Nonlinear Differ. Equ. Appl. (2018) 25:2
© 2017 Springer International Publishing AG
1021-9722/18/010001-35
published online December 11, 2017
https://doi.org/10.1007/s00030-017-0485-3

Nonlinear Differential Equations and Applications NoDEA

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Annular rearrangements, incompressible axi-symmetric whirls and L^1-local minimisers of the distortion energy

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Abstract. In this paper we consider a variational problem consisting of an energy functional defined by the integral,

$$\mathbb{F}[u, \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2}\, dx,$$

and an associated mapping space, here, the space of incompressible Sobolev mappings of the symmetric annular domain in the Euclidean n-space $\mathbf{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$:

$$\mathcal{A}_\phi(\mathbf{X}) = \left\{ u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : \det \nabla u = 1 \ a.e. \text{ and } u|_{\partial \mathbf{X}} \equiv x \right\}.$$

The goal is then twofold. Firstly to establish and highlight an unexpected difference in symmetries of the critical points and local minimisers of $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbf{X})$ in the two special cases $n = 2$ and $n = 3$. More specifically, that when $n = 3$, despite the inherent rotational symmetry in the problem, there are NO non-trivial rotationally symmetric critical points of $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbf{X})$, whereas in sharp contrast, when $n = 2$, not only that there is an infinitude of rotationally symmetric critical points of the energy but also there is an infinitude of local minimisers of $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbf{X})$ with respect to the L^1-metric. At the heart of this analysis is an investigation into the rich homotopy structure of the space of self-mappings of annuli. The second aim is to introduce and implement a novel symmetrisation technique in the planar case $n = 2$ for Sobolev mappings u in $\mathcal{A}_\phi(\mathbf{X})$ that lowers the energy whilst keeping the homotopy class of u invariant. We finally generalise and extend some of these results to higher dimensions, in particular, we show that only in even dimensions do we have an infinitude of non-trivial rotationally symmetric critical points.

Mathematics Subject Classification. 35J57, 35Q74, 49J19, 49K20, 49Q20, 22E30, 58C35.

Keywords. Annular rearrangements, L^1-local minimisers, Homotopy classes, Distortion energy, Symmetric extremisers, Incompressible whirl mappings.

1. Introduction

In this paper we set ourselves the task of finding *symmetric* minimisers and critical points (or equivalently extremisers) of the energy functional

$$\mathbb{F}[u, \mathbf{X}] = \frac{1}{2}\int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2}\, dx. \tag{1.1}$$

Here u is a weakly differentiable incompressible mapping of the rotationally symmetric annular domain $\mathbf{X} \subset \mathbb{R}^n$ centred at the origin and n is taken primarily as 2 or 3 but we later generalise some of our results to arbitrary $n \geq 2$. Our aim is to extremise the energy functional $\mathbb{F}[u, \mathbf{X}]$ (also called the *distortion* energy functional when $n = 2$ for reasons that will become clear shortly) where $u \in \mathcal{A}_\phi(\mathbf{X})$, $\mathbf{X} = \mathbf{X}[a, b] = \{x \in \mathbb{R}^n : 0 < a < |x| < b < \infty\}$ and

$$\mathcal{A}_\phi(\mathbf{X}) = \left\{u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) \colon \det \nabla u = 1 \text{ a.e. in } \mathbf{X} \text{ and } u|_{\partial \mathbf{X}} = \phi\right\}. \tag{1.2}$$

Hereafter ϕ is the identity mapping of $\overline{\mathbf{X}}$ onto itself, $\det \nabla u$ denotes the Jacobian determinant of ∇u and the last condition in (1.2) means that $u = \phi$ on $\partial \mathbf{X}$ in the sense of traces. Now the Euler–Lagrange system associated with the energy functional $\mathbb{F}[u, \mathbf{X}]$ over the class of admissible mappings $\mathcal{A}_\phi(\mathbf{X})$ is seen to take the form (*cf.* Sect. 3 for details)

$$\mathbf{EL}[u; \mathbf{X}, \mathbb{F}] := \begin{cases} |\nabla u|^2 u + |u|^4 \operatorname{div} \mathscr{C}[x, |u|, \nabla u] = 0, & \text{in } \mathbf{X}, \\ \det \nabla u = 1, & \text{in } \mathbf{X}, \\ u = \phi, & \text{on } \partial \mathbf{X}, \end{cases} \tag{1.3}$$

where the matrix field $\mathscr{C} = \mathscr{C}[x, y, \xi]$ with $x \in \mathbf{X}$, $y > 0$ and $\xi \in \mathrm{SL}(n, \mathbb{R})$ is given by

$$\mathscr{C}[x, y, \xi] = y^{-2}\xi - p(x)\xi^{-t}, \tag{1.4}$$

for a suitable Lagrange multiplier $p = p(x)$ while the divergence operator is understood to act on the matrix field $\mathscr{C}[x, |u|, \nabla u]$ row-wise.

The terminology *distortion* energy is prompted by close links with geometric function theory and the theory of quasiregular mappings (*see*, e.g., [13, 19, 20]). Recall that a mapping $f \in W^{1,1}_{loc}(\mathbf{Y}, \mathbb{R}^n)$ with $\det \nabla f \in L^1_{loc}(\mathbf{Y})$ (here $\mathbf{Y} \subset \mathbb{R}^n$ a connected open set) is said to have finite distortion *iff* for some measurable function $K = K(y)$ with $1 \leq K(y) < \infty$ *a.e.* on $\mathbf{Y}$,[1]

$$|\nabla f|^n \leq K(y) \det \nabla f. \tag{1.5}$$

The smallest such K is called the *outer* distortion of f and denoted by $K_O(y, f)$. Likewise the *inner* distortion $K_I(y, f)$ of f is defined by the quotient,

$$K_I(y, f) = \frac{n^{-n/2}|\operatorname{cof} \nabla f|^n}{\det(\operatorname{cof} \nabla f)}, \tag{1.6}$$

when $\det \nabla f(y) \neq 0$ and $K_I(y, f) = 1$ otherwise. Now mappings with minimum distortion or those extremising a suitably defined distortion energy, for

[1] Throughout this paper for a square matrix the symbol $|A|$ denotes the Hilbert–Schmidt norm: $|A| = \sqrt{tr(A^t A)}$.

instance, the L^p-norm of the inner distortion $K_I(y,f)$ for $1 \le p \le \infty$ are of particular interest in the theory (*cf.*, e.g., [1,2]). A classical example is Teichmüller theory where one seeks mappings between Riemann surfaces that minimise the *sup*-norm of $K_I(y,f)$ or yet another example, and at the opposite extreme, mappings that minimise the L^1-norm of $K_I(y,f)$ with respect to the weighted measure $|y|^{-n}dy$. As a matter of fact it is this latter case that ties in with the work in this paper and for which the distortion energy takes the form

$$\mathbb{K}[f,\mathbf{X}] = \int_{\mathbf{X}} \frac{K_I(x,f)}{|x|^n}\,dx. \tag{1.7}$$

For homeomorphisms $f \in W^{1,n}(\mathbf{X},\mathbf{X})$ with finite distortion and L^1-integrable inner distortion $K_I(x,f)$ over $\mathbf{X}$, i.e.,

$$||K_I(\cdot,f)||_{L^1(\mathbf{X},d\mathcal{L}^n)} = \int_{\mathbf{X}} K_I(x,f)\,dx = n^{-n/2}\int_{\mathbf{X}} \frac{|\mathrm{cof}\,\nabla f|^n}{(\det\nabla f)^{n-1}}\,dx < \infty,$$

it follows (*cf.* [1,14]) that the distortion energy integral (1.7) can be written as

$$\mathbb{K}[f,\mathbf{X}] = \int_{\mathbf{X}} \frac{K_I(x,f)}{|x|^n}\,dx = n^{-n/2}\int_{\mathbf{X}} \frac{|\nabla h|^n}{|h|^n}\,dx =: \mathbb{F}_n[h,\mathbf{X}], \tag{1.8}$$

where $h = f^{-1} \in W^{1,n}(\mathbf{X},\mathbf{X})$. Thus it is evident that for $n=2$ the $\mathbb{K}$ energy of the Sobolev homeomorphism f agrees with the $\mathbb{F}=\mathbb{F}_2$ energy of its inverse mapping h. Note also that if f satisfies the additional incompressibility constraint as set in $\mathcal{A}_\phi(\mathbf{X})$, the distortion energy and the above can also be reformulated as

$$\mathbb{K}[f,\mathbf{X}] = \int_{\mathbf{X}} \frac{|\mathrm{cof}\,\nabla f|^n}{|x|^n}\,dx = n^{-n/2}\int_{\mathbf{X}} \frac{|\nabla h|^n}{|h|^n}\,dx = \mathbb{F}_n[h,\mathbf{X}]. \tag{1.9}$$

The identities (1.8)–(1.9) when $n=2$ give one reason for studying (1.1) and the structure of its extremisers in this paper. For other reasons and motivations mostly pertaining to considerations of invariance and symmetry in nonlinear elasticity,[2] in particular, the various symmetries of extremisers as well as local and global energy minimisers, in the presence of an incompressibility constraint, *see* [3,4,6,24,25] as well as [9,17,21–23,27–29].

Returning now to (1.1) it is the nonlinear system (1.3) and the existence versus non-existence of non-trivial symmetric solutions to this system that will be the primary focus of the paper. Generally we think of a mapping $u \in \mathcal{A}_\phi(\mathbf{X})$ as being *rotationally* symmetric *iff* it is invariant under all rotations R, that is, *iff* it satisfies $u(x) = Ru(R^t x)$, for all $x \in \mathbf{X}$ and $R \in \mathbf{SO}(n)$.

For the sake of this paper however we shall considerably weaken this condition and refer to a mapping u as being *rotationally* symmetric *iff* u is

[2] Another closely related model would be the Dirichlet energy over $\mathcal{A}_\phi(\mathbf{X})$ as in [9,21,27], however, we point out that the symmetrisation technique and energy bounds developed here do not immediately extend to this case. In a forthcoming paper we generalise these techniques and results to the broader context of energy integrals of the form $\int_{\mathbf{X}} |\nabla u|^p/|u|^q$ with $p \ge n$, $q > 1$, $n \ge 2$ and over the space of admissible mappings $\mathcal{A}^p_\phi(\mathbf{X}) = \{u \in W^{1,p}(\mathbf{X},\mathbb{R}^n)$: $\det\nabla u = 1$ *a.e.* in $\mathbf{X}$, $u=\phi$ on $\partial\mathbf{X}\}$.

invariant under all rotations $R \in \mathbb{T} \subset \mathbf{SO}(n)$, that is, $u(x) = Ru(R^t x)$ for all $x \in \mathbf{X}$ and $R \in \mathbb{T}$, where $\mathbb{T}$ is a maximal torus in $\mathbf{SO}(n)$, that is, a maximal commutative subgroup in $\mathbf{SO}(n)$.

With this introduction in mind and by specialising for the sake of clarity and definiteness to $n = 3$ for the moment we define a *whirl* mapping u of an annulus $\mathbf{X} = \mathbf{X}[a, b] \subset \mathbb{R}^3$ as a continuous self-mapping of $\overline{\mathbf{X}}$ onto itself that agrees with the identity mapping $\phi \equiv x$ on the boundary $\partial\mathbf{X}$ and has the form

$$u: x \mapsto Q(\rho, x_3)x = \begin{bmatrix} \cos g(\rho, x_3) & -\sin g(\rho, x_3) & 0 \\ \sin g(\rho, x_3) & \cos g(\rho, x_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{1.10}$$

Here $\rho = \sqrt{x_1^2 + x_2^2}$ and $Q(\rho, x_3) \in \mathbb{T}$ for (ρ, x_3) in $\overline{\Omega}$ [as (1.11)] where $\mathbb{T} = \mathbb{T}_y$ is the subgroup of rotations that fix $y = (0, 0, 1)$.[3] The function $g \in \mathbf{C}(\overline{\Omega}, \mathbb{R})$ is called the *whirl* function associated with the whirl mapping u and

$$\Omega = \left\{ (\rho, x_3) \in \mathbb{R}^2 : \rho > 0 \text{ and } a < \sqrt{\rho^2 + x_3^2} < b \right\}. \tag{1.11}$$

Thus it is clear that $Q \in \mathbf{C}(\overline{\Omega}, \mathbf{SO}(3))$ with $Q \equiv \mathrm{I}_3$ on $\partial\Omega_a \cup \partial\Omega_b$, where $\partial\Omega_a = \{(\rho, x_3) \in \partial\Omega : \rho^2 + x_3^2 = a^2\}$ and $\partial\Omega_b = \{(\rho, x_3) \in \partial\Omega : \rho^2 + x_3^2 = b^2\}$. Note that $\rho \equiv 0$ corresponds to the segment $\{x = (0, 0, x_3) : a < |x_3| < b\} \subset \mathbf{X}$ where by inspection any whirl mapping u verifies $u \equiv x$ on this segment. Now translating the boundary conditions of u and Q onto the whirl function g it must be that $g \equiv 2\pi k_1$ on $\partial\Omega_a$ and $g \equiv 2\pi k_2$ on $\partial\Omega_b$ for some $k_1, k_2 \in \mathbb{Z}$ respectively while along the flat boundary segment $\partial\Omega_\rho = \{(\rho, x_3) \in \partial\Omega : \rho = 0\}$ the whirl function g is left free.

Next assuming further differentiability on the whirl function g one can easily show that the class of whirl mappings is contained in $\mathcal{A}_\phi(\mathbf{X})$, that is, any whirl mapping is an admissible mapping. Furthermore a straightforward calculation shows that the energy of a whirl mapping $u = Q(\rho, x_3)x$ (*see* Sect. 4 for notation and detail) can be expressed as

$$\begin{aligned} \mathbb{F}[Q(\rho, x_3)x, \mathbf{X}] &= \frac{1}{2} \int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2} \, dx = \frac{1}{2} \int_a^b \int_{\mathbb{S}^2} \frac{|\nabla u|^2}{|u|^2} \, d\mathcal{H}^2 r^2 dr \\ &= 6\pi \ln(b/a) + \pi \int_\Omega \frac{\rho^3 |\nabla g|^2}{\rho^2 + x_3^2} \, d\rho dx_3. \end{aligned} \tag{1.12}$$

Now in seeking extremisers of $\mathbb{F}$ in the form of whirl mappings we first examine the Euler–Lagrange equation associated with the restricted energy functional

$$\mathbb{H}[g, \Omega] = \pi^{-1} \mathbb{F}[Q(\rho, x_3)x, \mathbf{X}] - 6 \ln(b/a) = \int_\Omega \frac{\rho^3 |\nabla g|^2}{\rho^2 + x_3^2} \, d\rho dx_3, \tag{1.13}$$

[*see* (1.10) and (1.12)] over the space of admissible whirl functions given for each $j \in \mathbb{Z}$ fixed by

[3]Note that when $n = 3$ this subgroup is both a copy of $\mathbf{SO}(2)$ and a maximal torus in the rotation group $\mathbf{SO}(3)$.

$$\mathscr{H}_j(\Omega) = \left\{ g \in W^{1,2}(\Omega) \colon g|_{\partial\Omega_a} = 0, \text{ and } g|_{\partial\Omega_b} = 2\pi j \right\}. \tag{1.14}$$

It is quite remarkable to see that in contrast to the *full* nonlinear Euler–Lagrange system (1.3) associated with $\mathbb{F}[u, \mathbf{X}]$ over $\mathcal{A}_\phi(\mathbf{X})$ this restricted form of the Euler–Lagrange equation takes the convenient *linear* divergence form equation

$$\mathbf{EL}[g; \Omega, \mathbb{H}] := \begin{cases} \operatorname{div}\left[\rho^3 \nabla g/(\rho^2 + x_3^2)\right] = 0, & \text{in } \Omega, \\ g = 0, & \text{on } \partial\Omega_a, \\ g = 2\pi j, & \text{on } \partial\Omega_b, \\ \rho^3/(\rho^2 + x_3^2)\, \partial_\nu g = 0 & \text{on } \partial\Omega \backslash [\partial\Omega_a \cup \partial\Omega_b]. \end{cases} \tag{1.15}$$

Judging based on the above set of equations and considering the cases $n = 2$ and 3 only it can be shown that in each case there is a unique solution for every fixed $j \in \mathbb{Z}$ given explicitly in turn by the extremising whirl functions $g = g_j$:

- $n = 2$ (note that in two dimensions we have $\rho \equiv r$)

$$g(r) = 2\pi j \frac{\log(r/a)}{\log(b/a)}, \quad a \le r \le b, \tag{1.16}$$

- $n = 3$

$$g(\rho, x_3) = 2\pi j \frac{ab}{b - a} \left[\frac{1}{a} - \frac{1}{\sqrt{\rho^2 + x_3^2}} \right], \quad (\rho, x_3) \in \overline{\Omega}. \tag{1.17}$$

We shall demonstrate in the proceeding sections that unless $j = 0$ the whirl function g from (1.17) does NOT correspond to a whirl mapping $u \in \mathcal{A}_\phi(\mathbf{X})$ that is a critical point of $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbf{X})$. This is then shown to be in stark contrast to the two dimensional case ($n = 2$) where we show by utilising the rich topological structure of $\mathcal{A}_\phi(\mathbf{X})$, namely, the infinitude of its homotopy classes ($\mathcal{A}_j \colon j \in \mathbb{Z}$), that the whirl mapping u_j corresponding to the whirl function g from (1.16) is a critical point and a local minimiser of $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbf{X})$.

In summary we are able to prove that in three dimensions critical points of $\mathbb{F}$ cannot have the *rotational* symmetry one naturally expects (the *weak* symmetry as was defined earlier) whereas in two dimensions there is an infinitude of such symmetric critical points. As a matter of fact we prove that in two dimensions any minimiser of $\mathbb{F}$ in the *homotopy* class $\mathcal{A}_j$ (with $j \in \mathbb{Z}$)—that incidentally is also a local minimiser of $\mathbb{F}$ in $\mathcal{A}_\phi(\mathbf{X})$ with respect to the L^1-metric—must be *rotationally* symmetric. We finally end by discussing the counterparts in higher dimensions.

2. The rich homotopy structure of the space $\mathfrak{A}(\mathbf{X})$

In this section we pause briefly to describe the homotopy structure of the space of continuous self-mappings $\mathfrak{A}(\mathbf{X})$ of the annular region $\mathbf{X}$. This will enable us later to prove the existence of local energy minimisers for $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbf{X})$. To this end let $\mathbf{X} = \mathbf{X}[a,b]$ and consider the space of continuous self-mappings $\mathfrak{A} = \mathfrak{A}(\mathbf{X}) = \{f \in \mathbf{C}(\overline{\mathbf{X}}, \overline{\mathbf{X}}) \colon f|_{\partial \mathbf{X}} = \phi\}$ equipped with the uniform topology. A pair of mappings $f_0, f_1 \in \mathfrak{A}$ are said to be *homotopic*, denoted $f_0 \cong f_1$, *iff* there exists $F \in \mathbf{C}([0,1] \times \overline{\mathbf{X}}, \overline{\mathbf{X}})$ such that,

$$\begin{cases} F(0,x) = f_0(x), & \forall x \in \overline{\mathbf{X}}, \\ F(1,x) = f_1(x), & \forall x \in \overline{\mathbf{X}}, \\ F(t,x) = \phi(x) = x, & \forall t \in [0,1] \text{ and } \forall x \in \partial \mathbf{X}. \end{cases} \tag{2.1}$$

The set of all mappings in $\mathfrak{A}$ homotopic to a given one, say, $f \in \mathfrak{A}$—referred to as the homotopy class of f, is denoted $[f]$. As homotopy classes of $\mathfrak{A}$ partition $\mathfrak{A}$ into pairwise disjoint subsets we next show how to effectively enumerate the set $\{[f] \colon f \in \mathfrak{A}\}$ and in doing so we note a difference between $n = 2$ and $n \geq 3$.

- ($n = 2$) We use the winding number of closed plane curves about the origin to enumerate the homotopy classes $\{[f] \colon f \in \mathfrak{A}\}$. To do this we fix $f \in \mathfrak{A}$ and use polar co-ordinates to write for $\theta \in [0, 2\pi]$ the circle mapping
$$\omega_\theta(r) = f|f|^{-1}(r,\theta) \colon [a,b] \to \mathbb{S}^1, \tag{2.2}$$
where $\omega_\theta(a) = \omega_\theta(b) = \phi$. This circle mapping has a well defined winding number that as a result of f being continuous is independent of the choice of $\theta \in [0, 2\pi]$. The latter correspondence will be denoted hereafter by
$$f \mapsto \mathbf{deg}(f|f|^{-1}). \tag{2.3}$$
Note that this integer agrees with the Brouwer *degree* of the map resulting from identifying $\mathbb{S}^1 \cong [a,b]/\{a,b\}$, justified as a result of $\omega_\theta(a) = \omega_\theta(b)$.
- ($n \geq 3$) Fix $f \in \mathfrak{A}$. Then using the identification $\overline{\mathbf{X}} \cong [a,b] \times \mathbb{S}^m$ (where for convenience we put $m = n - 1$) it is not difficult to see[4]
$$\omega[r](\cdot) = f|f|^{-1}(r, \cdot) \colon [a,b] \to \mathbf{C}_\phi(\mathbb{S}^m, \mathbb{S}^m), \tag{2.4}$$
with
$$\omega[a] = \omega[b] = \phi, \tag{2.5}$$
uniquely defines an element of the fundamental group $\pi_1[\mathbf{C}_\phi(\mathbb{S}^m, \mathbb{S}^m)]$. Now by considering the action of the special orthogonal group $\mathbf{SO}(n)$ on the sphere $\mathbb{S}^m$—viewed as its group of orientation preserving isometries—specifically,
$$\mathbf{E} \colon \xi \in \mathbf{SO}(n) \mapsto \omega \in \mathbf{C}(\mathbb{S}^m, \mathbb{S}^m), \tag{2.6}$$
where the assignment $\mathbf{E} \colon \xi \mapsto \omega$ works by setting
$$\omega(x) = \mathbf{E}[\xi](x) = \xi x, \quad x \in \mathbb{S}^m,$$
it can be shown that the latter induces an isomorphism on the level of the fundamental groups, i.e.,
$$\mathbf{E}^\star \colon \pi_1[\mathbf{SO}(n)] \cong \pi_1[\mathbf{C}_\phi(\mathbb{S}^m, \mathbb{S}^m)] \cong \mathbb{Z}_2. \tag{2.7}$$

[4]Here as usual ϕ denotes the *identity* mapping of the m-sphere onto itself and $\mathbf{C}_\phi(\mathbb{S}^m, \mathbb{S}^m)$ denotes the connected component of $\mathbf{C}(\mathbb{S}^m, \mathbb{S}^m)$ containing ϕ, that is, the component containing all mappings with Brouwer–Hopf degree $+1$.

Thus we are lead to the correspondence

$$f \mapsto \mathbf{deg}_2(f|f|^{-1}) \in \mathbb{Z}_2. \tag{2.8}$$

Proposition 2.1. (Enumeration) *Let $\mathfrak{A} = \mathfrak{A}(\mathbf{X})$ be as described above. Then*

- $(n = 2)$ *The degree map* $\mathbf{deg}\colon \{[f]\colon f \in \mathfrak{A}\} \to \mathbb{Z}$ *is a bijection. Moreover, for a pair of maps $f_0, f_1 \in \mathfrak{A}$,*

$$\begin{aligned}[f_0] = [f_1] &\iff f_0 \cong f_1 \\ &\iff \mathbf{deg}(f_0|f_0|^{-1}) = \mathbf{deg}(f_1|f_1|^{-1}).\end{aligned}$$

- $(n \geq 3)$ *The degree mod 2 map* $\mathbf{deg}_2\colon \{[f]\colon f \in \mathfrak{A}\} \to \mathbb{Z}_2$ *is a bijection. Moreover, for a pair of maps $f_0, f_1 \in \mathfrak{A}$,*

$$\begin{aligned}[f_0] = [f_1] &\iff f_0 \cong f_1 \\ &\iff \mathbf{deg}_2(f_0|f_0|^{-1}) = \mathbf{deg}_2(f_1|f_1|^{-1}).\end{aligned}$$

For a proof of the above statement and more detail on the subject of this section the reader is referred to [26] or [28].

3. The Euler–Lagrange analysis and the unconstrained energy $n \geq 2$

The aim of this section is to formulate the Euler–Lagrange equation associated with $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbf{X})$. Recall that in principle the $\mathbb{F}$ energy can become infinite if $|u|$ is too small or zero, however, for whirl mappings which we are considering as potential critical points of the energy (or even more generally for mappings u in $\mathcal{A}_\phi(\mathbf{X})$ with an L^n-integrable gradient ∇u), $|u|$ is bounded away from zero by virtue of u being a self-mapping of $\overline{\mathbf{X}}$ onto itself. Thus in this case $\mathbb{F}[u] < \infty$. Additionally we remark that for the class of mappings just described $\det \nabla u$ is L^1-integrable; a conclusion that may not hold in general for the unconstrained Sobolev mappings u of class $W^{1,2}$ when $n \geq 3$.

Now to formally derive the Euler–Lagrange equations we use the method of Lagrange multipliers and consider instead the unconstrained energy functional

$$\mathbb{E}[u, \mathbf{X}] = \int_{\mathbf{X}} \left[\frac{|\nabla u|^2}{2|u|^2} - p(x) \left(\det \nabla u - 1\right) \right] dx. \tag{3.1}$$

Here $p = p(x)$ is the stated Lagrange multiplier while it is evident that whenever $u \in \mathcal{A}_\phi(\mathbf{X})$ we have $\mathbb{E}[u, \mathbf{X}] = \mathbb{F}[u, \mathbf{X}]$. We can next calculate the first variation of the unconstrained energy in the usual way by setting $d/d\varepsilon(\mathbb{E}[u + \varepsilon\varphi])|_{\varepsilon=0} = 0$, where $u \in \mathcal{A}_\phi(\mathbf{X})$ is sufficiently regular and satisfies $|u| \geq c > 0$ in $\mathbf{X}$, while $\varphi \in \mathbf{C}_c^\infty(\mathbf{X}, \mathbb{R}^n)$ and $\varepsilon \in \mathbb{R}$ is sufficiently small, hence obtaining,

$$
\begin{aligned}
0 &= \int_{\mathbf{X}} \left\{ \sum_{i,j=1}^{n} \left[\frac{1}{|u|^2} \frac{\partial u_i}{\partial x_j} - p(x)[\mathrm{cof}\nabla u]_{ij} \right] \frac{\partial \varphi_i}{\partial x_j} - \sum_{i=1}^{n} \frac{|\nabla u|^2}{|u|^4} u_i \varphi_i \right\} dx \\
&= \int_{\mathbf{X}} \left\{ - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left[\frac{1}{|u|^2} \frac{\partial u_i}{\partial x_j} - p(x)[\mathrm{cof}\nabla u]_{ij} \right] \varphi_i - \sum_{i=1}^{n} \frac{|\nabla u|^2}{|u|^4} u_i \varphi_i \right\} dx \\
&= \int_{\mathbf{X}} - \sum_{i=1}^{n} \left\{ \frac{|\nabla u|^2}{|u|^4} u_i + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[\frac{1}{|u|^2} \frac{\partial u_i}{\partial x_j} - p(x)[\mathrm{cof}\nabla u]_{ij} \right] \right\} \varphi_i \, dx.
\end{aligned}
$$

As this is true for every compactly supported φ an application of the fundamental lemma of the calculus of variations results that for each $1 \le i \le n$ we have the equation:

$$
\frac{|\nabla u|^2}{|u|^4} u_i + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[\frac{1}{|u|^2} \frac{\partial u_i}{\partial x_j} - p(x)[\mathrm{cof}\nabla u]_{ij} \right] = 0.
$$

Proceeding formally or assuming that the desired solution u is at least twice continuously differentiable an application of the Piola identity to the cofactor term gives (*see*, e.g., [3,4] or [18,30] for the relevant jargon)

$$
\frac{|\nabla u|^2}{|u|^4} u_i + \sum_{j=1}^{n} \left[\frac{\partial}{\partial x_j} \left(\frac{1}{|u|^2} \frac{\partial u_i}{\partial x_j} \right) - [\mathrm{cof}\nabla u]_{ij} \frac{\partial p(x)}{\partial x_j} \right] = 0. \tag{3.2}
$$

Therefore expanding the derivative and rearranging terms allows us to write

$$
\begin{aligned}
0 &= \frac{|\nabla u|^2}{|u|^4} u_i + \sum_{j=1}^{n} \left[\frac{1}{|u|^2} \frac{\partial^2 u_i}{\partial x_j^2} - \frac{2}{|u|^4} \frac{\partial u_i}{\partial x_j} \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_j} u_k - [\mathrm{cof}\nabla u]_{ij} \frac{\partial p(x)}{\partial x_j} \right] \\
&= \frac{|\nabla u|^2}{|u|^4} u_i + \sum_{j=1}^{n} \left[\frac{1}{|u|^2} \frac{\partial^2 u_i}{\partial x_j^2} - \frac{2}{|u|^4} \frac{\partial u_i}{\partial x_j} [\nabla u^t u]_j - [\mathrm{cof}\nabla u]_{ij} \frac{\partial p(x)}{\partial x_j} \right].
\end{aligned} \tag{3.3}
$$

Hence transferring the latter into vector notation with $u = (u_1, \ldots, u_n)$ we can re-write the above system in the convenient form

$$
\frac{\Delta u}{|u|^2} + \frac{|\nabla u|^2}{|u|^4} u - \frac{2}{|u|^4} \nabla u (\nabla u)^t u = (\mathrm{cof}\, \nabla u) \nabla p, \tag{3.4}
$$

and finally invoking the pointwise condition $\det \nabla u = 1$ results in the system

$$
\frac{(\nabla u)^t}{|u|^2} \left[\Delta u + \frac{|\nabla u|^2}{|u|^2} u - \frac{2}{|u|^2} \nabla u (\nabla u)^t u \right] = \nabla p. \tag{3.5}
$$

The Euler–Lagrange system is thus equivalent to (3.5). This is the form of the equation we shall deal with hereafter in the quest for finding extremising whirl mappings of the energy functional $\mathbb{F}[u, \mathbf{X}]$ over $\mathcal{A}_\phi(\mathbf{X})$.

4. Whirl mappings and the restricted $\mathbb{F}$ energy: a glimpse at $n = 3$

As previously outlined in the introduction, one of the principal aims of this paper is to consider and examine a particular class of geometrically motivated incompressible self-mappings of the annulus $\mathbf{X}$ [*onto* itself] as possible solutions to the nonlinear system (1.3). These mappings are hereafter called whirl mappings and by definition for $n = 3$ are required to admit the representation

$$u : x \mapsto Q[g](\rho, z)x, \tag{4.1}$$

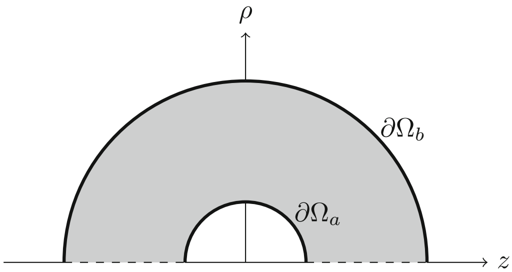
where $x = (x_1, x_2, x_3)$, $\rho = \sqrt{x_1^2 + x_2^2}$ and for brevity and convenience we set $z = x_3$. Here Q is rotation valued, i.e., $Q(\cdot) \in \mathbf{SO}(3)$, and additionally has the specific block diagonal form

$$Q = Q[g](\rho, z) = \operatorname{diag}(\mathbf{R}[g], 1) \quad \mathbf{R}[g] = \begin{bmatrix} \cos g & -\sin g \\ \sin g & \cos g \end{bmatrix}, \tag{4.2}$$

where $\mathbf{R}$ is a planar rotation matrix and $g = g(\rho, z)$ is called the whirl function. The whirl functions g are defined on the *half* vertical open annulus Ω

$$\Omega = \left\{ (\rho, z) \in \mathbb{R}^2 : \rho > 0 \text{ and } a < r = \sqrt{\rho^2 + z^2} < b \right\}, \tag{4.3}$$

whose closure upon a 2π rotation about the z-axis gives $\overline{\mathbf{X}}$. Also the curved parts of the boundary are denoted by $\partial\Omega_a$ and $\partial\Omega_b$ respectively (see the figure below).



Now it is not difficult to see that a whirl function $g = g(\rho, z)$ lying in any of the infinite scale of function spaces defined by

$$\mathscr{H}_j(\Omega) = \left\{ g \in W^{1,2}(\Omega) : g = 0 \text{ on } \partial\Omega_a, g = 2\pi j \text{ on } \partial\Omega_b \right\}, \quad j \in \mathbb{Z}, \tag{4.4}$$

results in a corresponding whirl mapping u via (1.10) in the class of admissible mappings $\mathcal{A}_\phi(\mathbf{X})$. Indeed it is clear from the boundary conditions imposed on g that u satisfies the identity boundary conditions required by $\mathcal{A}_\phi(\mathbf{X})$. Additionally referring to (4.1)–(4.2), u maps $\overline{\mathbf{X}}$ onto itself and so it remains to establish the incompressibility constraint, i.e.,the determinant condition on ∇u. To see this note firstly that by referring to (4.1)–(4.2) [see also (1.10)] we have $u(x_1, x_2, z) = (x_1 \cos g - x_2 \sin g, x_1 \sin g + x_2 \cos g, z)$ and so the gradient of u in $\mathbf{X}$ can now be obtained by a straightforward differentiation hence giving

$$\nabla u = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \tag{4.5}$$
$$= \begin{bmatrix} \cos g - x_1(x_1 \sin g + x_2 \cos g) g_\rho/\rho & -\sin g - x_2(x_1 \sin g + x_2 \cos g) g_\rho/\rho & -(x_1 \sin g + x_2 \cos g) g_z \\ \sin g + x_1(x_1 \cos g - x_2 \sin g) g_\rho/\rho & +\cos g + x_2(x_1 \cos g - x_2 \sin g) g_\rho/\rho & +(x_1 \cos g - x_2 \sin g) g_z \\ 0 & 0 & 1 \end{bmatrix},$$

or equivalently, referring to the explicit form of $\mathbf{R}[g](\rho, z)$ and using tensor notation for brevity,

$$\nabla u(x) = \mathrm{diag}(\mathbf{R}[g], 1) + \mathrm{diag}(\dot{\mathbf{R}}[g], 0)\, x \otimes (x_1 g_\rho/\rho, x_2 g_\rho/\rho, g_z). \tag{4.6}$$

Now by noting that the determinant is a quasiaffine function and so verifies the identity $\det(\mathrm{I}_n + a \otimes b) = 1 + \langle a, b \rangle$ for all vectors $a, b \in \mathbb{R}^n$ we can write

$$\begin{aligned} \det \nabla u(x) &= \det \Big(\mathrm{diag}(\mathbf{R}[g], 1) + \mathrm{diag}(\dot{\mathbf{R}}[g], 0)\, x \otimes (x_1 g_\rho/\rho, x_2 g_\rho/\rho, g_z) \Big) \\ &= \det \left(\mathrm{diag}(\mathbf{R}[g], 1) \right) \\ &\quad \times \det \Big(\mathrm{I}_3 + \mathrm{diag}(\mathbf{R}[g]^t, 1)\, \mathrm{diag}(\dot{\mathbf{R}}[g], 0)\, x \otimes (x_1 g_\rho/\rho, x_2 g_\rho/\rho, g_z) \Big) \\ &= 1 + \langle \mathrm{diag}(\mathbf{R}[g]^t \dot{\mathbf{R}}[g], 0)\, x, (x_1 g_\rho/\rho, x_2 g_\rho/\rho, g_z) \rangle = 1 \end{aligned} \tag{4.7}$$

where in deducing the last equality we have taken advantage of the fact that the product matrix $\mathbf{R}[g]^t \dot{\mathbf{R}}[g]$ is skew-symmetric hence forcing the inner product term in the last line to vanish. Next note that using (4.6) we have

$$|\nabla u|^2 = \mathrm{tr}[\nabla u]^t [\nabla u] = 3 + \rho^2 (g_\rho^2 + g_z^2), \tag{4.8}$$

and so in particular $g \in W^{1,2}(\Omega) \implies u = Q[g](\rho, z)x \in W^{1,2}(\mathbf{X}, \mathbb{R}^3)$. Hence in conclusion by putting all the above together it follows that any whirl function g of class $\mathscr{H}_j$ ($j \in \mathbb{Z}$) results in a corresponding admissible whirl mapping u, i.e.,

$$g \in \mathscr{H}_j(\Omega) \implies u = Q[g](\rho, z)x \in \mathcal{A}_\phi(\mathbf{X}). \tag{4.9}$$

Using the same calculation the energy of the whirl mapping $u = Q[g](\rho, z)x$ can be seen to be given by

$$\begin{aligned} \mathbb{F}[Q[g](\rho, z)x, \mathbf{X}] &= \frac{1}{2} \int_{\mathbf{X}} \frac{|\nabla Q[g](\rho, z)x|^2}{|Q[g](\rho, z)x|^2}\, dx \\ &= \frac{1}{2} \int_{\mathbf{X}} \frac{\left[3 + \rho^2 \left(g_\rho^2 + g_z^2\right)\right]}{|x|^2}\, dx \end{aligned}$$

$$= \pi \int_\Omega \frac{\left[3+\rho^2\left(g_\rho^2+g_z^2\right)\right]}{\rho^2+z^2}\rho\, d\rho dz$$
$$= 6\pi \ln(b/a) + \pi \int_\Omega \frac{\rho^3|\nabla g|^2}{\rho^2+z^2}\, d\rho dz. \tag{4.10}$$

As the logarithm term in the last line does not contribute to the variational structure of the energy, in searching for extremisers $g = g(\rho, z)$ to the above and thus u to the $\mathbb{F}$ energy, it suffices to restrict solely to the quadratic energy integral term in (4.10).

It is then seen that the resulting Euler–Lagrange equation to this fragment of the energy over the space $\mathscr{H}_j(\Omega)$ takes the form[5]

$$\mathbf{EL}[g;\Omega,\mathbb{H}] := \begin{cases} \operatorname{div}\left(\rho^3(\rho^2+z^2)^{-1}\nabla g\right) = 0 \text{ in } \Omega, \\ g = 0 & \text{on } \partial\Omega_a, \\ g = 2\pi j & \text{on } \partial\Omega_b, \\ \rho^3(\rho^2+z^2)^{-1}\partial_\nu g = 0 & \text{on } \partial\Omega\backslash[\partial\Omega_a \cup \partial\Omega_b]. \end{cases} \tag{4.11}$$

Note that the equation in the last line above is a result of the whirl function g being free on the boundary segment $\partial\Omega\backslash[\partial\Omega_a \cup \partial\Omega_b]$. In what follows a function g is referred to as a classical solution to (4.11) *iff* $g \in \mathbf{C}^2(\Omega)\cap\mathbf{C}^1(\overline{\Omega})$ and (4.11) is satisfied in the usual pointwise sense. Also for the sake of clarity we recall that the divergence and gradient operators here are in reference to the (ρ, z) variables, i.e.,

$$\operatorname{div}\left(\frac{\rho^3\nabla g}{\rho^2+z^2}\right) = \partial_\rho\left(\frac{\rho^3 g_\rho}{\rho^2+z^2}\right) + \partial_z\left(\frac{\rho^3 g_z}{\rho^2+z^2}\right). \tag{4.12}$$

5. No non-trivial whirl solutions $u = Q[g](\rho, z)x$ to the system (1.3)–(3.5) for $n = 3$

The aim of this section is to show that in three dimensions there are no critical points of the $\mathbb{F}$ energy in the form of whirl mappings except for the trivial identity mapping. The route we follow here is firstly to explicitly compute the solution to the restricted Euler–Lagrange equation as given above by (4.11) and then to show that for each $j \in \mathbb{Z}$ this solution is unique. The ultimate aim of proving the non-existence of non-trivial whirl solutions to the full Euler–Lagrange (3.5) then comes down to showing that the whirl mappings corresponding to these explicit solutions do not grant solutions to the system (3.5).

Proposition 5.1. *The restricted Euler–Lagrange equation* (4.11) *has a unique classical solution $g = g(\rho, z; j)$ for each fixed $j \in \mathbb{Z}$ given explicitly by*

$$g(\rho, z) = \frac{2\pi j b}{b-a}\left[1 - \frac{a}{\sqrt{\rho^2+z^2}}\right], \quad (\rho, z) \in \overline{\Omega}, j \in \mathbb{Z}. \tag{5.1}$$

[5]Note that for $n = 2$ the restricted Euler–Lagrange equation takes the form $\dot{g}(\rho)+\rho\ddot{g}(\rho) = 0$ with the boundary conditions of $g(a) = 0$ and $g(b) = 2\pi k$ while $\rho \equiv r$.

Proof. That g as given above has the required degree of regularity for a classical solution is evident. For the uniqueness part suppose that g_1, g_2 are solutions to this boundary valued problem. Then $g = g_1 - g_2$ is a classical solution to (4.11) with zero boundary conditions. Therefore applying the divergence theorem and taking into account the boundary condition on g gives

$$\begin{aligned}\int_\Omega \operatorname{div}\left(\frac{\rho^3 g \nabla g}{\rho^2+z^2}\right) d\rho dz &= \int_\Omega \left\{\frac{\rho^3|\nabla g|^2}{\rho^2+z^2} + g \operatorname{div}\left(\frac{\rho^3 \nabla g}{\rho^2+z^2}\right)\right\} d\rho dz \\ &= \int_\Omega \frac{\rho^3|\nabla g|^2}{\rho^2+z^2} d\rho dz = \int_{\partial\Omega} \frac{\rho^3 g \nabla g \cdot \nu}{\rho^2+z^2} d\sigma = 0.\end{aligned}$$

Note that in view of g_1, g_2 being classical solutions to (4.11) the vector field $U = (\rho^2+z^2)^{-1}\rho^3 g\nabla g$ lies in $\mathbf{C}^1(\Omega, \mathbb{R}^3) \cap \mathbf{C}(\overline{\Omega}, \mathbb{R}^3)$ with $\operatorname{div} U \in L^1(\Omega)$ and this justifies the application of the divergence theorem. Therefore,

$$\int_\Omega \frac{\rho^3|\nabla g|^2}{\rho^2+z^2} d\rho dz = \int_\Omega \operatorname{div}\left(\frac{\rho^3 g \nabla g}{\rho^2+z^2}\right) d\rho dz = 0. \tag{5.2}$$

Now as $\rho > 0$ for $\rho \in \Omega$ and $|\nabla g|^2 \geq 0$ it follows from (5.2) that $g \equiv c$ for some constant c and due to the zero boundary conditions this gives $g \equiv 0$ which in turn implies $g_1 \equiv g_2$. Hence the solution g to (4.11) for fixed $j \in \mathbb{Z}$ is unique. It remains to show that $g = g(\rho, z)$ as given by (5.1) is a solution to (4.11) and this follows from direct calculations. Indeed, since here we have,

$$g_\rho = \frac{\partial g}{\partial \rho} = \frac{2\pi jab}{b-a}\frac{\rho}{(\rho^2+z^2)^{3/2}}, \quad g_z = \frac{\partial g}{\partial z} = \frac{2\pi jab}{b-a}\frac{z}{(\rho^2+z^2)^{3/2}}, \tag{5.3}$$

it is evident that

$$\begin{aligned}\operatorname{div}\left(\frac{\rho^3 \nabla g}{\rho^2+z^2}\right) &= \frac{\partial}{\partial \rho}\frac{\rho^3 g_\rho}{\rho^2+z^2} + \frac{\partial}{\partial z}\frac{\rho^3 g_z}{\rho^2+z^2} \\ &= \frac{2\pi jab}{b-a}\left[\frac{\partial}{\partial \rho}\frac{\rho^4}{(\rho^2+z^2)^{5/2}} + \frac{\partial}{\partial z}\frac{\rho^3 z}{(\rho^2+z^2)^{5/2}}\right].\end{aligned}$$

Then as a result we can write and verify that

$$\begin{aligned}\frac{\partial}{\partial \rho}\frac{\rho^4}{(\rho^2+z^2)^{5/2}} + \frac{\partial}{\partial z}\frac{\rho^3 z}{(\rho^2+z^2)^{5/2}} &= \frac{4\rho^3}{(\rho^2+z^2)^{5/2}} - \frac{5\rho^5}{(\rho^2+z^2)^{7/2}} \\ &\quad + \frac{\rho^3}{(\rho^2+z^2)^{5/2}} - \frac{5\rho^3 z^2}{(\rho^2+z^2)^{7/2}} = 0.\end{aligned}$$

Thus it follows at once that the vector field $\rho^3\nabla g/(\rho^2+z^2)$ is divergence free in Ω and so in conclusion g as given by (5.1) is the unique solution to (4.11). □

We are in a position to prove that in three dimensions there are no non-trivial solutions to the Euler–Lagrange system (3.5) in the form of whirl mappings.

Theorem 5.1. *There are no non-trivial critical points of* $\mathbb{F}$ *over* $\mathcal{A}_\phi(\mathbf{X})$ *in the form of a whirl mapping when* $n = 3$.

Proof. We begin by deriving the formulation of the full Euler–Lagrange equation in terms of the whirl function g for an assumed whirl mapping $u = Q[g](\rho, z)$. Towards this end we make note of the useful differential identities

$$(\nabla u)(\nabla u)^t u = Qx + \langle \nabla g, x\rangle \dot{Q}x, \tag{5.4}$$

$$|\nabla u|^2 |u|^{-2} = (3 + \rho^2 |\nabla g|^2)|x|^{-2}, \tag{5.5}$$

$$\Delta u = 2\rho^{-1} g_\rho \dot{Q}x + \Delta g \dot{Q}x + |\nabla g|^2 \ddot{Q}x. \tag{5.6}$$

Therefore using (5.4)–(5.6) and referring to the Euler–Lagrange system (3.5) a basic calculation gives

$$\Delta u + \frac{|\nabla u|^2}{|u|^2} u - \frac{2}{|u|^2} \nabla u (\nabla u)^t u = \left(\frac{2g_\rho}{\rho} + \Delta g - \frac{2\langle \nabla g, x\rangle}{|x|^2} \right) \dot{Q}x + |\nabla g|^2 \ddot{Q}x + \frac{1 + \rho^2 |\nabla g|^2}{|x|^2} Qx. \tag{5.7}$$

Now since we have $\Delta g = \Delta_{\rho,z} g + g_\rho/\rho$ where the $\Delta_{\rho,z}$ denotes the Laplacian with respect to the (ρ, z) variables and Δ with respect to the (x_1, x_2, z) variables we can rewrite this as

$$\Delta u + \frac{|\nabla u|^2}{|u|^2} u - \frac{2}{|u|^2} \nabla u (\nabla u)^t u = \left(\frac{3g_\rho}{\rho} + \Delta_{\rho,z} g - \frac{2\langle \nabla g, x\rangle}{|x|^2} \right) \dot{Q}x + |\nabla g|^2 \ddot{Q}x + \frac{1 + \rho^2 |\nabla g|^2}{|x|^2} Qx. \tag{5.8}$$

However recall that,

$$\operatorname{div} \left(\frac{\rho^3 \nabla g}{\rho^2 + z^2} \right) = \frac{\rho^3 \Delta_{\rho,z} g}{\rho^2 + z^2} + \left(\frac{3\rho^2}{\rho^2 + z^2} - \frac{2\rho^4}{(\rho^2 + z^2)^2} \right) g_\rho - \frac{2\rho^3 x g_z}{(\rho^2 + z^2)^2} = 0.$$

Thus dividing both sides by $\rho^3/(\rho^2 + z^2)$ and taking the negative terms to one side gives

$$\Delta_{\rho,z} g + \frac{3g_\rho}{\rho} = 2 \left(\frac{\rho g_\rho + z g_z}{|x|^2} \right) = 2 \frac{\langle \nabla_{\rho,z} g, (\rho, z)\rangle}{|x|^2},$$

where $\nabla_{\rho,z}$ denotes the gradient with respect to the (ρ, z) variables. Now since $\langle \nabla g, x\rangle = \langle \nabla_{\rho,z} g, (\rho, z)\rangle$ we obtain,

$$\Delta_{\rho,z} g + \frac{3g_\rho}{\rho} = 2 \left(\frac{\rho g_\rho + z g_z}{|x|^2} \right) = 2 \frac{\langle \nabla g, x\rangle}{|x|^2}.$$

Hence referring to (5.7)–(5.8) and using the above calculations we can write

$$\begin{aligned} \Delta u + \frac{|\nabla u|^2}{|u|^2} u - \frac{2}{|u|^2} \nabla u (\nabla u)^t u &= \left(\frac{3g_\rho}{\rho} + \Delta_{\rho,z} g - \frac{2\langle \nabla g, x\rangle}{|x|^2} \right) \dot{Q}x \\ &\quad + |\nabla g|^2 \ddot{Q}x + \frac{1 + \rho^2 |\nabla g|^2}{|x|^2} Qx \\ &= |\nabla g|^2 \ddot{Q}x + \frac{1 + \rho^2 |\nabla g|^2}{|x|^2} Qx. \end{aligned} \tag{5.9}$$

Upon multiplying (5.9) by $(\nabla u)^t|u|^{-2}$ and referring to (3.5) we obtain after some cancellation that,

$$\nabla p = \begin{bmatrix} p_{,1} \\ p_{,2} \\ p_{,3} \end{bmatrix} = \frac{(\nabla u)^t}{|u|^2}\left[\Delta u + \frac{|\nabla u|^2}{|u|^2}u - \frac{2}{|u|^2}\nabla u(\nabla u)^t u\right]$$
$$= \frac{(\nabla u)^t}{|u|^2}\left[|\nabla g|^2\ddot{Q}x + \frac{1+\rho^2|\nabla g|^2}{|x|^2}Qx\right]$$
$$= \frac{1}{|x|^4}\begin{bmatrix}(1-z^2|\nabla g|^2)x_1 \\ (1-z^2|\nabla g|^2)x_2 \\ (1+\rho^2|\nabla g|^2)z\end{bmatrix} = \nabla\left(-\frac{1}{2|x|^2}\right) + \frac{|\nabla g|^2}{|x|^4}\begin{bmatrix}-z^2x_1 \\ -z^2x_2 \\ \rho^2 z\end{bmatrix}.$$

Evidently a necessary condition for the solvability of the above system for a pressure field p, is for the vector field on the *right* to be *curl-free*. This is therefore seen to lead to the system of equations

$$\nabla\times\nabla p = 0 \iff \nabla\times\left\{\nabla\left(-\frac{1}{2|x|^2}\right) + \frac{|\nabla g|^2}{|x|^4}\begin{bmatrix}-z^2x_1 \\ -z^2x_2 \\ \rho^2 z\end{bmatrix}\right\} = 0 \tag{5.10}$$

$$\iff$$

$$\frac{\partial}{\partial x_2}\left[x_1\left(g_\rho^2+g_z^2\right)\right] - \frac{\partial}{\partial x_1}\left[x_2\left(g_\rho^2+g_z^2\right)\right] = 0, \tag{5.11}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\rho^2|\nabla g|^2}{|x|^4}z\right) - \frac{\partial}{\partial z}\left(-\frac{z^2|\nabla g|^2}{|x|^4}x_1\right) = 0, \tag{5.12}$$

$$\frac{\partial}{\partial x_2}\left(\frac{\rho^2|\nabla g|^2}{|x|^4}z\right) - \frac{\partial}{\partial z}\left(-\frac{z^2|\nabla g|^2}{|x|^4}x_2\right) = 0. \tag{5.13}$$

Next upon writing $x_1 = \rho\cos\phi$ and $x_2 = \rho\sin\phi$ and invoking the differential identities,

$$\frac{\partial}{\partial x_1} = \frac{x_1}{\rho}\frac{\partial}{\partial\rho} - \frac{x_2}{\rho^2}\frac{\partial}{\partial\phi}, \qquad \frac{\partial}{\partial x_2} = \frac{x_2}{\rho}\frac{\partial}{\partial\rho} + \frac{x_1}{\rho^2}\frac{\partial}{\partial\phi}, \tag{5.14}$$

it is a straightforward matter to verify that the latter two equations in the above, namely, (5.12) and (5.13), transform into

$$\frac{x_1 z}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho^2|\nabla g|^2}{|x|^4}\right) + x_1\frac{\partial}{\partial z}\left(\frac{z^2|\nabla g|^2}{|x|^4}\right) = 0, \tag{5.15}$$

$$\frac{x_2 z}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho^2|\nabla g|^2}{|x|^4}\right) + x_2\frac{\partial}{\partial z}\left(\frac{z^2|\nabla g|^2}{|x|^4}\right) = 0. \tag{5.16}$$

The above set of equations after expanding out the brackets and differentiation reduce to a single equation, namely,

$$\rho\frac{\partial|\nabla g|^2}{\partial\rho} + z\frac{\partial|\nabla g|^2}{\partial z} = 0. \tag{5.17}$$

We now examine if the whirl function g as given by (5.1) verifies the above condition at least for $\mathbf{X}\backslash\{\rho\equiv 0\}$. Indeed a basic calculation gives

$$g_\rho^2 + g_z^2 = \left(\frac{2\pi jab}{b-a}\right)^2\frac{1}{(\rho^2+z^2)^2}, \tag{5.18}$$

and subsequently

$$\frac{\partial}{\partial \rho}\Big[\big(g_\rho^2+g_z^2\big)\Big] = -4\frac{\rho|\nabla g|^2}{\rho^2+z^2}, \quad \frac{\partial}{\partial z}\Big[\big(g_\rho^2+g_z^2\big)\Big] = -4\frac{z|\nabla g|^2}{\rho^2+z^2}. \tag{5.19}$$

Therefore by substituting into the expression on the left in (5.17) it follows that

$$\rho\frac{\partial|\nabla g|^2}{\partial\rho} + z\frac{\partial|\nabla g|^2}{\partial z} = -4|\nabla g|^2 \neq 0, \quad \text{if } j \neq 0. \tag{5.20}$$

Thus as claimed there are no critical points of the $\mathbb{F}$ energy in the form of a whirl mapping other than the identity mapping corresponding to $g = 0$. □

6. L^1-local minimisers and the homotopy classes of $\mathcal{A}_\phi(\mathrm{X})$ for $n = 2$

Having dealt with the problem of non-existence in three dimensions we now turn to the other end of our analysis and consider the case $n = 2$. Here by referring to the earlier discussion on enumeration and classification of the homotopy classes of $\mathcal{A}_\phi(\mathbf{X})$ we begin writing

$$\mathcal{A}_\phi(\mathbf{X}) = \bigcup_{j\in\mathbb{Z}} \mathcal{A}_j(\mathbf{X}), \tag{6.1}$$

where the sets on the right, hereafter the homotopy classes, are defined by

$$\mathcal{A}_j(\mathbf{X}) = \Big\{u \in \mathcal{A}_\phi(\mathbf{X}) \colon \mathbf{deg}(u|u|^{-1}) = j\Big\}, \quad j \in \mathbb{Z}, \tag{6.2}$$

and are pairwise disjoint. This is possible since any $u \in \mathcal{A}_\phi(\mathbf{X})$ has a continuous representative in $\mathfrak{A}$, again denoted by u, which in turn is a consequence of the Lebesgue type monotonicity of u and degree theory (*cf.*, e.g., [26,30,32]).

Proposition 6.1. *The homotopy class $\mathcal{A}_j(\mathbf{X})$ with $j \in \mathbb{Z}$ is $W^{1,2}$-sequentially weakly closed. Furthermore for $u \in \mathcal{A}_j(\mathbf{X})$ and $s > 0$ there exists $\delta = \delta(u,s) > 0$ such that*

$$\{v \colon \mathbb{F}[v] < s\} \cap \mathbb{B}_\delta^{L^1}(u) \subset \mathcal{A}_j(\mathbf{X}) \tag{6.3}$$

where $\mathbb{B}_\delta^{L^1}(u) = \{v \in \mathcal{A}_\phi(\mathbf{X}) \colon ||v-u||_{L^1} < \delta\}$.

Proof. Fix $j \in \mathbb{Z}$ and pick $(u_k \colon k \geq 1) \subset \mathcal{A}_j(\mathbf{X})$ so that $u_k \rightharpoonup u$ in $W^{1,2}(\mathbf{X}, \mathbb{R}^2)$. Then by a classical result of Reshetnyak $\det\nabla u_k \overset{\star}{\rightharpoonup} \det\nabla u$ (as measures) and so $u \in \mathcal{A}_\phi(\mathbf{X})$ while $u_k \to u$ uniformly on $\overline{\mathbf{X}}$ gives by Proposition 2.1 $u \in \mathcal{A}_j(\mathbf{X})$. For the second assertion one can argue indirectly. Indeed assuming the contrary there exists $u \in \mathcal{A}_j(\mathbf{X})$, $s > 0$ and $(v_k \colon k \geq 1)$ such that

$$\begin{cases} v_k \in \mathcal{A}_\phi(\mathbf{X}), \\ ||v_k - u||_{L^1} \to 0, \\ \mathbb{F}[v_k, \mathbf{X}] < s, \\ \text{while } v_k \notin \mathcal{A}_j(\mathbf{X}). \end{cases} \tag{6.4}$$

However by passing to a subsequence (not re-labeled) $v_k \rightharpoonup u$ in $W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ and as above $v_k \to u$ uniformly on $\overline{\mathbf{X}}$. Therefore again by Proposition 2.1, $v_k \in \mathcal{A}_j(\mathbf{X})$ for large enough k which is a contradiction. □

Theorem 6.1. (Local minimisers) *Let $\mathbf{X} = \mathbf{X}[a, b] \subset \mathbb{R}^2$ and for $j \in \mathbb{Z}$ consider $\mathcal{A}_j(\mathbf{X})$ as defined by (6.2). Then there exists $u = u(x; j) \in \mathcal{A}_j(\mathbf{X})$ such that*

$$\mathbb{F}[u, \mathbf{X}] = \inf_{v \in \mathcal{A}_j(\mathbf{X})} \mathbb{F}[v, \mathbf{X}]. \tag{6.5}$$

Furthermore for each such minimiser u there exists $\delta = \delta(u) > 0$ such that $\mathbb{F}[u, \mathbf{X}] \leq \mathbb{F}[v, \mathbf{X}]$ for all $v \in \mathcal{A}_\phi(\mathbf{X})$ satisfying $||u - v||_{L^1} < \delta$. Thus u is a local minimiser of $\mathbb{F}$ in $\mathcal{A}_\phi(\mathbf{X})$ with respect to the L^1-metric.

Proof. Fix $j \in \mathbb{Z}$ and pick $(v_k : k \geq 1) \subset \mathcal{A}_j(\mathbf{X})$ an *infimizing* sequence:

$$\mathbb{F}[v_k, \mathbf{X}] \downarrow \alpha := \inf_{v \in \mathcal{A}_j(\mathbf{X})} \mathbb{F}[v, \mathbf{X}]. \tag{6.6}$$

Then as $\alpha < \infty$ and $a \leq |v(x)| \leq b$ for all $v \in \mathcal{A}(\mathbf{X})$ it follows that by passing to a subsequence (not re-labeled) $v_k \rightharpoonup u$ in $W^{1,2}(\mathbf{X}, \mathbb{R}^2)$ and uniformly in $\overline{\mathbf{X}}$ where by the above discussion $u \in \mathcal{A}_j(\mathbf{X})$. Now

$$\left| \int_{\mathbf{X}} \frac{|\nabla v_k|^2}{|v_k|^2} - \int_{\mathbf{X}} \frac{|\nabla v_k|^2}{|u|^2} \right| \leq \int_{\mathbf{X}} |\nabla v_k|^2 \left| \frac{1}{|v_k|^2} - \frac{1}{|u|^2} \right| \to 0 \tag{6.7}$$

together with

$$\int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2} \leq \underline{\lim} \int_{\mathbf{X}} \frac{|\nabla v_k|^2}{|u|^2} \tag{6.8}$$

gives the desired lower semicontinuity of the distortion energy $\mathbb{F}$ on $\mathcal{A}_\phi(\mathbf{X})$, i.e.,

$$\mathbb{F}[u, \mathbf{X}] = \int_{\mathbf{X}} \frac{|\nabla u|^2}{2|u|^2} \leq \underline{\lim} \int_{\mathbf{X}} \frac{|\nabla v_k|^2}{2|v_k|^2} = \underline{\lim}\, \mathbb{F}[v_k, \mathbf{X}] \tag{6.9}$$

and so $\alpha \leq \mathbb{F}[u] \leq \liminf \mathbb{F}[v_k] \leq \alpha$ and therefore u as required is a minimiser.

To justify the second assertion fix $j \in \mathbb{Z}$ and u as above and with $s = 1 + \mathbb{F}[u]$ pick $\delta > 0$ as in the previous proposition. Then any $v \in \mathcal{A}$ with $||u - v||_{L^1} < \delta$ verifies $\mathbb{F}[u] \leq \mathbb{F}[v]$ as otherwise $\mathbb{F}[v] < \mathbb{F}[u] < s$ would give $v \in \mathcal{A}_j(\mathbf{X})$ and hence in view of u being a minimiser, $\mathbb{F}[v] \geq \mathbb{F}[u]$ which is a contradiction. □

7. Rotational symmetry of an infinitude of L^1-local minimisers via symmetrisation for $n = 2$

The goal of this section is to strengthen and further improve the results of the previous section by showing that when $n = 2$ there is an infinitude of L^1-local minimisers of the distortion energy $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbf{X})$ in the form of whirl mappings. We do this by invoking a symmetrisation argument, a consequence of which is that each homotopy class $\mathcal{A}_j(\mathbf{X})$ $(j \in \mathbb{Z})$ has a unique mapping

$u = u_j$ (a whirl) that minimises $\mathbb{F}$ over $\mathcal{A}_j(\mathbf{X})$. This mapping u_j is in fact the one given by

$$u_j : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} \cos g_j(r) & -\sin g_j(r) \\ \sin g_j(r) & \cos g_j(r) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{7.1}$$

where the corresponding whirl function $g = g_j$ has the explicit description

$$g_j(x) = 2\pi j \frac{\log(r/a)}{\log(b/a)}, \quad a \le r \le b, \tag{7.2}$$

for $j \in \mathbb{Z}$ and $r = \sqrt{x_1^2 + x_2^2}$. Firstly we observe that whirl mappings defined generally by $u(x) = Q[g](r)x$ with $x \in \overline{\mathbf{X}}$ and for $Q(\cdot) \in \mathbf{SO}(2)$ are rotationally symmetric, specifically,

$$Ru(R^t x) = RQ[g](r)R^t x = Q[g](r)x, \tag{7.3}$$

for every $R \in \mathbf{SO}(2)$ as $\mathbf{SO}(2)$ is commutative. This therefore combined with u_j being an L^1-local minimisers of $\mathbb{F}$ in particular means that $\mathbb{F}$ has an infinitude of rotationally symmetric L^1-local minimisers. We now verify that the mapping u_j lies in $\mathcal{A}_j(\mathbf{X})$. For this to be the case we must show that the whirl mapping u_j satisfies the incompressible constraint $\det \nabla u_j = 1$ for *a.e.* $x \in \mathbf{X}$ and that

$$\mathbf{deg}(u_j|u_j|^{-1}) = \frac{1}{2\pi} \int_a^b \frac{u_j \times (u_j)_r}{|u_j|^2} \, dr = j. \tag{7.4}$$

It is a straightforward matter to observe that for any whirl mapping u its gradient ∇u takes the form

$$\nabla u(x) = Q[g](r) + \frac{\dot g}{r} \dot Q[g](r) x \otimes x, \tag{7.5}$$

and therefore in virtue of $Q^t[g]\dot Q[g]$ being skew-symmetric we have

$$\begin{aligned} \det\left(Q[g](r) + \frac{\dot g}{r} \dot Q[g](r) x \otimes x \right) &= \det Q[g] \det \left(\mathrm{I}_2 + \frac{\dot g}{r} Q^t[g] \dot Q[g] x \otimes x \right) \\ &= 1 + \frac{\dot g}{r} \langle Q^t[g] \dot Q[g] x, x \rangle = 1. \end{aligned} \tag{7.6}$$

Thus whirl mappings satisfy the required incompressibility constraint. Next by referring to earlier discussions a straightforward calculation here results in the identity $(u \times u_r)|u|^{-2} = \dot g(r)$ and therefore [note that $g \in W^{1,2}(a,b)$]

$$\mathbf{deg}(u_j|u_j|^{-1}) = \frac{1}{2\pi} \int_a^b \frac{u_j \times (u_j)_r}{|u_j|^2} \, dr = \frac{1}{2\pi} \int_a^b \dot g_j(r) \, dr = \frac{g_j(b) - g_j(a)}{2\pi}.$$

Thus using the explicit form of the whirl function g_j from (7.2) it follows at once that,

$$\mathbf{deg}(u_j|u_j|^{-1}) = \frac{1}{2\pi} \int_a^b \dot g_j(r) \, dr = j. \tag{7.7}$$

Next the whirl mapping u_j from (7.1) is a critical point of $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbf{X})$. Indeed this follows by substituting for u in the full Euler–Lagrange (3.5) and showing that the right-hand side as required is a gradient. A straightforward calculation with $u = u_j$ here gives

$$\begin{aligned}\nabla p &= \frac{(\nabla u)^t}{|u|^2}\left[\Delta u + \frac{|\nabla u|^2}{|u|^2}u - \frac{2}{|u|^2}\nabla u(\nabla u)^t u\right]\\ &= \frac{1}{|x|^2}\left[\dot g_j + r\ddot g_j\right](\nabla u)^t \dot Q[g]\frac{x}{|x|}\\ &= \frac{2\pi j}{r^2}\left[\frac{d}{dr} + r\frac{d^2}{dr^2}\right]\frac{\log(r/a)}{\log(b/a)}(\nabla u)^t\dot Q[g]\frac{x}{|x|} = 0. \end{aligned} \tag{7.8}$$

Hence u_j are solutions to the full Euler–Lagrange equation (3.5) and are thus critical points of $\mathbb{F}$ over $\mathcal{A}_\phi(\mathbf{X})$ as claimed. Additionally it is not hard to prove (as seen earlier in the case $n = 3$) that u_j is the unique whirl mapping in $\mathcal{A}_j(\mathbf{X})$ which satisfies the full Euler–Lagrange, and hence is a critical point of $\mathbb{F}$ over $\mathcal{A}_j(\mathbf{X})$.

The goal is now to prove that u_j is the unique minimiser of $\mathbb{F}$ in $\mathcal{A}_j(\mathbf{X})$ for each $j \in \mathbb{Z}$. To do this the idea is to first prove that for each $u \in \mathcal{A}_j(\mathbf{X})$ there exists a whirl map $\bar u \in \mathcal{A}_j(\mathbf{X})$ satisfying

$$\mathbb{F}[u, \mathbf{X}] \geq \mathbb{F}[\bar u, \mathbf{X}] \tag{7.9}$$

with equality only possible if u is a whirl mapping. This would then imply that any minimiser must be a whirl mapping and since u_j is also the unique minimiser amongst all whirl mappings we conclude that any minimiser must coincide with u_j and therefore the minimiser is unique. (Note that Theorem 6.1 guarantees the existence of at least one minimiser $u \in \mathcal{A}_j(\mathbf{X})$.) Towards this end we find it more convenient to use polar co-ordinates and write

$$u\colon (r,\theta) \mapsto (f, \theta + g), \quad a \leq r \leq b, \tag{7.10}$$

or more specifically, for the cartesian counterparts,[6]

$$u\colon x \mapsto \begin{bmatrix} f\cos(\theta+g)\\ f\sin(\theta+g)\end{bmatrix}, \quad x \in \overline{\mathbf{X}}, f = f(x), g = g(x). \tag{7.11}$$

Here $f, g \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$, as $u \in \mathcal{A}_\phi(\mathbf{X})$, with $f(x) = a$ when $|x| = a$ and $f(x) = b$ when $|x| = b$. At times we shall abuse notation and write f, g as functions of (r, θ) in place of x. The identity boundary conditions of u dictates that $g(a,\theta) = 2\pi k_1$ and $g(b,\theta) = 2\pi k_2$ for some $k_1, k_2 \in \mathbb{Z}$. Note also that $g(r,\theta) = g(r,\theta+2\pi)$ and $f(r,\theta) = f(r,\theta+2\pi)$. Now if $u \in \mathcal{A}_j(\mathbf{X})$ then,

$$2\pi j = \int_a^b \frac{u \times u_r}{|u|^2}\,dr = \int_a^b g_r(r,\theta)\,dr = g(b,\theta) - g(a,\theta) = 2\pi(k_2 - k_1),$$

[6] Note that the lifting above is possible since each $u \in \mathcal{A}_\phi(\mathbf{X})$ has a continuous representative and $u = \phi$ on $\partial\mathbf{X}$. Indeed in view of the standard identification of the closed annulus $\overline{\mathbf{X}}[a,b]$ with the closed rectangle $\mathbf{R} = \{(r,\theta) : a \leq r \leq b, 0 \leq \theta \leq 2\pi\}$ any such u can be regarded as a mapping defined on $\mathbf{R}$ with $u(r,0) = u(r,2\pi)$. The existence of a lifting [here g] now follows as a result of the real line $\mathbb{R}$ being the covering space for the unit circle $\mathbb{S}^1$. See also [7] for more general results on liftings in Sobolev spaces.

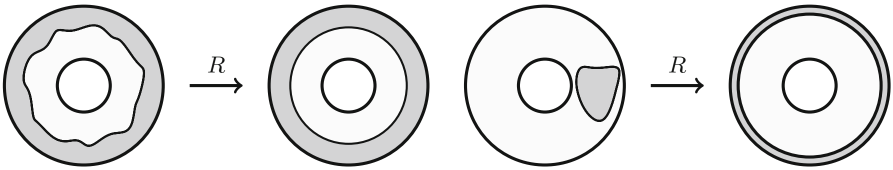


Figure 1. The image illustrates how the annular rearrangement for functions transforms their super-level sets. Indeed the rearrangement $R\colon f \mapsto f^\star$ spreads the super-level sets $\mathscr{A}_t[f] = \{x \in \mathbf{X} : f > t\}$ around $\mathbf{X}$ and make them *annularly* symmetric.

as $j = \mathbf{deg}(u/|u|)$ which in turn is the winding number of the continuous closed curve $\gamma_\theta(r) = (\cos(\theta + g), \sin(\theta + g))$ for any $\theta \in [0, 2\pi]$. Since a shift of g by a constant $2\pi k$ for $k \in \mathbb{Z}$ does not change u we can assume without loss generality from now on that $g(a, \theta) = 0$ and $g(b, \theta) = 2\pi j$. In the forthcoming proposition we introduce a particular rearrangement for functions defined on an annulus that relates to the classical Schwarz rearrangement (*see*, e.g., [15]).

Definition 7.1. (*Annular rearrangement*) Let $\mathbf{X} = \mathbf{X}[a, b] \subset \mathbb{R}^2$ be an annulus as before and $0 \le f \in \mathbf{C}(\overline{\mathbf{X}})$ with $f = 0$ on $|x| = a$. The annular rearrangement $f^\star$ of f is the radially symmetric function defined for $x \in \mathbf{X}$ by (see Fig. 1)

$$f^\star(x) = \alpha - h^\sharp(x), \quad \alpha = \sup_{\overline{\mathbf{X}}} f. \tag{7.12}$$

Here $h^\sharp$ denotes the Schwarz rearrangement of the function $h = \alpha - f$ in $\overline{\mathbb{B}}_b$, with f regarded as extended to $\overline{\mathbb{B}}_b$ by zero off $\overline{\mathbf{X}}$, that is,

$$h = \begin{cases} \alpha - f & \text{on } \overline{\mathbf{X}}, \\ \alpha & \text{on } \mathbb{B}_a. \end{cases} \tag{7.13}$$

Let us briefly recall for the ease of the reader that in this two dimensional context the Schwarz rearrangement of a function $h \in L^1(\mathbb{B}_b)$ is defined by,

$$h^\sharp(x) = \sup_t \{t \ge 0\colon |\mathscr{A}_t[h]| > \pi |x|^2\}, \tag{7.14}$$

for $x \in \mathbb{B}_b$ and where $\mathscr{A}_t[h] = \{x \in \mathbb{B}_b\colon h(x) > t\}$. We now state and prove a proposition that is the counterpart of the Polya–Szegö inequality for annular rearrangements. This will assist us in proving the main result of the section.

Proposition 7.1. *Let* $0 \le f \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$, $f = 0$ *on* $|x| = a$ *and* $f = \alpha$ *on* $|x| = b$ *with* α *as in* (7.12). *Then the annular rearrangement* $f^\star$ *of* f *satisfies* $f^\star \in W^{1,2}(\mathbf{X})$, $f^\star = 0$ *on* $|x| = a$, $f^\star = \alpha$ *on* $|x| = b$ *and*

$$\int_{\mathbf{X}} |\nabla f|^2 \, dx \ge \int_{\mathbf{X}} |\nabla f^\star|^2 \, dx. \tag{7.15}$$

Proof. Firstly as $h \in W_0^{1,2}(\mathbb{B}_b)$ we have $h^\sharp \in W_0^{1,2}(\mathbb{B}_b)$ and thus $f^\star \in W^{1,2}(\mathbf{X})$ with $f^\star(x) = \alpha$ on $|x| = b$. Furthermore as for the weak derivatives we have the *a.e.* relation

$$\nabla h = \begin{cases} -\nabla f & \text{on } \mathbf{X}, \\ 0 & \text{on } \mathbb{B}_a, \end{cases} \tag{7.16}$$

we obtain after utilising the well known Polya–Szegö inequality for Schwarz rearrangements (*see*, e.g., [15]) that

$$\int_{\mathbf{X}} |\nabla f|^2 \, dx = \int_{\mathbb{B}_b} |\nabla h|^2 \, dx \geq \int_{\mathbb{B}_b} |\nabla h^\sharp|^2 \, dx = \int_{\mathbf{X}} |\nabla f^\star|^2 \, dx, \tag{7.17}$$

which gives (7.15). Finally since $|\{x \in \mathbb{B}_b \colon h = \alpha\}| \geq \pi a^2$ we obtain by the definition of the Schwarz rearrangement that $h^\sharp(x) = \alpha$ for $x \in \overline{\mathbb{B}}_a$ which in turn gives $f^\star(x) = 0$ when $|x| = a$. □

With this proposition now at hand we are in a position to prove the main result of this section on the minimality of whirl mappings in homotopy classes.

Theorem 7.1. *For $u \in \mathcal{A}_j(\mathbf{X})$ there exists a whirl mapping $\bar{u} \in \mathcal{A}_j(\mathbf{X})$ defined by $\overline{u}(x) = Q[\bar{g}](r)x$ $(x \in \overline{\mathbf{X}})$ with associated whirl function $\bar{g} \in W^{1,2}(\mathbf{X})$ such that*[7]

$$\mathbb{F}[\bar{u}, \mathbf{X}] \leq \mathbb{F}[u, \mathbf{X}]. \tag{7.18}$$

If u is not a whirl mapping the energy inequality (7.18) is strict.

Proof. Firstly by a straightforward calculation using the polar representation of u we have

$$|\nabla u|^2 = |\nabla f|^2 + \frac{f^2}{r^2}(1 + 2g_\theta) + f^2 |\nabla g|^2, \tag{7.19}$$

where as before $r = |x|$ and we have set $g_\theta = \nabla g \cdot x^\perp = \nabla g \cdot (-x_2, x_1)$. Hence it is evident that the distortion energy can be written as

$$\mathbb{F}[u, \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2} dx = \frac{1}{2} \int_{\mathbf{X}} \left[\frac{|\nabla f|^2}{f^2} + \frac{1 + 2g_\theta}{r^2} + |\nabla g|^2 \right] dx. \tag{7.20}$$

We now consider each of the three terms in the above integral separately. Indeed regarding the first term, using the coarea formula for Sobolev functions (*see*, e.g., [8,16]) we can write

$$\int_{\mathbf{X}} \frac{|\nabla f|}{f^2} dx = \int_a^b \int_{\{x \in \mathbf{X} \colon f(x) = t\}} d\mathcal{H}^1 \frac{dt}{t^2}. \tag{7.21}$$

Next since $f = |u| \in W^{1,2}(\mathbb{B}_b)$ (here $\mathbb{B}_b = \{|x| < b\}$ and we have extended u by identity inside $\mathbb{B}_a = \{|x| < a\}$) the set $\mathscr{D}_t = \{|u(x)| \leq t\} \subset \mathbb{B}_b$ is of finite perimeter for almost every $t \in (a, b)$. In virtue of u being incompressible and hence measure-preserving in this two dimensional setting, we have $|\mathscr{D}_t| = |\mathbb{B}_t|$. Therefore by an application of the isoperimetric inequality in the context of sets of finite perimeter (*see*, e.g., [8,10] for terminology and notation) we have the chain of relations

$$\mathcal{H}^1 \left(\{x \in \mathbf{X} \colon f(x) = t\}\right) = \mathcal{H}^1(\partial^\star \mathscr{D}_t) \geq \mathcal{H}^1(\partial^\star \mathbb{B}_t) = \mathcal{H}^1 \left(\partial \mathbb{B}_t\right) = 2\pi t, \tag{7.22}$$

[7] The whirl function $\bar{g}$ as will be seen below is a suitably modified monotone rearrangement of g. As a result $\bar{u}$ is also a *monotone* whirl mapping on $\overline{\mathbf{X}}$.

for *a.e.* $a \leq t \leq b$ where as usual $\partial^\star$ denotes the reduced boundary. Thus returning to (7.21) we obtain, in light of the above inequality, the lower bound

$$\begin{aligned}\int_{\mathbf{X}} \frac{|\nabla f|}{f^2}\,dx &= \int_a^b \int_{\{x\in\mathbf{X}:\, f(x)=t\}} d\mathcal{H}^1 \frac{dt}{t^2}\\ &\geq \int_a^b \int_{\{x\in\mathbf{X}:\, |x|=t\}} d\mathcal{H}^1 \frac{dt}{t^2}\\ &\geq \int_a^b \mathcal{H}^1(\partial\mathbb{B}_t)\frac{dt}{t^2} = 2\pi\ln(b/a). \end{aligned} \tag{7.23}$$

Now upon noting the equimeasurablity of the two functions f and $|x|$ (i.e., that $|\{x \in \mathbf{X}\colon f(x) > t\}| = |\{x \in \mathbf{X}\colon |x| > t\}|$) with $a \leq f(x) \leq b$ on $\overline{\mathbf{X}}$ we find that,[8]

$$\int_{\mathbf{X}} \frac{1}{f^2} dx = 2\pi\ln(b/a). \tag{7.24}$$

Thus by an application of Hölder inequality and putting the above together we arrive at the following lower bound on the first term in (7.20):

$$\int_{\mathbf{X}} \frac{|\nabla f|^2}{f^2} dx \geq 2\pi\ln(b/a), \tag{7.25}$$

with equality occurring in (7.25), due to the equality case of the isoperimetric inequality, *iff* $f(x) \equiv |x|$, that is, when u is a whirl mapping. Hence we have that (7.18) is strict if u is not a whirl mapping. Next we note that the second term on the right in (7.20) verifies

$$\int_{\mathbf{X}} \frac{1+2g_\theta}{r^2} dx = 2\pi\ln(b/a), \tag{7.26}$$

and hence by a further reference to (7.20) we can write

$$\begin{aligned}\mathbb{F}[u,\mathbf{X}] = \frac{1}{2}\int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2}\,dx &= \frac{1}{2}\int_{\mathbf{X}} \left[\frac{|\nabla f|^2}{f^2} + \frac{1+2g_\theta}{r^2} + |\nabla g|^2\right] dx\\ &\geq 2\pi\ln(b/a) + \frac{1}{2}\int_{\mathbf{X}} |\nabla g|^2\,dx.\end{aligned}$$

Finally we are left with the proof of the following inequality

$$\frac{1}{2}\int_{\mathbf{X}} |\nabla g|^2\,dx \geq \frac{1}{2}\int_{\mathbf{X}} |\nabla \bar{g}|^2\,dx, \tag{7.27}$$

for some whirl function $\bar{g}$ associated with g in a way that $\bar{g} \in W^{1,2}(\mathbf{X})$ is radial, $\bar{g} = 0$ on $\partial\mathbf{X}_a$ and $\bar{g} = 2\pi j$ on $\partial\mathbf{X}_b$. Towards this end we offer two different approaches.

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[8] Alternatively the identity (7.24) can be seen as an application of Theorem 5.34 on page 145 in [11] for the mappings $u \in \mathcal{A}_\phi(\mathbf{X})$. Note that in the notation of [11] $E = \mathbf{X}$ and $N(u,\mathbf{X},y) = 1$ for *a.e.* $y \in \mathbf{X}$ with $N(u,\mathbf{X},y) = 0$ otherwise. Additionally in the notation of [11] let $v(x) = |x|^{-2}\chi_{\mathbf{X}}$ then the stated theorem gives the desired result.

Approach 1. (*Annular rearrangement*) In this approach we firstly put $h = \min(|g|, 2\pi|j|)$. Then $h \in W^{1,2}(\mathbf{X}) \cap \mathbf{C}(\overline{\mathbf{X}})$, $0 \le h \le 2\pi|j|$ in $\overline{\mathbf{X}}$ and $h = |g|$ on $\partial\mathbf{X}$, while

$$\frac{1}{2}\int_{\mathbf{X}} |\nabla g|^2 \, dx \ge \frac{1}{2}\int_{\mathbf{X}} |\nabla h|^2 \, dx. \tag{7.28}$$

Thus to prove (7.27) it is enough to show that there exists a radial $\bar{g} \in W^{1,2}(\mathbf{X})$ with the described boundary conditions such that

$$\frac{1}{2}\int_{\mathbf{X}} |\nabla h|^2 \, dx \ge \frac{1}{2}\int_{\mathbf{X}} |\nabla \bar{g}|^2 \, dx. \tag{7.29}$$

Now in order to prove (7.29) we recall from Proposition 7.1 that $h^\star \in W^{1,2}(\mathbf{X})$ with $h^\star(x) = 0$ when $|x| = a$ and $h^\star(x) = 2\pi|j|$ when $|x| = b$. Furthermore, by (7.15),

$$\frac{1}{2}\int_{\mathbf{X}} |\nabla h|^2 \, dx \ge \frac{1}{2}\int_{\mathbf{X}} |\nabla h^\star|^2 \, dx. \tag{7.30}$$

As a result the stated Polya–Szegö inequality combined together with (7.28) give

$$\frac{1}{2}\int_{\mathbf{X}} |\nabla g|^2 \, dx \ge \frac{1}{2}\int_{\mathbf{X}} |\nabla h|^2 \, dx \ge \frac{1}{2}\int_{\mathbf{X}} |\nabla h^\star|^2 \, dx = \frac{1}{2}\int_{\mathbf{X}} |\nabla \bar{g}|^2 \, dx \tag{7.31}$$

where depending on the sign of $j \in \mathbb{Z}$ we have set

$$\bar{g} = \begin{cases} h^\star, & j \ge 0 \\ -h^\star, & j \le 0. \end{cases} \tag{7.32}$$

This therefore gives (7.27) as required. Note that the difference in choice of $\bar{g}$ for $j \ge 0$ and $j \le 0$ is due to the fact that $h^\star(b) = 2\pi|j|$ and we wish to construct a $\bar{g}$ such that $\bar{g}(b) = 2\pi j$. Therefore as $\bar{g}(a) = 0$ and $\bar{g}(b) = 2\pi j$ it is plain to see that the whirl mapping $\bar{u} \in \mathcal{A}_j(\mathbf{X})$ if $u \in \mathcal{A}_j(\mathbf{X})$. In particular,

$$\mathbf{deg}(\bar{u}|\bar{u}|^{-1}) = \frac{1}{2\pi}\int_a^b \frac{\bar{u} \times (\bar{u})_r}{|\bar{u}|^2} \, dr = \frac{1}{2\pi}\int_a^b \dot{\bar{g}}(r) \, dr = \frac{\bar{g}(b) - \bar{g}(a)}{2\pi} = j \tag{7.33}$$

and so we have reached our desired conclusion.

Approach 2. (*Averaging*) In this approach we find a desired $\bar{g} \in W^{1,2}(a,b)$ with $\bar{g}(a) = 0$ and $\bar{g}(b) = 2\pi j$ which satisfies (7.27) by averaging the initial function $g \in W^{1,2}(\mathbf{X})$. To this end let

$$\bar{g}(r) = \frac{1}{2\pi}\int_a^r \int_0^{2\pi} \frac{\partial g}{\partial r}(r,\theta) \, dr d\theta, \tag{7.34}$$

which gives that $\bar{g}(a) = 0$ and

$$\begin{aligned} 2\pi\bar{g}(b) &= \int_a^b \int_0^{2\pi} \frac{\partial g}{\partial r}(r,\theta) \, dr d\theta = \int_{\mathbf{X}} \nabla g \cdot \frac{x}{|x|^2} \, dx \\ &= \int_{\partial\mathbf{X}} g \, \frac{x}{|x|^2} \cdot \nu \, d\mathcal{H}^1, \end{aligned} \tag{7.35}$$

with ν denoting the unit outward normal on $\partial\mathbf{X}$ and g in (7.35) being understood in the sense of traces. Therefore as $g = 0$ when $|x| = a$ and $g = 2\pi j$ when $|x| = b$ it follows that $\bar{g}(b) = 2\pi j$. Furthermore, by (7.34) it is plain that $\bar{g} \in W^{1,2}(a, b)$ and an application of Jensen's inequality gives,

$$\left|\frac{\partial \bar{g}}{\partial r}\right|^2 (r) \leq \frac{1}{2\pi} \int_0^{2\pi} |\nabla g|^2 d\theta, \tag{7.36}$$

for almost every $r \in (a, b)$. This in turn gives the desired inequality (7.27).

Thus summarising, upon using either approach, we obtain a $\bar{g} \in W^{1,2}(a, b)$ with $\bar{g}(a) = 0$ and $\bar{g}(b) = 2\pi j$ which satisfies (7.27). Therefore defining the whirl mapping $\bar{u}$ by setting $\bar{u}(x) = \mathbf{Q}[\bar{g}]x$ for $x \in \overline{\mathbf{X}}$ with $\bar{g}$ resulting from either of the approaches, gives, upon using the corresponding inequalities,

$$\mathbb{F}[u, \mathbf{X}] \geq 2\pi \ln(b/a) + \frac{1}{2} \int_{\mathbf{X}} |\nabla g|^2 \, dx \geq 2\pi \ln(b/a) + \frac{1}{2} \int_{\mathbf{X}} |\nabla \bar{g}|^2 \, dx = \mathbb{F}[\bar{u}, \mathbf{X}].$$

This therefore concludes the proof. □

Remark 7.1. The above result asserts that for each $u \in \mathcal{A}_j(\mathbf{X})$ $(j \in \mathbb{Z})$ there is a corresponding whirl mapping $\bar{u} \in \mathcal{A}_j(\mathbf{X})$ with strictly less $\mathbb{F}$ energy if u is not a whirl mapping. Thus any minimiser of $\mathbb{F}$ in the homotopy class $\mathcal{A}_j(\mathbf{X})$ is a whirl mapping and as the restriction of $\mathbb{F}$ to whirl mappings in $\mathcal{A}_j(\mathbf{X})$ admits a unique minimiser we conclude that the minimiser of $\mathbb{F}$ in $\mathcal{A}_j(\mathbf{X})$ is unique. In particular the (rotationally symmetric) whirl minimiser of $\mathbb{F}$ in $\mathcal{A}_j(\mathbf{X})$ is indeed an L^1-local minimiser of $\mathbb{F}$ in $\mathcal{A}_\phi(\mathbf{X})$.

Using the above theorem one can establish similar minimising properties on whirl mappings for a variety of closely related energies. The following is one such example whose integrand is Dirichlet type and replaces $|\nabla u|^2/|u|^2$ with $|\nabla u|^2/|x|^2$. (See also the footnotes in the introduction.)

Corollary 7.1. *For $u \in \mathcal{A}_j(\mathbf{X})$ there exists a whirl mapping $\bar{u} = Q[\bar{g}](r)x$ $(x \in \mathbf{X})$ in $\mathcal{A}_j(\mathbf{X})$ such that,*

$$\mathbb{G}[\bar{u}, \mathbf{X}] = \int_{\mathbf{X}} \frac{|\nabla \bar{u}|^2}{|x|^2} \, dx \leq \int_{\mathbf{X}} \frac{|\nabla u|^2}{|x|^2} \, dx = \mathbb{G}[u, \mathbf{X}]. \tag{7.37}$$

Furthermore inequality is strict if u is not a whirl mapping.

Proof. The proof of this corollary is a consequence of Theorem 7.1 above and the fact that mappings $u \in \mathcal{A}_\phi(\mathbf{X})$ are Sobolev homeomorphisms. In particular it is shown in [17] (when $n = 2$) that any $u \in \mathcal{A}_j(\mathbf{X})$ is a Sobolev homeomorphism with $u^{-1} \in \mathcal{A}_{-j}(\mathbf{X})$ and,

$$\nabla u(u^{-1}(x)) = (\nabla u^{-1}(x))^{-1}, \tag{7.38}$$

for almost every $x \in \mathbf{X}$. Then,

$$\mathbb{F}[u, \mathbf{X}] = \int_{\mathbf{X}} \frac{|\nabla u|^2}{|u|^2} = \int_{\mathbf{X}} \frac{|\nabla u(u^{-1}(x))|^2}{|x|^2} \, dx, \tag{7.39}$$

where in the last equality we have applied a change of variables formula from [12] (*cf.* Theorem 1.8, pp. 280). Now since we know from Theorem 7.1 that $\mathbb{F}[\bar{u}, \mathbf{X}] \le \mathbb{F}[u, \mathbf{X}]$ we obtain as a result

$$\int_{\mathbf{X}} \frac{|\nabla \bar{u}(\bar{u}^{-1}(x))|^2}{|x|^2}\, dx \le \int_{\mathbf{X}} \frac{|\nabla u(u^{-1}(x))|^2}{|x|^2}\, dx. \tag{7.40}$$

Next by an application of (7.38) along with the identity $|\mathbf{A}| = |\mathbf{A}^{-1}|$ for any 2×2 matrix with $\det \mathbf{A} = 1$ we obtain,

$$\int_{\mathbf{X}} \frac{|\nabla \bar{u}^{-1}(x)|^2}{|x|^2}\, dx \le \int_{\mathbf{X}} \frac{|\nabla u^{-1}(x)|^2}{|x|^2}\, dx. \tag{7.41}$$

Finally noting that $\bar{u}^{-1} = Q(-\bar{g})x \in \mathcal{A}_{-j}(\mathbf{X})$ is a whirl mapping leads to the desired result. □

8. Whirl mappings in higher dimensions $n \ge 3$ and a decomposition of the restricted $\mathbb{F}$ energy

In line with what was done earlier we now move on to the higher dimensional case and investigate whether suitable generalisations of whirl mappings can serve as critical points of the energy $\mathbb{F}[u, \mathbf{X}]$ over $\mathcal{A}_\phi(\mathbf{X})$ and formulate the counterparts of the results for $n = 2$ and $n = 3$ in this context. Towards this end let us begin by asserting that a whirl mapping is a continuous self-mapping of the annulus $\overline{\mathbf{X}}$ onto itself agreeing with the identity mapping ϕ on the boundary $\partial\mathbf{X}$ and having the form

$$u(x) = Q(\rho_1, \ldots, \rho_d)\, x, \quad x \in \overline{\mathbf{X}}. \tag{8.1}$$

Here $x = (x_1, \ldots, x_n) \in \mathbf{X} = \mathbf{X}[a, b]$ and for $1 \le j \le d$ when n is even, indeed $n = 2d$, and $1 \le j \le d-1$ when n is odd, indeed $n = 2d-1$, we have introduced and denoted the 2-plane variables

$$\rho_j = \sqrt{x_{2j-1}^2 + x_{2j}^2}. \tag{8.2}$$

Furthermore for convenience and uniformity in notation we set $\rho_d = x_n$ when $n = 2d - 1$. Next the continuous mapping $Q : \overline{\Omega}_n \subset \mathbb{R}^d \to \mathbf{SO}(n)$ here has as its domain of definition the closure of the semi-annular region

$$\Omega_n = \left\{ \rho = (\rho_1, \ldots, \rho_d) \in \mathbb{R}^d_+ : a < |\rho| = \sqrt{\rho_1^2 + \cdots + \rho_d^2} < b \right\}, \tag{8.3}$$

when $n = 2d$ and

$$\Omega_n = \left\{ \rho = (\rho_1, \ldots, \rho_d) \in \mathbb{R}^{d-1}_+ \times \mathbb{R} : a < |\rho| = \sqrt{\rho_1^2 + \cdots + \rho_d^2} < b \right\}, \tag{8.4}$$

when $n = 2d-1$ respectively. Now in keeping with the earlier lower dimensional definition of whirl mappings we require u to be invariant under a fixed maximal torus in $\mathbf{SO}(n)$. Indeed we demand any whirl mapping u to be invariant under the subgroup $\mathbb{T} \subset \mathbf{SO}(n)$ of all planar rotations in the (x_{2j-1}, x_{2j})-planes with j ranging as described above. It is well known that here $\mathbb{T}$ is a maximal torus in

$\mathbf{SO}(n)$ and as such is maximally commutative. This therefore fixes the range of Q and gives $Q \in \mathbf{C}(\overline{\Omega}_n, \mathbb{T})$, since if u is invariant under $\mathbb{T}$, then

$$Ru(R^t x) = RQ(\rho)R^t x = Q(\rho)x = u(x), \quad \forall x \in \overline{\mathbf{X}}, \forall R \in \mathbb{T}, \tag{8.5}$$

and so for each $\rho \in \Omega_n$, $Q(\rho)$ commutes with $\mathbb{T}$, which by definition of $\mathbb{T}$ being maximal commutative implies that $Q(\rho) \in \mathbb{T}$. Note that in the above we have used the fact that $\rho(Rx) = \rho$ for all $R \in \mathbb{T}$. In conclusion the whirl mapping takes the form

$$u(x) = Q(\rho_1, \dots, \rho_d)x, \quad \rho = (\rho_1, \dots, \rho_d) \in \overline{\Omega}_n, \quad x \in \overline{\mathbf{X}},$$

where the mapping $Q = Q(\rho_1, \dots, \rho_d)$ admits the specific block diagonal matrix form as given by $Q(\rho_1, \dots, \rho_d) = \mathrm{diag}(\mathbf{R}[g_1], \dots, \mathbf{R}[g_d])$, that is,

$$Q(\rho_1, \dots, \rho_d) = \begin{bmatrix} \mathbf{R}[g_1](\rho) & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{R}[g_2](\rho) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{R}[g_{d-1}](\rho) & 0 \\ 0 & 0 & \cdots & 0 & \mathbf{R}[g_d](\rho) \end{bmatrix} \tag{8.6}$$

when $n = 2d$ and likewise (and only with the exception of the last block) $Q(\rho_1, \dots, \rho_d) = \mathrm{diag}(\mathbf{R}[g_1], \dots, \mathbf{R}[g_{d-1}], 1)$, that is,

$$Q(\rho_1, \dots, \rho_d) = \begin{bmatrix} \mathbf{R}[g_1](\rho) & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{R}[g_2](\rho) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{R}[g_{d-1}](\rho) & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{8.7}$$

when $n = 2d - 1$. Note that in both the above cases we have for suitable real-valued whirl functions $g_j = g_j(\rho_1, \dots, \rho_d)$ on $\overline{\Omega}_n$ the description of the 2×2 rotation blocks $\mathbf{R}[g_j]$ as in (4.2).

Now to see if and when whirl mappings are admissible mappings, that is, u as in (8.1) with Q as formulated above lies in $\mathcal{A}_\phi(\mathbf{X})$, we first check the boundary condition $u \equiv \phi$ on $\partial\mathbf{X}$, for $u = Q(\rho_1, \dots, \rho_d)x$. It is easily seen that this will be subject to the whirl functions g_j satisfying $g_j(\rho) = 2\pi k_b$ for $\rho \in \partial(\Omega_n)_b$ and $g_j(\rho) = 2\pi k_a$ on $\partial(\Omega_n)_a$, where $k_a, k_b \in \mathbb{Z}$. We note that here and below $\partial(\Omega_n)_a$ and $\partial(\Omega_n)_b$ denote the segments of the boundary $\partial\Omega_n$ where $|\rho|$ takes on the values a and b respectively. Next we note that upon a further differentiability assumption on the $\mathbf{SO}(n)$-valued map Q we have

$$\nabla u = Q + \sum_{j=1}^{d} Q_{,j} x \otimes \nabla \rho_j. \tag{8.8}$$

Here $Q_{,j}$ denotes the partial derivative of $Q = Q(\rho_1, \ldots, \rho_d)$ with respect to the ρ_j variable. Therefore to verify the incompressibility constraint we proceed by directly calculating

$$\det \nabla u = \det \left(Q + \sum_{j=1}^{d} Q_{,j} x \otimes \nabla \rho_j \right) = \det \left(\mathrm{I}_n + \sum_{j=1}^{d} Q^t Q_{,j} x \otimes \nabla \rho_j \right) = 1, \tag{8.9}$$

where the last equality follows from the set of identities $\langle Q^t Q_{,j} x, \nabla \rho_i \rangle = 0$ for $1 \le i, j \le d$. Indeed to elaborate further we recall the relation

$$\det \left(\mathrm{I}_n + \sum_{j=1}^{d} a_j \otimes b_j \right) = 1, \tag{8.10}$$

for when the set of vectors (a_i), (b_j) satisfy the mutual orthogonality relations $\langle a_i, b_j \rangle = 0$ for $1 \le i, j \le d$. Under these assumptions the above identity follows upon invoking the basic properties of the determinant function and a standard induction argument on d. Finally arguing as above we have

$$\begin{aligned} |\nabla u|^2 &= tr \left[\nabla u (\nabla u)^t \right] = tr \left[\left(Q + \sum_{j=1}^{d} Q_{,j} x \otimes \nabla \rho_j \right) \left(Q^t + \sum_{j=1}^{d} \nabla \rho_j \otimes Q_{,j} x \right) \right] \\ &= tr \left[\mathrm{I}_n + \sum_{j=1}^{d} [Q_{,j} x \otimes Q \nabla \rho_j + Q \nabla \rho_j \otimes Q_{,j} x] + \sum_{j=1}^{d} Q_{,j} x \otimes Q_{,j} x \right] \\ &= n + \sum_{j=1}^{d} |Q_{,j} x|^2 + \sum_{j=1}^{d} tr \left(Q_{,j} x \otimes Q \nabla \rho_j + Q \nabla \rho_j \otimes Q_{,j} x \right). \end{aligned} \tag{8.11}$$

Now as here we evidently have the identity

$$tr \left(Q_{,j} x \otimes Q \nabla \rho_j \right) = tr \left(Q \nabla \rho_j \otimes Q_{,j} x \right) = \langle Q \nabla \rho_j, Q_{,j} x \rangle = \langle \nabla \rho_j, Q^t Q_{,j} x \rangle = 0,$$

it follows that the third term on the right in (8.11) vanishes and therefore we can rewrite (below we introduce $s = d$ when $n = 2d$ and $s = d - 1$ when $n = 2d - 1$)

$$\begin{aligned} |\nabla u|^2 &= n + \sum_{j=1}^{d} |Q_{,j} x|^2 + \sum_{j=1}^{d} tr \left(Q_{,j} x \otimes Q \nabla \rho_j + Q \nabla \rho_j \otimes Q_{,j} x \right) \\ &= n + \sum_{j=1}^{d} |Q_{,j} x|^2 = n + \sum_{j=1}^{d} \sum_{k=1}^{s} (g_k)_{,j} \rho_k^2 \\ &= n + \sum_{k=1}^{s} |\nabla g_k|^2 \rho_k^2. \end{aligned} \tag{8.12}$$

Therefore using the above expansion for the square Hilbert–Schmidt norm of ∇u and after integration over $\mathbf{X}$ we have that

$$\begin{aligned}\int_{\mathbf{X}} |\nabla u|^2\,dx - n|\mathbf{X}| &= \sum_{k=1}^{s} \int_{\mathbf{X}} |\nabla g_k|^2 \rho_k^2\,dx \\ &= (2\pi)^s \int_{\Omega_n} \sum_{k=1}^{s} |\nabla g_k|^2 \rho_k^2 \prod_{j=1}^{s} \rho_j\,d\rho \\ &\le b^{s+2}(2\pi)^s \sum_{k=1}^{s} \int_{\Omega_n} |\nabla g_k|^2\,d\rho < \infty, \end{aligned} \tag{8.13}$$

provided for each $1 \le k \le s$ we have $g_k \in W^{1,2}(\Omega_n)$. In a similar manner it is seen that the $\mathbb{F}$ energy associated with a whirl mapping u takes the form

$$\begin{aligned}\mathbb{F}[u, \mathbf{X}] - \frac{n}{2}\int_{\mathbf{X}} |x|^{-2}\,dx &= \frac{2^s\pi^s}{2}\int_{\Omega_n} \sum_{k=1}^{s} |\nabla g_k|^2 \frac{\rho_k^2}{|x|^2} \prod_{j=1}^{s} \rho_j\,d\rho \\ &= \frac{2^s\pi^s}{2}\sum_{k=1}^{s} \mathbb{H}_k[g_k, \Omega_n] = \frac{2^s\pi^s}{2}\mathbb{H}[\mathbf{g}, \Omega_n], \end{aligned} \tag{8.14}$$

where in the penultimate identity we have introduced the auxiliary quadratic energies $\mathbb{H}_k$ $(1 \le k \le s)$ and their direct sum the $\mathbb{H}$ energy given by

$$\mathbb{H}[\mathbf{g}, \Omega_n] = \sum_{k=1}^{s} \mathbb{H}_k[g_k, \Omega_n]. \tag{8.15}$$

Here we are using the notation $\mathbf{g} = (g_1, \ldots, g_s)$ and each quadratic energy summand $\mathbb{H}_k$ has the explicit form

$$\mathbb{H}_k[g, \Omega_n] = \int_{\Omega_n} |\nabla g|^2 \frac{\rho_k^2}{|x|^2} \prod_{j=1}^{s} \rho_j\,d\rho = \int_{\Omega_n} |\nabla g|^2 \omega_k(\rho)\,d\rho, \tag{8.16}$$

where we have set $\omega_k(\rho) = \rho_k^2 \prod_{j=1}^{s} \rho_j / |x|^2$ noting that $|x|^2 = \sum_{j=1}^{d} \rho_j^2$. Now since in the formulation of the $\mathbb{F}$ energy (8.14), each individual energy summand $\mathbb{H}_k$ depends only on the whirl function $g = g_k$, in the analysis of the resulting Euler–Lagrange equation we can focus on each of the summands $\mathbb{H}_k$ separately and independently. Now the admissible class of functions for each of the energies $\mathbb{H}_k$ is given by

$$\mathscr{H}(\Omega_n) = \bigcup_{j\in\mathbb{Z}} \mathscr{H}_j(\Omega_n), \tag{8.17}$$

where for each fixed $j \in \mathbb{Z}$ the function space $\mathscr{H}_j(\Omega_n)$ on the right is given by

$$\mathscr{H}_j(\Omega_n) = \Big\{ g \in W^{1,2}(\Omega_n) : g = 0 \text{ on } \partial(\Omega_n)_a, g = 2\pi j \text{ on } \partial(\Omega_n)_b \Big\}. \tag{8.18}$$

Seeking critical points of the $\mathbb{F}$ energy in the form of whirl mappings now leads to the formulation of the Euler–Lagrange equation associated with $\mathbb{H}_k$ over each $\mathscr{H}_j(\Omega_n)$ which is given by

$$\mathbf{EL}[g;\Omega_n,\mathbb{H}_k] := \begin{cases} \operatorname{div}(\omega_k(\rho)\nabla g) = 0 & \rho\in\Omega_n, \\ g=0 & \rho\in\partial(\Omega_n)_a, \\ g = 2\pi j & \rho\in\partial(\Omega_n)_b, \\ \omega(\rho)\partial_\nu g = 0 & \rho\in\partial\Omega_n\backslash[\partial(\Omega_n)_a\cup\partial(\Omega_n)_b]. \end{cases} \tag{8.19}$$

Proposition 8.1. *For each $1\le k\le s$ and $j\in\mathbb{Z}$ the Euler–Lagrange equation (8.19) has a unique smooth solution $g=g(\rho)=g(\rho_1,\dots,\rho_d)$ given explicitly by*

$$g(\rho) = 2\pi j\frac{a^{n-2}b^{n-2}}{b^{n-2}-a^{n-2}}\left(\frac{1}{a^{n-2}} - \frac{1}{\left(\sqrt{\sum_{i=1}^d \rho_i^2}\right)^{n-2}}\right), \quad \rho\in\overline{\Omega}_n, j\in\mathbb{Z}. \tag{8.20}$$

Proof. Firstly the proof for uniqueness works exactly as in the case $n=3$ and the details are left to the reader. We are left with verifying that g as given by (8.20) satisfies (8.19). As the boundary conditions on all three boundary segments are clearly satisfied in what follows we focus on the first equation in (8.19), namely, the verification of the fact that the vector field $\omega_k(\rho_1,\dots,\rho_d)\nabla g$ is divergence free in Ω_n for $1\le k\le s$. Towards this end note that

$$\partial_{\rho_l} g(\rho) = 2\pi j\frac{(n-2)a^{n-2}b^{n-2}}{b^{n-2}-a^{n-2}}\frac{\rho_l}{\left(\sqrt{\sum_{i=1}^d \rho_i^2}\right)^{n}}, \quad 1\le l\le d, \tag{8.21}$$

and from now on $c_n = (2\pi j(n-2)a^{n-2}b^{n-2})/(b^{n-2}-a^{n-2})$.

Due to the subtle differences in the calculations we treat the cases of n even and n odd separately. In the first case where we consider $n=2d$ upon using (8.21) and setting $U_k\equiv\omega_k(\rho_1,\dots,\rho_d)\nabla g$ we can write

$$\begin{aligned} \operatorname{div} U_k &= c_n\sum_{r=1}^d \frac{\partial}{\partial\rho_r}\left(\rho_k^2\prod_{\substack{j=1\\ j\neq r}}^d \rho_j\frac{\rho_r^2}{\left(\sum_{i=1}^d\rho_i^2\right)^{d+1}}\right) \\ &= c_n\left\{\sum_{\substack{r=1\\ r\neq k}}^d \rho_k^2\prod_{\substack{j=1\\ j\neq r}}^d\rho_j\frac{\partial}{\partial\rho_r}\frac{\rho_r^2}{\left(\sum_{i=1}^d\rho_i^2\right)^{d+1}} + \prod_{\substack{j=1\\ j\neq k}}^d\rho_j\frac{\partial}{\partial\rho_k}\frac{\rho_k^4}{\left(\sum_{i=1}^d\rho_i^2\right)^{d+1}}\right\} \\ &= c_n\left\{\sum_{\substack{r=1\\ r\neq k}}^d \frac{\rho_k^2\prod_{j=1,j\neq r}^d\rho_j}{\left(\sum_{i=1}^d\rho_i^2\right)^{d+1}}\left(2\rho_r - \frac{(2n+2)\rho_r^3}{\left(\sum_{i=1}^d\rho_i^2\right)}\right) + \frac{\prod_{j=1,j\neq k}^d\rho_j}{\left(\sum_{i=1}^d\rho_i^2\right)^{d+1}}\right. \\ &\quad\left.\times\left(4\rho_k^3 - \frac{(2d+2)\rho_k^5}{\left(\sum_{i=1}^d\rho_i^2\right)}\right)\right\}. \end{aligned} \tag{8.22}$$

Therefore, it is plain that,

$$\operatorname{div} U_k = c_n \left\{ \sum_{r=1}^{d} \frac{\omega_k(\rho)}{\left(\sum_{i=1}^{d} \rho_i^2\right)^d} \left(2 - \frac{(2d+2)\rho_r^2}{\left(\sum_{i=1}^{d} \rho_i^2\right)}\right) + 2\frac{\omega_k(\rho)}{\left(\sum_{i=1}^{d} \rho_i^2\right)^d} \right\}$$
$$= \frac{c_n \omega_k(\rho)}{\left(\sum_{i=1}^{d} \rho_i^2\right)^d} \{(2d - (2d+2)) + 2\} = 0. \tag{8.23}$$

Thus we have that (8.20) solves (8.19) and is thus the unique solution in the even dimensional case $n = 2d$. In the second case $n = 2d - 1$ the main difference stems from $\rho_d = x_n$ and here the calculations proceed by setting $U_k = \omega_k(\rho_1, \ldots, \rho_d)\nabla g$ and writing

$$\operatorname{div} U_k = c_n \left\{ \sum_{r=1}^{d-1} \frac{\partial}{\partial \rho_r} \left(\rho_k^2 \prod_{\substack{j=1\\ j\neq r}}^{d-1} \rho_j \frac{\rho_r^2}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{d+1/2}} \right) \right.$$
$$\left. + \ \rho_k^2 \prod_{j=1}^{d-1} \rho_j \frac{\partial}{\partial \rho_d} \left(\frac{\rho_d}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{d+1/2}} \right) \right\} = c_n \times (\mathbf{I} + \mathbf{II}). \tag{8.24}$$

Then

$$\mathbf{I} = \frac{1}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{1/2}} \sum_{r=1}^{d-1} \frac{\partial}{\partial \rho_r} \left(\rho_k^2 \prod_{\substack{j=1\\ j\neq r}}^{d-1} \rho_j \frac{\rho_r^2}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{d}} \right)$$
$$+ \sum_{r=1}^{d-1} \frac{\partial}{\partial \rho_r} \left(\frac{1}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{1/2}} \right) \left(\rho_k^2 \prod_{\substack{j=1\\ j\neq r}}^{d-1} \rho_j \frac{\rho_r^2}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{d}} \right)$$
$$= \mathbf{I}_1 + \mathbf{I}_2, \tag{8.25}$$

where the term $\mathbf{I}_1$ simplifies to

$$\mathbf{I}_1 = \sum_{r=1}^{d-1} \frac{\omega_k(\rho)}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{d-1/2}} \left(2 - \frac{2d\rho_r^2}{\left(\sum_{i=1}^{d} \rho_i^2\right)}\right) + 2\frac{\omega_k(\rho)}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{d-1/2}}$$
$$= \frac{\omega_k(\rho)}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{d-1/2}} \left(2d - 2d\frac{\sum_{i=1}^{d} \rho_i^2 - \rho_d^2}{\sum_{i=1}^{d} \rho_i^2}\right) = 2d\frac{\omega_k(\rho)\rho_d^2}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{d+1/2}}, \tag{8.26}$$

and $\mathbf{I}_2$ is given by,

$$\mathbf{I}_2 = -\sum_{r=1}^{d-1} \frac{\omega_k(\rho)\rho_r^2}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{d+1/2}} = -\frac{\omega_k(\rho)}{\left(\sum_{i=1}^{d} \rho_i^2\right)^{d+1/2}} \left(\sum_{r=1}^{d-1} \rho_r^2\right). \tag{8.27}$$

Then it can be easily seen via a straightforward calculation that

$$\mathbf{II} = \frac{\omega_k(\rho)}{\left(\sum_{i=1}^d \rho_k^2\right)^{d-1/2}} \left(1 - \frac{(2d+1)\rho_d^2}{\sum_{i=1}^d \rho_i^2}\right). \tag{8.28}$$

Hence by putting the above calculations and derivations together it is seen at once that

$$\begin{aligned} \operatorname{div} U_k &= c_n \times (\mathbf{I} + \mathbf{II}) \\ &= \frac{c_n \omega_k(\rho)}{\left(\sum_{i=1}^d \rho_k^2\right)^{d-1/2}} \left(1 - \frac{(2d+1)\rho_d^2 - 2d\rho_d^2 + \sum_{r=1}^{d-1} \rho_r^2}{\sum_{i=1}^d \rho_i^2}\right) = 0. \end{aligned} \tag{8.29}$$

Therefore (8.20) is also the unique solution to (8.19) when $n = 2d - 1$. □

Remark 8.1. Upon writing $r = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{\rho_1^2 + \cdots + \rho_d^2}$ it is easily seen that for each fixed $j \in \mathbb{Z}$ the solution $g = g_j$ as given by (8.20) to (8.19) can be expressed with a slight abuse of notation as

$$g_j(\rho_1, \ldots, \rho_d) = g_j(r) = d(j) - \frac{c(j)}{r^{n-2}}, \tag{8.30}$$

with the coefficients $c(j)$ and $d(j)$ given by

$$c(j) = 2\pi j \frac{a^{n-2} b^{n-2}}{b^{n-2} - a^{n-2}}, \quad d(j) = 2\pi j \frac{b^{n-2}}{b^{n-2} - a^{n-2}}. \tag{8.31}$$

Thus for every $\mathbf{j} = (j_1, \ldots, j_s) \in \mathbb{Z}^s$ there is a critical point of $\mathbb{H}$ in the form $\mathbf{g} = (g_{j_1}(r), \ldots, g_{j_s}(r))$ with a corresponding whirl mapping of the specific form $u(x) = Q(r)x$, $x = (x_1, \ldots, x_n) \in \overline{\mathbf{X}}$ with $Q \in \mathbf{C}^\infty([a, b], \mathbb{T})$ as in (8.6) and (8.7).

9. Higher dimensional whirls as solutions to the Euler–Lagrange system (1.3)–(3.5) for $n \geq 3$

The goal of this section is to prove Theorem 9.1 below which is the higher dimensional counterpart of what was seen earlier for $n = 2$ and $n = 3$. (Compare also with [21,27,29].)

Theorem 9.1. *Consider the energy $\mathbb{F}[u, \mathbf{X}]$ over the space of admissible mappings $\mathcal{A}_\phi(\mathbf{X})$ with the associated system of Euler–Lagrange equations* (3.5). *Then the following hold:*

- (n *even*) *Here* (3.5) *admits an infinitude of plane-symmetric whirl mappings as solutions. These are solutions in the form $u = Q(\rho_1, \ldots, \rho_d)x$ with Q as in* (8.6) *and $g_1, \ldots, g_{n/2} \in \{\pm g\}$ with the whirl function g given by* (8.20)
- (n *odd*) *The only whirl solution to* (3.5) *is the identity mapping $u = \phi$.*

Proof. This follows by direct verification. First recall that the system of Euler–Lagrange equations takes the form

$$\frac{(\nabla u)^t}{|u|^2}\left[\Delta u+\frac{|\nabla u|^2}{|u|^2}u-\frac{2}{|u|^2}\nabla u(\nabla u)^t u\right]=\nabla p. \tag{9.1}$$

The proof now comes to showing that when u is a whirl mapping as described in the last remark the resulting left hand side in (9.1) is a gradient only when the conditions given in the theorem on $\mathbf{g}=(g_1,\dots,g_s)$ are met. Towards this end we first note that for such u we have

$$\dot{Q}=\frac{BQ}{r^{n-1}},\quad \ddot{Q}=-\left[(n-1)B-r^{2-n}B^2\right]\frac{Q}{r^n}, \tag{9.2}$$

where the $n\times n$ skew-symmetric matrix B has the specific block-diagonal form

$$B=\mathrm{diag}(c_1\mathcal{J},\dots,c_{d-1}\mathcal{J},c_d\mathcal{J}),\quad n=2d, \tag{9.3}$$

and

$$B=\mathrm{diag}(c_1\mathcal{J},\dots,c_{d-1}\mathcal{J},0),\quad n=2d-1, \tag{9.4}$$

for suitable scalars $c_1,\dots,c_{d-1},c_d$ and $\mathcal{J}=\mathbf{R}[\pi/2]$, that is the rotation matrix by angle $g=\pi/2$. Now after a set of lengthy but straightforward calculations starting with $u=Q(r)x$ as described we have

$$\begin{aligned}\nabla u&=Q+r^{2-n}BQ\theta\otimes\theta,\\(\nabla u)^t&=Q^t+r^{2-n}\theta\otimes BQ\theta,\\|\nabla u|^2&=tr[(\nabla u)(\nabla u)^t]=n+\frac{|BQ\theta|^2}{r^{2(n-2)}},\end{aligned} \tag{9.5}$$

and subsequently

$$\Delta u=\left[\frac{(n+1)B}{r^{n-1}}-r\left(\frac{(n-1)B}{r^n}-\frac{B^2}{r^{2n-2}}\right)\right]Q\theta=\left[\frac{2B}{r^{n-1}}+\frac{B^2}{r^{2n-3}}\right]Q\theta. \tag{9.6}$$

Likewise we obtain the following where for the ease of notation we shall write $\omega=Q\theta$

$$\begin{aligned}\frac{\nabla u(\nabla u)^t}{|u|^2}u&=\frac{1}{r}\left[Q+r^{2-n}B\omega\otimes\theta\right]\left[Q^t+r^{2-n}\theta\otimes B\omega\right]\omega\\&=\frac{1}{r}\left[\mathrm{I}+\frac{B\omega\otimes\theta Q^t+Q\theta\otimes B\omega}{r^{n-2}}+\frac{B\omega\otimes B\omega}{r^{2n-4}}\right]\omega\\&=\frac{1}{r}\left[\mathrm{I}+\frac{B\omega\otimes\omega+\omega\otimes B\omega}{r^{n-2}}+\frac{B\omega\otimes B\omega}{r^{2n-4}}\right]\omega\\&=\frac{1}{r}(\mathrm{I}+r^{2-n}B)\omega.\end{aligned} \tag{9.7}$$

Note that in concluding the last identity here we have used the basic relations $\langle\omega,\omega\rangle=\langle Q\theta,Q\theta\rangle=1$ and $\langle B\omega,\omega\rangle=0$ in virtue of B being skew-symmetric. Now using (9.5)

$$\Delta u+\frac{|\nabla u|^2}{|u|^2}u=\left[2\frac{B}{r^{n-1}}+\frac{B^2}{r^{2n-3}}+\frac{1}{r}\left(n+\frac{|B\omega|^2}{r^{2n-4}}\right)\mathrm{I}\right]\omega, \tag{9.8}$$

and so together with (9.7) and (9.8) we obtain

$$\Delta u + \frac{|\nabla u|^2}{|u|^2} u - 2\frac{\nabla u(\nabla u)^t}{|u|^2} u = \left[\frac{B^2 + |B\omega|^2 \mathrm{I}}{r^{2n-3}} + \frac{n-2}{r} \mathrm{I} \right] \omega. \tag{9.9}$$

Finally another calculation using (9.5) results in

$$\begin{aligned} \frac{(\nabla u)^t}{|u|^2}(9.9) &= \frac{1}{r^2} \left[Q^t + \frac{\theta \otimes B\omega}{r^{n-2}} \right] \left[\frac{B^2 + |B\omega|^2 \mathrm{I}}{r^{2n-3}} + \frac{n-2}{r} \mathrm{I} \right] \omega \\ &= Q^t \left[\frac{B^2 + |B\omega|^2 \mathrm{I}}{r^{2n-1}} + \frac{n-2}{r^3} \mathrm{I} \right] \omega + \frac{(\theta \otimes B\omega) B^2 \omega}{r^{3n-3}} \\ &= Q^t \left[\frac{B^2 + |B\omega|^2 \mathrm{I}}{r^{2n-1}} + \frac{n-2}{r^3} \mathrm{I} \right] \omega. \end{aligned} \tag{9.10}$$

Notice that the last equality here results from $(\theta \otimes B\omega)B^2\omega = \langle B\omega, B^2\omega \rangle \theta = 0$ as B is skew-symmetric. Now

$$|B\omega|^2 = |BQ\theta|^2 = |B\theta|^2 = \sum_{i=1}^{d} c_i^2 \frac{\rho_i^2}{r^2}, \tag{9.11}$$

as B and Q are block diagonal with each corresponding block commuting. Additionally we have that,

$$\nabla \left(\frac{|Bx|^2}{|x|^{2n}} \right) = -2\frac{B^2 x}{|x|^{2n}} - 2n\frac{|Bx|^2 x}{|x|^{2n+2}}. \tag{9.12}$$

Therefore (9.10) can be written as,

$$\frac{(\nabla u)^t}{|u|^2}(9.9) = \nabla \left(-\frac{|Bx|^2}{2n|x|^{2n}} \right) + \frac{n-1}{n} \frac{B^2 x}{|x|^{2n}} + \nabla \left(-\frac{n-2}{|x|} \right). \tag{9.13}$$

By inspection it is evident that (9.13) is a gradient *iff* the term $B^2|x|^{-2N}x$ is a gradient. However since

$$\frac{x}{|x|^{2n}} = \nabla \left(-\frac{1}{2n|x|^{2n-2}} \right), \tag{9.14}$$

and B^2 is diagonal: $B^2 = -\mathrm{diag}(c_1^2, \dots, c_d^2)$ upon invoking the *curl-free* constraint on gradients it is easily verified that this can happen only if all the entries of B^2 are equal, i.e., $c_1^2 = c_2^2 = \cdots = c_d^2$. But when $n = 2d - 1$ we have $c_d = 0$ giving $B = 0$ and so the only whirl mapping satisfying the Euler–Lagrange equation is the identity mapping. In contrast when $n = 2d$ the condition $c_1^2 = c_2^2 = \cdots = c_d^2$ corresponds to the whirl functions $g_1, \dots, g_d$ each agreeing, modulo a factor ± 1, with the whirl function g as given by (8.20). This therefore justifies the two alternatives in the statement of the theorem and thus completes the proof. □

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Received: 11 January 2017.
Accepted: 21 October 2017.