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A reduced model for the polarization in a ferroelectric thin wire

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Abstract. In this paper, starting from a non-convex and nonlocal 3D-variational model for the electric polarization in a ferroelectric material, via an asymptotic process we obtain a rigorous $1D$-variational model for a thin wire.

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1. Introduction

The ferroelectricity is a property of some materials to have a spontaneous electrical polarization that can be reversed by the application of an external electric field. Ferroelectricity was demonstrated for first time in 1920 in Rochelle salt by J. Valasek. The chemical and crystallographic complexity of this salt restrained the research and suggested that the ferroelectricity was an exotic property requiring very specific conditions. Moreover, this property did not present any practical interest. A leap in the study of the ferroelectricity happened in the early 1950s with the discovery of ferroelectric oxides with perovskite structure: barium titanate (BaTiO3), lead titanate (PbTiO3), etc. These simpler materials enabled the development of theories of ferroelectricity and opened the way for industrial use of ferroelectric materials. Dielectric and piezoelectric properties of ferroelectric materials are now used in a wide variety of contexts. In particular, thin ferroelectric materials are receiving great interest due to their various applications to memory and storage devices, electronic circuits with miniaturized and integrated forms, etc. as, for instance, radio frequency identification cards (RFID) and ferroelectric tunnel junction (FTJ) (see [7, 22]).

In this paper, we start from a non-convex and nonlocal 3D-variational model for the electric polarization in a ferroelectric material. Via an asymptotic process based on dimensional reduction, we obtain a rigorous 1D-variational model for the electric polarization in a ferroelectric thin wire. In this study, we do not consider any deformation of the ferroelectric material.

Precisely, let

$$\Omega_n = h_n\omega \times \left]-\frac{1}{2}, \frac{1}{2}\right[, \quad n \in \mathbb{N},$$

be a 3D ferroelectric cylindrical device with small cross-section $h_n\omega$ and thickness 1, where $h_n \in]0,1[$, $n \in \mathbb{N}$, is a parameter tending to zero and $\omega \subset \mathbb{R}^2$ is an open polygonal set. In this material, the response to an applied electric field changes the electric displacement as $\mathbf{D} = \varepsilon E + 4\pi p$, where $\varepsilon > 0$ is the dielectric permeability, p is the spontaneous electric polarization field and $E = -D\varphi$ is the electric field which satisfies the electrostatic equation

$$\begin{cases} \operatorname{div}(-\varepsilon D\varphi + 4\pi p) = 0 & \text{in } \Omega_n, \\ (-\varepsilon D\varphi + 4\pi p)\cdot \nu = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{1.1}$$

with ν denoting the unit outer normal on $\partial\Omega_n$. The free energy associated with Ω_n is non-convex, nonlocal and it is given by (see [5,21,23] for an explanation of the model)

$$\mathcal{E}_n : p = (p_1,p_2,p_3) \in (H^1(\Omega_n))^3 \to \frac{1}{h_n^2}\left[\beta\int_{\Omega_n}|rotp|^2dx + \int_{\Omega_n}|divp|^2dx + \right.$$
$$\left. +\alpha\int_{\Omega_n}(|p|^2-1)^2dx + \int_{\Omega_n}|D\varphi|^2dx - \int_{\Omega_n}(g_n\cdot p)dx\right], \tag{1.2}$$

where α and β are positive constants, $g_n \in \left(L^2(\Omega_n)\right)^3$ is an external electric field, $x = (x_1,x_2,x_3) = (x',x_3)$ denotes the generic point of $\mathbb{R}^3$ and $\cdot$ denotes the inner product in $\mathbb{R}^3$.

Remark that $\beta\int_{\Omega_n}|rotp|^2dx + \int_{\Omega_n}|divp|^2dx$ reduces to the classical energy $\int_{\Omega_n}|Dp|^2dx$ when $\beta = 1$ (see Lemma 2.1), so roughly speaking this term penalizes the spatial variation of p, while the term $\alpha\int_{\Omega_n}(|p|^2-1)^2dx$ induces a phase transition of p. So the body is driven to have regions of uniform polarization separated by thin transition layers. The external energy $\int_{\Omega_n}(g_n\cdot p)dx$ favors the polarization parallel to an externally applied field.

Imposing appropriate convergence assumptions on the rescaled exterior field in $\Omega = \omega \times \left]-\frac{1}{2},\frac{1}{2}\right[$ [see (3.1) and (3.6)], we prove that (see Theorem 5.1)

$$\begin{aligned} &\lim_n \min\{\mathcal{E}_n(p) : p \in (H^1(\Omega_n))^3,\ p\cdot\nu = 0 \quad \text{on } \partial\Omega_n\} = \\ &\min\left\{E_\infty(q) : q \in H_0^1\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[\right)\right\} \end{aligned} \tag{1.3}$$

where, in (1.3), $p\cdot\nu$ means that inner product between the trace of p on $\partial\Omega_n$ and the unit outer normal on $\partial\Omega_n$, and

$$E_\infty : q \in H_0^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right) \to \beta|\omega|\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{dq}{dx_3}\right|^2dx_3 + \alpha|\omega|\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(|q|^2-1\right)^2dx_3 +$$
$$+\left(\frac{4\pi}{\varepsilon}\right)^2|\omega|\int_{-\frac{1}{2}}^{\frac{1}{2}}|q|^2dx_3 - \int_{-\frac{1}{2}}^{-\frac{1}{2}}\overline{g}_3 q\,dx_3,$$

with $\overline{g} = (\overline{g_1}, \overline{g}_2, \overline{g}_3)$ denoting the integral over x' on cross section ω of the $\left(L^2(\Omega)\right)^3$-weak limit of the rescaled external field. More precisely, we obtain $(H^1(\Omega))^3$-strong convergence for a subsequence of rescaled polarizations. The limit polarization depends only on $x_3 \in]-\frac{1}{2}, \frac{1}{2}[$, it is parallel to the wire (i.e. the first two components are zero), while its third component solves the limit problem in (1.3). We remark that the nonlocal term in (1.2) transforms into $\left(\frac{4\pi}{\varepsilon}\right)^2 |\omega| \int_{-\frac{1}{2}}^{\frac{1}{2}} |q|^2 dx_3$. So the limit problem becomes local. The reduced model is justified by reasons of simplicity and economy, especially by a numerical point of view.

We explicitly point out that this phenomenon does not appear in the 3D-2D dimensional reduction, where the limit problem depending only on x' and defined on the cross section of the thin film retains the same properties of the 3D starting problem, i.e. it remains non-convex, nonlocal and with the same boundary condition (see Theorem 5.1 in [14]).

On the other side, the limit problem in (1.3) has some similarities with the 3D-2D dimensional reduction in the case where in the left hand side of (1.3) boundary condition $p \cdot \nu = 0$ is replaced by

$$p \wedge \nu = 0 \quad \text{on } \partial\Omega_n, \tag{1.4}$$

with $\wedge$ denoting the cross product in $\mathbb{R}^3$. In fact, in the 3D-2D dimensional reduction with boundary condition (1.4), one obtains an uniaxial local limit problem, i.e. the first two components of the limit polarization are zero, while the third component depending only on x' solves a local problem on the cross section of the thin film with the homogeneous Dirichlet boundary condition (see Theorem 5.3 in [14]).

In the case of a wire with boundary condition (1.4), we prove that the rescaled polarization converges strongly in $(H^1(\Omega))^3$ to $(0, 0, 0)$ (see Theorem 5.2).

Although the prefix "ferro" was borrowed from ferromagnetism and the 3D model of ferromagnetic micro devices is close to our model, the limit behavior of a ferromagnetic thin wire is completely different. In fact, the ferromagnetic energy associated with $\Omega_n = h_n\omega\times]-\frac{1}{2}, \frac{1}{2}[$ is given by

$$\begin{aligned}&\mathcal{J}_n : m \in H^1(\Omega_n, S^2)\\ &\quad\to \frac{1}{h_n^2}\left[\beta \int_{\Omega_n} |Dm|^2 dx + \int_{\Omega_n} \psi(m) dx + \int_{\mathbb{R}^3} |D\varphi|^2 dx - \int_{\Omega_n} (g_n \cdot m) dx\right],\end{aligned}$$

where S^2 denotes the unit sphere of $\mathbb{R}^3$, $\psi : S^2 \to [0, +\infty[$ is a continuous and even function and

$$\operatorname{div}(-D\varphi + m) = 0 \quad \text{in } \mathbb{R}^3,$$

understanding $m = 0$ in $\mathbb{R}^3 \setminus \Omega_n$. Then, as the cross section of Ω_n vanishes, it results (see [4,13,19])

$$\begin{aligned}\lim_n \min\{\mathcal{J}_n(m) : m \in H^1(\Omega_n, S^2)\} \\ = \min\Bigg\{ \beta|\omega| \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{d\mu}{dx_3}\right|^2 dx_3 + |\omega| \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi(\mu) dx_3 - \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{g}\mu dx_3 \\ + \int_{\mathbb{R}^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_1(x_3) Dp(y,z) + \mu_2(x_3) Dq(y,z)|^2 dx_3 dy dz : \\ \mu = (\mu_1, \mu_2, \mu_3) \in H^1\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[, S^2\right) \Bigg\},\end{aligned} \tag{1.5}$$

where $p \in W^1(\mathbb{R}^2)$ and $q \in W^1(\mathbb{R}^2)$ are the weak solutions of

$$\begin{cases} \Delta p = 0 & \text{in } \omega, \\ \Delta p = 0 & \text{in } \mathbb{R}^2 \setminus \omega, \\ \left[\dfrac{\partial p}{\partial \nu}\right] = \nu e_1 & \text{on } \partial\omega, \end{cases} \qquad \begin{cases} \Delta q = 0 & \text{in } \omega, \\ \Delta q = 0 & \text{in } \mathbb{R}^2 \setminus \omega, \\ \left[\dfrac{\partial q}{\partial \nu}\right] = \nu e_2 & \text{on } \partial\omega, \end{cases}$$

respectively, with $W^1(\mathbb{R}^2)$ denoting the Beppo-Levi space on $\mathbb{R}^2$, ν the exterior unit normal to $\partial\omega$, $\left[\frac{\partial \cdot}{\partial \nu}\right]$ the jump of $\frac{\partial \cdot}{\partial \nu}$ on $\partial\omega$, and $e_1 = (1, 0)$, $e_2 = (0, 1)$.

Unlike the ferroelectric 1D limit, generally this magnetization limit is not uniaxial, i.e. the first two components of the solutions of the limit problem in (1.5) are not zero. In some particular cases, it becomes uniaxial. For instance, it is easy to check that the unique solutions of the limit problem in (1.5) are $\mu = (1, 0, 0)$ or $\mu = (-1, 0, 0)$, if $\psi = 0$ and $\overline{g} = 0$.

For the study of ferromagnetic thin structures, we refer to [1–4,8–13,16–19]. For the asymptotic partial domain decomposition in thin wires, see recent results in [15].

Mainly, we consider the case $p \cdot \nu = 0$ on $\partial\Omega_n$. The proofs of our results are based on accurate *a priori* estimates and on the given boundary conditions which provide a first characterization of the $H^1(\Omega)$-weak limit of the rescaled polarization. These informations allows us to identify the limit of the nonlocal term, i.e. the limit of the electric field satisfying the electrostatic equation (1.1). Then, using the main idea of Γ-convergence method, we characterize completely the limit polarization as solution of a minimization problem. Finally, we prove that the convergences are $H^1(\Omega)$-strong. At the end of the paper, we sketch the proof for the case $p \wedge \nu = 0$ on $\partial\Omega_n$.

To simplify computations, in the sequel we assume $\varepsilon = 4\pi$.

2. The setting of the problems

For every $n \in \mathbb{N}$, let

$$P_n = \left\{ p \in \left(H^1(\Omega_n)\right)^3 : p \cdot \nu = 0 \text{ on } \partial\Omega_n \right\}$$
$$\text{and } S_n = \left\{ p \in \left(H^1(\Omega_n)\right)^3 : p \wedge \nu = 0 \text{ on } \partial\Omega_n \right\}.$$

Lemma 2.1. *It results that*

$$\|Dp\|^2_{(L^2(\Omega_n))^9} = \|rotp\|^2_{(L^2(\Omega_n))^3} + \|divp\|^2_{L^2(\Omega_n)}, \qquad \forall p \in P_n \cup S_n, \qquad \forall n \in \mathbb{N}. \tag{2.1}$$

Proof. Fix $n \in \mathbb{N}$. It is known that this equality holds true in $(P_n \cup S_n) \cap (H^2(\Omega_n))^3$ (see the last three lines in the proof of Lemma 2.2 in [6]). Consequently, using the density of $P_n \cap (H^\infty(\Omega_n))^3$ in P_n and the density of $S_n \cap (H^\infty(\Omega_n))^3$ in S_n (see Lemma 2.6 in [6]) one obtains (2.1). Precisely, apply Lemma 2.6 in [6] (with its notations) in the simple case where $\{\Omega_j\}$ is the trivial polyhedral partition of Ω_n composed on Ω_n only, $\varepsilon = \mu = 1$, and Σ is the skeleton formed by the union of the closed edges of Ω_n. Then, in this case one has $H^\infty_T(\Omega_n, 1) = P_n \cap (H^\infty(\Omega_n))^3$, $H^\infty_N(\Omega_n, 1) = S_n \cap (H^\infty(\Omega_n))^3$, $H_T(\Omega_n, 1) = P_n$, and $H_N(\Omega_n, 1) = S_n$. □

For every $n \in \mathbb{N}$ and $p \in \left(L^2(\Omega_n)\right)^3$, Lax-Milgram Theorem ensures that the following problem:

$$\varphi_p \in H^1(\Omega_n), \int_{\Omega_n} \varphi_p dx = 0, \quad \int_{\Omega_n} ((-D\varphi_p + p) \cdot D\varphi)\, dx = 0, \quad \forall \varphi \in H^1(\Omega_n), \tag{2.2}$$

admits a unique solution.

For every $n \in N$, let

$$\mathcal{E}_n : p \in (H^1(\Omega_n))^3 \to \frac{1}{h_n^2} \int_{\Omega_n} [\beta |rotp|^2 + |divp|^2 + \alpha(|p|^2 - 1)^2 + |D\varphi_p|^2 - (g_n \cdot p)] dx, \tag{2.3}$$

where $g_n \in \left(L^2(\Omega_n)\right)^3$ and φ_p is the unique solution of (2.2). By using (2.1) and the direct method of Calculus of Variations, it is easy to see that the following problems

$$\min\{\mathcal{E}_n(p) : p \in P_n\}, \tag{2.4}$$

$$\min\{\mathcal{E}_n(p) : p \in S_n\} \tag{2.5}$$

admit solution. The aim of this paper is to study the asymptotic behavior, as n diverges, of problems (2.4) and (2.5).

3. The rescaling

As it is usual, problems (2.4) and (2.5) can be reformulated on a fixed domain through the following rescaling:

$$x = (x_1, x_2, x_3) \in \Omega = \omega \times]-\frac{1}{2}, \frac{1}{2}[\to (h_n x_1, h_n x_2, x_3) \in \Omega_n = h_n\omega \times]-\frac{1}{2}, \frac{1}{2}[.$$

Precisely, setting

$$P = \left\{p \in \left(H^1(\Omega)\right)^3 : p \cdot \nu = 0 \quad \text{on } \partial\Omega\right\} \quad \text{and}$$
$$S = \left\{p \in \left(H^1(\Omega)\right)^3 : p \wedge \nu = 0 \quad \text{on } \partial\Omega\right\}$$

where ν denotes also the unit outer normal on $\partial\Omega$, and for every $n \in N$

$$f_n : x = (x_1, x_2, x_3) \in \Omega \to g_n(h_n x_1, h_n x_2, x_3), \tag{3.1}$$

$$D_n : p \in \left(H^1(\Omega)\right)^3 \ \text{(resp. } H^1(\Omega)) \\ \to \left(\frac{1}{h_n}\frac{\partial p}{\partial x_1}, \frac{1}{h_n}\frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}\right) \in (L^2(\Omega))^9 \ \left(\text{resp. } (L^2(\Omega))^3\right),$$

$$div_n : p = (p_1, p_2, p_3) \in \left(H^1(\Omega)\right)^3 \to \frac{1}{h_n}\frac{\partial p_1}{\partial x_1} + \frac{1}{h_n}\frac{\partial p_2}{\partial x_2} + \frac{\partial p_3}{\partial x_3} \in L^2(\Omega),$$

$$rot_n : p = (p_1, p_2, p_3) \in \left(H^1(\Omega)\right)^3 \to \\ \left(\frac{1}{h_n}\frac{\partial p_3}{\partial x_2} - \frac{\partial p_2}{\partial x_3}, \frac{\partial p_1}{\partial x_3} - \frac{1}{h_n}\frac{\partial p_3}{\partial x_1}, \frac{1}{h_n}\frac{\partial p_2}{\partial x_1} - \frac{1}{h_n}\frac{\partial p_1}{\partial x_2}\right) \in (L^2(\Omega))^3,$$

functional $\mathcal{E}_n$ defined in (2.3) is rescaled in the following one:

$$E_n : p \in (H^1(\Omega))^3 \to \int_\Omega [\beta|rot_n p|^2 + |div_n p|^2 + \alpha(|p|^2 - 1)^2 + |D_n\phi_p|^2 - (f_n \cdot p)]dx, \tag{3.2}$$

where ϕ_p is the unique solution of the following problem:

$$\phi_p \in H^1(\Omega), \quad \int_\Omega \phi_p dx = 0, \quad \int_\Omega ((-D_n\phi_p + p) \cdot D_n\phi)\, dx = 0, \quad \forall \phi \in H^1(\Omega), \tag{3.3}$$

which rescales problem (2.2). Then, the goal of this paper turns in studying the asymptotic behavior, as n diverges, of the following problems:

$$\min\{E_n(p) : p \in P\}, \tag{3.4}$$

$$\min\{E_n(p) : p \in S\}. \tag{3.5}$$

To this aim, we assume that

$$f_n \rightharpoonup f = (\overline{f}_1, \overline{f}_2, \overline{f}_3) \text{ weakly in } (L^2(\Omega))^3. \tag{3.6}$$

We conclude this section recalling that (2.1) transforms into the following one:

$$\|D_n p\|^2_{(L^2(\Omega))^9} = \|rot_n p\|^2_{(L^2(\Omega))^3} + \|div_n p\|^2_{L^2(\Omega)}, \ \forall p \in P \cup S, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

4. A convergence result for problem (3.3)

Proposition 4.1. *Let $\{p_n\}_{n\in\mathbb{N}} \subset \left(L^2(\Omega)\right)^3$ be such that*

$$p_n \to (0, 0, q) \text{ strongly in } \left(L^2(\Omega)\right)^3, \tag{4.1}$$

with $q \in L^2(\Omega)$ independent of x'. Moreover, let ϕ_{p_n} be the unique solution of (3.3) with $p = p_n$. Then, it results that

$$\phi_{p_n} \longrightarrow \int_{-\frac{1}{2}}^{x_3} q(t)dt - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{x_3} q(t)dt\right) dx_3 \text{ strongly in } H^1(\Omega), \tag{4.2}$$

$$\frac{1}{h_n} D_{x'}\phi_{p_n} \to 0 \text{ strongly in } \left(L^2(\Omega)\right)^2. \tag{4.3}$$

Proof. Since $\{p_n\}_{n\in\mathbb{N}}$ is bounded in $\left(L^2(\Omega)\right)^3$, there exists a positive constant c such that

$$\|D_n\phi_{p_n}\|_{(L^2(\Omega))^3} \le c, \quad \forall n \in \mathbb{N}.$$

Consequently, by virtue of the Poincaré-Wirtinger inequality, there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence of the subsequence) $\tau \in H^1(\Omega)$, independent of x' and with zero average, and $\xi \in \left(L^2(\Omega)\right)^2$ such that

$$\phi_{p_n} \rightharpoonup \tau \text{ weakly in } H^1(\Omega), \tag{4.4}$$

$$\frac{1}{h_n} D_{x'}\phi_{p_n} \rightharpoonup \xi \text{ weakly in } \left(L^2(\Omega)\right)^2. \tag{4.5}$$

Now, passing to the limit in the equation satisfied by ϕ_{p_n} with test functions ψ independent of x', that is $\psi \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)$, convergences (4.1) and (4.4) entail that

$$\begin{cases} \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left(-\frac{d\tau}{dx_3}+q\right)\frac{d\psi}{dx_3}\right)dx_3 = 0, \quad \forall \psi \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right), \\ \tau \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right), \quad \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \tau dx_3 = 0. \end{cases}$$

Since this problem admits the unique solution

$$\tau(x_3) = \int_{-\frac{1}{2}}^{x_3} q(t)dt - \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{-\frac{1}{2}}^{x_3} q(t)dt\right)dx_3, \quad \text{in } \left]-\frac{1}{2},\frac{1}{2}\right[,$$

from (4.4) it follows that

$$\phi_{p_n} \rightharpoonup \int_{-\frac{1}{2}}^{x_3} q(t)dt - \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{-\frac{1}{2}}^{x_3} q(t)dt\right)dx_3 \text{ weakly in } H^1(\Omega). \tag{4.6}$$

By using the equation satisfied by ϕ_{p_n}, (4.1), (4.5), (4.6) and a l.s.c. argument, one obtains that

$$\begin{aligned} \int_\Omega |\xi|^2 dx + |\omega|\int_{-\frac{1}{2}}^{\frac{1}{2}} q^2 dx_3 &\le \lim_n \int_\Omega |D_n\phi_{p_n}|^2 dx \\ &= \lim_n \int_\Omega \left(D_n\phi_{p_n}\cdot p_n\right)dx = |\omega|\int_{-\frac{1}{2}}^{\frac{1}{2}} q^2 dx_3, \end{aligned} \tag{4.7}$$

which provides that $\xi = (0,0)$, and that convergences (4.5) and (4.6) are strong and hold true for the whole sequence since the limits are uniquely identified. □

Remark 4.2. We point out that from (4.2) and (4.3) it follows that

$$\lim_n \int_\Omega |D_n\phi_{p_n}|^2 dx = |\omega|\int_{-\frac{1}{2}}^{\frac{1}{2}} |q|^2 dx_3.$$

5. The main results

At first, we consider the case $p \cdot \nu = 0$ on $\partial\Omega$.

Let

$$P_\infty = \left\{ q \in H^1(\Omega) : q \text{ is independent of } x' \text{ and } q = 0 \text{ on } \omega \times \left\{ -\frac{1}{2}, \frac{1}{2} \right\} \right\} \simeq H_0^1 \left(\left] -\frac{1}{2}, \frac{1}{2} \right[\right) \tag{5.1}$$

and

$$E_\infty : q \in P_\infty \to \beta |\omega| \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{dq}{dx_3} \right|^2 dx_3 + \alpha |\omega| \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(|q|^2 - 1 \right)^2 dx_3 + |\omega| \int_{-\frac{1}{2}}^{\frac{1}{2}} |q|^2 dx_3 - \int_{-\frac{1}{2}}^{-\frac{1}{2}} \left(\int_\omega \overline{f}_3 dx' q \right) dx_3, \tag{5.2}$$

where f_3 is defined in (3.6).

The following statement contains the main result of this section:

Theorem 5.1. *Assume (3.6). For every $n \in \mathbb{N}$, let E_n be defined in (3.2), p_n be a solution of (3.4) and ϕ_{p_n} be the unique solution of (3.3) with $p = p_n$. Moreover, let P_∞ and E_∞ be defined in (5.1) and (5.2), respectively. Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i \in \mathbb{N}}$ and (in possible dependence of the subsequence) $\overline{q} \in P_\infty$ such that*

$$p_n \to (0, 0, \overline{q}) \text{ strongly in } \left(H^1(\Omega) \right)^3 \text{ and strongly in } \left(L^4(\Omega) \right)^3, \tag{5.3}$$

$$\frac{1}{h_n} D_{x'} p_n \to 0 \text{ strongly in } \left(L^2(\Omega) \right)^6, \tag{5.4}$$

$$\phi_{p_n} \to \int_{-\frac{1}{2}}^{x_3} \overline{q}(t) dt - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{x_3} \overline{q}(t) dt \right) dx_3 \text{ strongly in } H^1(\Omega), \tag{5.5}$$

$$\frac{1}{h_n} D_{x'} \phi_{p_n} \to 0 \text{ strongly in } \left(L^2(\Omega) \right)^2, \tag{5.6}$$

where $\overline{q}$ is a solution of the following problem

$$E_\infty(\overline{q}) = \min\{ E_\infty(q) : q \in P_\infty \}. \tag{5.7}$$

Moreover, the convergence of the energies holds true, that is

$$\lim_n E_n(p_n) = E_\infty(\overline{q}). \tag{5.8}$$

Proof. The proof of this theorem will be performed in several steps. The first step is devoted to prove that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence of the subsequence) $\overline{q} \in P_\infty \simeq H_0^1 \left(\left] -\frac{1}{2}, \frac{1}{2} \right[\right)$ and $z = (z_1, z_2, z_3) \in \left(L^2(] - \frac{1}{2}, \frac{1}{2}[, H_m^1(\omega)) \right)^3$ such that

$$p_n \rightharpoonup (0, 0, \overline{q}) \text{ weakly in } \left(H^1(\Omega) \right)^3 \text{ and strongly in } \left(L^4(\Omega) \right)^3, \tag{5.9}$$

$$\frac{1}{h_n} D_{x'} p_n \rightharpoonup D_{x'} z \text{ weakly in } \left(L^2(\Omega) \right)^6, \tag{5.10}$$

$$\int_\omega \left(\frac{\partial z_1}{\partial x_1} + \frac{\partial z_2}{\partial x_2}\right) dx' = 0, \text{ for } x_3 \text{ a.e. in } \left]-\frac{1}{2}, \frac{1}{2}\right[, \tag{5.11}$$

where the subscript "m" means zero average.

Since $0 \in P$, it results that

$$\int_\Omega (\beta|rot_n p_n|^2 + |div_n p_n|^2 + \alpha|p_n|^4 - 2\alpha|p_n|^2)dx \leq \|f_n\|_{(L^2(\Omega))^3}\|p_n\|_{(L^2(\Omega))^3}, \quad \forall n \in \mathbb{N}. \tag{5.12}$$

By using the continuous embedding of $(L^4(\Omega))^3$ into $(L^2(\Omega))^3$, estimate (5.12) gives

$$\frac{\alpha}{|\Omega|}\|p_n\|^3_{L^2(\Omega)} - 2\alpha\|p_n\|_{L^2(\Omega)} \leq \|f_n\|_{(L^2(\Omega))^3}, \quad \forall n \in \mathbb{N}, \tag{5.13}$$

from which, by virtue of (3.6), it follows the existence of a positive constant c such that

$$\|p_n\|_{(L^2(\Omega))^3} \leq c, \quad \forall n \in \mathbb{N}. \tag{5.14}$$

Then, combining (5.14), (3.6), (5.12) and (3.7), one obtains also the existence of a positive constant c such that

$$\|D_n p_n\|_{(L^2(\Omega))^9} \leq c, \quad \forall n \in \mathbb{N}. \tag{5.15}$$

Estimates (5.14) and (5.15) provide the existence of a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence of the subsequence) $\overline{p} = (\overline{p}_1, \overline{p}_2, \overline{p}_3) \in P$, independent of x', such that

$$p_n \rightharpoonup \overline{p} \text{ weakly in } \left(H^1(\Omega)\right)^3 \text{ and strongly in } \left(L^4(\Omega)\right)^3.$$

In particular, since $\overline{p}$ is independent of x' and $\overline{p} \cdot \nu = 0$ on $\partial\Omega$, it results that $\overline{p}_3 \in H^1_0(]-\frac{1}{2}, \frac{1}{2}[)$ and $(\overline{p}_1, \overline{p}_2) = (0, 0)$. Hence, (5.9) holds true with $\overline{q} = \overline{p}_3$.

To prove (5.10), for $i = 1, 2, 3$ and for every $n \in \mathbb{N}$ set

$$m_{n,i} : x_3 \in \left]-\frac{1}{2}, \frac{1}{2}\right[\longrightarrow \int_\omega p_{n,i}(x', x_3)dx'.$$

By using the Poincaré-Wirtinger inequality, there exists a positive constant c such that, for x_3 a.e. in $\left]-\frac{1}{2}, \frac{1}{2}\right[$,

$$\left\|\frac{1}{h_n}\left(p_{n,i}(\cdot, x_3) - m_{n,i}(x_3)\right)\right\|_{H^1_m(\omega)} \leq \frac{c}{h_n}\left\|D_{x'}p_{n,i}(\cdot, x_3)\right\|_{L^2(\omega)}, \quad \forall n \in \mathbb{N}, i = 1, 2, 3.$$

Thus, integrating these inequalities over $x_3 \in \left]-\frac{1}{2}, \frac{1}{2}\right[$, estimate (5.15) gives (5.10).

To prove (5.11), we remark that $\left(\frac{1}{h_n}p_{n,1}, \frac{1}{h_n}p_{n,2}\right) \cdot \nu' = 0$ on $\partial\omega \times]-\frac{1}{2}, \frac{1}{2}[$, where ν' denotes the unit outer normal on $\partial\omega$. Consequently, it results that

$$\begin{aligned}&\int_\Omega \left(\varphi(x_3)\left(\frac{1}{h_n}\frac{\partial p_{n,1}}{\partial x_1}(x', x_3) + \frac{1}{h_n}\frac{\partial p_{n,2}}{\partial x_2}(x', x_3)\right)\right) dx \\ &= \int_{\partial\omega \times]-\frac{1}{2}, \frac{1}{2}[} \left(\varphi(x_3)\left(\frac{1}{h_n}p_{n,1}, \frac{1}{h_n}p_{n,2}\right) \cdot \nu'\right) d\sigma = 0, \quad \forall \varphi \in C^\infty_0\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right).\end{aligned} \tag{5.16}$$

Then, (5.11) is obtained passing to the limit in (5.16) and using (5.10).

The second step is devoted to identify $\overline{q}$ and z. Since $(0, 0, q) \in P$ for every $q \in H^1_0\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right)$, it results that

$$E_n(p_n) \leq E_n((0,0,q)), \quad \forall q \in H^1_0\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right), \quad \forall n \in \mathbb{N}. \tag{5.17}$$

Then, passing to the limit in (5.17), by virtue of (3.6), Proposition 4.1, (5.9), (5.10) and a l.s.c. argument one obtains that

$$\begin{aligned}&\beta \int_\Omega \left(\left|\frac{\partial z_3}{\partial x_2}\right|^2 + \left|\frac{\partial z_3}{\partial x_1}\right|^2 + \left|\frac{\partial z_2}{\partial x_1} - \frac{\partial z_1}{\partial x_2}\right|^2 \right) dx + \int_\Omega \left|\frac{\partial z_1}{\partial x_1} + \frac{\partial z_2}{\partial x_2} + \frac{d\overline{q}}{dx_3}\right|^2 dx \\ &+\alpha|\omega| \int_{-\frac{1}{2}}^{\frac{1}{2}} (|\overline{q}|^2 - 1)^2 dx_3 + |\omega| \int_{-\frac{1}{2}}^{\frac{1}{2}} |\overline{q}|^2 dx_3 - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_\omega \overline{f}_3 dx' \overline{q} \right) dx_3 \\ &\leq \liminf_n E_n(p_n) \leq \limsup_n E_n(p_n) \leq E_\infty(q), \quad \forall q \in H^1_0\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right).\end{aligned} \tag{5.18}$$

On the other hand, taking into account that $\overline{q}$ is independent of x' and (5.11), it results that

$$\int_\Omega \left|\frac{\partial z_1}{\partial x_1} + \frac{\partial z_2}{\partial x_2} + \frac{d\overline{q}}{dx_3}\right|^2 dx = \int_\Omega \left|\frac{\partial z_1}{\partial x_1} + \frac{\partial z_2}{\partial x_2}\right|^2 dx + |\omega| \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{d\overline{q}}{dx_3}\right|^2 dx_3. \tag{5.19}$$

Hence, inserting (5.19) in (5.18), one has that

$$\begin{aligned}&\beta \int_\Omega \left(\left|\frac{\partial z_3}{\partial x_2}\right|^2 + \left|\frac{\partial z_3}{\partial x_1}\right|^2 + \left|\frac{\partial z_2}{\partial x_1} - \frac{\partial z_1}{\partial x_2}\right|^2 \right) dx + \int_\Omega \left|\frac{\partial z_1}{\partial x_1} + \frac{\partial z_2}{\partial x_2}\right|^2 dx + E_\infty(\overline{q}) \\ &\leq \liminf_n E_n(p_n) \leq \limsup_n E_n(p_n) \leq E_\infty(q), \quad \forall q \in H^1_0\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right),\end{aligned} \tag{5.20}$$

which entails that

$$\begin{cases} \dfrac{\partial z_1}{\partial x_1} + \dfrac{\partial z_2}{\partial x_2} = 0, & \text{a.e. in } \Omega, \\ \dfrac{\partial z_2}{\partial x_1} - \dfrac{\partial z_1}{\partial x_2} = 0, & \text{a.e. in } \Omega, \\ \dfrac{\partial z_3}{\partial x_1} = \dfrac{\partial z_3}{\partial x_2} = 0, & \text{a.e. in } \Omega \end{cases} \tag{5.21}$$

(in particular, $z_3 = 0$ a.e. in Ω since $z_3 \in L^2(]-\frac{1}{2}, \frac{1}{2}[, H^1_m(\omega)))$. Consequently, inserting (5.21) in (5.20), one obtains that $\overline{q}$ solves problem (5.7) and convergence (5.8) holds true. We remark that convergence in (5.8) holds true for the whole sequence since the limit is uniquely identified. Moreover, (5.5) and (5.6) follow from (5.9) and Proposition 4.1.

In the last step, we identify (z_1, z_2) in (5.10) and prove that convergences in (5.9) and (5.10) are strong. To this aim, by combining (5.8) with (3.6), (5.5), (5.6) and (5.9), one obtains that

$$\lim_n \int_\Omega \left(\beta|rot_n p_n|^2 + |div_n p_n|^2\right) dx = |\omega| \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{d\overline{q}}{dx_3}\right|^2 dx_3$$
$$= \int_\Omega \left(\beta \left|rot(0,0,\overline{q})\right|^2 + \left|div(0,0,\overline{q})\right|^2\right) dx. \tag{5.22}$$

Moreover, from (5.9), (5.10) and (5.21) it follows that

$$\begin{cases} rot_n p_n \rightharpoonup \left(\dfrac{\partial z_3}{\partial x_2}, -\dfrac{\partial z_3}{\partial x_1}, \dfrac{\partial z_2}{\partial x_1} - \dfrac{\partial z_1}{\partial x_2}\right) = (0,0,0) = rot(0,0,\overline{q}) \text{ weakly in } \left(L^2(\Omega)\right)^3 \\ div_n p_n \rightharpoonup \dfrac{\partial z_1}{\partial x_1} + \dfrac{\partial z_2}{\partial x_2} + \dfrac{d\overline{q}}{dx_3} = \dfrac{d\overline{q}}{dx_3} = div(0,0,\overline{q}) \text{ weakly in } L^2(\Omega). \end{cases} \tag{5.23}$$

Consequently, combining (5.22) with (5.23), one derives that

$$\begin{cases} rot_n p_n \to rot(0,0,\overline{q}) \text{ strongly in } \left(L^2(\Omega)\right)^3 \\ div_n p_n \to div(0,0,\overline{q}) \text{ strongly in } L^2(\Omega). \end{cases} \tag{5.24}$$

Finally, from (3.7) and (5.24) one deduces that

$$D_n p_n \to D(0,0,\overline{q}) \text{ strongly in } \left(L^2(\Omega)\right)^9,$$

which entails that $D_{x'} z = 0$ (in particular, also $(z_1, z_2) = (0,0)$ a.e. in Ω since $(z_1, z_2) \in \left(L^2(]-\frac{1}{2}, \frac{1}{2}[, H^1_m(\omega))\right)^2$) and that convergences in (5.9) and (5.10) are strong. We remark that also convergence in (5.10) holds true for the whole sequence since the limit is uniquely identified. □

Now, we consider the case $p \wedge \nu = 0$ on $\partial\Omega$.

Theorem 5.2. *Assume (3.6). For every $n \in \mathbb{N}$, let E_n be defined in (3.2), p_n be a solution of (3.5) and ϕ_{p_n} be the unique solution of (3.3) with $p = p_n$. Then, it results that*

$$p_n \to 0 \text{ strongly in } \left(H^1(\Omega)\right)^3 \text{ and strongly in } \left(L^4(\Omega)\right)^3, \tag{5.25}$$

$$\frac{1}{h_n} D_{x'} p_n \to 0 \text{ strongly in } \left(L^2(\Omega)\right)^6, \tag{5.26}$$

$$\phi_{p_n} \to 0 \text{ strongly in } H^1(\Omega), \tag{5.27}$$

$$\frac{1}{h_n} D_{x'} \phi_{p_n} \to 0 \text{ strongly in } \left(L^2(\Omega)\right)^2. \tag{5.28}$$

Moreover, the convergence of the energies holds true, that is

$$\lim_n E_n(p_n) = 0. \tag{5.29}$$

Proof. We sketch the proof. By arguing as in the proof of Theorem 5.1, there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence of the subsequence) $\overline{p} = (\overline{p}_1, \overline{p}_2, \overline{p}_3) \in S$, independent of x', and $z = (z_1, z_2, z_3) \in \left(L^2(]-\frac{1}{2}, \frac{1}{2}[, H^1_m(\omega))\right)^3$ such that such that

$$p_n \rightharpoonup \overline{p} \text{ weakly in } \left(H^1(\Omega)\right)^3 \text{ and strongly in } \left(L^4(\Omega)\right)^3.$$

$$\frac{1}{h_n} D_{x'} p_n \rightharpoonup D_{x'} z \text{ weakly in } \left(L^2(\Omega)\right)^6.$$

In particular, since $\overline{p}$ is independent of x' and $\overline{p} \wedge \nu = 0$ on $\partial\omega \times]-\frac{1}{2}, \frac{1}{2}[$, it results that $\overline{p} = 0$, that is

$$p_n \rightharpoonup 0 \text{ weakly in } \left(H^1(\Omega)\right)^3 \text{ and strongly in } \left(L^4(\Omega)\right)^3.$$

Finally, the proof proceeds as in Theorem 5.1. We remark explicitly that in this case we do not have (5.11), but really we do not need it, since the limit of p_n is $(0, 0, 0)$. □

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