partial_X \mathcal{E}_\varepsilon$ in (2.26) implies (2.18).*

Proof. Let ξ_ε converge weakly in $\mathrm{L}^2(0,T;X^*)$ to ξ and $\xi_\varepsilon(t) \in \partial_X \mathcal{E}_\varepsilon(u_\varepsilon(t))$ for almost all $t \in [0,T]$. According to [45, Thm. 3.2] there exists a subsequence $\varepsilon_k \to 0$ and a family of Young measures μ_t on X^* (see e.g. [45, Def. 3.1]) such that $\xi(t) = \int_{X^*} \eta\, \mu_t(\mathrm{d}\eta)$ and μ_t is concentrated on the set

$$L(t) = \bigcap_{n=1}^{\infty} \overline{\{\xi_{\varepsilon_k}(t) \,|\, k \geq n\}}^{w} \subset X^*,$$

where the superscript w refers to the weak closure in X^*. Hence, the strong-weak closedness (2.26) implies $L(t) \subset \partial_X \mathcal{E}_0(u(t))$ for almost all t and the convexity of $\partial_X \mathcal{E}_0$ yields $\xi(t) \in \partial_X \mathcal{E}_0(u(t))$. ☐

Finally, let us remark that in the Λ-convex setting of Sect. 2.2, condition (2.26) and hence also (2.18) are always satisfied.

Proposition 2.8. *Let* $u \mapsto \mathcal{E}_\varepsilon(u) - \Lambda\mathcal{R}_\varepsilon(u)$ *be convex,* $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ *in* X*, and* $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{C}} \mathcal{R}_0$ *in* X*. Then, the Fréchet-subdifferential* $\partial_{\mathrm{F}}\mathcal{E}_\varepsilon$ *satisfies* (2.26).

Proof. The proof follows along the lines of [36, Prop. 2.9] and [6, Thm. 3.66]. Due to the quadratic structure of $\mathcal{R}_\varepsilon$ and the convexity of $\mathcal{E}_\varepsilon$ any element $\xi_\varepsilon \in \partial_{\mathrm{F}}\mathcal{E}_\varepsilon(u_\varepsilon)$ satisfies

$$\text{for all } w \in X: \quad \mathcal{E}_\varepsilon(w) \geq \mathcal{E}_\varepsilon(u_\varepsilon) + \langle \xi_\varepsilon, w - u_\varepsilon \rangle + \Lambda\mathcal{R}_\varepsilon(w - u_\varepsilon).$$

The strong Γ-convergence of $\mathcal{E}_\varepsilon$ implies: For arbitrarily fixed $\widehat{u} \in X$, there exists a sequence $\widehat{u}_\varepsilon$ such that $\widehat{u}_\varepsilon \to \widehat{u}$ in X and $\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \to \mathcal{E}_0(\widehat{u})$. Choosing $w = \widehat{u}_\varepsilon$ and passing to the limit $\varepsilon \to 0$, we obtain $\mathcal{E}_0(\widehat{u}) \geq e_0 + \langle \xi, \widehat{u} - u \rangle + \Lambda\mathcal{R}_0(\widehat{u} - u)$, where we also used that $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{C}} \mathcal{R}_0$. Setting $\widehat{u} = u$, yields $\mathcal{E}_0(u) \geq e_0$. Finally, we employ the lim inf-estimate for $u_\varepsilon \to u$ in X, which gives $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u)$, and hence we arrive at $e_0 = \mathcal{E}_0(u)$. Altogether, we have shown $\mathcal{E}_0(w) \geq \mathcal{E}_0(u) + \langle \xi, w - u \rangle + \Lambda\mathcal{R}_0(w - u)$ for all $w \in X$, and therefore, we conclude with $\xi \in \partial_{\mathrm{F}}\mathcal{E}_0(u)$. ☐

3. Homogenization of a Cahn–Hilliard-type equation

In this section we apply the two approaches established in Sect. 2 to derive homogenization limits of a Cahn–Hilliard-type equation with a microscopic and a macroscopic length scale. In the bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, we consider the fourth order equation written formally as

$$\partial_t u_\varepsilon = \operatorname{div}\big[M_\varepsilon(x)\nabla\big(\partial_u W_\varepsilon(x, u_\varepsilon) - \operatorname{div}(A_\varepsilon(x)\nabla u_\varepsilon)\big)\big]. \tag{3.1}$$

subject to the usual homogeneous Neumann boundary conditions for u and the thermodynamic driving force (also called chemical potential) ξ, namely $A_\varepsilon(x)\nabla u \cdot \nu = 0$ and $M_\varepsilon(x)\nabla\xi \cdot \nu = 0$ with ν denoting the unit outer normal vector to $\partial\Omega$. The multiple scales of the problem are encoded in the periodically oscillating tensors $M_\varepsilon : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $A_\varepsilon : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ as well as the potential $W_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ (see subsequent subsection).

Using Theorem 2.5 and Theorem 2.6, we show that solutions u_ε of the multiscale Cahn–Hilliard equation (3.1) converge in a suitable sense to a solution u of an effective equation that reads

$$\partial_t u = \operatorname{div}\big[M_{\mathrm{eff}}(x)\nabla\big(\partial_u W_{\mathrm{eff}}(x, u) - \operatorname{div}(A_{\mathrm{eff}}(x)\nabla u)\big)\big]. \tag{3.2}$$

with M_{eff}, A_{eff}, and W_{eff} being effective (homogenized) quantities, see Propositions 3.3 and 3.6 in Sect. 3.3 for the precise definition.

3.1. Notation and assumptions

In this subsection, we introduce the notation and the assumptions on the given data, that we will use in the subsequent sections to apply the abstract results from Sect. 2. Let us remark that we do not claim that these assumptions are sufficient to prove existence of solutions. In fact, our basic assumption is that solutions of the Cahn–Hilliard equation (3.1) always exist (see Definition 3.2 for the precise notion of solution). We refer to [1,21,25,26] and the survey article [44] for results in this direction.

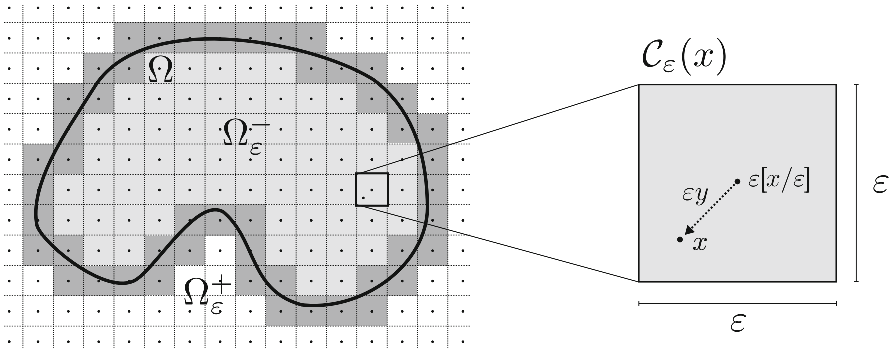

Figure 1. Covering of the domain Ω with microscopic cells. The light gray region contains all points in Ω_ε^-. The dark gray region depicts all points in $\Omega_+^\varepsilon \setminus \Omega_\varepsilon^-$

Following [34], we denote by $\mathcal{Y} = \mathbb{R}^d/_{\mathbb{Z}^d}$ the torus (also called periodicity cell), which can also be obtained by identifying the opposite faces of the unit cell $Y = [-\frac{1}{2}, \frac{1}{2})^d$. For a given point $x \in \Omega$, we define $[\![x/\varepsilon]\!] \in \mathbb{Z}^d$ as the lattice point closest to $x/\varepsilon \in \mathbb{R}^d$. Thus, we can decompose any $x \in \Omega$ via $x = \varepsilon([\![x/\varepsilon]\!] + y)$ into the macroscopic center $\varepsilon[\![x/\varepsilon]\!]$ and the fine-scale part $y = x/\varepsilon - [\![x/\varepsilon]\!] \in \mathcal{Y}$ of the microscopic cell $\mathcal{C}_\varepsilon(x) = \varepsilon([\![x/\varepsilon]\!] + Y) \subset \mathbb{R}^d$. We emphasize that $\mathcal{C}_\varepsilon(x)$ is in general not fully contained in Ω. In particular, we introduce the sets

$$\Omega_\varepsilon^- = \operatorname{int}\big(\{x \in \Omega \,|\, \mathcal{C}_\varepsilon(x) \subset \Omega\}\big) \quad \text{and} \quad \Omega_\varepsilon^+ = \operatorname{int}\big(\{x \in \mathbb{R}^d \,|\, \Omega \cap \mathcal{C}_\varepsilon(x) \neq \emptyset\}\big)$$

such that $\Omega_\varepsilon^- \subset \Omega \subset \Omega_\varepsilon^+$, see Fig. 1. Obviously, the set Ω_ε^+ is contained in an ε-neighborhood of Ω.

We are given two-scale tensors $\mathbb{M}$ and $\mathbb{A}$ which are elements of the space $\mathrm{L}^\infty(\Omega\times\mathcal{Y}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, symmetric, and uniformly elliptic with respect to all $(x, y) \in \Omega\times\mathcal{Y}$, i.e.

$$\exists\, \alpha,\, \beta > 0,\ \forall\, \eta \in \mathbb{R}^d: \quad \begin{cases} \alpha|\eta|^2 \leq \eta\cdot\mathbb{M}(x,y)\eta \leq \beta|\eta|^2, \\ \alpha|\eta|^2 \leq \eta\cdot\mathbb{A}(x,y)\eta \leq \beta|\eta|^2. \end{cases} \tag{3.3}$$

With $\mathbb{M}$ and $\mathbb{A}$ we then define $M_\varepsilon \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $A_\varepsilon \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ via

$$\begin{aligned} &M_\varepsilon(x) := \widehat{M}_\varepsilon(x, x/\varepsilon) \quad \text{and} \quad A_\varepsilon(x) := \widehat{A}_\varepsilon(x, x/\varepsilon), \text{ where} \\ &\widehat{M}_\varepsilon(x,y) := \begin{cases} \displaystyle\fint_{\mathcal{C}_\varepsilon(x)} \mathbb{M}(z,y)\,\mathrm{d}z & \text{if } x \in \Omega_\varepsilon^-, \\ \alpha\mathbb{I} & \text{otherwise,} \end{cases} \quad \text{and} \\ &\widehat{A}_\varepsilon(x,y) := \begin{cases} \displaystyle\fint_{\mathcal{C}_\varepsilon(x)} \mathbb{A}(z,y)\,\mathrm{d}z & \text{if } x \in \Omega_\varepsilon^-, \\ \alpha\mathbb{I} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.4}$$

Here, x/ε as second argument is understood modulo 1 in each component and $\mathbb{I}$ denotes the identity tensor in $\mathbb{R}^{d\times d}$. Since $\mathbb{M}$ and $\mathbb{A}$ satisfy (3.3) for all $(x, y) \in \Omega \times \mathcal{Y}$, it is immediate that M_ε and A_ε satisfy the same estimates in (3.3) uniformly with respect to $\varepsilon > 0$ and all $x \in \Omega$. In particular, the extension with $\alpha > 0$ guarantees the uniform ellipticity up to the boundary of Ω.

Finally, for a prescribed two-scale potential $\mathbb{W} : \Omega \times \mathcal{Y} \times \mathbb{R} \to [0, \infty)$ we introduce its macroscopic counterpart $W_\varepsilon : \Omega \times \mathbb{R} \to [0, \infty)$ via

$$W_\varepsilon(x, u) := \widehat{W}_\varepsilon(x, x/\varepsilon, u) \quad \text{with} \quad \widehat{W}_\varepsilon(x, y, u) := \fint_{\mathcal{C}_\varepsilon(x)} \mathbb{W}_{\mathrm{ex}}(z, y, u)\, \mathrm{d}z, \tag{3.5}$$

for all $u \in \mathbb{R}$, where for $\mathbb{F} \in \mathrm{L}^1(\Omega\times\mathcal{Y})$ the function $\mathbb{F}_{\mathrm{ex}} \in \mathrm{L}^1(\mathbb{R}^d\times\mathcal{Y})$ denotes the extension by 0 on $(\mathbb{R}^d\backslash\Omega) \times \mathcal{Y}$.

The potential $\mathbb{W} : \Omega \times \mathcal{Y} \times [a, b] \to [0, \infty)$ is a Carathéodory function, i.e. for all $u \in [a, b]$ the function $(x, y) \mapsto \mathbb{W}(x, y, u)$ is measurable and for a.e. $(x, y) \in \Omega \times \mathcal{Y}$ the function $u \mapsto \mathbb{W}(x, y, u)$ is continuous on $[a, b]$. If $u \notin [a, b]$ we set $\mathbb{W}(x, y, u) := +\infty$. Moreover, we make the following assumptions on growth and uniform continuity. Let $\mathbb{W}$ satisfy uniformly for all $(x, y) \in \Omega \times \mathcal{Y}$

Growth condition:
$\exists\, C_W \geq 0, \forall\, u \in [a, b] : \quad |\mathbb{W}(x, y, u)| \leq C_W(1 + |u|^p),$ (3.6a)
where $p < 2^*$ and $2^* \in [1, \infty)$ for $d = 1, 2$ and $2^* = \frac{2d}{d-2}$, for $d \geq 3$;

Uniform modulus of continuity:
$\exists\, \omega \in \mathrm{C}(\mathbb{R}; [0, \infty))$ with $\omega(\bar{u}) \to 0$ for $\bar{u} \to 0$, $\forall\, u_1, u_2 \in [a, b]$: (3.6b)
$|\mathbb{W}(x, y, u_1) - \mathbb{W}(x, y, u_2)| \leq \omega(|u_1 - u_2|)$.

Observe that for p as in (3.6a), the space $\mathrm{H}^1(\Omega)$ is compactly embedded in $\mathrm{L}^p(\Omega)$. The assumptions (3.3)–(3.6) suffice to prove the Γ-convergence of the energies $\mathcal{E}_\varepsilon$ in the weak topology of $\mathrm{H}^1(\Omega)$ (see Proposition 3.6).

Remark 3.1. *Note that the usual ansatz $A_\varepsilon(x) = \mathbb{A}(x, x/\varepsilon)$ for the oscillation coefficients is not well-defined for a general function $\mathbb{A} \in \mathrm{L}^\infty(\Omega\times\mathcal{Y}; \mathbb{R}^{d\times d})$ since $\{(x, x/\varepsilon) \in \mathbb{R}^d \times \mathcal{Y}\}$ has null Lebesgue measure. Hence, we are averaging on the microscopic cells $\mathcal{C}_\varepsilon$ with respect to the macroscopic variable x.*

Let us remark that by assuming for all u that $(x, y) \mapsto \mathbb{W}(x, y, u) \in \mathrm{C}(\overline{\Omega}\times\mathcal{Y})$ we can set $W_\varepsilon(x, u) := \mathbb{W}(x, x/\varepsilon, u)$, which would allow us to drop the assumption in (3.6b) and make some of the following proofs more straightforward. However, we want to deal with macroscopic heterostructures and hence, we consider the more general case (see also Remark 2.14 in [34]).

3.2. Gradient structure of the Cahn–Hilliard equation

The gradient structure of the Cahn–Hilliard equation in (3.1) respective (3.2) is well-known (cf. [1,7,26,28,49]). However, in this section we recall its definition within the framework described in Sect. 2. We allow for $\varepsilon \in [0, 1]$ and we identify with $\varepsilon = 0$ the effective quantities M_{eff}, A_{eff}, and W_{eff}.

Obviously, the Cahn–Hilliard equation leaves the average $\fint_\Omega u(t, x)\, \mathrm{d}x$ constant in time. Hence, given an initial value u_0 we set $\varrho := \fint_\Omega u_0(x)\, \mathrm{d}x$ and define the natural spaces

$$\mathrm{L}^2_\varrho(\Omega) := \big\{u \in \mathrm{L}^2(\Omega) \,|\, \fint_\Omega u(x)\, \mathrm{d}x = \varrho\big\} \quad \text{and} \quad \mathscr{Z}_\varrho := \mathrm{H}^1(\Omega) \cap \mathrm{L}^2_\varrho(\Omega). \tag{3.7}$$

The space $\mathcal{Z}_\varrho$ is an affine (and closed) subspace of $\mathrm{H}^1(\Omega)$. On $\mathcal{Z}_\varrho$ the driving functional $\mathcal{E}_\varepsilon : \mathcal{Z}_\varrho \to \mathbb{R}$ is given by the classical Allen–Cahn energy

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \left[\frac{1}{2}\nabla u \cdot A_\varepsilon(x)\nabla u + W_\varepsilon(x,u)\right] \mathrm{d}x. \tag{3.8}$$

We denote the linear space associated with $\mathcal{Z}_\varrho$ by $Z_0 = \mathrm{H}^1(\Omega) \cap \mathrm{L}^2_0(\Omega)$ such that $\mathcal{Z}_\varrho = \varrho + Z_0$. On Z_0 we introduce the (flat) Riemannian structure g_ε via

$$\forall\, v_1, v_2 \in Z_0 : \quad g_\varepsilon(v_1,v_2) = \int_\Omega \nabla\xi_{v_1} \cdot M_\varepsilon(x)\nabla\xi_{v_2}\,\mathrm{d}x,$$

$$\text{where } \xi_{v_i} \in \mathrm{H}^1(\Omega) \text{ is the unique solution of } -\operatorname{div}(M_\varepsilon(x)\nabla\xi_{v_i}) = v_i \text{ in } \Omega, \\ \text{satisfying } (M_\varepsilon(x)\nabla\xi_{v_i})\cdot\nu = 0 \text{ on } \partial\Omega \text{ and } ⨍_\Omega \xi_{v_i}(x)\,\mathrm{d}x = 0. \tag{3.9}$$

Assuming that M_ε is symmetric and positive definite, g_ε clearly defines a scalar product on Z_0. We denote the closure of Z_0 with respect to g with X_0 and easily verify that it is given via

$$X_0 := \left\{ v \in \mathrm{H}^1(\Omega)^* \mid \langle v, \mathbb{1}\rangle = 0 \right\}, \tag{3.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $\mathrm{H}^1(\Omega)^*$ and $\mathrm{H}^1(\Omega)$ and $\mathbb{1}$ is the constant function with value 1. On X_0 we define the (primal) dissipation potential via

$$\mathcal{R}_\varepsilon(v) = \frac{1}{2}\int_\Omega \nabla\xi_v \cdot M_\varepsilon(x)\nabla\xi_v\,\mathrm{d}x, \tag{3.11}$$

where $\xi_v \in \mathrm{H}^1(\Omega)$ is defined as in (3.9). By the usual embedding of $\mathrm{L}^2(\Omega)$ into $\mathrm{H}^1(\Omega)^*$ we have that Z_0 is densely and compactly embedded in X_0.

Let us remark that there are other choices for the space X_0, e.g. by considering $\xi \in \mathrm{H}^1(\Omega)/_\mathbb{R}$ and taking $(\mathrm{H}^1(\Omega)/_\mathbb{R})^*$ as state space. However, the space $(\mathrm{H}^1(\Omega)/_\mathbb{R})^*$ is isomorph to X_0 which becomes clear by uniquely identifying an equivalence class in $\mathrm{H}^1(\Omega)/_\mathbb{R}$ with an element in $\xi \in \mathrm{H}^1_{\mathrm{av}}(\Omega) = \{\xi \in \mathrm{H}^1(\Omega) \mid ⨍_\Omega \xi\,\mathrm{d}x = 0\}$. As a consequence we identify X_0^* with the space $\mathrm{H}^1(\Omega)/_\mathbb{R}$ and consider the dual dissipation potential $\mathcal{R}^*_\varepsilon$ on X_0^*

$$\mathcal{R}^*_\varepsilon(\xi) = \frac{1}{2}\int_\Omega \nabla\xi \cdot M_\varepsilon(x)\nabla\xi\,\mathrm{d}x, \tag{3.12}$$

which obviously does not depend on the choice of a representative ξ for an equivalence class in $\mathrm{H}^1(\Omega)/_\mathbb{R}$. In particular, we define the map $P_0 : \mathrm{H}^1(\Omega) \to \mathrm{H}^1_{\mathrm{av}}(\Omega)$ via $P_0\xi = \xi - ⨍_\Omega \xi\,\mathrm{d}x$, which provides the canonical representative for ξ.

On X_0^* we define the norm $\|\eta\|_{X_0^*} = \|\nabla\eta\|_{\mathrm{L}^2}$, which induces the norm $\|v\|_{X_0} = \|\eta_v\|_{X_0^*}$ on X_0. In particular, we immediately obtain the following uniform estimates for all $\varepsilon \in [0,1]$, cf. (2.1),

$$\tfrac{1}{2\beta}\|v\|^2_{X_0} \le \mathcal{R}_\varepsilon(v) \le \tfrac{1}{2\alpha}\|v\|^2_{X_0} \quad \text{and} \quad \tfrac{\alpha}{2}\|\xi\|^2_{X_0^*} \le \mathcal{R}^*_\varepsilon(\xi) \le \tfrac{\beta}{2}\|\xi\|^2_{X_0^*}. \tag{3.13}$$

For arbitrary functions $u \in \mathrm{L}^2(0,T;\mathcal{Z}_\varrho)$ with $\dot u \in \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))^*)$, we have $0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\,\mathrm{d}x = \langle \dot u(t), \mathbb{1}\rangle$, i.e. $\dot u(t) \in X_0$ for almost every $t \in [0,T]$. Therefore, we can consider the projection $P_0(u) = u - \varrho\mathbb{1}$ onto the space

$\mathrm{L}^2(0,T;\mathcal{Z}_0)\cap \mathrm{H}^1(0,T;X_0)$. In particular, without loss of generality and for notational consistency with Sect. 2, we set $\varrho=0$ from now on and consider the function spaces

$$Z := \mathcal{Z}_0 \quad \text{and} \quad X := X_0. \tag{3.14}$$

Moreover, the driving functional $\mathcal{E}_\varepsilon$ is extended to the space X in the usual way by extending it with infinity outside of Z. We recall, that for $u\in X$ we denote by $\partial_{\mathrm{F}}^X\mathcal{E}_\varepsilon(u)\subset X^*$ the Fréchet subdifferential of $\mathcal{E}_\varepsilon$ at u with respect to X, which is given via the formula in (2.10).

A solution of the Cahn–Hilliard equation is understood in the following sense.

Definition 3.2. *Given an initial value $u_0\in Z$ we call a curve $t\mapsto u(t)\in X$ a solution of the multiscale Cahn–Hilliard equation (3.1), if it satisfies $0\in \mathrm{D}\mathcal{R}_\varepsilon(\dot u(t))+\partial_{\mathrm{F}}^X\mathcal{E}_\varepsilon\big(u(t)\big)$ in X^* for a.a. $t\in[0,T]$ with $u\in \mathrm{L}^\infty(0,T;Z)\cap \mathrm{H}^1(0,T;X)$ and $u(0)=u_0$.*

3.3. Γ-Convergence of the energy and dissipation functionals

The theory for homogenization problems is vast. Here, we use the notion of two-scale convergence, which was introduced in [41] and further developed in [2]. It provides a better description of sequences of oscillating functions and thus gives rise to the derivation of a new homogenization method. In [32], an overview of the main homogenization problems which have been studied by this technique is given. In particular, an important tool from two-scale homogenization, that we are going to use, is the *periodic unfolding operator*, see also [14,15,34]. The latter is defined as a mapping $\mathcal{T}_\varepsilon:\mathrm{L}^q(\Omega)\to\mathrm{L}^q(\mathbb{R}^d{\times}\mathcal{Y})$, for $1\leq q\leq\infty$, with

$$(\mathcal{T}_\varepsilon\, u)(x,y)=u_{\mathrm{ex}}(\varepsilon[\![\tfrac{x}{\varepsilon}]\!]+\varepsilon y), \tag{3.15}$$

where $u_{\mathrm{ex}}\in\mathrm{L}^q(\mathbb{R}^d)$ denotes as before the extension with 0 outside of Ω. The unfolding operator $\mathcal{T}_\varepsilon:\mathrm{L}^q(\Omega)\to\mathrm{L}^q(\mathbb{R}^d{\times}\mathcal{Y})$ is linear, continuous, and norm preserving. For $u_\varepsilon\to u$ in $\mathrm{L}^q(\Omega)$, we obtain $\mathcal{T}_\varepsilon\, u_\varepsilon\to Eu$ in $\mathrm{L}^q(\mathbb{R}^d{\times}\mathcal{Y})$, where $E:\mathrm{L}^p(\Omega)\to\mathrm{L}^p(\mathbb{R}^d{\times}\mathcal{Y})$ denotes the canonical embedding via $(Eu)(x,y):=u_{\mathrm{ex}}(x)$, see e.g. [34, Prop. 2.4].

The Γ-convergence of the dual dissipation potentials $\mathcal{R}_\varepsilon^*:X^*\to[0,\infty)$ (cf. (3.12)) in the weak topology of X^* is well-known. Below, we give a proof based on the periodic unfolding method.

Proposition 3.3. *The dual dissipation potentials $\mathcal{R}_\varepsilon^*:X^*\to[0,\infty)$ Γ-converge in the weak topology of X^* to the limit potential $\mathcal{R}_0^*:X^*\to[0,\infty)$ given via*

$$\mathcal{R}_0^*(\xi)=\frac{1}{2}\int_\Omega \nabla\xi\cdot M_{\mathrm{eff}}(x)\nabla\xi\,\mathrm{d}x,$$

where the effective mobility is given via the cell minimization problem

$$\eta\cdot M_{\mathrm{eff}}(x)\eta=\min_{\phi\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})}\fint_{\mathcal{Y}}(\nabla_y\phi+\eta)\cdot\mathbb{M}(x,y)(\nabla_y\phi+\eta)\,\mathrm{d}y. \tag{3.16}$$

Proof. Let us remark that the spaces $\mathrm{L}^q(\Omega\times\mathcal{Y})$ and $\mathrm{L}^q(\Omega\times Y)$ for $1 \le q \le \infty$ can be identified in the definition of $\mathcal{T}_\varepsilon$, whereas $\mathrm{H}^1(Y)$ and $\mathrm{H}^1(\mathcal{Y})$ clearly cannot. We make use of the following properties of $\mathcal{T}_\varepsilon$, cf. [34, Sect. 2]: Let $1 \le q_1, q_2 \le \infty$ such that $1/q_1 + 1/q_2 = 1/r \le 1$, then $\mathcal{T}_\varepsilon$ satisfies

$$\begin{array}{ll} \textit{product rule:} & \mathcal{T}_\varepsilon(g_1 g_2) = \mathcal{T}_\varepsilon(g_1)\,\mathcal{T}_\varepsilon(g_2) \in \mathrm{L}^r(\mathbb{R}\times\mathcal{Y}) \quad \forall\, g_i \in \mathrm{L}^{q_i}(\Omega), \\ \textit{integral identity:} & \int_\Omega F(x)\,\mathrm{d}x = \int_{\mathbb{R}^d\times\mathcal{Y}} (\mathcal{T}_\varepsilon\, F)(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \forall\, F \in \mathrm{L}^1(\Omega). \end{array} \tag{3.17}$$

The Lipschitz condition for $\partial\Omega$ guarantees $\mathrm{vol}(\{x \in \Omega \,|\, \mathcal{C}_\varepsilon(x) \not\subset \overline{\Omega}\}) \to 0$ as $\varepsilon \to 0$, see [15]. With this, Lebesgue's differentiation theorem yields the pointwise convergence

$$(\mathcal{T}_\varepsilon\, M_\varepsilon)(x,y) \to \mathbb{M}_{\mathrm{ex}}(x,y) \text{ for a.a. } (x,y) \in \mathbb{R}^d \times \mathcal{Y}, \tag{3.18}$$

see e.g. [38, Prop. 5.2]. Thus, the boundedness of $\mathcal{T}_\varepsilon\, M_\varepsilon$ in (3.3) and Lebesgue's dominated convergence theorem yield the strong convergence $\mathcal{T}_\varepsilon\, M_\varepsilon \to \mathbb{M}_{\mathrm{ex}}$ in $\mathrm{L}^q(\mathbb{R}^d \times \mathcal{Y})$ for all $1 \le q < \infty$. We now prove the Γ-convergence of $\mathcal{R}^*_\varepsilon$ to $\mathcal{R}^*_0$ in two steps.

1. lim inf-estimate. Let $(\xi_\varepsilon)_\varepsilon \subset X^*$ be a sequence such that $\xi_\varepsilon \rightharpoonup \xi$ in X^*. According to [34, Thm. 2.8], there exists a subsequence (not relabeled) and a function $\Xi \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ such that $\mathcal{T}_\varepsilon \nabla\xi_\varepsilon \rightharpoonup E\nabla\xi + \nabla_y \Xi_{\mathrm{ex}} \in \mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})$. Using the integral identity and the product rule in (3.17) in the definition of $\mathcal{R}^*_\varepsilon$ (cf. (3.12)), we obtain

$$\mathcal{R}^*_\varepsilon(\xi_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathcal{Y}} (\mathcal{T}_\varepsilon\nabla\xi_\varepsilon)\cdot(\mathcal{T}_\varepsilon M_\varepsilon)(x,y)(\mathcal{T}_\varepsilon\nabla\xi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y.$$

With Ioffe's lower semicontinuity result [27] and (3.18), we arrive at the lower estimate

$$\liminf_{\varepsilon\to 0} \mathcal{R}^*_\varepsilon(\xi_\varepsilon) \ge \frac{1}{2}\int_{\mathbb{R}^d\times\mathcal{Y}} [E\nabla\xi + \nabla_y\Xi_{\mathrm{ex}}]\cdot\mathbb{M}_{\mathrm{ex}}(x,y)[E\nabla\xi + \nabla_y\Xi_{\mathrm{ex}}]\,\mathrm{d}x\,\mathrm{d}y.$$

Finally, we can minimize with respect to the microscopic fluctuations $\nabla_y\Xi$ (see Definition of M_{eff} in (3.16)) to get $\liminf_{\varepsilon\to 0}\mathcal{R}^*_\varepsilon(\xi_\varepsilon) \ge \mathcal{R}_0(\xi)$.

2. Recovery sequence. For given $\widehat{\xi} \in X^*$ and $x \in \Omega$, let $\Phi(x,\cdot)$ denote the unique minimizer for $\eta = \nabla\widehat{\xi}(x)$ in the unit cell problem (3.16). In particular, we easily verify that $\Phi \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$. Exploiting Proposition 2.9 in [34], we can find a sequence $(\widehat{\xi}_\varepsilon)_\varepsilon \subset \mathrm{H}^1_{\mathrm{av}}(\Omega)$ such that $\widehat{\xi}_\varepsilon \rightharpoonup \widehat{\xi}$ in X^* and $\mathcal{T}_\varepsilon\,\nabla\widehat{\xi}_\varepsilon \to E\nabla\widehat{\xi} + \nabla_y\Phi_{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})$. Therefore, with (3.18) we arrive at

$$\begin{aligned} \lim_{\varepsilon\to 0}\mathcal{R}^*_\varepsilon(\widehat{\xi}_\varepsilon) &= \lim_{\varepsilon\to 0}\frac{1}{2}\int_{\mathbb{R}^d\times\mathcal{Y}} (\mathcal{T}_\varepsilon\,\nabla\widehat{\xi}_\varepsilon)\cdot(\mathcal{T}_\varepsilon\,M_\varepsilon)(\mathcal{T}_\varepsilon\,\nabla\widehat{\xi}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathbb{R}^d\times\mathcal{Y}} [E\nabla\widehat{\xi} + \nabla_y\Phi_{\mathrm{ex}}]\cdot\mathbb{M}_{\mathrm{ex}}[E\nabla\widehat{\xi} + \nabla_y\Phi_{\mathrm{ex}}]\,\mathrm{d}x\,\mathrm{d}y = \mathcal{R}^*_0(\widehat{\xi}). \end{aligned}$$

Here, the last identity holds since Φ is a minimizer for the minimization problem in the definition of M_{eff}. The lim inf-estimate and the existence of a recovery sequence yield $\mathcal{R}^*_\varepsilon \xrightarrow{\Gamma} \mathcal{R}^*_0$ in X^*. □

Remark 3.4. *The unique minimizer* $\phi_\eta \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$ *of the cell problem* (3.16) *solves* $-\mathrm{div}_y(\mathbb{M}(x,y)(\nabla_y\phi_\eta+\eta)) = 0$ *in* $\mathcal{Y}$. *It is called* corrector *as it "corrects" the macroscopic behavior by taking the local fluctuations due to the microscopic structure into account.*

We note that there also exist homogenization results for functionals in the case of stochastic microstructures, see e.g. [5,9].

The following result is a direct consequence of the Γ-convergence of $\mathcal{R}^*_\varepsilon$ and the continuity properties of the Legendre transform with respect to Γ-convergence, see Proposition 2.4(b).

Corollary 3.5. *The primal dissipation potentials* $\mathcal{R}_\varepsilon : X \to [0,\infty)$ Γ*-converge in the strong topology of* X *to*

$$v \mapsto \mathcal{R}_0(v) = \mathcal{R}^*_0(\xi_v), \quad \textit{where } -\mathrm{div}(M_{\mathrm{eff}}(x)\nabla\xi_v) = v.$$

The Γ-convergence result for the driving functionals $\mathcal{E}_\varepsilon : Z \to \mathbb{R}$ in (3.8) reads as follows.

Proposition 3.6. *The family of driving functionals* $\mathcal{E}_\varepsilon$ Γ*-converges in the weak topology of* Z *to the limit functional*

$$\mathcal{E}_0(u) = \int_\Omega \left[\frac{1}{2}\nabla u\cdot A_{\mathrm{eff}}(x)\nabla u + W_{\mathrm{eff}}(x,u)\right]\mathrm{d}x,$$

where the effective quantities are given via

$$\eta\cdot A_{\mathrm{eff}}(x)\eta = \min_{\phi\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})} \fint_{\mathcal{Y}} (\nabla_y\phi+\eta)\cdot\mathbb{A}(x,y)(\nabla_y\phi+\eta)\,\mathrm{d}y, \quad \textit{and}$$
$$W_{\mathrm{eff}}(x,u) = \fint_{\mathcal{Y}} \mathbb{W}(x,y,u)\,\mathrm{d}y.$$

Proof. For each $\varepsilon\in[0,1]$, we split the family of energy functionals into $\mathcal{E}_\varepsilon = \mathcal{F}_\varepsilon + \mathcal{W}_\varepsilon$, where

$$\mathcal{F}_\varepsilon(u) = \frac{1}{2}\int_\Omega \nabla u\cdot A_\varepsilon(x)\nabla u\,\mathrm{d}x \quad \text{and} \quad \mathcal{W}_\varepsilon(u) = \int_\Omega W_\varepsilon(x,u)\,\mathrm{d}x.$$

Here, we write A_0 and W_0 for A_{eff} and W_{eff}, respectively. The convergence $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}_0$ in Z can be shown analogously to that of the dual dissipation potentials in Proposition 3.3.

Step 1: Let $[a,b]=\mathbb{R}$. It remains to prove the convergence of the lower order term $\mathcal{W}_\varepsilon(u_\varepsilon)\to\mathcal{W}_0(u)$ for arbitrary sequences $u_\varepsilon \rightharpoonup u$ in Z. Let $(u_\varepsilon)_\varepsilon \subset Z$ be such a sequence and define $U_\varepsilon = \mathcal{T}_\varepsilon u_\varepsilon$. Since Z embeds compactly into $\mathrm{L}^p_0(\Omega)$ for $p<2^*$ as in (3.6a), we have $u_\varepsilon \to u$ in $\mathrm{L}^p(\Omega)$ as well as $U_\varepsilon \to Eu$ in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$. Thus, there exists a subsequence (not relabeled) such that $U_\varepsilon(x,y)\to Eu(x,y)$ pointwise for a.a. $(x,y)\in\mathbb{R}^d\times\mathcal{Y}$. Therefore, exploiting the modulus of continuity in assumption (3.6b) gives for a.a. $(x,y)\in\mathbb{R}^d\times\mathcal{Y}$ the convergence

$$\begin{aligned} &\fint_{\mathcal{C}_\varepsilon(x)} \big|\mathbb{W}_{\rm ex}\big(z,y,U_\varepsilon(x,y)\big)-\mathbb{W}_{\rm ex}\big(z,y,Eu(x,y)\big)\big|\,\mathrm{d}z \\ &\quad\leq \omega\big(|U_\varepsilon(x,y)-Eu(x,y)|\big)\to 0. \end{aligned} \tag{3.19}$$

Moreover, Lebesgue's differentiation theorem yields for a.a. $(x,y)\in\mathbb{R}^d\times\mathcal{Y}$

$$\lim_{\varepsilon\to 0}\fint_{\mathcal{C}_\varepsilon(x)} \mathbb{W}_{\rm ex}\big(z,y,Eu(x,y)\big)\,\mathrm{d}z = \mathbb{W}_{\rm ex}\big(x,y,Eu(x,y)\big). \tag{3.20}$$

Using the integral identity (3.17) for $\mathcal{T}_\varepsilon$ and the definition of W_ε in (3.5) (see also [34, Eq. (2.16)]), we have

$$\int_\Omega W_\varepsilon\big(x,u_\varepsilon(x)\big)\,\mathrm{d}x = \int_{\mathbb{R}^d\times\mathcal{Y}}\fint_{\mathcal{C}_\varepsilon(x)} \mathbb{W}_{\rm ex}\big(z,y,U_\varepsilon(x,y)\big)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

Exploiting the pointwise convergences (3.19)–(3.20) as well as Lebesgue's dominated convergence theorem with the integrable (strongly in $\mathrm{L}^1(\mathbb{R}^d\times\mathcal{Y})$ converging) majorant $C_W(1+|U_\varepsilon(x,y)|^p)$, we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\times\mathcal{Y}}\fint_{\mathcal{C}_\varepsilon(x)} \mathbb{W}_{\rm ex}\big(z,y,U_\varepsilon(x,y)\big)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= \int_{\mathbb{R}^d\times\mathcal{Y}}\lim_{\varepsilon\to 0}\left\{\fint_{\mathcal{C}_\varepsilon(x)} \mathbb{W}_{\rm ex}(z,y,U_\varepsilon(x,y)\pm\mathbb{W}_{\rm ex}(z,y,Eu(x,y)\,\mathrm{d}z\right\}\mathrm{d}x\,\mathrm{d}y \\ &\quad= \int_{\mathbb{R}^d\times\mathcal{Y}} \mathbb{W}_{\rm ex}(x,y,Eu(x,u))\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega W_{\rm eff}(x,u(x))\,\mathrm{d}x, \end{aligned}$$

which finishes the proof in the case $[a,b]=\mathbb{R}$.

Step 2: Let $[a,b]\subsetneq\mathbb{R}$. For arbitrary sequences $u_\varepsilon\rightharpoonup u$ in Z, the lim inf-estimate for $\mathcal{E}_\varepsilon$ holds as follows. Subsequences $(u_{\varepsilon_k})_{\varepsilon_k}\subset(u_\varepsilon)_\varepsilon$ satisfying the constraint $u_{\varepsilon_k}(x)\in[a,b]$ for a.a. $x\in\Omega$ can be treated as in Step 1. And for all other subsequences $(u_{\varepsilon_l})_{\varepsilon_l}$, we have $W_{\varepsilon_l}(u_{\varepsilon_l})=+\infty$ and the lower estimate is immediately satisfied.

However, for the lim sup-estimate we have to guarantee that the recovery sequence $(\widehat{u}_\varepsilon)_\varepsilon$ for $\widehat{u}\in Z$ is lying in $[a,b]$. Without loss of generality let $a<0<b$. Following the construction in [34, Prop. 2.9], we set

$$\widehat{u}_\varepsilon(x) = \delta_\varepsilon\big(\widehat{u}(x)-m_\varepsilon\big)+\varepsilon U(t_\varepsilon,x,\tfrac{x}{\varepsilon}), \tag{3.21}$$

where $U(t,x,y)=\int_{\mathbb{R}^d}\int_{\mathcal{Y}}K(t,x-\tilde{x},y-\tilde{y})\widehat{U}_{\rm ex}(\tilde{x},\tilde{y})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}$ is a smooth convolution with $U(t,\cdot)\in C^\infty(\mathbb{R}^d\times\mathcal{Y})$ for $t>0$. Here, K is the heat kernel on $\mathbb{R}^d\times\mathcal{Y}$, $\widehat{U}\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\rm av}(\mathcal{Y}))$ is the solution of the cell problem for $\eta=\nabla\widehat{u}$ in (3.19), and $t_\varepsilon\to 0$ for $\varepsilon\to 0$. Moreover, we choose $\delta_\varepsilon\to 1$ and $m_\varepsilon\to 0$ accordingly to get $a<u_\varepsilon(x)<b$ for a.a. $x\in\Omega$ as well as $\mathcal{T}_\varepsilon(\nabla\widehat{u}_\varepsilon)\to E\nabla\widehat{u}+\nabla_y U_{\rm ex}$ in $\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})$. With this, we obtain $\mathcal{W}_\varepsilon(\widehat{u}_\varepsilon)\to\mathcal{W}_0(\widehat{u})$ as in Step 1. □

3.4. Convergence result based on (EVE)

In this section we prove the evolutionary Γ-convergence of the Cahn–Hilliard gradient systems $(X,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ to the effective system $(X,\mathcal{E}_0,\mathcal{R}_0)$ by relying on the convexity of $\mathcal{E}_\varepsilon$ with respect to $\mathcal{R}_\varepsilon$. In particular, the key assumption is

$$\exists\,\lambda \in \mathbb{R}\ \forall\,(x,y) \in \Omega{\times}\mathcal{Y}: \quad u \mapsto \mathbb{W}(x,y,u) - \frac{\lambda}{2}|u|^2 \quad \text{is convex.} \tag{3.22}$$

The next lemma shows that the λ-convexity of $\mathbb{W}$ implies Λ-convexity of the driving functionals $\mathcal{E}_\varepsilon$ with respect to $\mathcal{R}_\varepsilon$.

Lemma 3.7. *Let (3.22) be satisfied, then there exists $\Lambda \in \mathbb{R}$ such that $u \mapsto \mathcal{E}_\varepsilon(u) - \Lambda\mathcal{R}_\varepsilon(u)$ is convex.*

Proof. In this proof, we abbreviate $\mathrm{L}^2(\Omega)$ with L^2. It is easy to see that (3.22) yields the convexity of $u \mapsto \mathcal{E}_\varepsilon(u) - \frac{\lambda}{2}\|u\|^2_{\mathrm{L}^2} - \frac{\alpha}{2}\|\nabla u\|^2_{\mathrm{L}^2}$ with $\alpha > 0$ from (3.3). Namely, for $\theta \in [0,1]$ and $u_0, u_1 \in Z$ we have

$$\mathcal{E}_\varepsilon(u_\theta) \le (1{-}\theta)\mathcal{E}_\varepsilon(u_0) + \theta\mathcal{E}_\varepsilon(u_1) - \frac{\theta(1{-}\theta)}{2}\Big(\alpha\|\nabla(u_0{-}u_1)\|^2_{\mathrm{L}^2} + \lambda\|u_0{-}u_1\|^2_{\mathrm{L}^2}\Big),$$

where $u_\theta = (1{-}\theta)u_0 + \theta u_1$. Hence, it remains to show that we can find a constant $\Lambda \in \mathbb{R}$ such that the estimate $\Lambda\mathcal{R}_\varepsilon(v) \le \alpha\|\nabla v\|^2_{\mathrm{L}^2} + \lambda\|v\|^2_{\mathrm{L}^2}$ is satisfied for all $v \in Z$. Indeed, due to the embedding $Z \subset \mathrm{L}^2_0(\Omega) \subset X$ and Cauchy's estimate we obtain

$$\forall\,\delta > 0: \quad \|v\|^2_{\mathrm{L}^2} \le \delta\|\nabla v\|^2_{\mathrm{L}^2} + C_\delta\|v\|^2_X.$$

Here, we used Poincaré's inequality, i.e. $\|v\|_{\mathrm{L}^2} \le C_{\mathrm{P}}\|\nabla v\|^2_{\mathrm{L}^2}$ for all $v \in Z$.

Hence, in the case $\lambda = -\lambda_- < 0$ we fix $0 < \delta < \alpha/(\lambda_-)$ and choose $\Lambda \in \mathbb{R}$ such that $\Lambda \le -\lambda_- C_\delta/\alpha$, whereas for $\lambda \ge 0$ we simply set $\Lambda = 0$. With (3.13) it is now easy to see that $\mathcal{E}_\varepsilon - \Lambda\mathcal{R}_\varepsilon$ is convex. □

We now state the first homogenization result, namely the E-convergence of the multiscale Cahn–Hilliard system in the semiconvex case.

Theorem 3.8. *Let $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ be as before and let $u_\varepsilon(0) \to u(0)$ in X. Under the additional convexity assumption (3.22) the solutions u_ε of (3.1) weakly converge in Z for each $t \in [0,T]$, $T > 0$, to the unique solution of the effective Cahn–Hilliard equation (3.2). Moreover, for each $t \in (0,T]$ the energies converge, i.e. $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u(t))$.*

Proof. We aim to apply Theorem 2.5. For this it remains to show that $\mathcal{R}_\varepsilon(v_\varepsilon) \to \mathcal{R}_0(v)$ for $v_\varepsilon \to v$ strongly in X. Indeed, let a sequence $v_\varepsilon \to v$ strongly in X be given. Moreover, let $\xi_\varepsilon \in X^*$ be the sequence associated with v_ε via solving $-\operatorname{div}(M_\varepsilon\nabla\xi_\varepsilon) = v_\varepsilon$. By standard estimates, we obtain $\xi_\varepsilon \rightharpoonup \xi$ in X^* with ξ such that $-\operatorname{div}(M_{\mathrm{eff}}\nabla\xi) = v$ as in (3.9). Thus, we arrive at

$$\lim_{\varepsilon\to0}\mathcal{R}_\varepsilon(v_\varepsilon) = \frac{1}{2}\lim_{\varepsilon\to0}\langle v_\varepsilon, \xi_\varepsilon\rangle = \frac{1}{2}\langle v, \xi\rangle = \frac{1}{2}\int_\Omega \nabla\xi\cdot M_{\mathrm{eff}}\nabla\xi\,\mathrm{d}x = \mathcal{R}_0(v),$$

where we have used the strong-weak convergence in the duality product. □

3.5. Convergence results based on (EDP)

In this section we prove the E-convergence of the multiscale system $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ using the energy-dissipation principle (EDP) discussed in Sect. 2.3. In contrast to the previous section we drop the λ-convexity of the potential $\mathbb{W}$. Thus, it is in general not clear whether the chain rule in (2.4) holds, and we have to additionally assume it to be satisfied here.

Regardless of the convexity properties of the energy $\mathcal{E}_\varepsilon$, the (EDP) formulation requires in any case the well-preparedness of the initial conditions, viz. $\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon(0)) = \mathcal{E}_0(u(0)) < \infty$. Moreover, the application of Theorem 2.6 rests upon the closedness of the subdifferential $\partial_X\mathcal{E}_\varepsilon$ in the sense of (2.18). In the following two propositions we provide sufficient conditions on the potential $\mathbb{W}$ that guarantee the closedness. In the first proposition, we assume that the potential $\mathbb{W}$ is λ-convex as in (3.22).

Proposition 3.9. *Assume that the potential* $\mathbb{W}$ *is* λ*-convex as in* (3.22)*, then the closedness of the subdifferential* (2.18) *holds.*

Proof. In Lemma 3.7 and Theorem 3.8 it is shown that $u \mapsto \mathcal{E}_\varepsilon(u) - \Lambda\mathcal{R}_\varepsilon(u)$ is convex and $\mathcal{R}_\varepsilon \xrightarrow{\mathrm{C}} \mathcal{R}_0$ in X. Thus, the Propositions 2.7 and 2.8 yield the closedness (2.18). □

In the second proposition we replace the convexity assumption with a growth and continuity condition for the derivative of $\mathbb{W}$. In particular, in this case the energies are Fréchet differentiable on $\mathrm{H}^1(\Omega)$ with $\mathrm{D}\mathcal{E}_\varepsilon(u) = -\operatorname{div}(A_\varepsilon(x)\nabla u) + \partial_u W_\varepsilon(x,u)$. Moreover, the growth condition on $\partial_u\mathbb{W}$ implies that for $\mathbb{W}$ in (3.6a) with the same exponent. We recall that $P_0 : \mathrm{L}^1(\Omega) \to \mathrm{L}^1_0(\Omega)$ denotes the canonical projection with $P_0(\varphi) = \varphi - ⨍_\Omega \varphi\,\mathrm{d}x$.

Proposition 3.10. *Assume that* $\mathbb{W} : \Omega\times\mathcal{Y}\times\mathbb{R} \to \mathbb{R}$ *satisfies* $\mathbb{W}(x,y,\cdot) \in \mathrm{C}^1(\mathbb{R})$ *for all* $(x,y) \in \Omega\times\mathcal{Y}$ *as well as*

$$\begin{array}{l} \text{Growth condition:} \\ \exists\, C \geq 0,\ \forall\, u \in \mathbb{R}: \quad \big|\partial_u\mathbb{W}(x,y,u)\big| \leq C(1+|u|^{p-1}), \\ \textit{where } p < 2^* \textit{ and } 2^* \in [1,\infty) \textit{ for } d = 1,2 \textit{ and } 2^* = \frac{2d}{d-2},\ \textit{for } d \geq 3; \\ \text{Uniform modulus of continuity:} \\ \exists\, \widehat{\omega} \in \mathrm{C}(\mathbb{R};[0,\infty)) \textit{ with } \widehat{\omega}(\bar{u}) \to 0 \textit{ for } \bar{u} \to 0,\ \forall\, u_1,u_2 \in \mathbb{R}: \\ |\partial_u\mathbb{W}(x,y,u_1) - \partial_u\mathbb{W}(x,y,u_2)| \leq \widehat{\omega}(|u_1-u_2|). \end{array} \tag{3.23}$$

Then, $\mathcal{E}_\varepsilon$ *is Fréchet differentiable on* $\mathrm{H}^1(\Omega)$ *for all* $\varepsilon \in [0,1]$ *with* $\mathrm{D}\mathcal{E}_\varepsilon$ *denoting the differential. The Fréchet subdifferential of* $\mathcal{E}_\varepsilon$ *with respect to* X *is given via*

$$\partial^X_{\mathrm{F}}\mathcal{E}_\varepsilon(u) = \begin{cases} \big\{ P_0\left(\mathrm{D}\mathcal{E}_\varepsilon(u)\right) \big\} & \textit{if } \mathrm{D}\mathcal{E}_\varepsilon(u) \in \mathrm{H}^1(\Omega), \\ \emptyset & \textit{otherwise.} \end{cases} \tag{3.24}$$

Moreover, $\partial^X_{\mathrm{F}}\mathcal{E}_\varepsilon$ *satisfies the closedness condition in* (2.18)*.*

Proof. The Fréchet differentiability on $\mathrm{H}^1(\Omega)$ follows directly from the compact embedding $\mathrm{H}^1(\Omega) \subset \mathrm{L}^p(\Omega)$ and the continuity of the associated Nemytskii operator (for fixed ε)

$$\mathcal{N}_\varepsilon : \begin{cases} \mathrm{L}^p(\Omega) \to \mathrm{L}^{p'}(\Omega), \\ u \mapsto \partial_u W_\varepsilon\big(\cdot, u(\cdot)\big), \end{cases}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. The characterization of the subdifferential follows immediately.

It remains to verify the closedness of the Fréchet subdifferential $\partial^X_{\mathrm{F}}\mathcal{E}_\varepsilon$. Since $\partial^X_{\mathrm{F}}\mathcal{E}_\varepsilon$ is convex it is sufficient to prove the strong-weak closedness in X as in (2.26) according to Proposition 2.7. Hence, let us consider sequences

$u_\varepsilon \to u$ in X and $\xi_\varepsilon \rightharpoonup \xi$ in X^* satisfying $\mathcal{E}_\varepsilon(u_\varepsilon) \to e_0$ and $\xi_\varepsilon \in \partial_{\mathrm{F}}^X \mathcal{E}_\varepsilon(u_\varepsilon)$. We follow the lines of the proof of Proposition 3.6. Since the energies are uniformly bounded, we can extract a (non-relabeled) subsequence such that $u_\varepsilon \rightharpoonup u$ in Z and $u_\varepsilon \to u$ in $\mathrm{L}^p(\Omega)$ as well as $\mathcal{T}_\varepsilon \nabla u_\varepsilon \rightharpoonup E\nabla u + \nabla_y U_{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$ with $U \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$. Moreover, u_ε converges to u almost everywhere in Ω.

We consider a sequence $v_\varepsilon \rightharpoonup v$ in Z, which additionally satisfies the strong convergence $\mathcal{T}_\varepsilon \nabla v_\varepsilon \to E\nabla v + \nabla_y V_{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R} \times \mathcal{Y})$ for arbitrary but fixed $V \in \mathrm{L}(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$. Let us abbreviate $\xi_\varepsilon^W(x) = \partial_u W_\varepsilon(x, u_\varepsilon(x))$. Due to the assumptions in (3.23) we can argue as in the proof of Theorem 3.6 to deduce $\lim_{\varepsilon\to 0} \int_\Omega \xi_\varepsilon^W v_\varepsilon \,\mathrm{d}x = \int_\Omega \xi_{\mathrm{eff}}^W v \,\mathrm{d}x$, where $\xi_{\mathrm{eff}}^W(x) = \partial_u W_{\mathrm{eff}}(x, u(x))$. Moreover, using the integral identity for the unfolding operator we obtain

$$\langle \xi_\varepsilon, v_\varepsilon \rangle = \int_{\mathbb{R}^d \times \mathcal{Y}} (\mathcal{T}_\varepsilon \nabla v_\varepsilon) \cdot (\mathcal{T}_\varepsilon A_\varepsilon)(\mathcal{T}_\varepsilon \nabla u_\varepsilon) \,\mathrm{d}x \,\mathrm{d}y + \langle \xi_\varepsilon^W, v_\varepsilon \rangle. \tag{3.25}$$

Passing to the limit $\varepsilon \to 0$ in (3.25) yields

$$\langle \xi, v \rangle = \int_{\Omega \times \mathcal{Y}} [E\nabla v + \nabla_y V] \cdot \mathbb{A}[E\nabla u + \nabla_y U] \,\mathrm{d}x \,\mathrm{d}y + \langle \xi_{\mathrm{eff}}^W, v \rangle, \tag{3.26}$$

where we have used $v_\varepsilon \to v$ in X due to the compact embedding $Z \subset X$. We point out that v and V are arbitrary test functions in (3.26). On the one hand, we can set $v \equiv 0$ which gives $\int_{\Omega\times\mathcal{Y}} \nabla_y V \cdot \mathbb{A}[\nabla u + \nabla_y U] \,\mathrm{d}x \,\mathrm{d}y = 0$ for all $V \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$. Thus, U is the unique corrector function associated with u. Indeed, U solves the local problem $-\operatorname{div}_y(\mathbb{A}(x,y)[\nabla u + \nabla_y U]) = 0$ in $\mathcal{Y}$ for a.e. $x \in \Omega$. On the other hand, setting $V \equiv 0$ yields for all $v \in Z$

$$\begin{aligned} \langle \xi, v \rangle &= \int_{\Omega \times \mathcal{Y}} \nabla v \cdot \mathbb{A}[\nabla u + \nabla_y U] + \partial_u \mathbb{W}(u) v \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_\Omega \nabla v \cdot A_{\mathrm{eff}} \nabla u + \partial_u W_{\mathrm{eff}}(u) v \,\mathrm{d}x. \end{aligned}$$

Thus, we conclude that $\xi = \mathrm{D}\mathcal{E}_0(u)$ and $\xi \in \partial_{\mathrm{F}}^X \mathcal{E}_0(u)$.

Finally, it remains to show $\mathcal{E}_\varepsilon(u_\varepsilon) \to \mathcal{E}_0(u)$. For this, it suffices to prove the strong convergence $\mathcal{T}_\varepsilon \nabla u_\varepsilon \to E\nabla u + \nabla_y U_{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$. Indeed, using the uniform ellipticity of $\mathcal{T}_\varepsilon A_\varepsilon$ and (3.25) gives for $\Xi_\varepsilon = \mathcal{T}_\varepsilon(\nabla u_\varepsilon)$ and $\Xi = E\nabla u + \nabla_y U_{\mathrm{ex}}$

$$\begin{aligned} \alpha \|\Xi_\varepsilon - \Xi\|^2_{\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})} &\le \int_{\mathbb{R}^d \times \mathcal{Y}} (\Xi_\varepsilon - \Xi) \cdot \mathcal{T}_\varepsilon A_\varepsilon (\Xi_\varepsilon - \Xi) \,\mathrm{d}x \,\mathrm{d}y \\ &= \langle \xi_\varepsilon - \xi_\varepsilon^W, u_\varepsilon \rangle - \int_{\mathbb{R}^d \times \mathcal{Y}} \left[2\Xi_\varepsilon \cdot (\mathcal{T}_\varepsilon A_\varepsilon)\Xi - \Xi \cdot (\mathcal{T}_\varepsilon A_\varepsilon)\Xi \right] \mathrm{d}x \,\mathrm{d}y. \end{aligned}$$

Now, as the right-hand side vanishes for $\varepsilon \to 0$ using (3.26), we obtain the strong convergence $\Xi_\varepsilon \to \Xi$ in $\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$. □

Having collected all sufficient assumptions, we are now in the position to apply Theorem 2.6 to the homogenization of the Cahn–Hilliard equation. In particular, the assumptions $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ and $\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0$ in X are satisfied according to the Propositions 3.6 and 3.3.

Theorem 3.11. *Let $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ be as before. We assume that $u_\varepsilon(0) \to u(0)$ in X, the well-preparedness of the initial conditions, i.e. $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0)) < \infty$, the closedness condition (2.18), and the chain rule condition (2.4) are satisfied. Then, the solutions u_ε of (3.1) weakly converge in Z for each $t \in [0,T]$, $T > 0$, to a solution u of the effective Cahn–Hilliard equation (3.2). Moreover, we have $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u(t))$ for each $t \in [0,T]$.*

We complete this subsection by commenting on the well-preparedness condition.

Remark 3.12. (Choice of the initial conditions) *The well-preparedness (2.17) in Theorem 3.11 is satisfied for the following choice of initial values. For given $u(0) \in Z$, let $u_\varepsilon(0) \in Z$ be the unique solution of the elliptic problem: find $\widehat{u} \in Z$ such that*

$$\operatorname{div}\big(A_\varepsilon(x)\nabla\widehat{u}\big) = \operatorname{div}\big(A_{\text{eff}}(x)\nabla u(0)\big) \ \text{in } \Omega, \quad \big(A_\varepsilon(x)\nabla\widehat{u}\big)\cdot\nu = 0 \ \text{on } \partial\Omega.$$

Then, standard results in periodic homogenization yield $u_\varepsilon(0) \rightharpoonup u(0)$ in Z as well as $\int_\Omega \frac{1}{2}\nabla u_\varepsilon(0)\cdot A_\varepsilon \nabla u_\varepsilon(0)\,\mathrm{d}x \to \int_\Omega \frac{1}{2}\nabla u(0)\cdot A_{\text{eff}}\nabla u(0)\,\mathrm{d}x$, see e.g. [2]. Employing the compact embedding $Z \subset \mathrm{L}^p_0(\Omega)$ and treating the nonlinearity $\mathbb{W}$ as in Proposition 3.6, gives the desired convergence of the initial energies $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0))$.

In contrast, in the (EVE) formulation in Theorem 3.8, the choice of constant initial values $u_\varepsilon(0) \equiv u_0$ is admissible, since it is not necessary to "recover" the microstructure at $t = 0$. Nevertheless, the convergence of the energies follows for all later times $t > 0$.

3.6. Exemplary potentials

In this subsection, we collect three generic potentials as examples which are covered by our theory.

1. We consider the classical *double-well potential*

$$W_{\text{dw}}(u) = \tfrac{1}{4}(u^2-1)^2, \tag{3.27}$$

which satisfies the growth estimates in (3.6) and (3.23) for the dimensions $d = 1,2,3$ (see also [20,22]). Moreover, W_{dw} is λ-convex for all $\lambda \leq -1$.
To include different spatial scales in the potential we can consider two-scale functions $\Phi_1, \Phi_2 \in \mathrm{L}^\infty(\Omega\times\mathcal{Y})$ and set $\mathbb{W}_\Phi(x,y,u) = \Phi_1(x,y)W_{\text{dw}}(u) + \Phi_2(x,y)$, which also satisfies the assumptions (3.22)–(3.23). Moreover, for $\theta \in \mathrm{L}^\infty(\mathcal{Y})$ with $\theta \geq 0$, our multiscale analysis allows us to consider the variant

$$\mathbb{W}_\theta(y,u) = \tfrac{1}{4}\big(u^2-\theta(y)\big)^2,$$

where the minima are oscillating, i.e. $u_{\min}(x) = \pm(\theta(x/\varepsilon))^{1/2}$. In the limit $\varepsilon \to 0$ we obtain according to Proposition 3.6 the effective potential

$$W_{\text{eff}}(u) = \fint_Y \tfrac{1}{4}(u^2-\theta(y))^2\,\mathrm{d}y = \tfrac{1}{4}u^4 - \tfrac{1}{2}\theta_{\text{arith}}u^2 + \tfrac{1}{4}\fint_Y \theta(y)^2\,\mathrm{d}y,$$

where $\theta_{\text{arith}} = \fint_Y \theta(y)\,\mathrm{d}y$ denotes the arithmetic mean and the limiting minima are $u_{\min} = \pm(\theta_{\text{arith}})^{1/2}$. Concluding, the Theorems 3.8 and 3.11 are applicable for $\mathbb{W}_\Phi$ and $\mathbb{W}_\theta$.

2. Another well-known prototypical example is the *logarithmic potential*, cf. [1,12,16], given via

$$W_{\log}(u) = \begin{cases} (u-a)\log(u-a) + (b-u)\log(b-u) - \frac{\kappa}{2}u^2 & \text{if } u \in [a,b], \\ \infty & \text{else,} \end{cases} \tag{3.28}$$

with $a < b$ and $\kappa > 0$. Obviously, $W_{\log}$ is λ-convex for all $\lambda \leq -\kappa$. Hence, the Theorems 3.8 and 3.11 apply to $W_{\log}$. We refer to [1] for a characterization of the single-valued Fréchet subdifferential.
An interesting variation of (3.28) is to consider oscillating boundaries $a_\varepsilon(x) = a(x/\varepsilon)$ and $b_\varepsilon(x) = b(x/\varepsilon)$, where $a, b \in \mathrm{L}^\infty(\mathcal{Y})$ are given with $a_{\max} < b_{\min}$. However, it is an open problem to determine the effective limit domain $[a_0, b_0]$ for $\varepsilon \to 0$.
3. As a *nonconvex example* we consider the potential

$$W_\gamma(u) = \tfrac{1}{2}u^2 - \tfrac{1}{\gamma+1}|u|^{\gamma+1} \quad \text{with} \quad \gamma \in (\tfrac{1}{2}, 1). \tag{3.29}$$

The function W_γ satisfies the assumptions in (3.6) and (3.23) with $W_\gamma'(u) = u - |u|^{\gamma-1}u$. Indeed, W_γ' is globally γ-Hölder continuous as we have

$$\forall\, u_0, u_1 \in \mathbb{R}: \quad \big||u_0|^{\gamma-1}u_0 - |u_1|^{\gamma-1}u_1\big| \leq C_\gamma |u_0 - u_1|^\gamma,$$

where $C_\gamma = 1$, if $u_0 u_1 \geq 0$, and $C_\gamma = 2^{1-\gamma}$, if $u_0 u_1 < 0$. The latter follows from the concavity of $u \mapsto |u|^\gamma$ and choosing $\theta = 1/2$ for $u_\theta = (1-\theta)u_0 + \theta(-u_1)$.

However, the function W_γ is clearly not λ-convex for any $\lambda \in \mathbb{R}$ since $W_\gamma''(u) = 1 - \gamma|u|^{\gamma-1} \to -\infty$ for $|u| \to 0$. In particular, there exists no $\Lambda \in \mathbb{R}$ such that $u \mapsto \mathcal{E}(u) - \Lambda\mathcal{R}(u)$ is convex. To see this, we consider an arbitrary $\Lambda \in \mathbb{R}$ and set $\mathcal{F}_\Lambda(u) := \mathcal{E}(u) - \Lambda\mathcal{R}(u) = \mathcal{Q}_\Lambda(u) - \int_\Omega \frac{1}{\gamma+1}|u|^{\gamma+1}\,\mathrm{d}x$, where $\mathcal{Q}_\Lambda(u) := \int_\Omega \frac{1}{2}[\nabla u \cdot A\nabla u + u^2]\,\mathrm{d}x - \Lambda\mathcal{R}(u)$ comprises the quadratic terms. For smooth functions v, the second variation reads $\mathrm{D}^2\mathcal{F}_\Lambda(u)[v,v] = 2\mathcal{Q}_\Lambda(v) - \gamma\int_\Omega |u|^{\gamma-1}v^2\,\mathrm{d}x$ and for each $\Lambda \in \mathbb{R}$ we can find some $u \in Z$ such that $\mathrm{D}^2\mathcal{F}_\Lambda(u)[v,v] < 0$. Hence, the convexity condition (2.9) for the (EVE) formulation is violated and W_γ is a counterexample, for which Theorem 3.8 is not applicable.

However, we can still exploit the (EDP) formulation and apply Theorem 3.11 provided we can verify the chain rule (2.4). We refer to [45,46] for gradient formulations of non-convex driving functionals and the role of the chain rule. For our particular example, we drop the subscripts and write A for the tensors A_ε and A_{eff}, respectively, and prove the following theorem for $\mathcal{E} \equiv \mathcal{E}_\varepsilon$ with $\varepsilon \in [0,1]$. The proof can be found in Appendix A.

Theorem 3.13. *Assume that $\partial\Omega$ is of class C^2, $A \in \mathrm{W}^{1,\infty}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{spd}})$, and that W_γ is as in (3.29). Then, the Fréchet subdifferential (with respect to X) of the energy functional $\mathcal{E} : X \to \mathbb{R}_\infty$ is given by*

$$\partial^X_{\mathrm{F}}\mathcal{E}(u) = \begin{cases} \{-\operatorname{div}(A\nabla u) + P_0 W_\gamma'(u)\} & \textit{if } \operatorname{div}(A\nabla u) \in \mathrm{H}^1(\Omega) \textit{ and } (A\nabla u)\cdot\nu = 0 \textit{ on } \partial\Omega, \\ \emptyset & \textit{otherwise.} \end{cases} \tag{3.30}$$

Moreover, $\mathcal{E}$ satisfies the chain rule condition (2.4).

We conclude that the homogenization result in Theorem 3.11 is applicable.

4. Conclusion

We conclude our text with a comparison of the approaches for evolutionary Γ-convergence of gradient systems $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ in Sect. 2 based on the evolutionary variational estimate (EVE) and the energy-dissipation principle (EDP).

1. Both abstract results rely on the strong Γ-convergence of the energy functionals $\mathcal{E}_\varepsilon$ in X. Let us remark that we even have Mosco convergence of $\mathcal{E}_\varepsilon$ for the homogenization of the Cahn–Hilliard equation.
2. While the strong Γ-convergence of the dissipation potentials $\mathcal{R}_\varepsilon$ in X is sufficient for (EDP), we have to assume additionally continuous convergence in the (EVE) formulation. The latter is satisfied for the homogenization of Cahn–Hilliard-type equations in Sect. 3.
3. The initial values, which are assumed to converge strongly in X, have to be well-prepared in the (EDP) case, i.e. $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u(0))$. In particular, this means that $u_\varepsilon(0) \in \mathrm{dom}(\mathcal{E}_\varepsilon)$ has to hold for $\varepsilon \in [0,1]$ for (EDP) while (EVE) only requires $u_\varepsilon(0) \in \overline{\mathrm{dom}\, \mathcal{E}_\varepsilon}^X$.
4. The identification of the limit system in the (EDP) formulation relies on the closedness of the subdifferential $\partial_X \mathcal{E}_\varepsilon$ (see (2.18)), which is automatically satisfied for Λ-convex energy functionals.
5. The (EVE) formulation is based on the convexity of $\mathcal{E}_\varepsilon - \Lambda \mathcal{R}_\varepsilon$, which is always satisfied for λ-convex potentials $\mathbb{W}$ in the Cahn–Hilliard setting, see Lemma 3.7. Moreover, the Λ-convexity of $\mathcal{E}_\varepsilon$ implies many desirable properties of the gradient system, see e.g. [18,45]. In particular, the well-known double-well and logarithmic potentials W_{dw} and $W_{\log}$ fit into this setting. The (EDP) formulation allows us to consider also energy functionals that are not Λ-convex. In this case, the chain rule condition is not automatically satisfied and its verification may be cumbersome. For instance, the potential W_γ in (3.29) is not λ-convex, though the associated energy functional fulfills the chain rule, see Theorem 3.13.
6. Within the (EDB) formulation, it is possible to consider concentration-dependent (non-degenerating) mobilities $\mathbb{M}(x, y, u)$ satisfying the uniform ellipticity (3.3) and a uniform continuity as in (3.6b). However, these mobilities lead to state-dependent dissipation potentials $\mathcal{R}_\varepsilon(u, \dot{u})$ which in turn lead to non-trivial metric spaces such that the (EVE) formulation is not immediately applicable.

Let us remark that our approach is related to [36]. There, Theorem 3.6 gives an abstract E-convergence result based on (EDP). Note, however, that more general dissipation potentials are considered, which are also allowed to depend on the state u. However, there it is assumed that the dissipation potentials satisfy $\liminf_{\varepsilon \to 0} \mathcal{R}_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \mathcal{R}_0(u, v)$ for sequences $u_\varepsilon \to u$ in X and $v_\varepsilon \rightharpoonup v$ in X. For the Cahn–Hilliard dissipation potential this lim inf-estimate is not satisfied: Indeed, for $v_\varepsilon \rightharpoonup v$ in X, we consider

$$\mathcal{R}_\varepsilon(v_\varepsilon) = \int_\Omega \tfrac{1}{2}\nabla\xi_{v_\varepsilon} \cdot M_\varepsilon(x)\nabla\xi_{v_\varepsilon}\,\mathrm{d}x, \quad \text{where} \quad -\operatorname{div}(M_\varepsilon(x)\nabla\xi_{v_\varepsilon}) = v_\varepsilon$$

as in (3.9). The boundedness of $(v_\varepsilon)_\varepsilon \subset X$ implies the boundedness of $(\xi_{v_\varepsilon})_\varepsilon \subset X^*$ and thus, we obtain $\xi_{v_\varepsilon} \rightharpoonup \xi$ in X^* (up to subsequence). For arbitrary test functions $\varphi_\varepsilon \in X^*$, we study the weak formulation

$$\int_\Omega \nabla\varphi_\varepsilon \cdot M_\varepsilon(x)\nabla\xi_{v_\varepsilon}\,\mathrm{d}x = \langle v_\varepsilon, \varphi_\varepsilon\rangle. \tag{4.1}$$

Since M_ε is oscillating and not strongly convergent, the test function φ_ε has to capture the "right oscillations" in order to pass to the limit in the left-hand side. In particular, φ_ε satisfies $\varphi_\varepsilon \rightharpoonup \varphi$ in X^* and $\mathcal{T}_\varepsilon[\nabla\varphi_\varepsilon] \to [E\nabla\varphi + \nabla_y\Phi]$ in $\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})$. However, since v_ε is also only weakly converging we cannot pass to the limit in the right-hand side to establish a connection between the limits ξ and v. Thus, from the lower estimate

$$\liminf_{\varepsilon\to 0} \mathcal{R}_\varepsilon(v_\varepsilon) = \liminf_{\varepsilon\to 0} \mathcal{R}_\varepsilon^*(\xi_{v_\varepsilon}) \geq \mathcal{R}_0(\xi)$$

we cannot conclude $\liminf_{\varepsilon\to 0} \mathcal{R}_\varepsilon(v_\varepsilon) \geq \mathcal{R}_0(v)$.

Finally, let us compare our approach to the well-known Sandier & Serfaty result for evolutionary Γ-convergence in [47]. There, also the (EDP) formulation (Sect. 2.3) is considered in the abstract setting. The crucial conditions can be formulated as

$$\text{(i) } \forall\, s \in [0,T): \quad \liminf_{\varepsilon\to 0} \int_0^s \mathcal{R}_\varepsilon(v_\varepsilon(s))\,\mathrm{d}s \geq \int_0^s \mathcal{R}_0(v(s))\,\mathrm{d}s$$
$$\text{(ii) } \liminf_{\varepsilon\to 0} \mathcal{R}_\varepsilon^*(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t))) \geq \mathcal{R}_0^*(-\mathrm{D}\mathcal{E}_0(u(t))).$$

In particular, the conditions are formulated in a very general manner, e.g. the precise notion of the convergence of u_ε and v_ε is not explicitly stated and depends on the concrete problem. In contrast, we provide "easy" to check conditions for $\mathcal{R}_\varepsilon$ and $\mathcal{E}_\varepsilon$. Moreover, we do not need an independent bound for each of the terms $\int_0^T \mathcal{R}_\varepsilon\,\mathrm{d}t$ and $\int_0^T \mathcal{R}_\varepsilon^*\,\mathrm{d}t$.

Appendix A. Proof of chain rule for a nonconvex energy

Here, we prove Theorem 3.13, i.e. that the energy functional $\mathcal{E}$ given by $\mathcal{E}(u) = \int_\Omega \frac{1}{2}\nabla u\cdot A\nabla u + W_\gamma(u)\,\mathrm{d}x$ with $W_\gamma(u) = \frac{1}{2}u^2 - \frac{1}{\gamma+1}|u|^{\gamma+1}$ satisfies the following chain rule: If $u \in \mathrm{H}^1(0,T;X)$, $\xi \in \mathrm{L}^2(0,T;X^*)$ such that $\xi(t) \in \partial_{\mathrm{F}}^X\mathcal{E}(u(t))$ for a.a. $t \in [0,T]$, and the function $t \mapsto \mathcal{E}(u(t))$ is bounded, then it is also absolutely continuous on $[0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\big(u(t)\big) = \langle \dot{u}(t), \xi(t)\rangle \quad \text{for a.e. } t \in [0,T]. \tag{A.1}$$

In the proof, we use the following integration by parts formula, which is proven in [33].

Theorem A.1. *([33], Thm. 3.1) Let $\Omega \subset \mathbb{R}^d$ with uniform C^2 boundary $\partial\Omega$ and $A \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ be given. Then, for $u \in \mathrm{W}^{2,r}(\Omega)$ with $1 < r < \infty$ we have*

$$-(r-1)\int_\Omega |u|^{r-2}\nabla u\cdot A(x)\nabla u\,\mathrm{d}x = \int_\Omega u|u|^{r-2}\,\mathrm{div}(A(x)\nabla u)\,\mathrm{d}x - \int_{\partial\Omega} u|u|^{r-2}\nabla u\cdot A(x)\nu\,\mathrm{d}S_x. \tag{A.2}$$

Proof of Theorem 3.13. The proof follows the basic ideas of [45, Thm. 4], where the sum of a convex functional and a concave perturbation is considered. Thus, we write $W_\gamma = W_1 - W_2$, where $W_1(u) = \frac{1}{2}u^2$ and $W_2(u) = \frac{1}{\gamma+1}|u|^{\gamma+1}$. Analogously, we decompose the energy into

$$\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2 \text{ on } Z \quad \text{and} \quad \mathcal{E} = +\infty \text{ on } X\backslash Z, \text{ where} \tag{A.3}$$

$$\mathcal{E}_1(u) := \int_\Omega \tfrac{1}{2}\nabla u\cdot A(x)\nabla u + W_1(u)\,\mathrm{d}x \quad \text{and} \quad \mathcal{E}_2(u) := \int_\Omega W_2(u)\,\mathrm{d}x. \tag{A.4}$$

We easily check that $\mathcal{E}$, $\mathcal{E}_1$, and $\mathcal{E}_2$ are Fréchet differentiable on Z. In particular, if $\mathcal{E}$ is Fréchet subdifferentiable in some $u \in X$ we have that

$$\partial^X_{\mathrm{F}}\mathcal{E}(u) = \big\{ -\mathrm{div}(A(x)\nabla u) + P_0 W_\gamma'(u)\big\} \subset X^* \quad \text{with} \quad A(x)\nabla u\cdot\nu = 0 \text{ on } \partial\Omega.$$

Moreover, since $\mathcal{E}_1$ and $\mathcal{E}_2$ are convex, they separately satisfy the chain rule in (A.1) according to e.g. [10, Chap. III Lem. 3.3] or [50, Chap. IV Lem. 4.3]. Hence, it remains to prove that $\xi \in \mathrm{L}^2(0,T;X^*)$, satisfying $\xi(t) \in \partial^X_{\mathrm{F}}\mathcal{E}(u(t))$ for a.e. $t \in [0,T]$ with $u \in \mathrm{H}^1(0,T;X)$, can be decomposed into $\xi = \xi_1 - \xi_2$, where $\xi_i \in \mathrm{L}^2(0,T;X^*)$ and $\xi_i(t) \in \partial^X_{\mathrm{F}}\mathcal{E}_i(u(t))$ is satisfied for a.e. $t \in [0,T]$.

First, let us note that the boundedness of $t \mapsto \mathcal{E}(u(t))$ implies $u \in \mathrm{L}^\infty(0,T;Z)$, which in turn means that $t \mapsto W_\gamma'(u(t)) = |u(t)|^{\gamma-1}u(t) \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ is satisfied for $\frac{1}{2} < \gamma < 1$.

Due to the smoothness of $\partial\Omega$ and A we obtain higher regularity of u, namely $u \in \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))$, see e.g. [31, Thm. 5.11]. Thus, we can apply Theorem A.1 with $r = 2\gamma \in (1,2)$ to obtain

$$\alpha(2\gamma-1)\int_0^T\int_\Omega |u|^{2(\gamma-1)}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_\Omega |u|^{2\gamma-1}|\,\mathrm{div}(A(x)\nabla u)|\,\mathrm{d}x\,\mathrm{d}t \le C\big(\|u\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} + \|u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))}\big),$$

where $\alpha > 0$ is from (3.3). Note that the boundary integral in (A.2) vanishes since u satisfies $(A(x)\nabla u)\cdot\nu = 0$ on $\partial\Omega$. Since the right-hand side in the above estimate is finite we obtain that $\xi_2 := W_2'(u) = |u|^{\gamma-1}u \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$. Thus, we have shown the decomposition and therefore also the chain rule. □

Acknowledgements

The research of S.R. was supported by *Deutsche Forschungsgesellschaft* within *Collaborative Research Center 910: Control of self-organizing nonlinear systems: Theoretical methods and concepts of application* via the project A5 *Pattern formation in systems with multiple scales*. The research of M.L. was supported by Research Center Matheon under the ECMath Project SE2 *Electrothermal modeling of large-area OLEDs*. The authors gratefully thank A. Mielke, K. Disser, and P. Gussmann for useful discussions and comments.

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Received: 14 December 2015.
Accepted: 6 January 2018.