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# Renormalized solutions of semilinear equations involving measure data and operator corresponding to Dirichlet form

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**Abstract.** We generalize the notion of renormalized solution to semilinear elliptic and parabolic equations involving operator associated with general (possibly nonlocal) regular Dirichlet form and smooth measure on the right-hand side. We show that under mild integrability assumption on the data a quasi-continuous function $u$ is a renormalized solution to an elliptic (or parabolic) equation in the sense of our definition if and only if $u$ is its probabilistic solution, i.e. $u$ can be represented by a suitable nonlinear Feynman–Kac functional. This implies in particular that for a broad class of local and nonlocal semilinear equations there exists a unique renormalized solution.

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**Keywords.** Semilinear equation, Dirichlet form and operator, Measure data, Renormalized solution, Feynman–Kac formula.

## 1. Introduction

The aim of this paper is to extend the notion of renormalized solution to encompass semilinear elliptic and parabolic equations involving measure data and operators associated with Dirichlet forms. The paper consists of two parts. In the first one we are concerned with elliptic equations of the form

$$-Lu = f(x,u) + \mu. \tag{1.1}$$

In (1.1), $L$ is the operator associated with a regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ and $f : E \times \mathbb{R} \to \mathbb{R}$ is a measurable function. As for $\mu$ we assume that it is a bounded smooth measure on $E$, i.e. a measure of bounded total

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variation on $E$ which charges no set of zero capacity associated with the form $(\mathcal{E}, D(\mathcal{E}))$. Note that the class of operators $L$ we consider is quite large. It contains many local as well as nonlocal operators. The model examples are Laplacian and fractional Laplacian (many other examples are to be found for instance in [1–4]).

An important problem one encounters when dealing with equations of the form (1.1) is to define properly a solution. In case $L$ is local and (1.1) is linear, i.e. $f$ does not depend on $u$, some definition, now called Stampacchia's definition by duality, was proposed in [5]. To deal with semilinear equations the definitions of entropy solution (see [6]) and of renormalized solution (see [7]) have been introduced. For a comparison of different forms of these definitions as well as remarks on other concepts of solutions see [7]. In case $E = D \subset \mathbb{R}^d$ is a bounded domain and $L$ is a uniformly elliptic operator in divergence form with Dirichlet boundary conditions associated with the classical form

$$\mathcal{E}(\varphi, \psi) = \int_D (a\nabla\varphi, \nabla\psi)\, dx, \quad \varphi, \psi \in H_0^1(D)$$

one of the equivalent definitions of a solution of (1.1) given in [7] says that a quasi-continuous $u : D \to \mathbb{R}$ is a renormalized solution of (1.1) if $f(\cdot, u) \in L^1(D)$, $T_k u \in H_0^1(D)$ for $k > 0$, where $T_k u = ((-k) \vee u) \wedge k$, and there exists a sequence $\{\nu_k\}$ of bounded smooth measures on $D$ such that $\|\nu_k\|_{TV} \to 0$ as $k \to \infty$ and for any bounded quasi-continuous $v \in H_0^1(D)$ and $k \in \mathbb{N}$,

$$\int_D (a\nabla(T_k u), \nabla v)\, dx = \int_D f(x, u) v(x)\, dx + \int_D v(x)\, dx + \int_D v(x)\, \nu_k(dx). \tag{1.2}$$

In fact, the notion of entropy or renormalized solution can be applied to deal with more general then (1.1) equations in which $L$ is a Leray–Lions type operator and $\mu$ is not necessarily smooth.

Another approach to (1.1), covering both local and nonlocal operators, have been proposed in [2,3]. In this probabilistic in nature approach, a quasi-continuous (with respect to the form $\mathcal{E}$) function $u : E \to \mathbb{R}$ is a solution of (1.1) if the following nonlinear Feynman–Kac formula

$$u(x) = E_x \left( \int_0^\zeta f(X_t, u(X_t))\, dt + \int_0^\zeta dA_t^\mu \right) \tag{1.3}$$

is satisfied for quasi-every $x \in E$. Here $\mathbb{M} = (X, P_x)$ is a Markov process with life time $\zeta$ associated with $\mathcal{E}$, $E_x$ denotes the expectation with respect to $P_x$ and $A^\mu$ is the additive functional of $\mathbb{M}$ associated with $\mu$ in the Revuz sense (see Sect. 2). In (1.3) we only assume that $u$ is quasi-continuous and the integrals make sense. In fact, if (1.3) holds and $f(\cdot, u) \in L^1(E; m)$ then using the probabilistic potential theory and the theory of Dirichlet forms one can show that $u$ has some additional regularity properties. Namely, $T_k u$ belongs to the extended Dirichlet space $D_e(\mathcal{E})$ for every $k > 0$.

In [2,3] also a purely analytical definition of a solution of (1.1) resembling Stampacchia's definition is proposed (see also [8,9] for another approach in case of linear equation with $L$ being a fractional Laplacian). We call it a solution in the sense of duality. In [2,3] it is shown that under quite general assumptions on $\mathcal{E}, f, \mu$ a function $u$ is a solution in the sense of duality if and only if it is a probabilistic solution defined by (1.3). However, the definition in the sense of duality seems to be not particularly handy tool for investigating (1.1).

The natural question arises whether the concept of renormalized solution can be carried over to general (possibly nonlocal) operators corresponding to $\mathcal{E}$ (for some partial results in this direction see [10]). An obvious related question to ask is what is the relation between (1.2) and (1.3), i.e. between renormalized and probabilistic solutions? It appears that (1.2) is the right form of the definition to be generalized to encompass wider class of operators. In the paper, under the assumption that $\mathcal{E}$ is transient, we define renormalized solution of (1.1) as a quasi-continuous function $u : E \to \mathbb{R}$ such that $f(\cdot, u) \in L^1(E; m)$, $T_k u \in D_e(\mathcal{E})$ for $k > 0$ and there is a sequence $\{\nu_k\}$ of bounded smooth measures on $E$ such that $\|\nu_k\|_{TV} \to 0$ and

$$\mathcal{E}(T_k u, v) = \int_E f(x, u)v(x)\, m(dx) + \int_E v(x)\, \mu(dx) + \int_E v(x)\, \nu_k(dx) \qquad (1.4)$$

for every $k \in \mathbb{N}$ and every bounded quasi-continuous $v \in D_e(\mathcal{E})$. Thus (1.4) is a direct extension of (1.2) to general transitive Dirichlet forms. Our main theorem says that for transitive forms (1.3) is equivalent to (1.4), or more precisely, that $u$ is a probabilistic solution of (1.1) if and only if it is a renormalized solution of (1.1). Since one can prove that under some assumptions on $f$ there exists a unique probabilistic solution of (1.1) for $L$ associated with $\mathcal{E}$ (see [2,3] and Section 3 for some examples), our result a fortiori says that (1.4) provides right definition of a solution. In particular, (1.4) ensures uniqueness for interesting classes of equations. In general, the equivalence of (1.3) and (1.4) sheds new light on the nature of both probabilistic and analytic (renormalized) solutions of (1.1). What is perhaps more important, it also says that in the study of (1.1) one can use both probabilistic and analytical methods from the theory of PDEs. Let us point out once again, that contrary to [7], in our theorem we assume that the measure $\mu$ is smooth. An interesting open problem is how to define renormalized solutions for general bounded measures, at least for some classes of nonlocal operators. Finally, let us note that in case $L = \Delta$ the equivalence between probabilistic and renormalized solutions to (1.1) was observed in [11].

In the second part of the paper we consider parabolic equation of the form

$$-\frac{\partial u}{\partial t} - L_t u = f(t, x, u) + \mu, \quad u(T) = \varphi, \qquad (1.5)$$

where $\varphi : E \to \mathbb{R}$, $f : [0, T] \times E \to \mathbb{R}$, the operators $\frac{\partial}{\partial t} + L_t$ correspond to some time dependent regular Dirichlet form $\mathcal{E}^{0,T}$ and $\mu$ is a bounded measure on $(0, T] \times E$ which is smooth with respect to the capacity associated with $\mathcal{E}^{0,T}$.

In case $L_t$ are local, a definition of a renormalized solution of equations of the form (1.5) involving more general nonlinear local operators $L_t$ of Leray–Lions type but with $f$ not depending on $u$ have been introduced in [12] (see also [13] for earlier existence results for equations with general bounded measure $\mu$ and [14] for uniqueness results in the case where $\mu$ is a function in $L^1$). In [15,16] definitions of renormalized solutions to (1.5) with Leray–Lions type operators and $f$ depending on $u$ have been proposed (in [16] equations with general, not necessarily smooth measures are considered). Another definition of a renormalized solution, which is suitable for handling equations with local operators and nonlinear $f$, have been introduced in [17]. It may be viewed as parabolic analogue of (1.2). Existence and uniqueness results for weak solutions to linear equations with fractional Laplacian and $\mu$ being a function in $L^1$ are proved in [9]. A probabilistic approach to (1.5) has been developed in [18]. A probabilistic solution of (1.5) is defined similarly to (1.3), but with $\mathbb{M}$ replaced by a time-space Markov process associated with $\mathcal{E}^{0,T}$. In [18] the existence, uniqueness and regularity of probabilistic solutions of (1.5) is proved for $f$ satisfying some natural conditions (monotonicity together with mild integrability conditions) and general operators associated with $\mathcal{E}^{0,T}$.

Similarly to the elliptic case, in the paper we generalize the notion of a renormalized solution of [17] to the case of general operators corresponding to $\mathcal{E}^{0,T}$. Then we show that the proposed definition is equivalent to the probabilistic definition considered in [18]. As in elliptic case, this shows that the renormalized solutions are properly defined and gives new information on the structure of solutions. We illustrate the utility of our result by stating some theorems on existence and uniqueness of renormalized solutions of parabolic equations with $f$ satisfying the monotonicity condition and mild integrability conditions.

For simplicity, in the paper we confine ourselves to equations with operators corresponding to regular forms, but our results can be generalized to quasi-regular forms (see remarks at the end of Sects. 3 and 4).

## 2. Preliminaries

In the paper we assume that $E$ is a locally compact separable metric space and $m$ is an everywhere dense Radon measure on $E$, i.e. $m$ is a non-negative Borel measure on $E$ finite on compact sets and strictly positive on non-empty open sets.

We set $E^1 = \mathbb{R} \times E$, $E_T = [0,T] \times E$, $E_{0,T} = (0,T] \times E$. By $\mathcal{B}(E)$ we denote the $\sigma$-field of Borel subsets of $E$. $\mathcal{B}_b(E)$ is the set of all real bounded Borel measurable functions on $E$ and $\mathcal{B}_b^+(E)$ is the subset of $\mathcal{B}_b(E)$ consisting of positive functions. The sets $\mathcal{B}(E^1)$, $\mathcal{B}_b(E^1)$, $\mathcal{B}_b^+(E^1)$ are defined analogously.

We set $H = L^2(E;m)$ and $\mathcal{H}_{0,T} = L^2(0,T;H)$. The last space we identify with $L^2(E_T;m_1)$, where $m_1 = dt \otimes m$. By $(\cdot,\cdot)_H$, $(\cdot,\cdot)_{\mathcal{H}_{0,T}}$ we denote the usual inner products in $H$ and $\mathcal{H}_{0,T}$, respectively.

### 2.1. Dirichlet forms

In what follows we assume that $(\mathcal{E}, D(\mathcal{E}))$ is a (non-symmetric) Dirichlet form on $H$, i.e. positive definite closed form satisfying the weak sector condition and such that $(\mathcal{E}, D(\mathcal{E}))$ has both the sub-Markov and the dual sub-Markov property. For the definitions we refer the reader to [4]. Here let us only recall that $(\mathcal{E}, D(\mathcal{E}))$ satisfies the weak sector condition if there is $K > 0$ (called the sector constant) such that

$$|\mathcal{E}_1(u,v)| \le K\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2}, \quad u,v \in D(\mathcal{E}),$$

where $\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v)_H$ for $\alpha \ge 0$. If there is $K > 0$ such that

$$|\mathcal{E}(u,v)| \le K\mathcal{E}(u,u)^{1/2}\mathcal{E}(v,v)^{1/2}, \quad u,v \in D(\mathcal{E}), \tag{2.1}$$

then we say that $(\mathcal{E}, D(\mathcal{E}))$ satisfies the strong sector condition.

By Theorems I.2.8 and I.4.4 in [4] every Dirichlet form on $H$ determines uniquely strongly continuous contraction resolvents $(G_\alpha)_{\alpha>0}$, $(\hat{G}_\alpha)_{\alpha>0}$ on $H$ such that $G_\alpha, \hat{G}_\alpha$ are sub-Markov, $G_\alpha(H) \subset D(\mathcal{E})$, $\hat{G}_\alpha(H) \subset D(\mathcal{E})$ and

$$\mathcal{E}_\alpha(G_\alpha f, u) = (f,u)_H = \mathcal{E}_\alpha(u, \hat{G}_\alpha f), \quad f \in H,\, u \in D(\mathcal{E}),\, \alpha > 0.$$

In fact, from the sub-Markov and the dual sub-Markov property of $(\mathcal{E}, D(\mathcal{E}))$ it follows that $(G_\alpha)_{\alpha>0}, (\hat{G}_\alpha)_{\alpha>0}$ may be extended to sub-Markov resolvents on $L^\infty(E;m)$ and on $L^1(E;m)$, respectively (see [19, Section 1.1]).

Let $f \in L^\infty(E;m)$ be a non-negative function. Since $G_{1/l}f$ increases as $l \uparrow \infty$, the potential operator

$$Gf = \lim_{l\to\infty} G_{1/l}f$$

is $m$-a.e. well defined but may take the value $\infty$. We say that $\mathcal{E}$ is transient if $Gf < \infty$ $m$-a.e. for every non-negative $f \in L^\infty(E;m)$.

Let $\tilde{\mathcal{E}}$ denote the symmetric part of $\mathcal{E}$, i.e. $\tilde{\mathcal{E}}(u,v) = \frac{1}{2}(\mathcal{E}(u,v) + \mathcal{E}(v,u))$. The extended Dirichlet space $D_e(\mathcal{E})$ associated with $(\mathcal{E}, D(\mathcal{E}))$ is the family of measurable functions $u : E \to \mathbb{R}$ such that $|u| < \infty$ $m$-a.e. and there exists an $\tilde{\mathcal{E}}$-Cauchy sequence $\{u_n\} \subset D(\mathcal{E})$ such that $u_n \to u$ $m$-a.e. The sequence $\{u_n\}$ is called an approximating sequence for $u \in D_e(\mathcal{E})$.

For $u \in D_e(\mathcal{E})$ we set $\mathcal{E}(u,u) = \lim_{n\to\infty} \mathcal{E}(u_n,u_n)$, where $\{u_n\}$ is an approximating sequence for $u$. If moreover $\mathcal{E}$ satisfies the strong sector condition (2.1) then we may extend $\mathcal{E}$ to $D_e(\mathcal{E})$ by putting $\mathcal{E}(u,v) = \lim_{n\to\infty} \mathcal{E}(u_n,v_n)$ with approximating sequences $\{u_n\}$ and $\{v_n\}$ for $u \in D_e(\mathcal{E})$ and $v \in D_e(\mathcal{E})$, respectively (see [19, Section 1.3]). This extension satisfies again the strong sector condition. By [19, Theorem 1.3.9], if $(\mathcal{E}, D(\mathcal{E}))$ is transient then $(D_e(\mathcal{E}), \tilde{\mathcal{E}})$ is a Hilbert space.

Given a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ we define quasi notions with respect to $\mathcal{E}$ (exceptional sets, nests and quasi-continuity) as in [4, Chapter III] (see also [19, Sections 2.1, 2.2]). We will say that a property of points in $E$ holds quasi everywhere (q.e. for short) if it holds outside some exceptional set.

In the paper we assume that $(\mathcal{E}, D(\mathcal{E}))$ is regular (see [4, Section IV.4] or [19, Section 1.2] for the definition). By [4, Proposition IV.3.3], if $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form then each element $u \in D(\mathcal{E})$ admits an quasi-continuous $m$-version, which we denote by $\tilde{u}$, and $\tilde{u}$ is q.e. unique for every $u \in D(\mathcal{E})$. If moreover $(\mathcal{E}, D(\mathcal{E}))$ is transient then such a unique $m$-version $\tilde{u}$ exists for every $u \in D_e(\mathcal{E})$. This follows from [1, Theorem 2.1.7] and the fact that $D_e(\mathcal{E})$ and the notion of quasi-continuity only depend on the symmetric part of $\mathcal{E}$.

A positive measure $\mu$ on $\mathcal{B}(E)$ is said to be smooth ($\mu \in S(E)$ in notation) if $\mu(B) = 0$ for all exceptional sets $B \in \mathcal{B}(E)$ and there exists an nest $\{F_k\}_{k\in\mathbb{N}}$ of compact sets such that $\mu(F_k) < \infty$ for $k \in \mathbb{N}$.

Given a transient form $(\mathcal{E}, D(\mathcal{E}))$ satisfying the strong sector condition we will denote by $S_0^{(0)}(E)$ the set of measures of finite 0-order energy integral, i.e. the subset of $S(E)$ consisting of all measures $\nu \in S(E)$ such that for some $c > 0$,

$$\int_E |\tilde{u}(x)|\,\nu(dx) \le c\mathcal{E}(u,u)^{1/2}, \quad u \in D_e(\mathcal{E}).$$

If $\nu \in S_0^{(0)}(E)$ then from the Lax-Milgram theorem it follows that there is a unique element $\hat{U}\nu$ (called a copotential of $\nu$) such that

$$\mathcal{E}(u, \hat{U}\nu) = \int_E \tilde{u}(x)\,\nu(dx), \quad u \in D_e(\mathcal{E}).$$

By $\hat{S}_{00}^{(0)}(E)$ we denote the subset of $S_0^{(0)}(E)$ consisting of all measures $\nu$ such that $\nu(E) < \infty$ and $\|\hat{U}\nu\|_\infty < \infty$.

### 2.2. Time dependent Dirichlet forms

We assume that we are given a family $\{B^{(t)}, t \in [0,T]\}$ of Dirichlet forms on $H$ with common domain $V$ and sector constant $K$ independent of $t$. We also assume that

(a) $[0,T] \ni t \mapsto B^{(t)}(\varphi,\psi)$ is measurable for every $\varphi,\psi \in V$,
(b) there is a constant $\lambda \ge 1$ such that $\lambda^{-1}B(\varphi,\varphi) \le B^{(t)}(\varphi,\varphi) \le \lambda B(\varphi,\varphi)$ for every $t \in [0,T]$ and $\varphi \in V$, where $B(\varphi,\varphi) = B^{(0)}(\varphi,\varphi)$.

By putting $B^{(t)} = B$ for $t \in \mathbb{R}\setminus[0,T]$ we may and will assume that $B^{(t)}$ is defined and satisfies (a), (b) for $t \in \mathbb{R}$.

By the definition of a Dirichlet form $V$ is a dense subspace of $H$ and $(B,V)$ is closed. Therefore $V$ is a real Hilbert space with respect to $\tilde{B}_1(\cdot,\cdot)$, which is densely and continuously embedded in $H$. By $\|\cdot\|_V$ we denote the norm in $V$, i.e. $\|\varphi\|_V^2 = B_1(\varphi,\varphi)$, $\varphi \in V$. By $V'$ we denote the dual space of $V$ and by $\|\cdot\|_{V'}$ the corresponding norm. We set $\mathcal{H} = L^2(\mathbb{R}; H)$, $\mathcal{V} = L^2(\mathbb{R}; V)$, $\mathcal{V}' = L^2(\mathbb{R}; V')$ and

$$\|u\|_{\mathcal{V}}^2 = \int_{\mathbb{R}} \|u(t)\|_V^2\,dt, \quad \|u\|_{\mathcal{V}'}^2 = \int_{\mathbb{R}} \|u(t)\|_{V'}^2\,dt. \tag{2.2}$$

We shall identify $H$ and its dual $H'$. Then $V \subset H \simeq H' \subset V'$ continuously and densely, and hence $\mathcal{V} \subset \mathcal{H} \simeq \mathcal{H}' \subset \mathcal{V}'$ continuously and densely.

For $u \in \mathcal{V}$ we denote by $\frac{\partial u}{\partial t}$ the derivative in the distribution sense of the function $t \mapsto u(t) \in V$ and we set

$$\mathcal{W} = \{u \in \mathcal{V} : \frac{\partial u}{\partial t} \in \mathcal{V}'\}, \quad \|u\|_{\mathcal{W}} = \|u\|_{\mathcal{V}} + \|\frac{\partial u}{\partial t}\|_{\mathcal{V}'} \tag{2.3}$$

We will consider time dependent Dirichlet forms $\mathcal{E}$ and $\mathcal{E}^{0,T}$ associated with the families $\{(B^{(t)}, V), t \in \mathbb{R}\}$ and $\{(B^{(t)}, V), t \in [0,T]\}$, respectively. We define $\mathcal{E}$ by

$$\mathcal{E}(u,v)=\begin{cases}\langle -\frac{\partial u}{\partial t},v\rangle+\mathcal{B}(u,v), & u\in\mathcal{W}, v\in\mathcal{V},\\ \langle\frac{\partial v}{\partial t},u\rangle+\mathcal{B}(u,v), & u\in\mathcal{V}, v\in\mathcal{W},\end{cases}\tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$ and

$$\mathcal{B}(u,v)=\int_{\mathbb{R}}B^{(t)}(u(t),v(t))\,dt.$$

Note that $\mathcal{E}$ can be identified with some generalized Dirichlet form (see [20, Example I.4.9.(iii)]).

Given a time dependent form (2.4) we define capacity as in [19, Section 6.2], and then using it we define quasi-notions (exceptional sets, nests and quasi-continuity) as in [19, Section 6.2]. Note that by [19, Theorem 6.2.11] each element $u$ of $\mathcal{W}$ has a quasi-continuous $m_1$-version. We will denote it by $\tilde{u}$.

To define $\mathcal{E}^{0,T}$, we set $\mathcal{H}_{0,T}=L^2(0,T;H)$, $\mathcal{V}_{0,T}=L^2(0,T;V)$, $\mathcal{V}'_{0,T}=L^2(0,T;V')$ and $\mathcal{W}_{0,T}=\{u\in\mathcal{V}_{0,T}:\frac{\partial u}{\partial t}\in\mathcal{V}'_{0,T}\}$ (the norms in $\mathcal{V}_{0,T}$, $\mathcal{V}'_{0,T}$, $\mathcal{W}_{0,T}$ are defined analogously to (2.2), (2.3)). Let $C([0,T];H)$ denote the space of all continuous functions on $[0,T]$ with values in $H$ equipped with the norm $\|u\|_C=\sup_{0\le t\le T}\|u(t)\|_H$. It is known (see, e.g., [21, Theorem 2] that there is a continuous embedding of $\mathcal{W}_{0,T}$ into $C([0,T];H)$, i.e. for every $u\in\mathcal{W}_{0,T}$ one can find $\bar{u}\in C([0,T];H)$ such that $u(t)=\bar{u}(t)$ for a.e. $t\in[0,T]$ (with respect to the Lebesgue measure) and

$$\|\bar{u}\|_C\le M\|u\|_{\mathcal{W}_{0,T}}\tag{2.5}$$

for some $M>0$. In what follows we adopt the convention that any element of $\mathcal{W}_{0,T}$ is already in $C([0,T];H)$. With this convention we may define the spaces

$$\mathcal{W}_0=\{u\in\mathcal{W}_{0,T}:u(0)=0\},\quad \mathcal{W}_T=\{u\in\mathcal{W}_{0,T}:u(T)=0\}.$$

By the definition of $\mathcal{W}_{0,T}$, $\partial/\partial t:\mathcal{W}_{0,T}\to\mathcal{V}'_{0,T}$ is bounded. Since $\mathcal{W}_0$ is dense in $\mathcal{V}_{0,T}$, we can regard the restriction of $\partial/\partial t$ to $\mathcal{W}_0$ as an unbounded operator from $\mathcal{V}_{0,T}$ to $\mathcal{V}'_{0,T}$ defined on $\mathcal{W}_0$. Its adjoint is defined on $\mathcal{W}_T$ and is given by $-\partial/\partial t$ (see, e.g., [21]). Finally, we set

$$\mathcal{E}^{0,T}(u,v)=\begin{cases}\langle -\frac{\partial u}{\partial t},v\rangle+\int_0^T B^{(t)}(u(t),v(t))\,dt, & u\in\mathcal{W}_T, v\in\mathcal{V}_{0,T},\\ \langle\frac{\partial v}{\partial t},u\rangle+\int_0^T B^{(t)}(u(t),v(t))\,dt, & u\in\mathcal{V}_{0,T}, v\in\mathcal{W}_0,\end{cases}\tag{2.6}$$

where now $\langle\cdot,\cdot\rangle$ denote the duality pairing between $\mathcal{V}'_{0,T}$ and $\mathcal{V}_{0,T}$. As in the case of $\mathcal{E}$, the form $\mathcal{E}^{0,T}$ can be identified with some generalized Dirichlet form (see [20, Example I.4.9.(iii)]).

By Propositions I.3.4 and I.3.6 in [20] the form $\mathcal{E}^{0,T}$ determines uniquely strongly continuous resolvents $(G^{0,T}_\alpha)_{\alpha>0}$, $(\hat{G}^{0,T}_\alpha)_{\alpha>0}$ on $\mathcal{H}_{0,T}$ such that $G^{0,T}_\alpha$, $\hat{G}^{0,T}_\alpha$ are sub-Markov, $G^{0,T}_\alpha(\mathcal{H}_{0,T})\subset\mathcal{W}_T$, $\hat{G}^{0,T}_\alpha(\mathcal{H}_{0,T})\subset\mathcal{W}_0$ and

$$\mathcal{E}^{0,T}_\alpha(G^{0,T}_\alpha\eta,u)=(u,\eta)_{\mathcal{H}_{0,T}},\quad \mathcal{E}^{0,T}_\alpha(u,\hat{G}^{0,T}_\alpha\eta)=(u,\eta)_{\mathcal{H}_{0,T}}$$

for $u\in\mathcal{V}_{0,T}$ and $\eta\in\mathcal{H}_{0,T}$, where $\mathcal{E}^{0,T}_\alpha(u,v)=\mathcal{E}^{0,T}(u,v)+\alpha(u,v)_{\mathcal{H}_{0,T}}$ for $\alpha\ge 0$.

### 2.3. Markov processes and additive functionals

In what follows $\Delta$ is a one-point compactification of $E$. If $E$ is already compact then we adjoin $\Delta$ to $E$ as an isolated point.

In the case of Dirichlet forms (and elliptic equations) we adopt the convention that every function $f$ on $E$ is extended to $E\cup\{\Delta\}$ by setting $f(\Delta)=0$.

In the case of time dependent Dirichlet forms (and parabolic equations) we adopt the convention that every function $\varphi$ on $E$ is extended to $E^1$ by setting $\varphi(t,x)=\varphi(x)$, $(t,x)\in E^1$, and every function $f$ on $E^1$ (resp. $E_{0,T}$) is extended to $E^1\cup\{\Delta\}$ by setting $f(\Delta)=0$ (resp. $f(z)=0$ for $z\in E^1\cup\{\Delta\}\setminus E_{0,T}$).

**Dirichlet forms.** Let $(\mathcal{E},D(\mathcal{E}))$ be a regular Dirichlet form on $H$. Then there exists a unique Hunt process $\mathbb{M}=(\Omega,(\mathcal{F}_t)_{t\ge0},(X_t)_{t\ge0},\zeta,(P_x)_{x\in E\cup\{\Delta\}})$ with state space $E$, life time $\zeta$ and cemetery state $\Delta$ properly associated with $(\mathcal{E},D(\mathcal{E}))$ (see Theorems IV.3.5, IV.6.4 and V.2.13 in [4]). The last statement means that for every $\alpha>0$ and $f\in\mathcal{B}_b(E)\cap H$ the resolvent of $\mathbb{M}$, that is the function

$$R_\alpha f(x)=E_x\int_0^\infty e^{-\alpha t}f(X_t)\,dt,\quad x\in E,\ \alpha>0$$

is a quasi-continuous $m$-version of $G_\alpha f$ (see [4, Proposition IV.2.8]).

It is known (see, e.g., [4, Theorem VI.2.4]) that there is a one to one correspondence (called Revuz correspondence) between smooth measures $\mu$ and positive continuous additive functionals (positive CAFs) $A$ of $\mathbb{M}$. It is given by the following relation

$$\lim_{t\to0^+}\frac{1}{t}E_m\int_0^t f(X_s)\,dA_s=\int_E f(x)\,\mu(dx),\quad f\in\mathcal{B}^+(E),\tag{2.7}$$

where $E_m$ denotes the expectation with respect to the measure $P_m(\cdot)=\int_E P_x(\cdot)\,m(dx)$. In what follows the positive CAF of $\mathbb{M}$ corresponding to $\mu\in S(E)$ will be denoted by $A^\mu$.

For $\mu\in S(E)$ we set

$$R\mu(x)=E_x\int_0^\zeta dA^\mu_t,\quad x\in E$$

and

$$\mathcal{R}(E)=\{\mu:|\mu|\in S(E),R|\mu|<\infty\quad m\text{-a.e.}\},$$

where $|\mu|$ denotes the total variation of $\mu$. Note that by [3, Lemma 2.3], in the above definition of the class $\mathcal{R}(E)$ one can replace $m$-a.e. by q.e. By $\mathcal{M}_{0,b}(E)$ we denote the space of all signed measures $\mu$ on $E$ such that $|\mu|\in S(E)$ and $|\mu|(E)<\infty$. By [3, Proposition 3.2], if $(\mathcal{E},D(\mathcal{E}))$ is transient then $\mathcal{M}_{0,b}(E)\subset\mathcal{R}(E)$.

**Time dependent Dirichlet forms.** Let us consider the time dependent Dirichlet form $\mathcal{E}$ defined by (2.4). Then by [19, Theorem 6.3.1] there exists a Hunt process $\mathbf{M}=(\Omega,(\mathcal{F}_t)_{t\ge0},(\mathbf{X}_t)_{t\ge0},\zeta,(P_z)_{z\in E^1\cup\{\Delta\}})$ with state space $E^1$, life time $\zeta$ and cemetery state $\Delta$ properly associated with $\mathcal{E}$ in the sense that for every $\alpha > 0$ and $f \in \mathcal{B}_b \cap L^2(E^1; m_1)$ the resolvent of $\mathbf{M}$ defined as

$$R_\alpha f(z) = E_z \int_0^\infty e^{-\alpha t} f(\mathbf{X}_t)\, dt, \quad x \in E,\, \alpha > 0$$

is a quasi-continuous version of the resolvent $G_\alpha f$ associated with $\mathcal{E}$. Moreover, by [19, Theorem 6.3.1],

$$\mathbf{X}_t = (\tau(t), X_{\tau(t)}), \quad t \geq 0,$$

where $\tau(t)$ is the uniform motion to the right, i.e. $\tau(t) = \tau(0) + t$, $\tau(0) = s$, $P_z$-a.s. for $z = (s, x)$.

Let $S(E^1)$ denote the set of smooth measures on $E^1$ (with respect to $\mathcal{E}$), which we define analogously to $S(E)$ (see, e.g., [18,20] for details). We say that a positive AF $A$ of $\mathbf{M}$ is in the Revuz correspondence with $\mu \in S(E^1)$ if

$$\lim_{\alpha \to \infty} \alpha E_{m_1} \int_0^\infty e^{-\alpha t} f(\mathbf{X}_t)\, dA_t = \int_{E^1} f(z)\, \mu(dz), \quad f \in \mathcal{B}_b^+(E^1),$$

where $E_{m_1}$ denotes the expectation with respect to $P_{m_1}(\cdot) = \int_{E^1} P_z(\cdot)\, m_1(dz)$ (see [22,23]).

It is known (see [18, Section 2]) that for every $\mu \in S(E^1)$ there exists a unique positive natural AF $A$ of $\mathbf{M}$, i.e. a positive AF of $\mathbf{M}$ such that $A$ and $\mathbf{M}$ have no common discontinuities, such that $A$ is in the Revuz correspondence with $\mu$. In what follows we will denote it by $A^\mu$. In fact, $A^\mu$ is a predictable process (see [24]). On the contrary, if $A$ is a positive natural AF of $\mathbf{M}$ then by Proposition in Section II.1 of [23] and [22, Theorem 5.6] there exists a smooth measure $\mu$ on $E^1$ such that $A$ is in the Revuz correspondence with $\mu$.

Let $S(E_{0,T})$ denote the set of all $\mu \in S(E^1)$ with support in $E_{0,T}$ and for $\mu \in S(E_{0,T})$ let

$$R^{0,T}\mu(z) = E_z \int_0^{\zeta_\tau} dA_t^\mu, \quad z \in E_{0,T},$$

where

$$\zeta_\tau = \zeta \wedge (T - \tau(0)). \tag{2.8}$$

We set

$$\mathcal{R}(E_{0,T}) = \{\mu : |\mu| \in S(E_{0,T}), R^{0,T}|\mu| < \infty \quad m_1\text{-a.e.}\}$$

and by $\mathcal{M}_{0,b}(E_{0,T})$ we denote the space of all signed measures $\mu$ on $E^1$ such that $|\mu| \in S(E_{0,T})$ and $|\mu|(E^1) < \infty$. Note that by [18, Proposition 3.8], $\mathcal{M}_{0,b}(E_{0,T}) \subset \mathcal{R}(E_{0,T})$.

## 3. Elliptic equations

Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form on $H$. We consider the problem

$$-Lu = f_u + \mu, \tag{3.1}$$

where $f : E \times \mathbb{R} \to \mathbb{R}$ is a measurable function, $f_u = f(\cdot, u)$, $\mu \in \mathcal{R}(E)$ and $L$ is the operator associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e.

$$D(L) = \{u \in D(\mathcal{E}) : v \mapsto \mathcal{E}(u, v) \text{ is continuous w.r.t.} (\cdot, \cdot)_H^{1/2} \text{on } D(\mathcal{E})\}$$

and

$$(-Lu, v)_H = \mathcal{E}(u, v), \quad u \in D(L), v \in D(\mathcal{E}) \tag{3.2}$$

(see [4, Proposition I.2.16]).

In what follows $\mathbb{M}$ is the Markov process of Sect. 2 associated with $(\mathcal{E}, D(\mathcal{E}))$. Let us recall that a càdlàg adapted (with respect to $(\mathcal{F}_t)$) process $Y$ is said to be of class (D) if the collection $\{Y_\tau, \tau \text{ is a finite } (\mathcal{F}_t)\text{-stopping time}\}$ is uniformly integrable.

**Definition 3.1.** Let $f : E \times \mathbb{R} \to \mathbb{R}$ be a measurable function and let $A^\mu$ be a CAF of $\mathbb{M}$ corresponding to some $\mu \in \mathcal{R}(E)$. We say that a pair $(Y^x, M^x)$ is a solution of the backward stochastic differential equation

$$Y_t^x = Y_{T\wedge\zeta}^x + \int_{t\wedge\zeta}^{T\wedge\zeta} f(X_s, Y_s^x)\, ds + \int_{t\wedge\zeta}^{T\wedge\zeta} dA_s^\mu - \int_{t\wedge\zeta}^{T\wedge\zeta} dM_s^x, \quad t \geq 0 \tag{3.3}$$

under the measure $P_x$ if

(a) $Y^x$ is an $(\mathcal{F}_t)$-progressively measurable càdlàg process such that $Y_{t\wedge\zeta}^x \to 0$, $P_x$-a.s. as $t \to \infty$, $Y^x$ is of class (D) under $P_x$ and $M^x$ is a càdlàg $(\mathcal{F}_t)$-local martingale under $P_x$,

(b) For every $T > 0$, $[0, T] \ni t \mapsto f(X_t, Y_t^x) \in L^1(0, T)$ and (3.3) is satisfied $P_x$-a.s.

The following definition is taken from [2,3].

**Definition 3.2.** Let $\mu \in \mathcal{R}(E)$. We say that a quasi-continuous function $u : E \to \mathbb{R}$ is a probabilistic solution to (1.1) if $f_u \cdot m \in \mathcal{R}(E)$ and for q.e. $x \in E$,

$$u(x) = E_x \left( \int_0^\zeta f_u(X_t)\, dt + \int_0^\zeta dA_t^\mu \right). \tag{3.4}$$

*Remark* 3.3. (i) The quasi-continuity requirement on $u$ in the above definition can be omitted, because if $\mu, f_u \cdot m \in \mathcal{R}(E)$ then from the very definition of the class $\mathcal{R}(E)$ it follows that the right-hand side of (3.4) is finite for $m$-a.e. $x \in E$, and, in consequence, it is a quasi-continuous function of $x$ (see [2, Lemma 4.3] and [3, Lemma 2.3]).

(ii) If $u$ is a probabilistic solution to (1.1) then there exists a martingale additive functional (MAF) $M$ of $\mathbb{M}$ such that $M$ is a martingale under $P_x$ for q.e. $x \in E$ and for q.e. $x \in E$ the pair

$$(Y_t, M_t) = (u(X_t), M_t), \quad t \geq 0$$

is a solution of (3.3) under $P_x$. Indeed, with our convention (see the beginning of Sect. 2.3),

$$u(x) = E_x \left( \int_0^\infty f_u(X_t)\, dt + \int_0^\infty dA_t^\mu \right).$$

Set

$$M^x_t=E_x\left(\int_0^\zeta f_u(X_s)\,ds+\int_0^\zeta dA^\mu_s|\mathcal{F}_{t\wedge\zeta}\right)-u(X_0),\quad t\ge 0.$$

By [1, Lemma A.3.6] there exists a MAF $M$ od $\mathbb{M}$ such that $M^x_t=M_t$, $t\ge 0$, $P_x$-a.s. for q.e. $x\in E$. Therefore

$$M_t=M_{t\wedge\zeta}=E_x\left(\int_0^\zeta f_u(X_s)\,ds+\int_0^\zeta dA^\mu_s|\mathcal{F}_{t\wedge\zeta}\right)-u(X_0),\quad t\ge 0 \tag{3.5}$$

under $P_x$ for q.e. $x\in E$. By the strong Markov property, under $P_x$ we have

$$\begin{aligned}M_{t\wedge\zeta}&=\int_0^{t\wedge\zeta}(f_u(X_s)\,ds+dA^\mu_s)+E_x\left(\int_{t\wedge\zeta}^\zeta(f_u(X_s)\,ds+dA^\mu_s)|\mathcal{F}_{t\wedge\zeta}\right)\\&\quad-u(X_0)\\&=\int_0^{t\wedge\zeta}(f_u(X_s)\,ds+dA^\mu_s)+E_{X_{t\wedge\zeta}}\int_0^\zeta(f_u(X_s)\,ds+dA^\mu_s)-u(X_0)\\&=\int_0^{t\wedge\zeta}(f_u(X_s)\,ds+dA^\mu_s)+u(X_{t\wedge\zeta})-u(X_0)\end{aligned}$$

for q.e. $x\in E$. Hence

$$u(X_{t\wedge\zeta})-u(X_{T\wedge\zeta})=\int_{t\wedge\zeta}^{T\wedge\zeta}(f_u(X_s)\,ds+dA^\mu_s)-\int_{t\wedge\zeta}^{T\wedge\zeta}dM_s,$$

which shows (3.3). Taking $t=0$ in the above equality and using (3.5) we get

$$u(X_{T\wedge\zeta})=-\int_0^{T\wedge\zeta}(f_u(X_s)\,ds+dA^\mu_s)+E_x\left(\int_0^\zeta f_u(X_s)\,ds+\int_0^\zeta dA^\mu_s|\mathcal{F}_{T\wedge\zeta}\right).$$

It follows that for q.e. $x\in E$, $u(X_{T\wedge\zeta})\to 0$, $P_x$-a.s. as $T\to\infty$.

In what follows we assume that $(\mathcal{E},D(\mathcal{E}))$ is transient and satisfies the strong sector condition. For a measure $\mu$ on $E$ and a function $u:E\to\mathbb{R}$ we use the notation

$$\langle\mu,u\rangle=\int_E u(x)\,\mu(dx)$$

whenever the integral is well defined.

We adopt the following definition of renormalized solution of (1.1). In the case of local operators, this is essentially [7, Definition 2.29].

**Definition 3.4.** Let $\mu\in\mathcal{M}_{0,b}(E)$. We say that $u:E\to\mathbb{R}$ is a renormalized solution of (1.1) if

(a) $u$ is quasi-continuous, $f_u\in L^1(E;m)$ and $T_ku\in D_e(\mathcal{E})$ for every $k>0$,
(b) there exists a sequence $\{\nu_k\}\subset\mathcal{M}_{0,b}(E)$ such that $\|\nu_k\|_{TV}\to 0$ as $k\to\infty$ and for every $k\in\mathbb{N}$ and every bounded $v\in D_e(\mathcal{E})$,

$$\mathcal{E}(T_ku,v)=\langle f_u\cdot m+\mu,\tilde v\rangle+\langle\nu_k,\tilde v\rangle. \tag{3.6}$$

**Theorem 3.5.** *Assume that* $(\mathcal{E}, D(\mathcal{E}))$ *is transient, satisfies the strong sector condition and that* $\mu \in \mathcal{M}_{0,b}(E)$.

(i) *If* $u$ *is a probabilistic solution of* (1.1) *and* $f_u \in L^1(E;m)$ *then* $u$ *is a renormalized solution of* (1.1).

(ii) *If* $u$ *is a renormalized solution of* (1.1) *then* $u$ *is a probabilistic solution of* (1.1).

*Proof.* (i) Let $u$ be a probabilistic solution of (1.1) and let $M$ be the martingale of Remark 3.3. (ii) For $k > 0$ put

$$Y_t = u(X_t), \quad Y_t^k = T_k u(X_t), \quad t \geq 0.$$

From the fact that $(Y, M)$ is a solution of (3.3) it follows that

$$Y_t = Y_{t\wedge\zeta} = Y_0 - \int_0^{t\wedge\zeta} f(X_s, Y_s)\,ds - \int_0^{t\wedge\zeta} dA_s^\mu + \int_0^{t\wedge\zeta} dM_s, \quad t \geq 0. \tag{3.7}$$

By the Meyer–Itô formula (see, e.g., [25, Theorem IV.70]),

$$(u \wedge k)(X_t) - (u \wedge k)(X_0) = \int_0^t \mathbf{1}_{\{Y_{s-} \leq k\}}\,dY_s - A_t^{1,k} \tag{3.8}$$

and

$$(u(X_t) + k) \wedge 0 - (u(X_0) + k) \wedge 0 = \int_0^t \mathbf{1}_{\{Y_{s-} \leq -k\}}\,dY_s - A_t^{2,k} \tag{3.9}$$

for some increasing processes $A^{1,k}, A^{2,k}$. Since $T_k y = y \wedge k - ((y + k) \wedge 0)$ for $y \in \mathbb{R}$, it follows from (3.8) and (3.9) that

$$Y_t^k - Y_0^k = \int_0^t \mathbf{1}_{\{-k < Y_{s-} \leq k\}}\,dY_s - (A_t^{1,k} - A_t^{2,k}). \tag{3.10}$$

From (3.8), it follows immediately that $A^{1,k}, A^{2,k}$ are AFs of $\mathbb{M}$. Since $u$ is a probabilistic solution, $u(X_t) \to 0$, $P_x$-a.s. as $t \to \infty$ for q.e. $x \in E$. Therefore from (3.8) and continuity of $A^\mu$ we conclude that for q.e. $x \in E$,

$$E_x A_\zeta^{1,k} = E_x(u \wedge k)(X_0) - E_x \int_0^\zeta \mathbf{1}_{\{Y_s \leq k\}}(f_u(X_s)\,ds + dA_s^\mu).$$

Since $E_x(u \wedge k)(X_0) = (u \wedge k)(x) \leq u(x)$ and $u$ is a probabilistic solution of (1.1), it follows from the above that

$$E_x A_\zeta^{1,k} \leq E_x \int_0^\zeta \mathbf{1}_{\{Y_s > k\}}(f_u(X_s)\,ds + dA_s^\mu). \tag{3.11}$$

Similarly, by (3.9) we have

$$E_x A_\zeta^{2,k} \leq -E_x \int_0^\zeta \mathbf{1}_{\{Y_s \leq -k\}}(f_u(X_s)\,ds + dA_s^\mu). \tag{3.12}$$

It follows that for q.e. $x \in E$, $E_x(A_\zeta^{1,k} + A_\zeta^{2,k}) < \infty$. Therefore by [1, Theorem A.3.16] there exists AFs $B^{1,k}, B^{2,k}$ of $\mathbb{M}$ such that $B^{i,k}$ is a compensator of $A^{i,k}$, $i = 1, 2$ under $P_x$ for q.e. $x \in E$. Since $A^{1,k}, A^{2,k}$ are increasing, $B^{1,k}, B^{2,k}$ are increasing, too. Furthermore, since by [1, Theorem A.3.2] the process $X$ has no predictable jumps, it follows from [1, Theorem A.3.5] that $B^{1,k}, B^{2,k}$ are continuous. Thus $B^{1,k}, B^{2,k}$ are increasing CAFs of $\mathbb{M}$ such that $A^{i,k} - B^{i,k}$, $i = 1, 2$, are martingales under $P_x$ for q.e. $x \in E$. Let $b^i_k$, $i = 1, 2$, denote the Revuz measure of $B^{i,k}$. Since for q.e. $x \in E$, $E_x A^{i,k}_t = E_x B^{i,k}_t$ for $t \geq 0$, from (3.11), (3.12) and [2, Lemma 5.4] we conclude that

$$b^1_k(E) \leq \|\mathbf{1}_{\{u>k\}} f_u\|_{L^1(E;m)} + \|\mathbf{1}_{\{u>k\}} \cdot \mu\|_{TV},$$

$$b^2_k(E) \leq \|\mathbf{1}_{\{u\leq -k\}} f_u\|_{L^1(E;m)} + \|\mathbf{1}_{\{u\leq -k\}} \cdot \mu\|_{TV}.$$

Hence $b^1_k, b^2_k \in \mathcal{M}_{0,b}(E)$ and $\|b^1_k\|_{TV} \to 0$, $\|b^2_k\|_{TV} \to 0$ as $k \to \infty$. Combining (3.7) with (3.10) we obtain

$$Y^k_t - Y^k_0 = -\int_0^t \mathbf{1}_{\{-k<Y_{s-}\leq k\}} (f_u(X_s)\,ds + dA^\mu_s) - (B^{1,k}_t - B^{2,k}_t) + M^k_t \tag{3.13}$$

with

$$M^k_t = \int_0^t \mathbf{1}_{\{-k<Y_{s-}\leq k\}}\,dM_s - (A^{1,k}_t - B^{1,k}_t) + A^{2,k}_t - B^{2,k}_t.$$

From (3.13) and the fact that $u(X_t) \to 0$, $P_x$-a.s. as $t \to \infty$ for q.e. $x \in E$ it follows that for q.e. $x \in E$,

$$T_k u(x) = E_x\left(\int_0^\zeta (f_u(X_t)\,dt + dA^\mu_t) + \int_0^\zeta dA^{\nu_k}_t\right) \tag{3.14}$$

with

$$\nu_k = -\mathbf{1}_{\{u\notin(-k,k]\}}(f_u \cdot m + \mu) + b^1_k - b^2_k.$$

Clearly $\nu_k \in \mathcal{M}_{0,b}(E)$ and $\|\nu_k\|_{TV} \to 0$ as $k \to \infty$. By [3, Theorem 4.2] (see also [2, Proposition 5.9] in the case of regular symmetric forms), $T_k u \in D_e(\mathcal{E})$ for every $k > 0$. Let $\lambda_k = f_u \cdot m + \mu + \nu_k$ and let $A = A^{\lambda_k}$. Since $\lambda_k \in \mathcal{M}_{0,b}(E)$, $R|\lambda_k|(x) < \infty$ for q.e. $x \in E$. By Fubini's theorem, for q.e. $x \in E$ we have

$$\begin{aligned} R\lambda_k(x) - R_\alpha\lambda_k(x) &= E_x \int_0^\infty (1 - e^{-\alpha t})\,dA_t \\ &= E_x \int_0^\infty \left(\int_0^t \alpha e^{-\alpha s}\,ds\right) dA_t \\ &= \alpha E_x \int_0^\infty e^{-\alpha t}\left(\int_t^\infty dA_s\right) dt. \end{aligned}$$

By the Markov property the right-hand side of the above equality equals

$$\begin{aligned} \alpha E_x \int_0^\infty e^{-\alpha t}\left(\int_0^\infty d(A_s \circ \theta_t)\right) dt &= \alpha E_x \int_0^\infty e^{-\alpha t} E_{X_t}\left(\int_0^\infty dA_s\right) dt \\ &= \alpha E_x \int_0^\infty e^{-\alpha t} R\lambda_k(X_t)\,dt \\ &= \alpha R_\alpha(R\lambda_k)(x). \end{aligned}$$

Hence

$$R\lambda_k(x) - R_\alpha\lambda_k(x) = \alpha R_\alpha(R\lambda_k)(x)$$

for q.e. $x \in E$. By (3.14), $T_k u = R\lambda_k$. Therefore from the above generalized resolvent equation it follows that for every bounded $v \in D(\mathcal{E})$ we have

$$\alpha(T_k u - \alpha R_\alpha(T_k u), v)_H = \alpha(R_\alpha \lambda_k, v)_H. \tag{3.15}$$

By [4, Theorem 2.13(iii)] the left-hand side of (3.15) converges to $\mathcal{E}(T_k u, v)$ as $\alpha \to \infty$. The right-hand side is equal to $\langle \lambda_k, \alpha \hat{R}_\alpha v\rangle$, where $\hat{R}_\alpha$ denotes the resolvent of a Hunt process associated with the form $(\hat{\mathcal{E}}, D(\mathcal{E}))$ defined as $\hat{\mathcal{E}}(u,v) = \mathcal{E}(v,u)$, $u, v \in D(\mathcal{E})$. Since the functions $\alpha\hat{R}_\alpha v$ are bounded uniformly in $\alpha > 0$ and by Propositions I.2.13(ii) and III.3.5 in [4] we may assume that $\alpha\hat{R}_\alpha v \to \tilde{v}$ q.e. as $\alpha \to \infty$, the right-hand side of (3.15) converges to $\langle \lambda_k, \tilde{v}\rangle$ as $\alpha \to \infty$. Thus (3.6) is satisfied for bounded $v \in D(\mathcal{E})$. Now assume that $v$ is bounded, say by $l$, and $v \in D_e(\mathcal{E})$. Let $v_n$ be an approximating sequence for $v$. Then $T_l(v_n) \to v$ $m$-a.e. and, by [1, Corollary 1.6.3], in $(D_e(\mathcal{E}), \tilde{\mathcal{E}})$ as $n \to \infty$. Taking a subsequence if necessary we may assume that $T_l v_n \to \tilde{v}$ q.e. By what has already been proved, for $n \in \mathbb{N}$ we have

$$\mathcal{E}(T_k u, T_l v_n) = \langle \lambda_k, T_l \tilde{v}_n\rangle.$$

Letting $n \to \infty$ we get (3.6), which completes the proof of (i).

(ii) If $u$ is a renormalized solution of (1.1) then $u$ is quasi-continuous and (3.6) is satisfied for all functions $v$ of the form $v = \hat{U}\nu$ with $\nu \in \hat{S}^{(0)}_{00}(E)$. Hence

$$\langle \nu, T_k u\rangle = \mathcal{E}(T_k u, \hat{U}\nu) = \langle f_u \cdot m + \mu + \nu_k, \widetilde{\hat{U}\nu}\rangle.$$

Therefore $T_k u$ is a solution in the sense of duality (see [3, Section 3.3] or [2, Section 5] for the definition) of the linear problem

$$-L(T_k u) = f_u + \mu + \nu_k. \tag{3.16}$$

By [3, Proposition 3.9] (or [2, Proposition 5.3] in the case of symmetric forms) $T_k u$ is a probabilistic solution of (3.16). In particular (3.14) (with the measure $\nu_k$ of (3.16)) is satisfied. Since $\|\nu_k\|_{TV} \to 0$ as $k \to \infty$, there is a subsequence (still denoted by $k$) such that

$$R\nu_k(x) = E_x \int_0^\zeta dA^{\nu_k}_t \to 0 \tag{3.17}$$

for $m$-a.e. $x \in E$. To see this, let us first observe that if $\mu \in S^{(0)}_0(E)$ and $\tilde{u} \le c$ $\mu$-a.e., where $u = \hat{U}\mu$, then $u \le c$ $m$-a.e. Indeed, we have

$$\mathcal{E}(u \wedge c, u) = \mathcal{E}(u \wedge c, \hat{U}\mu) = \int_E (\tilde{u} \wedge c)\, \mu(dx) = \int_E \tilde{u}\, \mu(dx) = \mathcal{E}(u,u).$$

Hence

$$\mathcal{E}(u - u\wedge c, u - u\wedge c) = \mathcal{E}(u - u\wedge c, u) - \mathcal{E}(u - u\wedge c, u\wedge c) \le 0, \tag{3.18}$$

the last inequality being a consequence of [4, Theorem I.4.4] and the fact that $\mathcal{E}$ is a Dirichlet form. By (3.18) and [1, Theorem 1.6.2], $u - u\wedge c = 0$ $m$-a.e., which shows that $u \le c$ $m$-a.e. Since $m \in S(E)$, by the 0-order version of [1, Theorem 2.2.4] (see the comment following [1, Corollary 2.2.2]) there exists a generalized nest $\{F_n\}$ such that $\mu_n := \mathbf{1}_{F_n} \cdot m \in S^{(0)}_0(E)$ for $n \in \mathbb{N}$

and $m(E\setminus\bigcup_{n=1}^{\infty}F_n)=0$. Let $F_{n,N}=\{x\in F_n:\widetilde{\hat{U}\mu_n}(x)\le N\}$ and $\mu_{n,N}=\mathbf{1}_{F_{n,N}}\cdot\mu_n=\mathbf{1}_{F_{n,N}}\cdot m$. Then $\mu_{n,N}\in S_0^{(0)}(E)$ and $\hat{U}\mu_{n,N}\le\widetilde{\hat{U}\mu_n}\le N$ $\mu_n$-a.e. Therefore by the observation made above, $\hat{U}\mu_{n,N}\le N$ $m$-a.e., and hence $\widetilde{\hat{U}\mu_{n,N}}\le N$ q.e. Moreover,

$$\int_{F_{n,N}}R|\nu_k|(x)\,m(dx)=\mathcal{E}(R|\nu_k|,\hat{U}\mu_{n,N})=\langle|\nu_k|,\widetilde{\hat{U}\mu_{n,N}}\rangle \le\|\nu_k\|_{TV}\|\hat{U}\mu_{n,N}\|_{\infty}.$$

Hence for every $n,N\in\mathbb{N}$,

$$\lim_{k\to\infty}\int_{F_{n,N}}R|\nu_k|(x)\,m(dx)=0. \tag{3.19}$$

Since $\widetilde{\hat{U}\mu_n}$ is q.e. finite, $m(F_n\setminus\bigcup_{N=1}^{\infty}F_{n,N})=m(\{x\in F_n:\widetilde{\hat{U}\mu_n}(x)=\infty\}=0$. Therefore from (3.19) one can deduce that (3.17) holds for $m$-a.e. $x\in F_n$ for each $n\in\mathbb{N}$. Since $m(E\setminus\bigcup_{n=1}^{\infty}F_n)=0$, we see that (3.17) holds for $m$-a.e. $x\in E$. Letting $k\to\infty$ in (3.14) and using (3.17) we conclude that (3.4) holds true for $m$-a.e. $x\in E$. In fact, since $u$ and the right-hand side of (3.4) are quasi-continuous, (3.4) holds for q.e. $x\in E$, which completes the proof. □

To illustrate the utility of Theorem 3.5 we now give some results on existence and uniqueness of renormalized solutions of (1.1) with $f$ satisfying the monotonicity condition and mild integrability conditions. To state the results we will need the following hypotheses.

(E1) $f:E\times\mathbb{R}\to\mathbb{R}$ is measurable and $y\mapsto f(x,y)$ is continuous for every $x\in E$,
(E2) $(f(x,y_1)-f(x,y_2))(y_1-y_2)\le 0$ for every $y_1,y_2\in\mathbb{R}$ and $x\in E$,
(E3) $\mu\in\mathcal{M}_{0,b}(E)$ and $f(\cdot,y)\in L^1(E;m)$ for every $y\in\mathbb{R}$.

In what follows we assume that $(\mathcal{E},D(\mathcal{E}))$ satisfies the assumptions of Theorem 3.5.

**Theorem 3.6.** *Let $u_1,u_2$ be renormalized solutions of* (1.1) *with the data* $(f^1,\mu_1)$ *and* $(f^2,\mu_2)$, *respectively. Assume that $\mu_1\le\mu_2$ and either that $f^1(x,u_1(x))\le f^2(x,u_1(x))$ $m$-a.e. and $f^2$ satisfies* (E2) *or $f^1(x,u_2(x))\le f^2(x,u_2(x))$ $m$-a.e. and $f^1$ satisfies* (E2). *Then $u_1(x)\le u_2(x)$ for q.e. $x\in E$.*

*Proof.* Follows from Theorem 3.5 and [2, Proposition 4.9]. □

**Corollary 3.7.** *If* (E2) *is satisfied then there exists at most one renormalized solution of* (1.1).

**Theorem 3.8.** *Assume* (E1)–(E3). *Then there exists renormalized solution of* (1.1).

*Proof.* Follows from Theorem 3.5 and [3, Theorem 3.8, Proposition 3.10] (see also [2, Theorem 5.14]). □

We close this section with some general remarks on possible generalization of our results and on their applicability.

An inspection of the proof of Theorem 3.5 reveals that it only makes use of some general results from the theory of stochastic processes that are valid for general semimartingals, some results from [3], which are proved for quasi-regular forms and the fact that $\mathbb{M}$ associated with $\mathcal{E}$ in the resolvent sense is a standard process (the fact that $\mathbb{M}$ a Hunt process is not needed). Therefore the proof of Theorem 3.5 carries over to quasi-regular Dirichlet forms.

By using probabilistic methods one can prove that for many interesting equations there exists a unique probabilistic solution $u$ such that $f_u \in L^1(E;m)$. This can be done for instance for $f$ satisfying (E1)–(E3). For specific examples of local and nonlocal operators $A$ satisfying our assumptions see, e.g., [1–4]. Then as in Corollary 3.7 and Theorems 3.8 above, a direct consequence of Theorem 3.5 is that $u$ is a renormalized solution to (1.1), i.e. has clear analytical meaning, and that $u$ is the unique renormalized solution. On the other hand, renormalized solutions to (1.1), which are obtained by analytical methods, automatically have stochastic representation of the form (3.4). We may then use (3.4) (and the theory of BSDEs; see Remark 3.3) to study further properties of the solution by probabilistic methods. For instance, probabilistic methods are quite effective in proving comparison results and hence uniqueness.

It would be desirable to define renormalized solutions for equations with general bounded measures $\mu$, at least for some classes of nonlocal operators. Another interesting open problem is to give other equivalent to Definition 3.4 analytical definitions of a solution (like in the local case considered in [7]).

## 4. Parabolic equations

In this section we assume that the family $\{B^{(t)}, t \in [0,T]\}$ satisfies the assumptions of Sect. 2.2 and $\mathcal{E}^{0,T}$ is the time dependent Dirichlet form defined by (2.6). By $L_t$ we denote the operator associated with $B^{(t)}$ in the sense of (3.2) and by $\frac{\partial u}{\partial t} + L_t$ the operator corresponding to $\mathcal{E}^{0,T}$, i.e. the generator of the strongly continuous contraction semigroup corresponding to $(G^{0,T}_\alpha)_{\alpha>0}$.

We consider the Cauchy problem

$$-\frac{\partial u}{\partial t} - L_t u = f_u + \mu, \quad u(T) = \varphi, \tag{4.1}$$

where $\varphi : E \to \mathbb{R}$ is a measurable function such that $\delta_{\{T\}} \otimes \varphi \cdot m \in \mathcal{R}(E_{0,T})$, $\mu \in \mathcal{R}(E_{0,T})$, $f : [0,T] \times E \times \mathbb{R} \to \mathbb{R}$ is measurable function and $f_u = f(\cdot,\cdot,u)$.

In what follows we maintain the notation of Sect. 2.2 and the second part of Sect. 2.3 concerning time dependent forms and associated Markov processes. In particular, $\mathbf{M}$ is a Markov process associated with $\mathcal{E}$ and $\zeta_\tau$ is defined by (2.8). By abuse of notation, in this section

$$\langle \mu, u \rangle = \int_{E_{0,T}} u(z)\,\mu(dz)$$

for $u: E_{0,T}\to\mathbb{R}$ and $\mu\in\mathcal{R}(E_{0,T})$.

We will say that a Borel measurable function $u$ on $E_{0,T}$ is quasi-càdlàg if for q.e. $z\in E_{0,T}$ the process $t\mapsto u(\mathbf{X}_t)$ is càdlàg on $[0,T-\tau(0)]$, $P_z$-a.s.

**Definition 4.1.** Let $\delta_{\{T\}}\otimes\varphi\cdot m\in\mathcal{R}(E_{0,T})$, $\mu\in\mathcal{R}(E_{0,T})$. We say that a quasi-càdlàg function $u: E_{0,T}\to\mathbb{R}$ is a probabilistic solution to (4.1) if $f_u\cdot m_1\in\mathcal{R}(E_{0,T})$ and for q.e. $z\in E_{0,T}$,

$$u(z)=E_z\left(\varphi(\mathbf{X}_{\zeta_\tau})+\int_0^{\zeta_\tau}f_u(\mathbf{X}_t)\,dt+\int_0^{\zeta_\tau}dA^\mu_t\right). \tag{4.2}$$

*Remark* 4.2. Let $\varphi,f_u,\mu$ be as in the above definition. Then by [18, Proposition 3.4] the right-hand side of (4.2) is a quasi-càdlàg function of $z$. Therefore the quasi-càdlàg requirement on $u$ in the above definition can be omitted.

(ii) From the proof of [18, Theorem 5.8] it follows that if $u$ is a probabilistic solution to (4.1) then there is an adapted process $M$ such that $M$ is a martingale under $P_z$ for q.e. $z\in E_{0,T}$ and

$$Y_t=\varphi(\mathbf{X}_{\zeta_\tau})+\int_t^{\zeta_\tau}f(\mathbf{X}_s,Y_s)\,ds+\int_t^{\zeta_\tau}dA^\mu_s-\int_t^{\zeta_\tau}dM_s,\quad t\in[0,\zeta_\tau],\quad P_z\text{-a.s.} \tag{4.3}$$

for q.e. $z\in E_{0,T}$, where $Y_t=u(\mathbf{X}_t)$, $t\ge0$.

*Remark* 4.3. (i) Let $\nu=\delta_{\{T\}}\otimes\varphi\cdot m$. If $\varphi\in L^1(E;m)$ then $\nu\in\mathcal{R}(E_{0,T})$.

(ii) One can check that $A^\nu_t=\mathbf{1}_{[T-\tau(0),\infty]\cap\{T>\tau(0)\}}(t)\varphi(\mathbf{X}_{\zeta_\tau})$, $t\ge0$, for $\nu$ defined in (i) (see the beginning of the proof of [18, Proposition 3.4]). Hence

$$E_z\varphi(\mathbf{X}_{\zeta_\tau})=E_z\int_0^{\zeta_\tau}dA^\nu_t.$$

Our definition of a renormalized solution is similar to [17, Definition 4.1].

**Definition 4.4.** Let $\varphi\in L^1(E;m)$, $\mu\in\mathcal{M}_{0,b}(E_{0,T})$. We say that a measurable function $u: E_{0,T}\to\mathbb{R}$ is a renormalized solution of (4.1) if

(a) $u$ is quasi-càdlàg, $f_u\in L^1(E_{0,T};m_1)$ and $T_ku\in\mathcal{V}_{0,T}$ for every $k>0$,

(b) there exists a sequence $\{\lambda_k\}\subset\mathcal{M}_{0,b}(E_{0,T})$ such that $\|\lambda_k\|_{TV}\to0$ as $k\to\infty$ and for every $k\in\mathbb{N}$ and every bounded $v\in\mathcal{W}_0$,

$$\mathcal{E}^{0,T}(T_ku,v)=(T_k\varphi,v(T))_H+\langle f_u\cdot m+\mu,\tilde v\rangle+\langle\nu_k,\tilde v\rangle. \tag{4.4}$$

**Theorem 4.5.** *Assume that $\varphi\in L^1(E;m)$, $\mu\in\mathcal{M}_{0,b}(E_{0,T})$.*

(i) *If $u$ is a probabilistic solution of (4.1) and $f_u\in L^1(E_{0,T};m_1)$ then $u$ is a renormalized solution of (4.1).*

*(ii) If $u$ is a renormalized solution of (4.1) then $u$ is a probabilistic solution of (4.1).*

*Proof.* (i) Let $u$ be a probabilistic solution of (4.1). For $k>0$ put

$$Y^k_t=T_ku(\mathbf{X}_t),\quad t\ge0.$$

By Remark 4.2 there is a martingale $M$ such that (4.3) is satisfied. As in the proof of Theorem 3.5, applying the Meyer–Itô formula we show that for $k>0$,

$$Y^k_t-Y^k_0=\int_0^t\mathbf{1}_{\{-k<Y_{s-}\le k\}}\,dY_s-(A^{1,k}_t-A^{2,k}_t) \tag{4.5}$$

for some increasing processes $A^{1,k},A^{2,k}$ such that

$$E_zA^{1,k}_{\zeta_\tau}\le E_z(\varphi-\varphi\wedge k)(\mathbf{X}_{\zeta_\tau})+E_z\int_0^{\zeta_\tau}\mathbf{1}_{\{Y_{s-}>k\}}(f_u(\mathbf{X}_s)\,ds+dA^\mu_s) \tag{4.6}$$

and

$$E_zA^{2,k}_{\zeta_\tau}\le -E_z((\varphi+k)\wedge 0)(\mathbf{X}_{\zeta_\tau})-E_z\int_0^{\zeta_\tau}\mathbf{1}_{\{Y_{s-}\le -k\}}(f_u(\mathbf{X}_s)\,ds+dA^\mu_s) \tag{4.7}$$

for q.e. $z\in E^1$. Hence $E_zA^{i,k}_{\zeta_\tau}<\infty$ for q.e. $z\in E_{0,T}$. From this and [1, Theorem A.3.16] it follows that there is a positive increasing AF $B^{i,k}$ of $\mathbf{M}$ such that $B^{i,k}$ is the compensator of $A^{i,k}$ under $P_z$ for q.e. $z\in E_{0,T}$. In particular,

$$E_zA^{i,k}_t=E_zB^{i,k}_t,\quad t\ge 0,\quad i=1,2 \tag{4.8}$$

for q.e. $z\in E_{0,T}$. Since $A^\mu$ is predictable, there exists a sequence $\{T_n\}$ of predictable stopping times exhausting the jumps of $A^\mu$, i.e.

$$\{\Delta A\neq 0\}=\bigcup_{n\ge 1}[T_n],\quad [T_n]\cap[T_m]=\emptyset,\quad n\neq m,$$

where $[T_n]$ denotes the graph of $T_n$. Let $A^{\mu,c}$ denote the continuous part of $A^\mu$ and let $A^{\mu,d}=A^\mu-A^{\mu,c}$. We have

$$\begin{aligned}E_z\int_0^{\zeta_\tau}\mathbf{1}_{\{Y_{s-}>k\}}\,dA^\mu_s&=E_z\int_0^{\zeta_\tau}\mathbf{1}_{\{Y_{s-}>k\}}\,dA^{\mu,c}_s+E_z\int_0^{\zeta_\tau}\mathbf{1}_{\{Y_{s-}>k\}}\,dA^{\mu,d}_s\\&=E_z\int_0^{\zeta_\tau}\mathbf{1}_{\{u(\mathbf{X}_s)+\Delta u(\mathbf{X}_s)>k\}}\,dA^{\mu,c}_s\\&\quad+\sum_{n\ge1}E_z\mathbf{1}_{\{u(\mathbf{X}_{T_n})+\Delta u(\mathbf{X}_{T_n})>k\}}\,\Delta A^\mu_{T_n}.\end{aligned} \tag{4.9}$$

Since the filtration $\{\mathcal{F}_t,t\ge 0\}$ is quasi-left continuous (see [26, Proposition IV.4.2]), $\Delta u(X_{T_n})=\Delta A^\mu_{T_n}$ by Theorem A.3.6 in [1]. On the other hand, by [24, Theorem 16.8], there exists a Borel function $a:E_{0,T}\to\mathbb{R}$ such that $\Delta A^\mu_t=a(\mathbf{X}_{t-})$, $t>0$, $P_z$-a.s. for q.e. $z\in E_{0,T}$. From this and the fact that $\mathbf{X}$ is quasi-left continuous it follows that $\Delta u(\mathbf{X}_{T_n})=a(\mathbf{X}_{T_n})$. By this and (4.9),

$$E_z\int_0^{\zeta_\tau}\mathbf{1}_{\{Y_{s-}>k\}}\,dA^\mu_s=E_z\int_0^{\zeta_\tau}\mathbf{1}_{\{u(\mathbf{X}_s)+a(\mathbf{X}_s)>k\}}\,dA^\mu_s. \tag{4.10}$$

Analogously to (4.10) we show that

$$E_z\int_0^{\zeta_\tau}\mathbf{1}_{\{Y_{s-}\le -k\}}\,dA^\mu_s=E_z\int_0^{\zeta_\tau}\mathbf{1}_{\{u(\mathbf{X}_s)+a(\mathbf{X}_s)\le -k\}}\,dA^\mu_s. \tag{4.11}$$

Combining (4.6)–(4.8) with (4.10), (4.11) we get

$$E_z B^{1,k}_{\zeta_\tau} \le E_z\Big(\mathbf{1}_{\{\varphi>k\}}\varphi(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} \mathbf{1}_{\{u(\mathbf{X}_s)>k\}} f_u(\mathbf{X}_s)\,ds + \int_0^{\zeta_\tau} \mathbf{1}_{\{(u+a)(\mathbf{X}_s)>k\}} dA^\mu_s\Big) \tag{4.12}$$

and

$$E_z B^{2,k}_{\zeta_\tau} \le -E_z\Big(\mathbf{1}_{\{\varphi<-k\}}\varphi(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} \mathbf{1}_{\{u(\mathbf{X}_s)\le -k\}} f_u(\mathbf{X}_s)\,ds + \int_0^{\zeta_\tau} \mathbf{1}_{\{(u+a)(\mathbf{X}_s)\le -k\}} dA^\mu_s\Big) \tag{4.13}$$

for q.e. $z \in E_{0,T}$. By [1, Theorem A.3.2] the jumps of $\mathbf{M}$ occur in totally inaccessible stopping times, while the jumps of $B^{i,k}$ in stopping times which are not totally inaccessible since $B^{i,k}$ is predictable. Therefore $\mathbf{M}$ and $B^{i,k}$ have no common discontinuities, and hence $B^{i,k}$ is a positive natural AF of $\mathbf{M}$. Let $b^i_k$, $i = 1, 2$, denote the Revuz measure of $B^{i,k}$. From (4.12), (4.13), Remark 4.2 and [18, Proposition 3.13] we conclude that

$$b^1_k(E_{0,T}) \le \|\mathbf{1}_{\{\varphi>k\}}\varphi\|_{L^1(E;m)} + \|f_u\|_{L^1(E_{0,T};m_1)} + \|\mathbf{1}_{\{u+a>k\}} \cdot \mu\|_{TV},$$

$$b^2_k(E_{0,T}) \le \|\mathbf{1}_{\{\varphi<-k\}}\varphi\|_{L^1(E;m)} + \|f_u\|_{L^1(E_{0,T};m_1)} + \|\mathbf{1}_{\{u+a\le -k\}} \cdot \mu\|_{TV}.$$

Hence $b^1_k, b^2_k \in \mathcal{M}_{0,b}(E_{0,T})$ and $\|b^1_k\|_{TV} \to 0$, $\|b^2_k\|_{TV} \to 0$ as $k \to \infty$. Combining (4.3) with (4.5) we see that

$$Y^k_t - Y^k_0 = -\int_0^t \mathbf{1}_{\{-k<Y_t\le k\}} f_u(\mathbf{X}_s)\,ds - \int_0^t \mathbf{1}_{\{-k<Y_{t-}\le k\}}\,dA^\mu_s - (B^{1,k}_t - B^{2,k}_t) + M^k_t$$

with

$$M^k_t = \int_0^t \mathbf{1}_{\{-k<Y_{s-}\le k\}}\,dM_s - (A^{1,k}_t - B^{1,k}_t) + A^{2,k}_t - B^{2,k}_t.$$

By the above and the definition of the measures $b^1_k, b^2_k$ we have

$$T_k u(z) = E_z\Big(T_k\varphi(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} \mathbf{1}_{\{-k<Y_t\le k\}} f_u(\mathbf{X}_t)\,dt + \int_0^{\zeta_\tau} \mathbf{1}_{\{-k<Y_{t-}\le k\}}\,dA^\mu_t + \int_0^{\zeta_\tau} d(A^{b^1_k}_t - A^{b^2_k}_t)\Big).$$

From this and (4.10), (4.11) we obtain

$$T_k u(z) = E_z\left(T_k\varphi(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} (f_u(\mathbf{X}_t)\,dt + dA^\mu_t) + \int_0^{\zeta_\tau} dA^{\nu_k}_t\right) \tag{4.14}$$

with

$$\nu_k = -\mathbf{1}_{\{u\notin(-k,k]\}} f_u \cdot m - \mathbf{1}_{\{u+a\notin(-k,k]\}} \cdot \mu + b^1_k - b^2_k.$$

By what has already been proved, $\nu_k \in \mathcal{M}_{0,b}(E_{0,T})$ and $\|\nu_k\|_{TV} \to 0$ as $k \to \infty$. Moreover, by [18, Theorem 3.12], $T_k u \in \mathcal{V}_{0,T}$, so what is left is to show that (4.4) is satisfied. We shall show that (4.4) follows from (4.14) by the same method as in elliptic case (see the proof of the fact that (3.6) follows from (3.14)). Let $\lambda_k = \delta_{\{T\}} \otimes \varphi \cdot m + f_u \cdot m + \mu + \nu_k$ and let $A = A^{\lambda_k}$. By Fubini's theorem,

$$\begin{aligned} R^{0,T}\lambda_k(z) - R^{0,T}_\alpha \lambda_k(z) &= E_z \int_0^{\zeta_\tau} (1 - e^{-\alpha t})\, dA_t \\ &= E_z \int_0^{\zeta_\tau} \left( \int_0^t \alpha e^{-\alpha s}\, ds \right) dA_t \\ &= \alpha E_z \int_0^{\zeta_\tau} e^{-\alpha t} \left( \int_t^{\zeta_\tau} dA_s \right) dt \end{aligned}$$

for q.e. $z \in E_{0,T}$. Using the definition of $\zeta_\tau$ and the fact that $A$ is an AF of $\mathbf{M}$ one can check that $A_{\zeta_\tau} - A_t = (A_{\zeta_\tau} - A_0) \circ \theta_t$. Therefore applying the Markov property shows that

$$\begin{aligned} R^{0,T}\lambda_k(z) - R^{0,T}_\alpha \lambda_k(z) &= \alpha E_z \int_0^{\zeta_\tau} e^{-\alpha t} E_{\mathbf{X}_t} \left( \int_0^{\zeta_\tau} dA_s \right) dt \\ &= \alpha R^{0,T}_\alpha (R^{0,T}\lambda_k)(z) \end{aligned}$$

for q.e. $z \in E_{0,T}$. Since by Remarks 4.3 and (4.14), $T_k u = R^{0,T}\lambda_k$, it follows from the above equation that

$$\alpha(T_k u - \alpha R^{0,T}_\alpha (T_k u), v)_{\mathcal{H}_{0,T}} = \alpha (R^{0,T}_\alpha \lambda_k, v)_{\mathcal{H}_{0,T}} \tag{4.15}$$

for every bounded $v \in \mathcal{W}_0$. Since the left-hand side of (4.15) is equal to $\mathcal{E}^{0,T}(\alpha G^{0,T}_\alpha T_k u, v)$, it converges to $\mathcal{E}^{0,T}(T_k u, v)$ as $\alpha \to \infty$. Let $\hat{R}^{0,T}_\alpha$ denote the resolvent associated with the dual form $\hat{\mathcal{E}}^{0,T}$. By [20, Corollary III.3.8] applied to the functions $\alpha \hat{R}^{0,T}_\alpha v$ we may assume that $\alpha \hat{R}^{0,T}_\alpha v$ converges to $\tilde{v}$ q.e. as $\alpha \to \infty$. It follows that the right-hand side of (4.15) converges to $\langle \lambda_k, \tilde{v} \rangle$ as $\alpha \to \infty$. Therefore letting $\alpha \to \infty$ in (4.15) we obtain (4.4), which completes the proof of (i).

(ii) Let $\eta \in L^2(E_{0,T}; m_1)$ be a bounded non-negative function. Then $\hat{G}^{0,T}\eta \in \mathcal{W}_0$ and

$$\mathcal{E}^{0,T}(T_k u, \hat{G}^{0,T}\eta) = (T_k u, \eta)_{\mathcal{H}_{0,T}}.$$

From this and (4.4) it follows that $T_k u$ is a solution in the sense of duality (see [18, Section 4] for the definition) of the linear problem

$$\left( -\frac{\partial}{\partial t} - L_t \right) T_k u = f_u + \mu + \nu_k, \quad T_k u(T) = T_k \varphi, \tag{4.16}$$

so by [18, Corollary 4.2] $T_k u$ is a probabilistic solution of the above equation. Therefore (4.14) (with the measure $\nu_k$ of (4.16)) is satisfied. Since $\|\nu_k\|_{TV} \to 0$ and for every Borel set $F \subset E_{0,T}$ such that $m(F) < \infty$ we have

$$(R^{0,T}|\nu_k|, \mathbf{1}_F)_{\mathcal{H}_{0,T}} = \langle |\nu_k|, \widetilde{\hat{G}^{0,T}\mathbf{1}_F} \rangle \le T \|\nu_k\|_{TV},$$

one can find a subsequence (still denoted by $k$) such that $R^{0,T}\nu_k(z) \to 0$ as $k \to \infty$ for $m_1$-a.e. $z \in E_{0,T}$. Therefore letting $k \to \infty$ in (4.14) we show that (4.2) holds true for $m_1$-a.e. $z \in E_{0,T}$, and hence for q.e. $z \in E_{0,T}$, because $u$ and the right-hand side of (4.2) are quasi-continuous. □

*Remark* 4.6. One can show that the function $u + a$ appearing in (4.12) and (4.13) is equal quasi everywhere to the precise version of $u$ (for the notion of a precise version of a parabolic potential see [27]).

We now illustrate the applicability of Theorem 4.5. Let us consider the following hypotheses.

(P1) $u \mapsto f(t, x, u)$ is continuous for every $(t, x) \in E_{0,T}$.
(P2) There is $\alpha \in \mathbb{R}$ such that $(f(t, x, y) - f(t, x, y'))(y - y') \le \alpha|y - y'|^2$ for every $(t, x) \in E_{0,T}$ and $y, y' \in \mathbb{R}$.
(P3) $\mu \in \mathcal{M}_{0,b}(E_{0,T})$ and $f(\cdot, y) \in L^1(E_{0,T}; m_1)$ for every $y \in \mathbb{R}$.

**Theorem 4.7.** *Let $u_i$ be renormalized solution of* (4.1) *with terminal condition $\varphi_i$, and right-hand side $(f^i, \mu_i)$, $i = 1, 2$. If $\varphi_1 \le \varphi_2$ $m_1$-a.e., $\mu_1 \le \mu_2$ and either $f^1$ satisfies* (P2) *and $f^1_{u_2} \le f^2_{u_2}$ $m_1$-a.e. or $f^2$ satisfies* (P2) *and $f^1_{u_1} \le f^2_{u_1}$ $m_1$-a.e., then $u_1(z) \le u_2(z)$ for q.e. $z \in E_{0,T}$.*

*Proof.* Follows from Theorem 4.5 and [18, Corollary 5.9]. □

**Corollary 4.8.** *If* (P2) *is satisfied then there exists at most one renormalized solution of* (4.1).

**Theorem 4.9.** *Assume* (P1)–(P3). *Then there exists renormalized solution of* (4.1).

*Proof.* Follows from Theorem 4.5 and [18, Theorem 5.8, Proposition 5.10]. □

The results of [18] used in the proof of Theorem 4.5 can be generalized to quasi-regular time dependent Dirichlet forms (see [18, Remark 4.4]). Moreover, if the forms $B^{(t)}$, $t \in [0, T]$, are quasi-regular, then by [20, Theorem IV.2.2] there exists a special standard process $\mathbf{M}$ properly associated in the resolvent sense with the time dependent form determined by $\{B^{(t)}, t \in [0, T]\}$. Since one can check that the results from the theory of stochastic processes used in the proof of Theorem 4.5 hold true for such process $\mathbf{M}$, Theorem 4.5 can be extended to quasi-regular time dependent Dirichlet forms.

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