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Generalized semiconcavity of the value function of a jump diffusion optimal control problem

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Abstract. Generalized semiconcavity results for the value function of a jump diffusion optimal control problem are established, in the state variable, uniformly in time. Moreover, the semiconcavity modulus of the value function is expressed rather explicitly in terms of the semiconcavity or regularity moduli of the data (Lagrangian, terminal cost, and terms comprising the controlled SDE), at least under appropriate restrictions either on the class of the moduli, or on the SDEs. In particular, if the moduli of the data are of power type, then the semiconcavity modulus of the value function is also of power type. An immediate corollary are analogous regularity properties for (viscosity) solutions of certain integro-differential Hamilton-Jacobi-Bellman equations, which may be represented as value functions of appropriate optimal control problems for jump diffusion processes.

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1. Introduction

This article should be seen as continuation of work initiated a long time ago—at least since Krushkov—on obtaining semiconcavity estimates (or one-sided estimates on second-order difference quotients) for deterministic and/or stochastic optimal control value functions. And since under reasonable assumptions, value functions may be interpreted as solutions (at least in some generalized sense such as viscosity solutions) of appropriate Hamilton-Jacobi-Bellman equations, and vice versa, the said estimates are in fact also regularity estimates about solutions of these equations.

Let us try to be more precise taking a PDE-est point of view. Consider a parabolic partial integro-differential equation (henceforth abbr. PIDE) of Hamilton-Jacobi-Bellman (abbr. HJB) type

$$\begin{cases} \frac{\partial u}{\partial t} + \inf_{\alpha \in A} \Big\{ b(t,x,\alpha) \cdot \nabla u + \frac{1}{2} tr \left(\sigma(t,x,\alpha)\sigma^t(t,x,\alpha) D^2 u \right) \\ + \int_{\|z\| \geq \delta} \big(u(\cdot, \cdot + H(t,x,z,\alpha)) - u - \nabla u \cdot H(t,x,z,\alpha) \big) \nu(dz) \\ + \int_{\|z\| < \delta} \big(u(\cdot, \cdot + K(t,x,z,\alpha)) - u \big) \nu(dz) \Big\} = 0 & \text{in } [0,T) \times I\!R^d, \\ u(T,\cdot) = \psi & \text{in } I\!R^d; \end{cases} \tag{1.1}$$

where data b, H, K, L are $I\!R^d$-valued maps, σ an $I\!R^{d\times m}$-valued map, $d, m \in \mathbb{N}$, defined for $0 \leq t \leq T$, $x \in I\!R^d$, $\alpha \in A$—here A is some fixed metric space to be interpreted as a control space–whereas H, K depend also on "small jumps" $\|z\| \leq \delta$, and "big jumps" $\|z\| > \delta$, respectively, ν is a Lévy measure on $I\!R^d \backslash \{0\}$ and $\delta > 0$ is some fixed parameter; for precise assumptions on data see Sect. 4 below. Let us first recall the following

Definition 1.1. [16] *Given an upper semicontinuous nondecreasing function* $\omega : I\!R_+ \to I\!R_+$ *such that* $\omega(0+) = \lim_{\rho\to 0+} \omega(\rho) = 0$ (*such a function is called a* semiconcavity modulus), *we say that a function* $u : K \to I\!R$, *where* K *is some subset of some normed space* $(X, \|\cdot\|_X)$, *is an* ω-semiconcave *function if*

$$\lambda u(x_1) + (1-\lambda)u(x_2) - u(\lambda x_1 + (1-\lambda)x_2) \\ \leq \lambda(1-\lambda)\|x_1 - x_2\|_X \omega(\|x_1 - x_2\|_X)$$

for all $x_1, x_2 \in K$ *such that the segment* $[x_1, x_2] \subset K$ *and* $0 \leq \lambda \leq 1$. *A function* u *is called* ω-semiconvex *if* $-u$ *is* ω*-semiconcave. We say that* u *is* of class $C^{1,\omega}$ *or* $C^{1,\omega}$-regular *if it is both* ω*-semiconcave and* ω*-semiconvex.*[1] *Finally, a vector-valued map* $u : K \to Y$, *where* Y *is another normed space, is said to be* of class $C^{1,\omega}$ *or* $C^{1,\omega}$-regular *if each "component of* u*", that is, if* $\langle u, y^* \rangle$ *is of class* $C^{1,\omega}$ *for all*[2] $y^* \in Y^*$ (*in other words if the inequality above holds for the left-hand side being replaced by its own* Y*-norm*).

Assume that "vector fields" b, σ, H, K are in order of class C^{1,ω_b}, C^{1,ω_σ}, C^{1,ω_H}, C^{1,ω_K}, respectively, and that "running cost" L and "terminal cost" ψ are, respectively, ω_L and ω_ψ-semiconcave, in the state variable, uniformly in time, control, and—when it occurs—in jump variables, where all the ω's are given semiconcavity moduli. Is it possible to conclude that a solution of (1.1) is also ω-semiconcave in state variable, uniformly in time, for an appropriate semiconcavity modulus ω? Further, in such a case, is it possible to express the modulus of the solution in terms of the moduli of the data? We see in this paper that the answer to these questions is "Yes" for a large class of PIDEs of the kind (1.1). As hinted by terminology or, by the very title of the article, the main idea of the proof consists in interpreting a solution as a value function of a stochastic optimal control problem for jump diffusion processes, that is, processes which are solutions of appropriate stochastic differential equations (abbr. SDE) of jump type, see, e.g., [25] and references therein.

[1] In finite-dimensional normed spaces, such a definition is justified, e.g., by [16, Theorem 3.3.7, p. 60]. Certain constants that appear in its proof are universal, that is, independent of dimension, and this fact hints to its validity or possibility of extension to a large class of infinite-dimensional normed spaces.

[2] Y^*—as it is standard nowadays—stands for the topological dual of Y.

The novelties of the paper are as follows. We pursue great generality on semiconcavity moduli of the data; part of these regularity results may be new even for "pure" diffusions ($H = K = 0$), if not also for the "deterministic" case ($\sigma = H = K = 0$, for this case consult also [16] and references therein). Further, although applications mentioned in this paper deal only with SDEs of jump type driven by Lévy noises, results are phrased in a rather abstract or general fashion which makes them potentially applicable to other classes of SDEs such as SDEs driven by general semimartingale-valued random measures, or backward SDEs, provided that one has appropriate moments or Burkholder-Davis-Gundy type inequalities. Last but not least, no ellipticity assumption of any kind is made.

In terms of regularity (under mild assumptions) ω-semiconcave functions stand between locally Lipschitz continuous functions and (classical) semiconcave functions. Although less regular than the latter, they share with them a number of interesting properties as outlined in the book [16]. We believe that this may justify research on generalized semiconcavity of value functions in optimal control problems.

Power type moduli and moduli with certain concavity properties (see Lemmas 2.5 and 3.2) admit a rather general treatment the main tool being Gronwall's and moments inequalities. While for general moduli, we use in addition Kolmogorov's continuity criterion which may be a new technique in obtaining regularity results about value functions or solutions of (1.1).

Regularity theory of PIDEs such as (1.1) or, even more general classes of equations, has of course attracted much attention since a long time. A review of literature would be a quite difficult endeavor and probably beyond the scope of this article. Nevertheless, we can say the following. Obviously, most of regularity results hold under some kind of uniform ellipticity assumption. A first group of results has been obtained assuming nondegenerate diffusions or elliptic second-order differential (local) terms as in [6,19,21] (just to mention a very few) and references therein. More recently there has been a revival of interest on the theory of PIDEs which is due to the work on one hand of Caffareli et al. [10–15], and Barles et al. on the other [3–5]. These authors, differently from the earlier ones, prove regularity results such as Hölder, Lipschitz, $C^{1,\alpha}$-estimates, working under a kind of ellipticity assumption, which they have to give appropriate meaning, and which is not any more due to the second-order local terms (or to the presence of nonsingular diffusions), but comes either from the nonlocal terms or from the combined effect of both local and nonlocal terms.

Our semiconcavity results concern only the state or space variable, but another issue that could or should be tackled is that of joint semiconcavity (or, more generally, regularity) in time-space variables. There has been recently considerable progress in this direction in the case of linear moduli of (that is, classical) semiconcavity. The case of diffusions without jumps ($H = K = 0$), that is of second-order PDEs of HJB type is treated in [8] and in [9]. The case of general PIDE of HJB type is treated in [23] but under the restrictive assumption that the Lévy measure ν should be finite; the case of a general Lévy measure with $\int_{\mathbb{R}^d} \min\{1, \|z\|^2\}\nu(dz) < \infty$, according to [23], is still open. Of course, we are not aware of any systematic treatment dedicated to joint generalized semiconcavity in the sense of Definition 1.1 in both time and space variables available in literature.

Related to our discussion here is also [7] where convexity preservation results (in space variable) for HJB PIDEs, and their significance to financial applications are addressed, however under important restrictions: equations are linear in the integro-differential part, and the second-order fully nonlinear local part of the equation is assumed to be strictly elliptic. Finally, for reader's benefit let us mention the following references regarding semiconcavity results in space variable: for semiconcavity results (even in the generalized sense of Definition 1.1) in deterministic optimal control (or first-order Hamilton-Jacobi PDEs) see [16], for classical semiconcavity estimates in optimal control of diffusions (or second-order HJB PDEs) see [17] or [27]; semiconcavity estimates can also be proved via comparison principles as in [20,22].

The paper is organized as follows. First section is dedicated to $C^{1,\omega}$-estimates for solutions of "SDE"s, with an application to SDEs of Itô-Skorokhod type, which complements work of Kunita [24]. In Sect. 3 a quite general finite-horizon (stochastic) optimal control problem is formulated, and Theorem 3.1, providing a general result about the semiconcavity of the value function, is probably the main result of this article. These results are stated in a quite general form and may be applicable to larger classes of SDEs, e.g., SDEs driven by general semimartingale-valued random measures (although this has the drawback of requiring the introduction of a rather large amount of notation). Then as a corollary to this theorem semiconcavity results are derived under suitable restrictions on either the class of equations or the class of semiconcavity moduli. Last section is dedicated to applications of results to the value function arising in optimal control of jump diffusions (or Itô-Skorokhod SDEs).

Notation. We stick to the habit of denoting by L_u a Lipschitz constant of a function or map u, and by ω_u its semiconcavity modulus. Throughout the paper we have done our best to allow readers to keep track of constants and semiconcavity moduli. A property is said to hold *locally* in a normed space, if it holds on bounded subsets of that normed space.

2. $C^{1,\omega}$-estimates for solutions of jump type stochastic differential equations

Given a stochastic differential equation with $C^{1,\omega}$ data, we show that corresponding solutions are $C^{1,\omega'}$ with respect to the initial condition, giving an explicit expression of ω' in terms of ω. We prefer to formulate results in a somewhat abstract fashion, the main benefit here being to shorten notation, but, they may also be applicable to a wider class of equations than the one considered here. These results complement work of Kunita [24] who has studied extensively the continuity and differentiability properties of solutions of (jump type) stochastic differential equations with respect to the initial condition. The method of proof is classical, based on Gronwall's inequality, L^2-isometry properties of stochastic integrals and on Kunita's moments inequalities (see again [24]).

2.1. The general setting

Let $s < T$, and let B and B' be normed spaces. Let $\Sigma(s,T)$ be a linear space of maps (or trajectories) $x = x(\cdot) : [s,T] \to B$. Let

$$\Phi : \Sigma(s,T) \to \Sigma(s,T), \tag{2.1}$$

$$f : [s,T] \times B \to B' \tag{2.2}$$

and assume also that f is *pathwise strongly measurable* with respect to $\Sigma(s,T)$, which, by definition, means that for all $x(\cdot) \in \Sigma(s,T)$, the trajectory $[s,T] \ni t \to f(t, x(t)) \in B'$ is strongly measurable. Let B_s—to be interpreted as the space of initial conditions—be a subspace of B, which may depend on s. Given $x^0 \in B_s$, consider the equation

$$x(\cdot) = x^0 + \Phi\big(x(\cdot)\big) \quad \text{in } [s,T], \tag{2.3}$$

by a *solution* of which we mean any trajectory $x(\cdot) \in \Sigma(s,T)$ such that the map $[s,T] \ni t \to x^0 + \Phi\big(x(\cdot)\big)(t)$ belongs also to $\Sigma(s,T)$ and coincides with $x(\cdot)$.

Before dealing with the $C^{1,\omega}$-dependence of solutions on initial condition, we need (to recall) a result regarding Lipschitz estimates of solutions in terms of initial conditions, which is useful also in our subsequent applications to Itô-Skorokhod equations.

Theorem 2.1. (Local Lipschitz estimates) *Assume that for all* $x_1(\cdot)$, $x_2(\cdot) \in \Sigma(s,T)$, *and for some fixed* $1 \leq p < \infty$,

$$\big\|\Phi\big(x_1(\cdot)\big)(t) - \Phi\big(x_2(\cdot)\big)(t)\big\|_B^p \leq \int_s^t \big\|f\big(r, x_1(r)\big) - f\big(r, x_2(r)\big)\big\|_{B'}^p \, dr, \tag{2.4}$$

$$\big\|\Phi\big(x_1(\cdot)\big)(t)\big\|_B^p \leq \int_s^t \big\|f\big(r, x_1(r)\big)\big\|_{B'}^p \, dr. \tag{2.5}$$

Assume that f *grows at most linearly and is locally Lipschitz uniformly in time, that is, for some* $C_f \geq 0$,

$$\|f(r,x)\|_{B'}^p \leq C_f \left(1 + \|x\|_B^p\right) \tag{2.6}$$

for all $s \leq r \leq T$, $x \in B$. *Finally, assume that for any bounded subset* $K \subset B$, *there exists* $L_{f,K} \geq 0$, *such that*

$$\|f(r,x_1) - f(r,x_2)\|_{B'} \leq L_{f,K} \|x_1 - x_2\|_B \tag{2.7}$$

for all $s \leq r \leq T$, $x_1, x_2 \in K$.

Let K *be a bounded subset of* B_s. *Then there exists* $L_{\Phi,K} \geq 0$ (*see* (2.12) *below for an explicit value of such a constant*), *such that, for all* $x_1^0, x_2^0 \in K$, *if* $x_i(\cdot)$ *are solutions to Eq.* (2.3) *with initial condition* $x^0 = x_i^0$, $i = 1,2$, *respectively, we have*

$$\|x_1(t) - x_2(t)\|_B \leq L_{\Phi,K} \|x_1^0 - x_2^0\|. \tag{2.8}$$

Proof. First of all, solutions, departing from any $x_0 \in K$, remain bounded in B. Indeed, by (2.3), (2.5), (2.6),

$$\|x(t)\|_B^p \le 2^{p-1} \left(\|x^0\|_B^p + C_f(t-s) + C_f \int_s^t \|x(r)\|_B^p dr \right),$$

and Gronwall's inequality implies

$$\|x(t)\|_B^p \le \left(2^{p-1}\|x^0\|_B^p + 1\right) e^{2^{p-1} C_f\,(t-s)}. \tag{2.9}$$

Let B_R be the the ball of B centered at the origin and radius R an upper bound of the $1/p$-th power of the quantities on the right-hand side of (2.9) for $t = T$, as $x^0 \in K$, for example, let

$$R = 2^{1-\frac{1}{p}} \left(\mathrm{diam}(K) + \mathrm{dist}(0, K) + 1\right) e^{2^{1-1/p} C_f\,(T-s)}. \tag{2.10}$$

By (2.3), (2.4), (2.7), for all $x_1^0, x_2^0 \in K$,

$$\|x_1(t)-x_2(t)\|_B^p \le 2^{p-1}\|x_1^0-x_2^0\|_B^p + 2^{p-1} L_{f,B_R}^p \int_s^t \|x_1(r)-x_2(r)\|_B^p dr, \tag{2.11}$$

which, again by Gronwall's inequality, yields the desired estimate (2.8) with

$$L_{\Phi,K} = 2^{1-1/p} L_{f,B_R} e^{2^{p-1} L_{f,B_R}^p (t-s)/p}. \tag{2.12}$$

□

Remark 2.2. (Global Lipschitz estimates) *If we wish to obtain global Lipschitz estimates, then we must assume that f is globally Lipschitz in state variable uniformly in time, that is, that assumption* (2.7) *holds for some fixed $L_{f,K} = L_f$ independent of K, and for all $x_1^0, x_2^0 \in B$, obviously now assumption* (2.5) *becomes superfluous. We can conclude that estimate* (2.8) *holds for all $x_1^0, x_2^0 \in B_s$, where L_Φ is given by the right-hand side of* (2.12) *with $L_{f,B_R} = L_f$. We write down L_Φ explicitly for future reference*

$$L_\Phi = 2^{1-1/p} L_f e^{2^{p-1} L_f^p (t-s)/p}. \tag{2.13}$$

In order to obtain $C^{1,\omega}$-estimates of solutions with respect to initial condition applicable to stochastic differential equations, we need a further hypothesis on the dynamic Φ in (2.1) and on the map f in (2.2).

Let be given another pair of normed spaces $\mathcal{C}$, $\mathcal{C}'$,

$$\mathcal{C} \hookrightarrow B, \quad \mathcal{C}' \hookrightarrow B' \tag{2.14}$$

(with embedding constants equal to one, this is not restrictive for one can always suitably renormalize norms) which is invariant for the map f, that is, $f(t, \mathcal{C}) \subset \mathcal{C}'$ for all $s \le t \le T$, so that we can see f as a map

$$f : [s, T] \times \mathcal{C} \to \mathcal{C}' \tag{2.15}$$

by restriction. We assume now that the space of trajectories $\Sigma(s, T)$ consists only of maps $x(\cdot) : [s, T] \to \mathcal{C}$, and moreover that f in (2.15) is pathwise strongly measurable also with respect to $\Sigma(s, T)$ (recall, this means that for all $x(\cdot) \in \Sigma(s, T)$, the map $[s, T] \ni t \mapsto f(t, x(t)) \in \mathcal{C}'$ is strongly measurable).

We fix also a subspace $\mathcal{C}_s \subset \mathcal{C}$, possibly depending on s, which is going to be the set of initial conditions x^0 for which we solve Eq. (2.3).

Theorem 2.3. (Local $C^{1,\omega}$-estimates) *Let $1 \le p_i < \infty$, $i = 1, 2$. Assume that for all $x_1(\cdot)$, $x_2(\cdot)$, $x_3(\cdot) \in \Sigma(s,T)$ and for all $0 \le \lambda \le 1$, $s \le t \le T$*

$$\begin{aligned}\big\|\lambda\Phi\big(x_1(\cdot)\big)(t) &+ (1-\lambda)\Phi\big(x_2(\cdot)\big)(t) - \Phi\big(x_3(\cdot)\big)(t)\big\|_B^{p_1} \\ &\le \int_s^t \big\|\lambda f\big(r, x_1(r)\big) + (1-\lambda) f\big(r, x_2(r)\big) - f\big(r, x_3(r)\big)\big\|_{B'}^{p_1} dr\,, \end{aligned} \tag{2.16}$$

$$\big\|\Phi\big(x_1(\cdot)\big)(t) - \Phi\big(x_2(\cdot)\big)(t)\big\|_{\mathcal{C}}^{p_2} \le \int_s^t \big\|f\big(r, x_1(r)\big) - f\big(r, x_2(r)\big)\big\|_{\mathcal{C}'}^{p_2} dr, \tag{2.17}$$

$$\big\|\Phi\big(x_1(\cdot)\big)(t)\big\|_{\mathcal{C}}^{p_2} \le \int_s^t \big\|f\big(r, x_1(r)\big)\big\|_{\mathcal{C}'}^{p_2} dr. \tag{2.18}$$

Assume that $f : [s,T] \times \mathcal{C} \to \mathcal{C}'$, $f : [s,T] \times B \to B'$ are locally Lipschitz continuous on $\mathcal{C}$, uniformly in time, that is, for any bounded subset $K \subset \mathcal{C}$, there exists $L_{f,K} \ge 0$, such that

$$\|f(r,x_1) - f(r,x_2)\|_{\mathcal{C}'} \le L_{f,K}\|x_1 - x_2\|_{\mathcal{C}}\,, \tag{2.19}$$

$$\|f(r,x_1) - f(r,x_2)\|_{B'} \le L_{f,K}\|x_1 - x_2\|_B \tag{2.20}$$

for all $s \le r \le T$, $x_1, x_2 \in K$. Let also $f : [s,T] \times \mathcal{C} \to \mathcal{C}'$ grow at most linearly, that is, for some $C_f \ge 0$,

$$\|f(r,x)\|_{\mathcal{C}'}^{p_2} \le C_f(1 + \|x\|_{\mathcal{C}}^{p_2}) \tag{2.21}$$

for all $s \le r \le T$, $x \in \mathcal{C}$.

Fix a bounded subset K of $\mathcal{C}_s$. If $f : [s,T] \times \mathcal{C} \to B'$ is of class C^{1,ω_f} in $x \in \mathcal{C}$ for some modulus ω_f, that is, if for all $x_1, x_2 \in \mathcal{C}$, $s \le r \le T$ and for all $0 \le \lambda \le 1$,

$$\begin{aligned}\big\|\lambda f(r,x_1) + (1-\lambda) f(r,x_2) - f\big(r, \lambda x_1 + (1-\lambda)x_2\big)\big\|_{B'} \\ \le \lambda(1-\lambda)\|x_1 - x_2\|_{\mathcal{C}}\omega_f(\|x_1 - x_2\|_{\mathcal{C}}),\end{aligned} \tag{2.22}$$

then, for all $x_1^0, x_2^0 \in K$, $0 \le \lambda \le 1$, if $x_3^0 = \lambda x_1^0 + (1-\lambda)x_2^0$, and if $x_i(\cdot)$ are solutions of (2.3) with initial conditions $x^0 = x_i^0$, $i = 1, 2, 3,$, respectively, we have

$$\|\lambda x_1(t) + (1-\lambda)x_2(t) - x_3(t)\|_B \le \lambda(1-\lambda)\|x_1^0 - x_2^0\|_{\mathcal{C}}\omega_{\Phi,K}(\|x_1^0 - x_2^0\|_{\mathcal{C}}) \tag{2.23}$$

for all $s \le t \le T$, where

$$\omega_{\Phi,K}(\rho) = c_{\Phi,K,1}\omega_f(c_{\Phi,K,2}\rho), \quad \rho \ge 0, \tag{2.24}$$

for some constants $c_{\Phi,K,1}, c_{\Phi,K,2} \ge 0$ that depend only on $T - s, p, C_f, K$ and ω_f (see (2.26) for an example of explicit values of such constants).

Proof. By our assumptions on Φ and $f : [s,T] \times \mathcal{C} \to \mathcal{C}'$ Theorem 2.1 implies that for any bounded subset K of $\mathcal{C}_s$ there exists $L_{\Phi,K} > 0$ such that for all $x_1^0, x_2^0 \in K$, if $x_i(\cdot)$ are solutions to Eq. (2.3) with initial conditions $x^0 = x_i^0$, $i = 1, 2$, respectively, we have

$$\|x_1(t) - x_2(t)\|_{\mathcal{C}} \le L_{\Phi,K}\|x_0^1 - x_0^2\|_{\mathcal{C}}; \tag{2.25}$$

the constant $L_{\Phi,K}$ in this estimate can be taken equal to the right-hand side of (2.12), where B_R is now the ball of $\mathcal{C}$ centered at the origin and radius R

given by (2.10) (of course, now $\operatorname{diam}(K)$, $\operatorname{dist}(0, K)$ are to be understood in $\mathcal{C}$-norm).

Moreover, as we saw during the proof of Theorem 2.1 solutions departing from points of K remain in the ball B_R. Therefore, we can use both estimates (2.19), (2.20) with $K = B_R$.

By applying in order (2.3), (2.16), triangle inequality, (2.22), (2.20), (2.19) and (2.25), we estimate as follows

$$
\begin{aligned}
&\|\lambda x_1(t) + (1-\lambda)x_2(t) - x_3(t)\|_B^{p_1}\\
&\quad\le 2^{p_1-1}\int_s^t \left\|\lambda f\big(r, x_1(r)\big) + (1-\lambda) f\big(r, x_2(r)\big) - f\left(r, \lambda x_1(r) + (1-\lambda)x_2(r)\right)\right\|_{B'}^{p_1} dr\\
&\qquad + 2^{p_1-1}\int_s^t \left\|f\left(r, \lambda x_1(r) + (1-\lambda)x_2(r)\right) - f\big(r, x_3(r)\big)\right\|_{B'}^{p_1} dr\\
&\quad\le 2^{p_1-1}\lambda^{p_1}(1-\lambda)^{p_1}\int_s^t \|x_1(r) - x_2(r)\|_{\mathcal{C}}^{p_1}\omega_f^{p_1}(\|x_1(r) - x_2(r)\|_{\mathcal{C}})dr\\
&\qquad + 2^{p_1-1}L_{f,B_R}^{p_1}\int_s^t \|\lambda x_1(t) + (1-\lambda)x_2(t) - x_3(t)\|_B^{p_1} dr\\
&\quad\le 2^{p_1-1}\lambda^{p_1}(1-\lambda)^{p_1}L_{\Phi,K}^{p_1}\|x_1^0 - x_2^0\|_{\mathcal{C}}^{p_1}\omega_f^{p_1}(L_{\Phi,K}\|x_1^0 - x_2^0\|_{\mathcal{C}})\\
&\qquad + 2^{p_1-1}L_{f,B_R}^{p_1}\int_s^t \|\lambda x_1(t) + (1-\lambda)x_2(t) - x_3(t)\|_B^{p_1} dr,
\end{aligned}
$$

where $L_{\Phi,K}$ is defined in (2.12). Gronwall's inequality strikes again and therefore (2.23), (2.24) hold with

$$
c_{\Phi,K,1} = 2^{1-1/p_1}L_{\Phi,K}e^{2^{p_1-1}L_{f,B_R}^{p_1}(T-s)/p_1}, \quad c_{\Phi,K,2} = L_{\Phi,K}. \tag{2.26}
$$

Thus, the proof is finished. □

Remark 2.4. (Global $C^{1,\omega}$-estimates) *If we wish to obtain global $C^{1,\omega}$-estimates, we must assume that f is globally Lipschitz in state variable uniformly in time, that is, that assumptions (2.20), (2.19) hold for some fixed $L_{f,K} = L_f$ which is the same for all bounded subsets K of $\mathcal{C}$, and for all $x_1, x_2 \in \mathcal{C}$, Obviously, now assumption (2.21) is superfluous. We can then conclude that estimate (2.23) holds for all $x_1^0, x_2^0 \in \mathcal{C}_s$, $0 \le \lambda \le 1$, where ω_Φ is given by the right-hand side of (2.24), with constants $c_{\Phi,K,1}$, $c_{\Phi,K,2}$, actually independent of K, given by (2.26) with $L_{f,K} = L_f$ and $L_{\Phi,K} = L_\Phi$ given by (2.13). We write down this ω_Φ explicitly for future reference:*

$$
\omega_\Phi(\rho) = 2^{1-1/p_1}L_\Phi e^{2^{p_1-1}L_f^{p_1}(T-s)/p_1}\omega_f(L_\Phi\rho) \tag{2.27}
$$

for all $\rho \ge 0$, where L_Φ is given by (2.13) with $p = p_2$.

2.2. Application of results to general stochastic differential equations

In applications we have in mind, spaces $B, B', \mathcal{C}, \mathcal{C}'$ are normed spaces of random variables in some fixed probability space. More precisely, let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ be a probability space, X and Y normed spaces (not necessarily finite-dimensional), let B, $\mathcal{C}$ be normed spaces of X-valued random variables and B', $\mathcal{C}'$ normed spaces of Y-valued random variables. In our applications we have to consider maps $f : B \to B'$ (such that $f(\mathcal{C}) \subset \mathcal{C}'$) which arise in a natural way through a "deterministic" map $g : X \to Y$, by setting[3] $f(x)(\omega) = g(x(\omega))$ for all $x \in B$, $\omega \in \Omega$. Assuming the C^{1,ω_g}-regularity of $g : X \to Y$ for a given modulus ω_g, we want to prove the C^{1,ω_f}-regularity of $f : \mathcal{C} \to B$ for some modulus ω_f, possibly expressing ω_f in terms of ω_g. We do not know whether this is possible in general, however, we have the following results.

Lemma 2.5. *Let $g : X \to Y$ be of class C^{1,ω_g} for some modulus ω_g. Then $f : \mathcal{C} \to B'$ is of class C^{1,ω_f} for some modulus ω_f under anyone of the following conditions:*

- *$\omega_g(\rho) = k\,\rho^\alpha$ for some $k \geq 0$, $0 < \alpha (\leq 1)$, and $L^p(\Omega; Y) \hookrightarrow B'$, $\mathcal{C} \hookrightarrow L^{p(1+\alpha)}(\Omega; X)$ for some $1 \leq p \leq \infty$;*
- *$\gamma_g(\rho) = \left(\rho^\beta \omega_g^2(\rho)\right)^q$, where $0 \leq \beta \leq 2$, $1 \leq q \leq \infty$, $r^{-1} + q^{-1} = 1$, is concave, and $L^2(\Omega; Y) \hookrightarrow B'$, $\mathcal{C} \hookrightarrow L^{(2-\beta)r}(\Omega; X)$, $\mathcal{C} \hookrightarrow L^1(\Omega; X)$.*

In both cases $\omega_f = \omega_g$.

Proof. The proof of the result under the first set of assumptions is trivial. As to the second, by Hölder's and Jensen's inequalities and assumptions, we can estimate as follows

$$\begin{aligned}
&\|\lambda f(x_1) + (1-\lambda) f(x_2) - f(\lambda x_1 + (1-\lambda)x_2)\|_{B'}^2 \\
&\quad \leq E[\|\lambda g(x_1) + (1-\lambda) g(x_2) - g(\lambda x_1 + (1-\lambda)x_2)\|_Y^2] \\
&\quad \leq \lambda^2 (1-\lambda)^2 E[\|x_1 - x_2\|_X^2 \omega_g^2(\|x_1 - x_2\|_X)] \\
&\quad \leq \lambda^2 (1-\lambda)^2 \left(E[\|x_1 - x_2\|_X^{r(2-\beta)}]\right)^{\frac{1}{r}} \left(E[\gamma(\|x_1 - x_2\|_X)]\right)^{\frac{1}{q}} \\
&\quad \leq \lambda^2 (1-\lambda)^2 \left(\left(E[\|x_1 - x_2\|_X^{r(2-\beta)}]\right)^{\frac{1}{(2-\beta)r}}\right)^{2-\beta} \left(\gamma\left(E[\|x_1 - x_2\|_X]\right)\right)^{\frac{1}{q}} \\
&\quad \leq \lambda^2 (1-\lambda)^2 \|x_1 - x_2\|_{\mathcal{C}}^{2-\beta} \left(\gamma(\|x_1 - x_2\|_{\mathcal{C}})\right)^{\frac{1}{q}} \\
&\quad = \lambda^2 (1-\lambda)^2 \|x_1 - x_2\|_{\mathcal{C}}^2 \omega_g^2(\|x_1 - x_2\|_{\mathcal{C}}).
\end{aligned}$$

Thus, the proof is over. □

For example, in the case of power type moduli (the first instance in the lemma above), Theorem 2.3 (see also Remark 2.4) has the following corollaries.

Corollary 2.6. (Global $C^{1,\omega}$-estimates, power type moduli) *Consider maps Φ and f as in (2.1), (2.15), respectively, and assume that Φ and f satisfy conditions (2.16), (2.17) for certain $1 \leq p_1, p_2 < \infty$. Let X, Y be normed spaces, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, and let[4] $B \hookrightarrow L^p(\Omega; X)$, $L^p(\Omega; Y) \hookrightarrow B'$, $\mathcal{C} \hookrightarrow L^{(1+\alpha)p}(\Omega; X)$, $L^{(1+\alpha)p}(\Omega; Y) \hookrightarrow \mathcal{C}'$, for some $1 \leq p \leq \infty$, $0 < \alpha (\leq 1)$, $f(r, x)(\omega) = g(r, x(\omega))$ for all $r \in [s, T]$, $x \in B$, $\omega \in \Omega$, where $g : [s, T] \times X \to$*

[3] This is one of the few instances in this paper where ω denotes an element of a probability space rather than a semiconcavity modulus.

[4] One may read these embeddings as equalities at a first reading; this is also sufficient for our applications to jump diffusions below.

Y is a given map which is Lipschitz continuous in $x \in X$, uniformly in time $r \in [s,T]$, that is, for some $L_g \geq 0$,

$$\|g(r,x_1) - g(r,x_2)\|_Y \leq L_g \|x_1 - x_2\|_X \tag{2.28}$$

for all $s \leq r \leq T$, and $x_1, x_2 \in X$. Assume that g is of class C^{1,ω_g} on X, uniformly in time $r \in [s,T]$ for some modulus ω_g, that is,

$$\begin{aligned} &\|\lambda g(r,x_1) + (1-\lambda)g(r,x_2) - g(r,\lambda x_1 + (1-\lambda)x_2)\|_Y \\ &\quad \leq \lambda(1-\lambda)\|x_1 - x_2\|_X \omega_g(\|x_1 - x_2\|_X) \end{aligned} \tag{2.29}$$

for all $s \leq r \leq T$, $x_1, x_2 \in X$, $0 \leq \lambda \leq 1$, and that the modulus ω_g is of power type, that is, $\omega_g(\rho) = k\rho^\alpha$ for $\rho \geq 0$, where $k \geq 0$. Then estimate (2.23) holds for all $s \leq t \leq T$, $x_1^0, x_2^0 \in \mathcal{C}_s$ and $0 \leq \lambda \leq 1$ with ω_Φ as in (2.27), where L_Φ is as in (2.13) and $\omega_f = \omega_g$.

Corollary 2.7. (Extension to Cartesian product maps) *Let the map g (and consequently f) in Corollary 2.6 above, come up as a Cartesian product. That is, let $Y = \prod_{i=1}^{\ell} Y_i$, transformed into a normed space via a given norm $\|\cdot\|_{\mathbb{R}^\ell}$ on $\mathbb{R}^\ell$ in a canonical way,[5] and $g = \prod_{i=1}^{\ell} g_i$, (which means $g(r,x) = (g_1(r,x), \ldots, g_\ell(r,x))$ for all $(r,x) \in [s,T] \times X$), $B' = \prod_{i=1}^{\ell} B_i'$, $\mathcal{C}' = \prod_{i=1}^{\ell} \mathcal{C}_i'$, where each B_i', $\mathcal{C}_i'$ are normed spaces of Y_i-valued random variables for $i = 1, \ldots, \ell$. Assume that each g_i, for $i = 1, \ldots, \ell$, is of class $C^{1,\omega_{g_i}}$ for some power type modulus $\omega_{g_i}(\rho) = k_i \rho^{\alpha_i}$, where $k_i \geq 0$, $0 < \alpha_i (\leq 1)$. If $B \hookrightarrow L^p(\Omega; X)$, $L^p(\Omega; Y_i) \hookrightarrow B_i'$, $\mathcal{C} \hookrightarrow L^{(1+\alpha)p}(\Omega; X)$, $L^{(1+\alpha_i)p}(\Omega; Y_i) \hookrightarrow \mathcal{C}_i'$ for $i = 1, \ldots, \ell$ for some $1 \leq p \leq \infty$, $\alpha = \max\{\alpha \, : \, i = 1, \ldots, \ell\}$, keeping the rest of assumptions unchanged in Corollary 2.6, then the conclusion of Corollary 2.6 still holds, with the only change that we must take now $\omega_g = \|(\omega_{g_1}, \ldots, \omega_{g_1})\|_{\mathbb{R}^\ell}$.*

Of course, similar results can be formulated for moduli with suitable concavity properties (the second set of assumptions in Lemma 2.5). Since this is rather straightforward we leave the task to the interested reader.

Remark 2.8. (Lipschitz and $C^{1,\omega}$-estimates with conditions only along solutions) For any $B_0 \subset B_s (\subset B)$, let

$$\mathrm{Sol}_B^\Phi(B_0) = \{x(\cdot) \in \Sigma(s,T) \, : \, \exists x^0 \in B_0 \text{ such that (2.3) is satisfied}\} \tag{2.30}$$

$$\mathrm{Acc}_B^\Phi(B_0; t) = \{x(t)(\in B) \, : \, x(\cdot) \in \mathrm{Sol}^\Phi(\mathcal{C}_0)\}. \tag{2.31}$$

The conclusion of Theorem 2.1 remains valid for any bounded set $K \subset B_0$ if we require that (2.4), (2.5) hold for all $x_1(\cdot), x_2(\cdot) \in \mathrm{Sol}_B^\Phi(B_0)$ (instead of $\Sigma(s,T)$), (2.6) holds for any $x \in \mathrm{Acc}_B^\Phi(B_0; t)$, and (2.7) holds for any bounded subset K of $\mathrm{Acc}_B^\Phi(B_0; t)$, with constants C_f, $L_{f,K}$ independent of $t \in [s,T]$. For the conclusion of Remark 2.2 to remain valid, we must require that the

[5] That is, with norm

$$\prod_{i=1}^{\ell} Y_i \ni (y_1, \ldots, y_\ell) \mapsto \|(\|y_1\|_{Y_1}, \ldots, \|y_\ell\|_{Y_\ell})\|_{\mathbb{R}^\ell} \in \mathbb{R}.$$

Lipschitz condition above hold with constant $L_{f,K} = L_f$ independent of K. (Recall that (2.5) and (2.6) are not needed for this remark.)

An analogous observation holds for Theorem 2.3. Let $\mathcal{C}_0 \subset \mathcal{C}_s(\subset \mathcal{C})$. If we require that (2.16), (2.17), (2.18) hold only for $x_1(\cdot), x_2(\cdot), x_3(\cdot) \in \mathrm{Sol}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0)$, $0 \leq \lambda \leq 1$, (2.19), (2.20) hold only for bounded sets $K \subset \mathrm{Acc}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0; t)$, and (2.21) holds for any $x \in \mathrm{Acc}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0; t)$ for $t \in [s, T]$ with constants $L_{K,f}$, C_f independent of t, then the conclusion of that theorem is still valid for any bounded set $K \subset \mathcal{C}_0$. For the conclusion of Remark 2.4 to remain valid it suffices that in addition the said Lipschitz constant be independent of K. (Recall that (2.18), (2.21) are unnecessary for the validity of this remark.)

These observations turn useful sometimes, for example, in proving local results: the fact is that if $g : [s, T] \times X \to Y$ is locally Lipschitz in $x \in X$, uniformly in time (which in turn may follow[6] by the $C^{1,\omega}$-regularity of g), then the map $f : [s, T] \times B \to B'$, defined by $f(r, x)(\omega) = g(r, x(\omega))$ for all $r \in [s, T]$, $x \in B$, $\omega \in \Omega$, is not necessarily locally Lipschitz continuous in $x \in B$ (or in $\mathcal{C}$). However, if we can prove that for some zero measure subset N of Ω, and some $\mathcal{C}_0 \subset \mathcal{C}_s$, the set

$$\{x(t)(\omega) \ : \ x(\cdot) \in \mathrm{Sol}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0), \quad s \leq t \leq T, \quad \omega \in \Omega \backslash N\}$$

is bounded in X, for example, by Kolmogorov's continuity criterion (Theorem 2.9 below) or some other method, then a locally Lipschitz g, uniformly in time, gives an f which is locally Lipschitz on each set $\mathrm{Acc}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0; t)$, uniformly in $t \in [s, T]$. Now the formulation of local results (under the assumption that g is only of class C^{1,ω_g} and locally bounded) should be routine.

In the derivation of $C^{1,\omega}$-regularity and ω-semiconcavity estimates in the sequel via Kolmogorov's continuity criterion. we are able to prove only this kind of $C^{1,\omega}$-regularity for the map f:

$$\begin{aligned} &\|\lambda f\big(r, x(r, x_1^0)\big) + (1-\lambda) f\big(r, x(r, x_1^0)\big) - f\big(r, \lambda x(r, x_1^0) + (1-\lambda) x(r, x_2^0)\big)\|_{B'} \\ &\quad \leq \lambda(1-\lambda)\|x(r, x_1^0) - x(r, x_2^0)\|_{\mathcal{C}}\, \omega_f(\|x_1^0 - x_2^0\|_{\mathcal{C}}). \end{aligned} \tag{2.32}$$

for all $x_1^0, x_2^0 \in \mathcal{C}_0$, where $\mathcal{C}_0 \subset \mathcal{C}_s$ and $x(\cdot, x^0)$ denotes the solution of (2.3) corresponding to the initial condition x^0. Yet, this is sufficient to obtain the conclusion of Theorem 2.3 on bounded subsets of $\mathcal{C}_0$ or of Remark 2.4 on $\mathcal{C}_0$, if the rest of their assumptions are unaltered or at most replaced by the weaker ones discussed at the beginning of this remark.

Assume that Eq. (2.3) has continuous flow in the following sense.

$$\|x(t, x_1^0) - x(t, x_2^0)\|_X \leq \omega_c(\|x_1^0 - x_2^0\|_{\mathcal{C}}) \quad \mathbb{P} - a.s. \tag{2.33}$$

for all $x_1^0, x_2^0 \in \mathcal{C}_0$ for some $\mathcal{C}_0 \subset \mathcal{C}_s$, where $\omega_c : \Omega \times [0, \infty[\to [0, \infty[$ is a random modulus.[7] Is is then easy to show, by Hölder's inequality, that estimate (2.32) holds with

$$\omega_f(\rho) = \big(E\left[\omega_g^q\left(\omega_c(\rho)\right)\right]\big)^{1/q} \quad \forall \rho \geq 0, \tag{2.34}$$

[6] See, for example, [16, Theorem 2.1.7, p. 33].

[7] That is, for each $\omega \in \Omega$, the map $\omega_c(\omega, \cdot) : [0, \infty[\to [0, \infty[$ is a modulus, and for each $\rho \in [0, \infty[$, the map $\omega(\cdot, \rho) : \Omega \to [0, \infty[$ is a random variable.

if $L^{p_0}(\Omega; Y) \hookrightarrow B'$ and $\mathcal{C} \hookrightarrow L^p(\Omega; X)$ for some $1 \leq p_0 \leq p, q \leq \infty$ such that $1/p_0 \geq 1/p + 1/q$. Of course, it is not known a priori whether the right-hand side of (2.34) is finite or not; however, if $\omega_f(\rho_0) < \infty$ for some $\rho_0 > 0$, then, by Lebesgue's dominated convergence theorem, $\omega_f(0^+) = 0$. If we assume that $\rho \mapsto \omega_g^q\left(\rho^{1/q}\right)$ is concave (a requirement which is not very restrictive on these moduli: for instance, all power type moduli satisfy this requirement) then ω_f can be estimated as follows:

$$\omega_f(\rho) \leq \omega_g\left(\left(E\left[\omega_c^q(\rho)\right]\right)^{1/q}\right) \quad \forall \rho \geq 0. \tag{2.35}$$

One can obtain continuity results about the flow $\mathcal{C}_0 \ni x^0 \to x(t, x^0) \in X$ by using Kolmogorov's continuity criterion, as do, e.g., Fujiwara and Kunita [18,24] or Protter [26]. However, it seems that not much attention is paid to obtaining explicit expressions for the continuity moduli. In order to comply with this necessity let us recall Kolmogorov's criterion in a slightly more precise form than it is usually stated in literature.

Theorem 2.9. (Kolmogorov's continuity criterion) *Let $X \subset \mathcal{C}_s$ be a d-dimensional linear space, where $d \in \mathbb{N}$, and assume that for some $p > d$, $q > 0$, $C_0 > 0$,*

$$E\left[\|x(t, x_1^0) - x(t, x_2^0)\|_X^q\right] \leq C_0 \|x_1^0 - x_2^0\|_{\mathcal{C}}^p$$

for all $x_1^0, x_2^0 \in X$, $s \leq t \leq T$. Then, for all $0 < \beta < (p - d)/q$, there exists a random variable $k \geq 0$ (that depends on d, p, q, β, C_0) with $E[k^q] < \infty$ such that (a modification of) $X \ni x^0 \to x(t, x^0) \in B$ satisfies (2.33) with

$$\omega_c(\rho) = k(\rho^\beta + \rho) \quad \forall \rho \geq 0, \tag{2.36}$$

for all $x_1^0, x_2^0 \in X$.

Proof. A standard proof uses the Sobolev-Hölder embedding $W^{s,q}(K) \hookrightarrow C^{0,\beta}(\overline{K})$ if $sq > d$ and $\beta = s - d/q$ where K is, for example, any unit ball in X, see [1]. (Of course, the embedding constant can be chosen to be the same for all unit balls in X.) Using this fact one proves (2.33) with ω_c given by (2.36) with k satisfying the claimed properties. □

Using this criterion, Theorem 2.3 (see also Remark 2.4 and estimate (2.32) with $\omega_f = \omega_g \circ \omega_c$) has the following corollary.

Corollary 2.10. (Global $C^{1,\omega}$-estimates on X with general moduli) *Consider maps Φ and f as in (2.1), (2.15), respectively, and assume that Φ and f satisfy conditions (2.16), (2.17) for some $1 \leq p_1, p_2 < \infty$. Let X, Y be normed spaces and assume that X is d-dimensional, where $d \in \mathbb{N}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $B \hookrightarrow L^{p_0}(\Omega; X)$, $L^{p_0}(\Omega; Y) \hookrightarrow B'$, $\mathcal{C} \hookrightarrow L^p(\Omega; X)$, $L^p(\Omega; Y) \hookrightarrow \mathcal{C}'$, for certain $1 \leq p_0 \leq p \leq \infty$, $p > d$. Let $f(r, x)(\omega) = g(r, x(\omega))$ for all $r \in [s, T]$, $x \in B$, $\omega \in \Omega$, where $g : [s, T] \times X \to Y$ is a given map which is Lipschitz continuous in $x \in X$, uniformly in time $r \in [s, T]$, that is, satisfies (2.28) for some $L_g \geq 0$, and of class C^{1,ω_g} in $x \in X$ for some modulus ω_g, uniformly in time $r \in [s, T]$, that is, satisfies (2.29).*

Then for all $0 < \beta < 1 - d/p$ estimate (2.23) holds for all $s \leq t \leq T$, $x_1^0, x_2^0 \in X$ and $0 \leq \lambda \leq 1$ with ω_Φ as in (2.27) with ω_f given by (2.34), where

$q \in [1,\infty]$ is such that $1/p_0 \geq 1/p + 1/q$, ω_c given by (2.36), where $k \geq 0$ is a random variable that depends on $T-s$, p_1, p_2, d, p, β, L_g with $E[k^p] < \infty$, and $L_f = L_g$.

If in addition $\rho \mapsto \omega_g^q\left(\rho^{1/q}\right)$ is concave, then ω_f can be estimated by

$$\omega_f(\rho) \leq \omega_g\left(\left(E\left[k^q\right]\right)^{1/q}\left(\rho^\beta + \rho\right)\right) \quad \forall \rho \geq 0, \tag{2.37}$$

(which is finite if also $q \leq p$).

In case of power type moduli ω_g, the result given by the corollary above is of course not as precise as Corollary 2.6. Assumptions are generally stronger, and the moduli obtained for the $C^{1,\omega}$-dependence of solutions on initial conditions are much "weaker". However, its advantage stands in the fact that it allows to deal with quite general moduli.

2.3. Application of results to jump diffusions

As an application of the above "abstract" results we obtain the $C^{1,\omega}$-estimates for solutions of a large class of jump type stochastic differential equations with respect to initial conditions.

Let $T > 0$ be a fixed time horizon, and let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$ be a complete filtered probability space. For any $(s,x^0) \in [0,T) \times \mathbb{R}^d$ consider a jump stochastic differential equation (or an Itô-Skorokhod equation as it is alternatively called in literature)

$$\begin{aligned} x(t) = x^0 &+ \int_s^t b\big(r, x(r-)\big)dr + \int_s^t \sigma\big(r, x(r-),\big)dW(r) \\ &+ \int_s^t \int_{\|z\|\leq\delta} H\big(r, x(r-), z\big)\tilde{N}(drdz) + \int_s^t \int_{\|z\|>\delta} K\big(r, x(r-), z\big)N(drdz) \end{aligned} \tag{2.38}$$

where notation has the following meaning. $W = W(\cdot)$ is a standard m-dimensional Brownian motion and N an independent Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d\backslash\{0\})$ with associated compensated measure $\tilde{N}$ and intensity measure ν, which we assume to be a Lévy measure. As usual, we also assume that W and N have increments $W(r) - W(s)$, $N(r) - N(s)$ independent of $\mathcal{F}_s$ for all $s \leq r \leq T$. The maps

$$\begin{aligned} b &: [0,T] \times \mathbb{R}^d \to \mathbb{R}^d, & \sigma &: [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}, \\ H &: [0,T] \times \mathbb{R}^d \times B_\delta \times A \to \mathbb{R}^d, & K &: [0,T] \times \mathbb{R}^d \times (\mathbb{R}^d\backslash B_\delta) \times A \to \mathbb{R}^d, \end{aligned}$$

are measurable (here $\delta > 0$ is some fixed parameter and B_δ is the ball of $\mathbb{R}^d$ centered at 0 and of radius δ) and in line with the purpose of this article we assume that they satisfy the following growth, Lipschitz and $C^{1,\omega}$ conditions.

Let $p \geq p_0 \geq 2$. We assume that there exist $L_b, L_\sigma, L_H, L_K \geq 0$ and semiconcavity moduli $\omega_b, \omega_\sigma, \omega_H, \omega_K$ such that

$$\begin{cases} \|b(r,x_1) - b(r,x_2)\| \leq L_b \|x_1 - x_2\|, \\ \|\sigma(r,x_1) - \sigma(r,x_2)\| \leq L_\sigma \|x_1 - x_2\|, \\ \displaystyle\int_{\|z\|\leq\delta} \|H(r,x_1,z) - H(r,x_2,z)\|^p \nu(dz) \leq (L_H\|x_1 - x_2\|)^p, \\ \displaystyle\int_{\|z\|>\delta} \|K(r,x_1,z) - K(r,x_2,z)\|^p \nu(dz) \leq (L_K\|x_1 - x_2\|)^p \end{cases} \tag{2.39}$$

$$\begin{cases} \|\lambda b(r,x_1) + (1-\lambda)b(r,x_2) - b\big(r,\lambda x_1 + (1-\lambda)x_2\big)\| \\ \qquad\qquad \leq \lambda(1-\lambda)\|x_1 - x_2\|\omega_b(\|x_1 - x_2\|) \\ \|\lambda\sigma(r,x_1) + (1-\lambda)\sigma(r,x_2) - \sigma\big(r,\lambda x_1 + (1-\lambda)x_2\big)\| \\ \qquad\qquad \leq \lambda(1-\lambda)\|x_1 - x_2\|\omega_\sigma(\|x_1 - x_2\|) \\ \displaystyle\int_{\|z\|\leq\delta} \|\lambda H(r,x_1,z) + (1-\lambda)H(r,x_1,z) - H\big(r,\lambda x_1 + (1-\lambda)x_2, z\big)\|^p \nu(dz) \\ \qquad\qquad \leq \lambda(1-\lambda)\left(\|x_1 - x_2\|\omega_H(\|x_1 - x_2\|)\right)^p \\ \displaystyle\int_{\|z\|>\delta} \|\lambda K(r,x_1,z) + (1-\lambda)K(r,x_1,z) - K\big(r,\lambda x_1 + (1-\lambda)x_2, z\big)\|^p \nu(dz) \\ \qquad\qquad \leq \lambda(1-\lambda)\left(\|x_1 - x_2\|\omega_K(\|x_1 - x_2\|)\right)^p \end{cases} \tag{2.40}$$

for all $0 \leq r \leq T$, $x_1, x_2 \in I\!R^d$, $0 \leq \lambda \leq 1$; if $p > 2$, we assume that estimates regarding "small jumps" H above hold also for $p = 2$.

Our results require that maps b, σ, H, K grow at most linearly[8] (see below), and for this purpose the Lipschitz continuity and generalized semiconcavity conditions stated above alone are not sufficient. We must assume in addition that

$$\begin{aligned} &\|b(r,0)\| \leq C_b^0, && \|\sigma(r,0)\| \leq C_\sigma^0, \\ &\int_{\|z\|\leq\delta} \|H(r,0,z)\|^p \nu(dz) \leq (C_H^0)^p, && \int_{\|z\|>\delta} \|K(r,0,z)\|^p \nu(dz) \leq (C_K^0)^p \end{aligned} \tag{2.41}$$

for all $0 \leq r \leq T$; if $p > 2$, condition on H is required to hold also for $p = 2$. From (2.39), (2.41) follow immediately the following estimates:

[8] Although this is not the case for global results, the definition of maps that we introduce below requires nevertheless this growth assumption.

$$\begin{cases} \|b(r,x)\|^p \le C_b(1+\|x_1\|)^p, \\ \|\sigma(r,x)\|^p \le C_\sigma(1+\|x_1\|)^p, \\ \int_{\|z\|\le\delta} \|H(r,x,z)\|^p \nu(dz) \le C_H(1+\|x_1\|)^p, \\ \int_{\|z\|\le\delta} \|H(r,x,z)\|^2 \nu(dz) \le C_H(1+\|x_1\|)^2, \\ \int_{\|z\|>\delta} \|K(r,x,z)\|^p \nu(dz) \le C_K(1+\|x_1\|)^p \end{cases} \tag{2.42}$$

for all $0 \le r \le T$, $x \in I\!R^d$, where

$$C_b = 2^{p-1}\left(\max\{C_b^0, L_b\}\right)^p, \quad C_\sigma = 2^{p-1}\left(\max\{C_\sigma^0, L_\sigma\}\right)^p$$
$$C_H = 2^{p-1}\left(\max\{C_H^0, L_H\}\right)^p, \quad C_K = 2^{p-1}\left(\max\{C_K^0, L_K\}\right)^p.$$

Let us point out explicitly that there is decreasing monotonicity in p, of the generality of our hypotheses (2.39), (2.40), (2.41) which means that the larger the p is the more restrictive our assumptions are. So we aim at proving results assuming that our conditions are satisfied for $p(\ge 2)$ as small as possible, ideally for $p = 2$. However, for $C^{1,\omega}$ (even in L^2-norm), or semiconcavity (for the value function, see Sect. 4) estimates our approach forces us to take always $p > 2$, for example, in the simplest case of power type moduli, it suffices to take $p = 4$ for $C^{1,\omega}$-estimates in L^2-norm. (Notice that this is not the case for Lipschitz estimates.)

In order to apply Theorem 2.3, or more precisely, Corollarys 2.6, 2.7, and 2.10 to solutions of Eq. (2.38), we take, for each $0 \le s \le T$, $X = I\!R^d$, $B = L^{p_0}(\Omega, \mathcal{F}, P; \mathbb{R}^d)$, $\mathcal{C} = L^p(\Omega, \mathcal{F}, P; \mathbb{R}^d)$, $\mathcal{C}_s = L^p(\Omega, \mathcal{F}_s, P; \mathbb{R}^d)$. (For the definition of Y, B' and $\mathcal{C}'$ see below.)

Let $\Sigma(s,T) = \Sigma^p(s,T)$ be the linear space of adapted (as usual, with respect to the already fixed filtration $(\mathcal{F}_t)_{0\le t\le T}$) càdlàg processes $x(\cdot)$ such that

$$E\left[\sup_{s\le t\le T} \|x(t)\|^p\right] < \infty.$$

In this application the map Φ in (2.1) is defined by setting, for all $x(\cdot) \in \Sigma(s,T)$, $\Phi\big(x(\cdot)\big)$ equal to the right hand side of (2.38). The definition of this map relies of course on the theory of stochastic integration which is exposed in many works, e.g., in [2,24].

We need some preliminary estimates for p-moments ($p \ge 2$) of stochastic processes, and deal first with the simpler case $p = 2$. We have

$$E\left[\left\|\int_s^t \sigma(r)dW(r)\right\|^2\right] = E\left[\int_s^t \|\sigma(r)\|^2 dr\right], \tag{2.43}$$

$$E\left[\left\|\int_s^t \int_E H(r,z)\tilde{N}(drdz)\right\|^2\right] = E\left[\int_s^t \int_E \|H(r,z)\|^2 dr\nu(dz)\right] \tag{2.44}$$

whenever $\sigma \in L^2\big([s,T]\times\Omega, dt\otimes\mathbb{P}; I\!R^{d\times m}\big)$, $H \in L^2\big([s,T]\times E\times\Omega, dt\otimes\nu|_E\otimes\mathbb{P}; I\!R^d\big)$ are predictable processes, where E is a Borel set in $I\!R^d$; e.g., see [2, Theorem 4.2.3, p. 224].

The estimation of the L^2-norm of an integral corresponding to "big jumps", that is of

$$E\left[\left\|\int_s^t\int_{\|z\|>\delta} K(r,z)N(drdz)\right\|^2\right],$$

where $K \in L^2\big([s,T]\times(I\!R^d\backslash B_\delta)\times\Omega, dt\otimes\nu|_{I\!R^d\backslash B_\delta}\otimes\mathbb{P}; I\!R^d\big)$ is a predictable process, causes some small problem, of which we take care by first compensating and then applying identities above and Hölder's inequality as follows. Since $\tilde{N} = N - dr\nu(dr)$, we have

$$\begin{aligned} &E\left[\left\|\int_s^t\int_{\|z\|>\delta} K(r,z)N(drdz)\right\|^2\right]\\ &\quad\le 2E\left[\left\|\int_s^t\int_{\|z\|>\delta} K(r,z)\tilde{N}(drdz)\right\|^2\right] + 2E\left[\left\|\int_s^t\int_{\|z\|>\delta} K(r,z)dr\nu(dz)\right\|^2\right]\\ &\quad\le 2\big(1+(t-s)\nu(I\!R^d\backslash B_\delta)\big)E\left[\int_s^t\int_{\|z\|>\delta}\|K(r,z)\|^2 dr\nu(dz)\right]. \end{aligned} \tag{2.45}$$

The fact that ν is a Lévy measure and hence $\nu(I\!R^d\backslash B_\delta) < \infty$ is essential for this last estimate to be useful.

For $p > 2$ we must replace the L^2-isometry identities (2.43), (2.44) by moments inequalities of Burkholder type. For any $p \ge 2$ there exist $c_p, c'_p, c''_p \ge 1$ such that

$$E\left[\left\|\int_s^t \sigma(r)dW(r)\right\|^p\right] \le c_p^p E\left[\left(\int_s^t \|\sigma(r)\|^2 dr\right)^{p/2}\right], \tag{2.46}$$

and

$$\begin{aligned} E\left[\left\|\int_s^t\int_E H(r,z)\tilde{N}(drdz)\right\|^p\right] &\le (c'_p)^p E\left[\left(\int_s^t\int_E \|H(r,z)\|^2 dr\nu(dz)\right)^{p/2}\right]\\ &\quad + (c''_p)^p E\left[\int_s^t\int_E \|H(r,z)\|^p dr\nu(dz)\right] \end{aligned} \tag{2.47}$$

for all predictable processes $\sigma \in L^p\big([s,T]\times\Omega, dt\otimes\mathbb{P}; I\!R^{d\times m}\big)$, $H \in L^p\big([s,T]\times E\times\Omega, dt\otimes\nu|_E\otimes\mathbb{P}; I\!R^d\big)$, where E is a Borel set in $I\!R^d$; see e.g., [2, Theroem 4.4.22, p. 263 and Theorem 4.4.23, p. 265]. As we said, for $p = 2$, in view of (2.43), (2.44), the above estimates hold with $c_p = c'_p = 1$, $c''_p = 0$.

The integral of "big jumps" still causes some small trouble of which we take care as above by first compensating, and then using inequality (2.47), and Hölder's inequality: we have

$$
\begin{aligned}
&E\left[\left\|\int_s^t\int_{\|z\|>\delta}K(r,z)N(drdz)\right\|^p\right]\\
&\quad\le 2^{p-1}E\left[\left\|\int_s^t\int_{\|z\|>\delta}K(r,z)\tilde N(drdz)\right\|^p\right]\\
&\qquad+2^{p-1}E\left[\left\|\int_s^t\int_{\|z\|>\delta}K(r,z)dr\nu(dz)\right\|^p\right]\\
&\quad\le 2^{p-1}(c_p')^pE\left[\left(\int_s^t\int_{\|z\|>\delta}\|K(r,z)\|^2dr\nu(dz)\right)^{p/2}\right]\\
&\qquad+2^{p-1}(c_p')^pE\left[\int_s^t\int_{\|z\|>\delta}\|K(r,z)\|^pdr\nu(dz)\right]\\
&\qquad+2^{p-1}\big((t-s)\nu(I\!R^d\setminus B_\delta)\big)^{p-1}E\left[\int_s^t\int_{\|z\|>\delta}\|K(r,z)\|^pdr\nu(dz)\right]\\
&\quad\le 2^{p-1}\left((c_p'')^p+(c_p')^p\big((t-s)\nu(I\!R^d\setminus B_\delta)\big)^{p/2-1}+\big((t-s)\nu(I\!R^d\setminus B_\delta)\big)^{p-1}\right)\\
&\qquad\times E\left[\int_s^t\int_{\|z\|>\delta}\|K(r,z)\|^pdr\nu(dz)\right]
\end{aligned}
\tag{2.48}
$$

On the other hand for an integral corresponding to small jumps we can only write

$$
\begin{aligned}
&E\left[\left\|\int_s^t\int_{\|z\|\le\delta}H(r,z)\tilde N(drdz)\right\|^p\right]\\
&\quad\le (c_p')^p(t-s)^{\frac p2-1}E\left[\int_s^t\left(\int_{\|z\|\le\delta}\|H(r,z)\|^2\nu(dz)\right)^{\frac p2}dr\right]\\
&\qquad+(c_p'')^pE\left[\int_s^t\int_{\|z\|\le\delta}\|H(r,z)\|^pdr\nu(dz)\right]
\end{aligned}
\tag{2.49}
$$

for any predictable process $H\in L^p\big([s,T]\times E\times\Omega, dt\otimes\nu|_E\otimes\mathbb{P}; I\!R^d\big)$; we cannot proceed further with majorization as we did for K in (2.48) for we do not know whether $\nu(B_\delta)<\infty$ or not.

Turning to our application of Theorem 2.3 (or better, of its corollaries) we take $Y_1=I\!R^d$, $Y_2=I\!R^{d\times m}$, $Y_3=L^2(B_\delta,\nu;I\!R^d)$, $Y_4=L^p(B_\delta,\nu;I\!R^d)$, $Y_5=L^p(I\!R^d\backslash B_\delta,\nu;I\!R^d)$, $Y=\prod_{i=1}^5 Y_i$ in the sense of normed spaces via a fixed norm[9] $\|\cdot\|_{I\!R^5}$ or $I\!R^5$, $B_i'=L^{p_0}(\Omega,\mathcal{F},\mathbb{P};Y_i)$, $\mathcal{C}_i'=L^p(\Omega,\mathcal{F},\mathbb{P};Y_i)$ for $i=1,\dots,5$, $B'=\prod_{i=1}^5 B_i'$, $\mathcal{C}'=\prod_{i=1}^5\mathcal{C}_i'$ again via the same norm $\|\cdot\|_{I\!R^5}$ or $I\!R^5$. We define the map $g:[s,T]\times X\to Y$ by setting, for all $s\le t\le T$, $x\in X(=I\!R^d)$,

[9] See footnote 5.

$$g(t,x,\alpha)=4^{1-\frac{1}{p}}\Big((T-s)^{1-\frac{1}{p}}b(t,x), c_p(T-s)^{\frac{1}{2}-\frac{1}{p}}\sigma(t,x),\, c'_p\,(T-s)^{\frac{1}{2}-\frac{1}{p}}H(t,x,\cdot),\\ c''_p H(t,x,\cdot), D_K K(t,x,\cdot)\Big), \tag{2.50}$$

where

$$D_K = 2^{1-\frac{1}{p}}\left(c''_p + c'_p\big((T-s)\nu(I\!R^d\backslash B_\delta)\big)^{\frac{1}{2}-\frac{1}{p}} + \big((t-s)\nu(I\!R^d\backslash B_\delta)\big)^{1-\frac{1}{p}}\right). \tag{2.51}$$

(It is clear that we can write $g=\prod_{i=1}^5 g_i$, for suitable maps $g_i:[s,T]\times X\to Y_i$, $i=1,\dots,5$.) The map g is well defined by the growth estimates (2.42). Finally, we define the map $f:[s,T]\times B\to B'$ by setting $f(r,x)(\omega)=g(r,x(\omega))$ for all $r\in[s,T]$, $x\in B$ and $\omega\in\Omega$. That the map f is well-defined follows again by the linear growth estimates (2.42). By (2.43), (2.44), and by (2.45), Φ and f clearly satisfy the compatibility relation (2.16) with $p_1=p_0$. Further, for all $r\in[s,T]$, $f(r,\mathcal{C})\subset\mathcal{C}'$ as a consequence of growth estimates (2.42). Of course, f is defined in such a way that the compatibility relation (2.17) with $p_2=p$ between Φ and f also holds; this is easily seen by using inequalities (2.46), (2.47), estimates (2.48) and (2.49).

The Lipschitz continuity assumptions (2.39) imply that $g:[s,T]\times X\to Y$ satisfies the Lipschitz continuity condition (2.28) with[10]

$$L_g=4^{1-\frac{1}{p}}\left\|\Big((T-s)^{1-\frac{1}{p}}L_b,\ c_p(T-s)^{\frac{1}{2}-\frac{1}{p}}L_\sigma,\ c'_p\,(T-s)^{\frac{1}{2}-\frac{1}{p}}L_H,\ c''_pL_H, D_KL_K\Big)\right\|_{I\!R^5}. \tag{2.52}$$

(and hence, f satisfies the Lipschitz continuity conditions (2.7), (2.20) with the same Lipschitz constant).

Linear growth estimates (2.42) imply also that $f:[0,T]\times\mathcal{C}\to\mathcal{C}'$ satisfies the linear growth condition (2.21) with $p_2=p$; an explicit value of the constant C_f can easily be computed but we do not need it here for global results. Moreover, it is easy to see that g is of class C^{1,ω_g} with

$$\omega_g=4^{1-\frac{1}{p}}\left\|\Big((T-s)^{1-\frac{1}{p}}\omega_b,\ c_p(T-s)^{\frac{1}{2}-\frac{1}{p}}\omega_\sigma,\ c'_p\,(T-s)^{\frac{1}{2}-\frac{1}{p}}\omega_H,\ c''_p\omega_H,\ D_K\omega_K\Big)\right\|_{I\!R^5}. \tag{2.53}$$

However, this does not necessarily lead to any kind of $C^{1,\omega}$-regularity result for f in general. Nevertheless, as we noticed, we can handle three cases: (i) power type moduli, (ii) moduli with suitable concavity properties, (Lemma 2.5), and (iii) equations with regular flows (last part of Remark 2.8).

For example, we have just verified that we can apply Corollarys 2.7, and 2.10 with $\mathcal{C}_0=I\!R^d$ and deduce the following results.

Theorem 2.11. (Global $C^{1,\omega}$-estimates, power moduli) *Let* (2.39), (2.40), (2.41) *hold for some $p\geq p_0\geq 2$ (if $p>2$ estimates on H are assumed to hold also for $p=2$), some constants $L_b,L_\sigma,L_H,L_K\geq 0$, and some moduli*

$$\omega_b(\rho)=k_1\rho^{\alpha_1},\quad \omega_\sigma(\rho)=k_2\rho^{\alpha_2},\quad \omega_H(\rho)=k_3\rho^{\alpha_3},\quad \omega_K(\rho)=k_4\rho^{\alpha_4}, \tag{2.54}$$

where $k_i\geq 0$, $0<\alpha_i(\leq 1)$, $i=1,\dots,4$. Assume that

[10] Because any norm $\|\cdot\|_{I\!R^5}$ on $I\!R^5$ is component-wise nondecreasing on vectors with non-negative components.

$$p \geq p_0\,(1+\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})\,.$$

Then for all $x_1^0, x_2^0 \in L^p(\Omega,\mathcal{F}_s,\mathbb{P};I\!R^d)$, $0 \leq \lambda \leq 1$, *if* $x_i(\cdot)$, $i = 1,2,3$, *are solutions of* (2.38) *with* $x^0 = x_i^0$, *respectively, where* $x_3^0 = \lambda x_1^0 + (1-\lambda)x_2^0$, *then*

$$\begin{aligned}&(E\,[\|\lambda x_1(t) + (1-\lambda)x_2(t) - x_3(t)\|^{p_0}])^{1/p_0}\\ &\quad\leq \lambda(1-\lambda)\left(E\left[\|x_1^0 - x_2^0\|^p\right]\right)^{1/p}\omega_\Phi\left(\left(E\left[\|x_1^0-x_2^0\|^p\right]\right)^{1/p}\right)\end{aligned}\tag{2.55}$$

where ω_Φ *is given by* (2.27), *with* $L_f = L_g$ *in* (2.52), *and* $\omega_f = \omega_g$ *in* (2.53), $p_1 = p_0$, $p_2 = p$.

It will suffice to take $p_0 = 2$ above for our applications to the generalized semiconcavity of the value function in optimal control of jump diffusions in Sect. 4.

Theorem 2.12. ($C^{1,\omega}$-estimates on $I\!R^d$, arbitrary moduli) *Let* b, σ, H, K *satisfy* (2.39), (2.40), (2.41) *for some* $p > d$, *some constants* $L_b, L_\sigma, L_H, L_K \geq 0$, *and some arbitrary semiconcavity moduli* $\omega_b, \omega_\sigma, \omega_H, \omega_K$. *Let* $0 < \beta < 1-d/p$. *Then for all* $x_1^0, x_2^0 \in I\!R^d$, $0 \leq \lambda \leq 1$, *if* $x_i(\cdot)$, $i = 1,2,3$, *are solutions of* (2.38) *with* $x^0 = x_i^0$, *where* $x_3^0 = \lambda x_1^0 + (1-\lambda)x_2^0$, *respectively, we have*

$$\begin{aligned}&(E\,[\|\lambda x_1(t) + (1-\lambda)x_2(t) - x_3(t)\|^{p}])^{1/p}\\ &\quad\leq \lambda(1-\lambda)\|x_1^0 - x_2^0\|\,\omega_\Phi\left(\|x_1^0-x_2^0\|\right)\end{aligned}\tag{2.56}$$

with ω_Φ *given by* (2.27), *with* $L_f = L_g$ *in* (2.52), *and* $\omega_f = \omega_g \circ \omega_c$, *where* ω_g *is given by* (2.53), *and* ω_c *by* (2.36) *for some* $k \geq 0$ *that depends only on* β, *and* $p_1 = p_2 = p$.

3. A general (stochastic) optimal control problem

3.1. The general setting

We formulate and study the value function of a rather general finite horizon (possibly stochastic) optimal control problem.

Fix a finite time horizon $[s,T]$, where $0 \leq s < T$. Let $B, B', \mathcal{C}, \mathcal{C}'$ be normed spaces such that embedding conditions (2.14) hold between them, with embedding constants $= 1$.

Let $\Sigma(s,T)$ be a collection of maps $x(\cdot) : [s,T] \to \mathcal{C}$, playing the role of *admissible trajectories*. Given a metric space $\mathcal{A}$–to be interpreted as the *set of controls*–let $\mathcal{A}(s,T)$ be a fixed collection of maps $\alpha(\cdot) : [s,T] \to \mathcal{A}$, to be interpreted as the set of *admissible (open loop) controls*.

Let

$$\Phi : \Sigma(s,T) \times \mathcal{A}(s,T) \to \Sigma(s,T).\tag{3.1}$$

We consider the following *controlled dynamic system*

$$x(\cdot) = x^0 + \Phi\big(x(\cdot),\alpha(\cdot)\big) \quad \text{in } [s,T],\tag{3.2}$$

where $x^0 \in \mathcal{C}_s \subset \mathcal{C}$ and $\alpha(\cdot) \in \mathcal{A}(s,T)$; of course, a map $x(\cdot) \in \Sigma(s,T)$ satisfying (3.2) is said to be a *solution* of Eq. (3.2) (or simply of Φ) for the given initial condition $x^0 \in \mathcal{C}_s$ and control $\alpha(\cdot) \in \mathcal{A}(s,T)$.

We want to study the following finite horizon optimal control problem. For any $x^0 \in \mathcal{C}_s$, $\alpha(\cdot) \in \mathcal{A}(s,T)$ we introduce the cost functional

$$J\big(s, x^0, \alpha(\cdot)\big) = \int_s^T \overline{L}\big(t, x(t), \alpha(t)\big)\,dt + \overline{\psi}\big(x(T)\big), \tag{3.3}$$

where

$$\overline{L} : [s,T] \times \mathcal{C} \times \mathcal{A} \to I\!R, \quad \overline{\psi} : \mathcal{C} \to I\!R \tag{3.4}$$

are given functionals.

Of course, Eq. (3.2) above may admit more than one solution or no solution at all: however, we assume[11] as an hypothesis that (3.2) admits a unique solution for any fixed initial condition $x^0 \in \mathcal{C}_s$ and admissible control $\alpha(\cdot) \in \mathcal{A}(s,T)$.

We investigate the value function of the following optimal control problem

$$V(s, x^0) = \inf_{\alpha(\cdot) \in \mathcal{A}(s,T)} J\big(s, x^0, \alpha(\cdot)\big). \tag{3.5}$$

Roughly speaking, the program is to assume that data are ω-semiconcave in the state variable uniformly in time and control variables, and show that the value function V is also ω-semiconcave in the state variable. We can do this but only under suitable restrictions either on the class of the semiconcavity moduli ω, which includes moduli of power type, or on the class of Eq. (3.2). The precise assumptions on the optimal control problem are outlined in the sequel. Let

$$f : [s,T] \times B \times \mathcal{A} \to B', \tag{3.6}$$

be such that $f(r, \mathcal{C}) \subset \mathcal{C}'$ for all $r \in [s,T]$. Assume that f seen as a map $f : [s,T] \times \mathcal{C} \times \mathcal{A} \to \mathcal{C}'$ is *pathwise strongly measurable* with respect to $\Sigma(s,T)$ and $\mathcal{A}(s,T)$, which, by definition, means that for all $x(\cdot) \in \Sigma(s,T)$ and $\alpha(\cdot) \in \mathcal{A}(s,T)$, the map $[s,T] \ni t \mapsto f(t, x(t), \alpha(t)) \in \mathcal{C}'$ is strongly measurable (this, of course, implies that $[s,T] \ni t \mapsto f(t, x(t), \alpha(t)) \in B'$ is also strongly measurable). We assume also that for all $x(\cdot) \in \Sigma(s,T)$ which is a solution to (3.2) for some $x^0 \in \mathcal{C}_s$ and $\alpha(\cdot) \in \mathcal{A}(s,T)$, the map $[s,T] \ni t \mapsto \overline{L}\big(t, x(t), \alpha(t)\big)$ is sommable so that the integral in the right-hand side of (3.23) makes sense. Probably, the main result of this paper is the following

[11] This hypothesis is consistent with the following ones.

Theorem 3.1. (ω-semiconcave value function) *Let $1 \le p_1, p_2 < \infty$. Let the following compatibility relations subsist between Φ and f:*

$$\begin{aligned}&\big\|\lambda\Phi\big(x_1(\cdot),\alpha(\cdot)\big)(t) + (1-\lambda)\Phi\big(x_2(\cdot),\alpha(\cdot)\big)(t) - \Phi\big(x_3(\cdot),\alpha(\cdot)\big)(t)\big\|_B^{p_1}\\ &\quad\le \int_s^t \big\|\lambda f\big(r,x_1(r),\alpha(\cdot)\big) + (1-\lambda) f\big(r,x_2(r),\alpha(\cdot)\big) - f\big(r,x_3(r),\alpha(\cdot)\big)\big\|_{B'}^{p_1}\,dr,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\big\|\Phi\big(x_1(\cdot),\alpha(\cdot)\big)(t) - \Phi\big(x_2(\cdot),\alpha(\cdot)\big)(t)\big\|_{\mathcal{C}}^{p_2}\\ &\quad\le \int_s^t \big\|f\big(r,x_1(r),\alpha(\cdot)\big) - f\big(r,x_2(r),\alpha(\cdot)\big)\big\|_{\mathcal{C}'}^{p_2}\,dr,\end{aligned} \tag{3.8}$$

$$\big\|\Phi\big(x_1(\cdot),\alpha(\cdot)\big)(t)\big\|_{\mathcal{C}}^{p_2} \le \int_s^t \big\|f\big(r,x_1(r),\alpha(\cdot)\big)\big\|_{\mathcal{C}'}^{p_2}\,dr. \tag{3.9}$$

for all $x_1(\cdot)$, $x_2(\cdot)$, $x_3(\cdot) \in \Sigma(s,T)$, $0 \le \lambda \le 1$, $s \le t \le T$ and for all $\alpha(\cdot) \in \mathcal{A}(s,T)$

Let the "vector field" $f : [s,T] \times \mathcal{C} \times \mathcal{A} \to \mathcal{C}'$ grow at most linearly in the state variable $x \in \mathcal{C}$, uniformly in time and control variables, that is, for some $C_f \ge 0$,

$$\|f(r,x,\alpha)\|_{\mathcal{C}'}^{p_2} \le C_f(1 + \|x\|_{\mathcal{C}}^{p_2}). \tag{3.10}$$

for all $r \in [s,T]$, $x \in \mathcal{C}$, $\alpha \in \mathcal{A}$. Let also maps $f : [s,T] \times \mathcal{C} \times \mathcal{A} \to \mathcal{C}'$, $f : [s,T] \times B \to B'$, $\overline{L} : [s,T] \times \mathcal{C} \times \mathcal{A} \to I\!R$ and $\overline{\psi} : \mathcal{C} \to I\!R$ be locally Lipschitz continuous in state variable $x \in \mathcal{C}$, uniformly in time and control variables, that is, for any bounded subset K of $\mathcal{C}$, there exist constants $L_{f,K}, L_{\overline{L},K}, L_{\overline{\psi},K} \ge 0$ such that

$$\|f(r,x_1,\alpha) - f(r,x_2,\alpha)\|_{B'} \le L_{f,K}\|x_1 - x_2\|_B, \tag{3.11}$$

$$\|f(r,x_1,\alpha) - f(r,x_2,\alpha)\|_{\mathcal{C}'} \le L_{f,K}\|x_1 - x_2\|_{\mathcal{C}}, \tag{3.12}$$

$$|\overline{L}(r,x_1,\alpha) - \overline{L}(r,x_2,\alpha)| \le L_{\overline{L},K}\|x_1 - x_2\|_{\mathcal{C}}, \tag{3.13}$$

$$|\overline{\psi}(x_1) - \overline{\psi}(x_2)| \le L_{\overline{\psi},K}\|x_1 - x_2\|_{\mathcal{C}} \tag{3.14}$$

for all $r \in [s,T]$, $x_1, x_2 \in K$, $\alpha \in \mathcal{A}$.

Finally, assume that map $f : [s,T] \times \mathcal{C} \times \mathcal{A} \to B'$ is of class C^{1,ω_f}, functional $\overline{L}$ is $\omega_{\overline{L}}$-semiconcave, and functional $\overline{\psi}$ is $\omega_{\overline{\psi}}$-semiconcave in state variable, for given semiconcavity moduli $\omega_f, \omega_{\overline{L}}, \omega_{\overline{\psi}}$ uniformly in time and control variables, that is,

$$\begin{aligned}&\big\|\lambda f(r,x_1,\alpha) + (1-\lambda) f(r,x_2,\alpha) - f\big(r,\lambda x_1 + (1-\lambda)x_2,\alpha\big)\big\|_{B'}\\ &\quad\le \lambda(1-\lambda)\|x_1 - x_2\|_{\mathcal{C}}\,\omega_f(\|x_1 - x_2\|_{\mathcal{C}});\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\lambda\overline{L}(r,x_1,\alpha) + (1-\lambda)\overline{L}(r,x_2,\alpha) - \overline{L}\big(r,\lambda x_1 + (1-\lambda)x_2,\alpha\big)\\ &\quad\le \lambda(1-\lambda)\|x_1 - x_2\|_{\mathcal{C}}\,\omega_{\overline{L}}(\|x_1 - x_2\|_{\mathcal{C}})\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\lambda\overline{\psi}(x_1) + (1-\lambda)\overline{\psi}(x_2) - \overline{\psi}\big(\lambda x_1 + (1-\lambda)x_2\big)\\ &\quad\le \lambda(1-\lambda)\|x_1 - x_2\|_{\mathcal{C}}\,\omega_{\overline{\psi}}(\|x_1 - x_2\|_{\mathcal{C}})\end{aligned} \tag{3.17}$$

for all $r \in [s,T]$, $x_1, x_2 \in \mathcal{C}$, $\alpha \in \mathcal{A}$, $0 \le \lambda \le 1$.

Then for any bounded set $K \subset \mathcal{C}_s$, the value function V is ω_K-semiconcave in state variable in K, uniformly in time, for a modulus ω_K defined below by (3.19).

If maps $f, \overline{L}, \overline{\psi}$ are globally Lipschitz in state variable uniformly in time and control variables, that is, the Lipschitz estimates above hold with constants independent of K, so that $L_{f,K} = L_f$, $L_{\overline{L},K} = L_{\overline{L}}$, $L_{\overline{\psi},K} = L_{\overline{\psi}}$ for any bounded $K \subset \mathcal{C}$, then V is globally ω-semiconcave on $\mathcal{C}_s$ uniformly in time with ω given by (3.20) below. In this case assumptions (3.9) and (3.10) are superfluous.

Proof. Fix a control $\alpha(\cdot) \in \mathcal{A}(s,T)$. We can apply Theorems 2.1 and 2.3 to maps $\Phi\big(\cdot, \alpha(\cdot)\big)$ and $f\big(\cdot, \cdot, \alpha(\cdot)\big)$, and to the bounded subset K of $\mathcal{C}_s$. Thus, there exist $L_{\Phi,K} \geq 0$ as in (2.12) with $p = p_2$ and a modulus $\omega_{\Phi,K}$ as in (2.24), (2.26), independent of $\alpha(\cdot)$ such that for all $x_1^0, x_2^0 \in K$, $0 \leq \lambda \leq 1$, if $x_1(\cdot)$, $x_2(\cdot)$, $x_3(\cdot)$ are solutions of (3.2) for initial conditions, respectively, $x_1^0, x_2^0, \lambda x_1^0 + (1-\lambda)x_2^0$, and control $\alpha(\cdot)$, then estimates (2.25) and (2.23) hold true.

Moreover, as we noted during the proof of Theorem 2.1 solutions departing from points belonging to a bounded set, remain bounded for all subsequent times $t \in [s,T]$. Thus applying this to our set K, there exists $R \geq 0$ as in (2.10) with $p = p_2$ such that for all $\alpha(\cdot) \in \mathcal{A}(s,T)$ and $x^0 \in K$

$$\|x(t)\|_{\mathcal{C}} \leq R, \tag{3.18}$$

where $x(\cdot)$ is the solution of (3.2) for the initial condition x^0 and control $\alpha(\cdot)$.

$L_{f,B_R}, L_{\overline{L},B_R}, L_{\overline{\psi},B_R} \geq 0$ are the Lipschitz constants of $f, \overline{L}, \overline{\psi}$ on the ball B_R of $\mathcal{C}$ centered at the origin and radius R. Applying (3.16), (3.17), Lipschitz conditions on $\overline{L}$, $\overline{\psi}$, and (2.25) and (2.23), we obtain for $t \in [s,T]$

$$\begin{aligned}
&\lambda J\big(t, x_1^0, \alpha(\cdot)\big) + (1-\lambda) J\big(t, x_2^0, \alpha(\cdot)\big) - J\big(t, x_3^0, \alpha(\cdot)\big) \\
&\quad = \int_t^T \Big(\lambda \overline{L}\big(r, x_1(r), \alpha(r)\big) + (1-\lambda)\overline{L}\big(r, x_2(r), \alpha(r)\big) \\
&\qquad - \overline{L}\big(r, \lambda x_1(r) + (1-\lambda)x_2(r), \alpha(r)\big)\Big) dr \\
&\qquad + \lambda \overline{\psi}\big(x_1(T)\big) + (1-\lambda)\overline{\psi}\big(x_2(T)\big) - \overline{\psi}\big(\lambda x_1(T) + (1-\lambda)x_2(T)\big) \\
&\qquad + \int_t^T \Big(\overline{L}\big(r, \lambda x_1(r) + (1-\lambda)x_2(r), \alpha(r)\big) - L\big(r, x_3(r), \alpha(r)\big)\Big) dr \\
&\qquad + \overline{\psi}\big(\lambda x_1(T) + (1-\lambda)x_2(T)\big) - \overline{\psi}\big(x_3(T)\big) \\
&\quad \leq \lambda(1-\lambda)\Big(\int_t^T \|x_1(r) - x_2(r)\|_{\mathcal{C}} \omega_{\overline{L}}(\|x_1(r) - x_2(r)\|_{\mathcal{C}}) dr \\
&\qquad + \|x_1(T) - x_2(T)\|_{\mathcal{C}} \omega_{\overline{\psi}}(\|x_1(T) - x_2(T)\|_{\mathcal{C}})\Big) \\
&\qquad + L_{\overline{L},B_R} \int_t^T \|\lambda x_1(r) + (1-\lambda)x_2(r) - x_3(r)\|_B dr \\
&\qquad + L_{\overline{\psi},B_R} \|\lambda x_1(T) + (1-\lambda)x_2(T) - x_3(T)\|_{\mathcal{C}} \\
&\quad \leq \lambda(1-\lambda)\|x_1^0 - x_2^0\|_{\mathcal{C}} \omega_K(\|x_1^0 - x_2^0\|_{\mathcal{C}}),
\end{aligned}$$

with

$$\omega_K(\rho) = (T-s)L_{\Phi,K}\omega_{\overline{L}}(L_{\Phi,K}\rho) + L_{\Phi,K}\omega_{\overline{\psi}}(L_{\Phi,K}\rho) + (L_{\overline{L},B_R}(T-s) + L_{\overline{\psi},B_R})\omega_{\Phi,K}(\rho) \tag{3.19}$$

for all $\rho \geq 0$, where—we recall—$L_{\Phi,K}$ is given by (2.12) for $p = p_2$, R by (2.10) for $p = p_2$, and $\omega_{\Phi,K}$ by (2.24), (2.26). This concludes the proof of the first part for $\alpha(\cdot)$ is arbitrary. In the second case of globally Lipschitz $f, \overline{L}, \overline{\psi}$, it is clear by the same proof above, that, for all $t \in [s,T]$, $V(t, \cdot$ is ω-semiconcave with ω given by

$$\omega_K(\rho) = (T-s)L_{\Phi}\omega_{\overline{L}}(L_{\Phi}\rho) + L_{\Phi}\omega_{\overline{\psi}}(L_{\Phi}\rho) + (L_{\overline{L}}(T-s) + L_{\overline{\psi}})\omega_{\Phi}(\rho) \tag{3.20}$$

for all $\rho \geq 0$, where L_{Φ} is given by (2.13) and ω_{Φ} by (2.27). □

3.2. Applications to stochastic optimal control

We consider the same optimal control problem introduced above, with same assumptions on Φ, f, $\overline{L}$, $\overline{\psi}$ etc. We continue to specialize further. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X, Y normed spaces, and let $B, \mathcal{C}$ be normed spaces of X-valued normed spaces, and $B', \mathcal{C}'$ normed spaces of Y-valued random variables. Let A be a metric space, and $\mathcal{A}$ a set of A-valued random variables. Further, let the cost functionals arise in the following manner

$$\overline{L}(t,x,\alpha) = E[L(t,x,\alpha)], \tag{3.21}$$

$$\overline{\psi}(x) = E[\psi(x)] \tag{3.22}$$

for all $x \in \mathcal{C}$, $\alpha \in \mathcal{A}$, $s \leq t \leq T$, where

$$L : [s,T] \times \mathbb{R}^d \times A \to I\!R, \quad \psi : \mathbb{R}^d \to \mathbb{R} \tag{3.23}$$

are suitable maps. Assume also that for all $r \in [s,T]$, $x \in B$, $\alpha \in \mathcal{A}$, $\omega \in \Omega$

$$f(r,x,\alpha)(\omega) = g(r, x(\omega), \alpha(\omega)) \tag{3.24}$$

for some map

$$g : [s,T] \times X \times A \to Y. \tag{3.25}$$

Let L be ω_L-semiconcave, ψ ω_ψ-semiconcave, and g of class C^{1,ω_g} in state variable, uniformly (in the case of L, g) in time and control variables, where ω_L, ω_ψ, ω_g are given semiconcavity moduli. That is, we have

$$\begin{aligned} \lambda L(t,x_1,\alpha) + (1-\lambda) \ & L(t,x_2,\alpha) - L\big(t, \lambda x_1 + (1-\lambda)x_2, \alpha\big) \\ & \leq \lambda(1-\lambda)\|x_1 - x_2\|_X \omega_L(\|x_1 - x_2\|_X), \quad (3.26) \\ \lambda\psi(x_1) + (1-\lambda)\psi \ & (x_2) - \psi\big(\lambda x_1 + (1-\lambda)x_2\big) \\ & \leq \lambda(1-\lambda)\|x_1 - x_2\|_X \omega_\psi(\|x_1 - x_2\|_X) \quad (3.27) \\ \|\lambda g(t,x_1,\alpha) + (1-\lambda) \ & g(t,x_2,\alpha) - g\big(t, \lambda x_1 + (1-\lambda)x_2, \alpha\big)\|_Y \\ & \leq \lambda(1-\lambda)\|x_1 - x_2\|_X \omega_g(\|x_1 - x_2\|_X), \quad (3.28) \end{aligned}$$

for all $x_1, x_2 \in X$ $\alpha \in A$, $s \leq t \leq T$, $0 \leq \lambda \leq 1$.

A delicate issue is that of establishing the semiconcavity of $\overline{L}$ and $\overline{\psi}$ from that of L and ψ. (We already dealt with the very similar problem of establishing the $C^{1,\omega}$-regularity of f from that of g in Subsect. 2.2.) So we do not "repeat" proofs here, but just state results. There are as we already saw three kinds of situations that we can handle: (i) power type moduli, (ii) moduli with suitable concavity properties, and (iii) dynamics with regular flows, that is, applying Kolmogorov's continuity criterion. We collect results in the following Lemma 3.2 and Lemma 3.4.

Lemma 3.2. *For every $t \in [s,T]$, $\alpha \in \mathcal{A}$, $\overline{L}(t,\cdot,\alpha)$ is $\omega_{\overline{L}}$-semiconcave with $\omega_{\overline{L}} = \omega_L$ if one of the following happens:*

- $\omega_L(\rho) = k\,\rho^\alpha$ *for* $k \geq 0$, $0 < \alpha (\leq 1)$ *and* $\mathcal{C} \hookrightarrow L^{1+\alpha}(\Omega; X)$;
- $\gamma_L(\rho) = \left(\rho^\beta \omega_L(\rho)\right)^q$, *where* $0 \leq \beta \leq 1$, $1 \leq q, r \leq \infty$, $q^{-1} + r^{-1} = 1$, *is concave and* $\mathcal{C} \hookrightarrow L^{(1-\beta)r}(\Omega; X)$, $\mathcal{C} \hookrightarrow L^1(\omega; X)$.

Similar results hold for $\overline{\psi}$.

Now using Lemmas 3.2 and 2.5, Theorem 3.1 has several corollaries. But before giving examples of such corollaries let us make a comment regarding the Lipschitz character of maps at hand.

If the state space X is finite-dimensional, (which is the case in applications to jump diffusions of this paper), assumptions (3.26), (3.27), (3.28) imply that L, ψ, g are locally Lipschitz, see [16, Theorem 2.1.7, p. 33]. We do not know whether or not such a result holds in a general normed (Banach) space. It would be interesting to investigate such a problem even assuming, if need be, some regularity on the structure of the underlying normed space X (such as a Hilbert, Asplund, uniformly convex, reflexive etc., structure). However, if we assume—as we do assume—that L and ψ are locally bounded below, while g is locally bounded then, by same ideas as in the proof of [16, Theorem 2.1.7, p. 33], we may prove that these maps are locally Lipschitz (uniformly in time and control variables in the case of L and g) even on an arbitrary normed space X.

(But if X is in addition finite-dimensional, then g is actually of class C^1, see [16, Theorem 3.3.7, p. 60].)

A somewhat troublesome fact is that local Lipschitzianity of g, L and ψ, which may follow form their $C^{1,\omega}$-regularity and ω-semiconcavity, is not sufficient to guarantee the local Lipschitzianity of f, $\overline{L}$, $\overline{\psi}$ even on the set of reachable values in $\mathcal{C}$ of solutions at a certain time $t \in [s,T]$ for an admissible control $\alpha(\cdot)$

$$\mathrm{Acc}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0; t; \alpha(\cdot)) = \{x(t) \in \mathcal{C} \ : \ x(\cdot) \in \Sigma(s,T) \text{ solution of (3.2) for some } x^0 \in K,\ a(\cdot) \in \mathcal{A}(s,T),\ s \leq t \leq T\} \tag{3.29}$$

departing from points of a (even bounded) subset $\mathcal{C}_0 \subset \mathcal{C}_s$ of $\mathcal{C}$, with Lipschitz constants independent of t and $\alpha(\cdot)$, which would be sufficient for proving Theorem 3.1. In fact, if assumptions (3.11), (3.12), (3.13), (3.14) are required to hold only on subsets of $\mathrm{Acc}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0; t; \alpha(\cdot))$ with Lipschitz constants independent of t and $\alpha(\cdot)$, then by the same proof as that of Theorem 3.1, we may conclude, that V is locally or globally (depending on the nature of Lipschitz assumptions on f, $\overline{L}$, $\overline{\psi}$) on $\mathcal{C}_0$ uniformly in time. One reason is that although solutions may remain bounded in $\mathcal{C}$-norm, that is,

$$\mathrm{Acc}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0) = \bigcup_{\alpha(\cdot)\in\mathcal{A}(s,T)} \bigcup_{s\le t\le T} \mathrm{Acc}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0; t; \alpha(\cdot)) \tag{3.30}$$

be bounded in $\mathcal{C}$, this does not imply that their values in X remain bounded in X almost surely, that is, whatever the zero measure subset N of Ω is,

$$\mathrm{Acc}^{\Phi}_{X}(\mathcal{C}_0; N) = \{x(\omega) \,:\, x \in \mathrm{Acc}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0),\ \omega \in \Omega\backslash N\} \tag{3.31}$$

is not necessarily a bounded subset of the state space X. Of course, this is not a problem if we assume that L, ψ are globally Lipschitz, that is,

$$|L(r, x_1, \alpha) - L(r, x_2, \alpha)| \le L_L \|x_1 - x_2\|_X, \tag{3.32}$$

$$|\psi(x_1) - \psi(x_2)| \le L_\psi \|x_1 - x_2\|_X, \tag{3.33}$$

$$\|g(r, x_1, \alpha) - g(r, x_2, \alpha)\|_Y \le L_g \|x_1 - x_2\|_X \tag{3.34}$$

for all $x_1, x_2 \in X$, $s \le t \le T$, $\alpha \in A$. In that case f, $\overline{L}$, $\overline{\psi}$ are globally Lipschitz in $x \in \mathcal{C}$, uniformly in $t \in [s, T]$, $\alpha \in \mathcal{A}$, that is, (3.11), (3.12), (3.13), (3.14) are satisfied for any bounded $K \subset \mathcal{C}$ with

$$L_{\overline{L},K} = L_{\overline{L}} = L_L, \quad L_{\overline{\psi},K} = L_{\overline{\psi}} = L_\psi \tag{3.35}$$

independent of K.

While these are reasonable assumptions for obtaining global generalized semiconcavity results, they are probably too much for local results. Alternatively, in some cases we can show that $\mathrm{Acc}^{\Phi}_{X}(\mathcal{C}_0; N)$ is indeed a bounded subset of $\mathcal{C}$ if so is $\mathcal{C}_0$, for a suitably chosen zero measure subset N of Ω. In this case we can conclude that f, $\overline{L}$, $\overline{\psi}$ are indeed locally Lipschitz on bounded (in $\mathcal{C}$-norm) subsets of $\mathrm{Acc}^{\Phi}_{\mathcal{C}}(\mathcal{C}_0; t; \alpha(\cdot))$, with Lipschitz constants independent of $t \in [s, T]$ and $\alpha(\cdot) \in \mathcal{A}(s, T)$ which is sufficient for proving Theorem 3.1 (the local part), provided we restrict to $\mathcal{C}_0$, as we already noticed.

We content ourselves here with the formulation of global generalized semiconcavity results. The formulation of local results is somewhat trickier, but it should be simpler now after the considerations made above. Actually, the most difficult part is keeping track of constants. We leave it to the interested reader.

Corollary 3.3. (Global ω-semiconcavity, power type moduli) *Let Φ, f satisfy assumptions (3.7), (3.8) for certain $1 \le p_1, p_2 < \infty$ (and also conditions stated before Theorem 3.1). Let maps f, g, $\overline{L}$, L, $\overline{\psi}$, ψ satisfy (3.21), (3.22), (3.24), and (3.32), (3.33), (3.34) for suitable $L_L, L_\psi, L_g \ge 0$. Let also g come up as Cartesian product map $g = \prod_{i=1}^{\ell} g_i$ for certain maps $g_i : [s, T] \times X \times A \to Y_i$ and normed spaces Y_i, for $i = 1, \ldots, \ell$, where $Y = \prod_{i=1}^{\ell} Y_i$ in the sense of normed spaces via a fixed norm[12] $\|\cdot\|_{\mathbb{R}^\ell}$ on $\mathbb{R}^\ell$. Assume that each component g_i is of class $C^{1,\omega_{g_i}}$, and that L, ψ satisfy (3.26), (3.27), for some moduli*

$$\omega_{g_i}(\rho) = k_i \rho^{\alpha_i}, \quad \omega_L(\rho) = k_{\ell+1}\rho^{\alpha_{\ell+1}}, \quad \omega_\psi(\rho) = k_{\ell+2}\rho^{\alpha_{\ell+2}}, \quad \rho \ge 0, \tag{3.36}$$

where $k_i \ge 0$, $0 < \alpha_i \le 1$ for $i = 1, \ldots, \ell + 2$. Finally, assume that for some $1 \le p \le \infty$, $B \hookrightarrow L^p(\Omega; X)$, $L^p(\Omega; Y_i) \hookrightarrow B'$, $\mathcal{C} \hookrightarrow L^{p(1+\alpha)}(\Omega; X)$, $\mathcal{C} \hookrightarrow$

[12] See footnote 5.

$L^{1+\alpha_{\ell+1}}(\Omega;X)$, $\mathcal{C}\hookrightarrow L^{1+\alpha_{\ell+2}}(\Omega;X)$, $L^{p(1+\alpha_i)}(\Omega;Y_i)\hookrightarrow\mathcal{C}'$, for $i=1,\dots,\ell$, where B'_i, $\mathcal{C}'_i$ are normed spaces such that $B'=\prod_{i=1}^{\ell}B'_i$, $\mathcal{C}'=\prod_{i=1}^{\ell}\mathcal{C}'_i$, and $\alpha=\max\{\alpha_i\ :\ 1\le i\le\ell\}$.

Then, for all $t\in[s,T]$, $V(t,\cdot)$ is ω-semiconcave on $\mathcal{C}_s$ with ω given by (3.20) with L_f, $L_{\overline{L}}$, $L_{\overline{\psi}}$ given by (3.35) and $\omega_f=\|(g_1,\dots,g_\ell)\|_{\mathbb{R}^\ell}$, $\omega_{\overline{L}}=\omega_L$, $\omega_{\overline{\psi}}=\omega_\psi$.

Similar results can be formulated for moduli $\omega_g,\omega_L,\omega_\psi$ satisfying suitable concavity properties as in the second part of Lemma 3.2. Even combinations can be considered, that is, some of the said moduli being of power type and the rest of them satisfying concavity properties. To save space and since it is rather routine we do not present such results here.

Let us assume now that Eq. (3.2) has regular flow in the following sense. If for all $x^0\in\mathcal{C}_s$ we indicate by $x(\cdot,x^0)$ the solution to (3.2), for some fixed $\alpha(\cdot)\in\mathcal{A}$, in order to emphasize its dependence on initial condition x^0, we assume that (2.33) holds for all $x_1^0,x_2^0\in\mathcal{C}_0$, where $\mathcal{C}_0$ is a subset of $\mathcal{C}_s$ and $\omega_c:\Omega\times[0,\infty[\to[0,\infty[$ is some random modulus of continuity (which in principle, may depend also on t and on admissible control $\alpha(\cdot)\in\mathcal{A}(s,T)$; however, in order to keep things simple, we assume that $\omega_c(\cdot)$ and does not depend on t, $\alpha(\cdot)$). Under this assumption it is easy to prove that maps f, $\overline{L}$, $\overline{\psi}$ satisfy the following semiconcavity properties.

Lemma 3.4. *If $B\hookrightarrow L^{p_0}(\Omega;X)$, $L^{p_0}(\Omega;Y)\hookrightarrow B'$, $\mathcal{C}\hookrightarrow L^p(\Omega;X)$, $L^p(\Omega;Y)\hookrightarrow\mathcal{C}'$, for certain $1\le p_0\le p\le\infty$, then*

$$\begin{aligned}&\|\lambda f\big(t,x(t,x_1^0),\alpha(t)\big)+(1-\lambda)f\big(t,x(t,x_2^0),\alpha(t)\big)-f\big(t,\lambda x(t,x_1^0)\\&\quad+(1-\lambda)\lambda x(t,x_2^0),\alpha(t)\big)\|_{B'}\le\lambda(1-\lambda)\|x(t,x_1^0)-x(t,x_2^0)\|_{\mathcal{C}}\omega_f(\|x_1^0-x_2^0\|_X)\end{aligned}\tag{3.37}$$

$$\begin{aligned}&\lambda\overline{L}\big(t,x(t,x_1^0),\alpha(t)\big)+(1-\lambda)\overline{L}\big(t,x(t,x_2^0),\alpha(t)\big)-\overline{L}\big(t,\lambda x(t,x_1^0)\\&\quad+(1-\lambda)\lambda x(t,x_2^0),\alpha(t)\big)\le\lambda(1-\lambda)\|x(t,x_1^0)-x(t,x_2^0)\|_{\mathcal{C}}\omega_{\overline{L}}(\|x_1^0-x_2^0\|_X)\end{aligned}\tag{3.38}$$

$$\begin{aligned}&\lambda\overline{\psi}\big(x(T,x_1^0)\big)+(1-\lambda)\overline{\psi}\big(x(T,x_2^0)\big)-\overline{\psi}\big(\lambda x(T,x_1^0)+(1-\lambda)\lambda x(T,x_2^0)\big)\\&\quad\le\lambda(1-\lambda)\|x(t,x_1^0)-x(t,x_2^0)\|_{\mathcal{C}}\omega_{\overline{\psi}}(\|x_1^0-x_2^0\|_X)\end{aligned}\tag{3.39}$$

for all $x_1^0,x_2^0\in\mathcal{C}_0$, $0\le\lambda\le1$, $t\in[s,T]$, with

$$\begin{cases}\omega_f(\rho)=\big(E\left[\omega_g^{q_f}\left(\omega_c(\rho)\right)\right]\big)^{1/q_f}\\ \omega_{\overline{L}}(\rho)=\left(E\left[\omega_L^{q_L}\left(\omega_c(\rho)\right)\right]\right)^{1/q_L}\\ \omega_{\overline{\psi}}(\rho)=\left(E\left[\omega_\psi^{q_\psi}\left(\omega_c(\rho)\right)\right]\right)^{1/q_\psi}\end{cases}\tag{3.40}$$

for all $\rho\ge0$, where $p_f,p_L,p_\psi,q_f,q_L,q_\psi\in[1,\infty]$ are such that $p_f,p_L,p_\psi\le p$ and $1/p_0\ge1/p_f+1/q_f$, $1\ge1/p_L+1/q_L$, $1\ge1/p_\psi+1/q_\psi$.

Nothing guarantees that the right-hand sides in (3.40) be finite, but if it happens that ω_f, $\omega_{\overline{L}}$, $\omega_{\overline{\psi}}$ are finite for some positive values of their arguments, then $\omega_f(0^+) = 0$, $\omega_{\overline{L}}(0^+) = 0$, $\omega_{\overline{\psi}}(0^+) = 0$. If it happens that $\rho \mapsto \omega_g^{q_f}(\rho^{1/q_f})$, $\rho \mapsto \omega_L^{q_L}(\rho^{1/q_L})$, $\rho \mapsto \omega_\psi^{q_\psi}(\rho^{1/q_\psi})$ are concave (a quite not restrictive assumption on regularity moduli; for example, power type moduli clearly satisfy it), then

$$\begin{cases} \omega_f(\rho) \le \omega_g\left(\left(E\left[\omega_c^{q_f}(\rho)\right]\right)^{1/q_f}\right) \\ \omega_{\overline{L}}(\rho) \le \omega_L\left(\left(E\left[\omega_c^{q_L}(\rho)\right]\right)^{1/q_L}\right) \\ \omega_{\overline{\psi}}(\rho) \le \omega_\psi\left(\left(E\left[\omega_c^{q_\psi}(\rho)\right]\right)^{1/q_\psi}\right) \end{cases} \tag{3.41}$$

for all $\rho \ge 0$.

Modifying slightly the proof of Theorem 3.1 we obtain the following result.

Theorem 3.5. (Global ω-semiconcavity, general moduli) *Let Φ, f satisfy assumptions (3.7), (3.8) for certain $1 \le p_1, p_2 < \infty$ (and also conditions stated before Theorem 3.1). Let maps f, g, $\overline{L}$, L, $\overline{\psi}$, ψ satisfy (3.21), (3.22), (3.24), and (3.26), (3.27), (3.28), (3.32), (3.33), (3.34) for suitable $L_L, L_\psi, L_g \ge 0$, and given semiconcavity moduli ω_g, ω_L, ω_ψ. Let $\mathcal{C}_0$ be a subset of $\mathcal{C}$ such that solutions of (3.2) satisfy (2.33) on $\mathcal{C}_0$ with some continuity modulus ω_c. Finally, let also $B \hookrightarrow L^{p_0}(\Omega; X)$, $L^{p_0}(\Omega; Y) \hookrightarrow B'$, $\mathcal{C} \hookrightarrow L^p(\Omega; X)$, $L^p(\Omega; Y) \hookrightarrow \mathcal{C}'$, for certain $1 \le p_0 \le p \le \infty$,*

Then for all $t \in [s, T]$, $V(t, \cdot)$ is ω-semiconcave on C_0 with ω given by (3.20) with L_f, $L_{\overline{L}}$, $L_{\overline{\psi}}$ given by (3.35), and ω_f, $\omega_{\overline{L}}$, $\omega_{\overline{\psi}}$ given by (3.40) (provided that they are finite).

Via Komogorov's continuity criterion (Theorem 2.9) Theorem 3.5 has the following corollary.

Corollary 3.6. (Global ω-semiconcavity on X, general moduli) *Let X, Y be normed spaces, X a d-dimensional one for some $d \in \mathbb{N}$, and let $B \hookrightarrow L^{p_0}(\Omega; X)$, $L^{p_0}(\Omega; Y) \hookrightarrow B'$, $\mathcal{C} \hookrightarrow L^p(\Omega; X)$, $L^p(\Omega; Y) \hookrightarrow \mathcal{C}'$, for given $1 \le p_0 \le p \le \infty$, $p > d$. Let Φ, f satisfy assumptions (3.7), (3.8) for certain $1 \le p_1, p_2 < \infty$ (and also conditions stated before Theorem 3.1). Let maps f, g, $\overline{L}$, L, $\overline{\psi}$, ψ satisfy (3.21), (3.22), (3.24), and (3.26), (3.27), (3.28), (3.32), (3.33), (3.34) for suitable $L_L, L_\psi, L_g \ge 0$, and given semiconcavity moduli ω_g, ω_L, ω_ψ.*

Then, if $0 < \beta < 1 - d/p$, there exists a random $k \ge 0$ (depending on β, T, s, p_1, p_2, L_f, L_L, L_ψ) with $E[k^p] < \infty$ such that, for all $t \in [s, T]$, $V(t, \cdot)$ is ω-semiconcave on X with ω given by (3.20) where L_f, $L_{\overline{L}}$, $L_{\overline{\psi}}$ are given by (3.35), ω_f, $\omega_{\overline{L}}$, $\omega_{\overline{\psi}}$ by (3.40), and ω_c by (2.36).

If in addition maps $\rho \mapsto \omega_g^{q_f}(\rho^{1/q_f})$, $\rho \mapsto \omega_g^{q_L}(\rho^{1/q_L})$, $\rho \mapsto \omega_g^{q_\psi}(\rho^{1/q_\psi})$ are concave we can take

$$\omega_f(\rho) = \omega_g\left(\left(E[k^{q_f}]\right)^{1/q_f}\left(\rho^\beta + \rho\right)\right), \quad \omega_{\overline{L}}(\rho) = \omega_L\left(\left(E[k^{q_L}]\right)^{1/q_L}\left(\rho^\beta + \rho\right)\right),$$

$$\omega_{\overline{\psi}}(\rho) = \omega_\psi\left(\left(E[k^{q_\psi}]\right)^{1/q_\psi}\left(\rho^\beta + \rho\right)\right) \quad \forall \rho \ge 0$$

above (each of which is finite if also $q_f \le p$, $q_L \le p$, $q_\psi \le p$, respectively).

In case of power type moduli of semiconcavity ω_L, ω_ψ the result given by the preceding corollary is not as precise as that of Corollary 3.3: hypothesis are generally stronger, and semiconcavity modulus obtained for V is "weaker". On the other hand, it allows to deal with general semiconcavity moduli without any restriction (on them) at all.

4. The case of jump diffusions optimal control

Let $T > 0$ be fixed time horizon, let A be a metric space—the control space—and let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ be a complete probability space. For any $(s, x^0) \in [0, T) \times \mathbb{R}^d$ consider a controlled jump stochastic differential equation (or an Itô-Skorokhod equation as it is alternatively called in literature)

$$\begin{aligned} x(t) = x^0 + \int_s^t b\big(r, x(r-), \alpha(r)\big) dr + \int_s^t \sigma\big(r, x(r-), \alpha(r)\big) dW(r) \\ + \int_s^t \int_{\|z\| \le \delta} H\big(r, x(r-), z, \alpha(r)\big) \tilde{N}(drdz) \\ + \int_s^t \int_{\|z\| > \delta} K\big(r, x(r-), z, \alpha(r)\big) N(drdz) \end{aligned} \tag{4.1}$$

where notation has the following meaning. $W = W(\cdot)$ is a standard m-dimensional Brownian motion and N an independent Poisson random measure on $I\!R^+ \times (I\!R^d \backslash \{0\})$ with associated compensated measure $\tilde{N}$ and intensity measure ν, which we assume to be a Lévy measure. As usual, we also assume that W and N are adapted with respect to some right-continuous complete filtration $(\mathcal{F}_t)_{s \le t \le T}$ of $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ which means that $W(t)$, $N(t)$ are $\mathcal{F}_t$-measurable, and have increments $W(t) - W(r)$, $N(t) - N(r)$ that are independent of $\mathcal{F}_r$ for all $s \le r \le t \le T$. Fixed a Lévy measure ν on $I\!R^d$, (that is, we recall a Borel measure such that $\nu(\{0\}) = 0$ and $\int_{I\!R^d} \min\{1, \|z\|^2\} \nu(dz) < \infty$) let us call a system

$$R = (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{s \le t \le T}, W, N) \tag{4.2}$$

that satisfies conditions described above an *admissible reference probability system* in time frame $[s, T]$ (with respect to the Lévy measure ν, which is kept fixed).

For any $s \in [0, T]$, we define the set of admissible controls $\mathcal{A}(s, T)$ to be the set of stochastic processes $\alpha : [s, T] \to A$, for which there exists an admissible reference probability system R as in (4.2) such that $\alpha(\cdot)$ is a predictable process with respect to the filtration $(\mathcal{F}_t)_{s \le t \le T}$.

The maps

$$b : [0, T] \times \mathbb{R}^d \times A \to I\!R^d, \quad \sigma : [0, T] \times \mathbb{R}^d \times A \to I\!R^{d \times m},$$

$$H : [0, T] \times \mathbb{R}^d \times B_\delta \times A \to I\!R^d, \qquad K : [0, T] \times \mathbb{R}^d \times (I\!R^d \backslash B_\delta) \times A \to I\!R^d,$$

are measurable, $\delta > 0$ is some fixed parameter, and B_δ is the ball of $I\!R^d$ centered at 0 and of radius δ.

Under standard assumptions on b, σ, H, K that are explicitly recalled below, fixed any $\alpha(\cdot) \in \mathcal{A}(s,T)$, Eq. (4.1) admits a unique solution $x(\cdot)$ for any given $s, \in [0,T]$, and $\mathcal{F}_s$-measurable $I\!R^d$-valued random variable x^0 with finite second moment. Then for any such pair (s, x^0) and $\alpha(\cdot) \in \mathcal{A}(s,T)$ we can compute the cost

$$J\big(s, x^0, \alpha(\cdot)\big) = E\left[\int_s^T L\big(t, x(t), \alpha(t)\big)dt + \psi\big(x(T)\big)\right] \tag{4.3}$$

and the corresponding optimal value function as in (3.5); here L, ψ are measurable functions as in (3.23).

We collect in the theorem below some of the results of our analysis about the generalized semiconcavity of the value function.

Theorem 4.1. (Global ω-semiconcavity) *Let for all $\alpha \in A$ maps $b = b(\cdot,\cdot,\alpha)$, $\sigma = \sigma(\cdot,\cdot,\alpha)$, $H = H(\cdot,\cdot,\cdot,\alpha)$ and $K = K(\cdot,\cdot,\cdot,\alpha)$ satisfy (2.39), (2.40), (2.41) for certain $p \geq 2$, and maps $L = L(t,\cdot,\alpha)$, ψ satisfy (3.32), (3.33), (3.26), (3.27) for given nonnegative constants $L_b = L_1$, $L_\sigma = L_2$, $L_H = L_3$, $L_K = L_4$, $L_L = L_5$, $L_\psi = L_6$ and semiconcavity moduli $\omega_b = \omega_1$, $\omega_\sigma = \omega_2$, $\omega_H = \omega_3$, $\omega_K = \omega_4$, $\omega_L = \omega_5$, $\omega_\psi = \omega_6$ (with these data all independent of $\alpha \in A$).*

- *(Power type moduli.) If these moduli are all of power type, that is, $\omega_i(\rho) = k_i\rho^{\alpha_i}$, $\rho \geq 0$, for given $k_i \geq 0$, $0 < \alpha_i (\leq 1)$, for $i = 1,\dots,6$, with $p \geq 2(1 + \max\{\alpha_1,\alpha_2,\alpha_3,a_4\})$ ($p \geq 1 + \max\{\alpha_5,\alpha_6\}$), then $V(s,\cdot)$ is ω-semiconcave on $L^p\left(\Omega, \mathcal{F}_s, \mathbb{P}; I\!R^d\right)$ with*

$$\omega(\rho) = \sum_{i=1}^{6} \bar{k}_i \rho^{\alpha_i} \qquad \forall \rho \geq 0,$$

  where $\bar{k}_i$ can be chosen to depend only on d, T, ν, p, L_i, α_i, k_i for $i = 1,\dots,6$. More precisely, ω can be taken as in (3.20) with $L_f = L_g$ given by (2.52), $\omega_f = \omega_g$ by (2.53), $L_{\overline{L}} = L_L$, $L_{\overline{\psi}}$ by (3.35), $p_1 = 2$, $p_2 = p$, and[13] *$s = 0$.*
- *(General moduli.) If $p > d$, then, for all $0 < \beta < 1 - d/p$, $V(s,\cdot)$ is ω-semiconcave on $I\!R^d$ with*

$$\omega(\rho) = \sum_{i=1}^{6} k_i' \left(E\left[\omega_i\left(\left(k_i(\rho^\beta + \rho)\right)^{q_i}\right)\right]\right)^{1/q_i} \qquad \forall \rho \geq 0,$$

  (provided that it is finite) where q_i's are such that $1/2 \geq 1/q_i + 1/p$ for $i = 1,2,3,4$, and $1 \geq 1/q_i + 1/p$ for $i = 5,6$, constants $k_i' \geq 0$ and random variables $k_i \geq 0$ with $E[k_i^p] < \infty$ for $i = 1,\dots,6$ depend only on d, T, ν, p, L_i, α_i, k_i, $i = 1,\dots,6$. An example of such an ω can be constructed as above, by (3.20) with $L_f = L_g$ given by (2.52), ω_f, ω_L, $\omega_{\overline{\psi}}$ given by (3.40)

[13] We make ω independent of s by increasing it "slightly".

with ω_g *given by* (2.53), $L_{\overline{L}} = L_L$, $L_{\overline{\psi}}$ *by* (3.35), $p_1 = 2$, $p_2 = p$, $s = 0$. *If in addition maps* $\rho \mapsto \omega_i^{q_i}\left(\rho^{1/q_i}\right)$ *for* $i = 1, \ldots, 6$ *are concave we can take*

$$\omega(\rho) = \sum_{i=1}^{6} k_i' \omega_i \left(\left(E\left[k_i^{q_i}\right] \right)^{1/q_i} \left(\rho^\beta + \rho\right) \right) \quad \forall \rho \geq 0,$$

which is finite if also $q_i \leq p$ *for* $i = 1, \ldots, 6$.

Proof. We show that first result is a consequence of Corollary 3.3, and the second of Corollary 3.6. Indeed, for the given reference probability system R in (4.2) with respect to the fixed Lévy measure ν, and time frame $[s, T]$, where $0 \leq s \leq T$, we adhere to notations set forth in Subsect. 2.3. So the reader is referred to that subsection for the definition of X, Y_i, Y, B, B_i' with $p_0 = 2$, $\mathcal{C}$, $\mathcal{C}_s$, $\mathcal{C}_i$, $\mathcal{C}_i'$, $i = 1, \ldots, 5$, $\Sigma(s, T)$. Let the set of admissible controls $\mathcal{A}(s, T)$ be the set of A-valued stochastic processes $\alpha(\cdot)$ that are predictable with respect to filtration $(\mathcal{F}_t)_{s \leq t \leq T}$. The map Φ in (3.1) is defined by setting, for all $x(\cdot) \in \Sigma(s, T)$, $\alpha(\cdot) \in \mathcal{A}(s, T)$, $\Phi\big(x(\cdot), \alpha(\cdot)\big)$ equal to the right hand side of (4.1). Let

$$\begin{aligned} g(t, x, \alpha) = 4^{1-\frac{1}{p}} \Big(& (T-s)^{1-\frac{1}{p}} b(t, x, \alpha), c_p (T-s)^{\frac{1}{2}-\frac{1}{p}} \sigma(t, x, \alpha), \\ & c_p' (T-s)^{\frac{1}{2}-\frac{1}{p}} H(t, x, \cdot, \alpha), c_p'' H(t, x, \cdot), D_K K(t, x, \cdot, \alpha) \Big) \end{aligned} \tag{4.4}$$

for all $t \in [s, T]$, $x \in \mathbb{R}^d$, $\alpha \in A$, where c_p, c_p', c_p'' are constants that appear in moment inequalities (2.46), (2.47), and D_p is given by (2.51). We can write $g = \prod_{i=1}^5 g_i$, for certain $g_i : [s, T] \times X \times A \to Y_i$, $i = 1, \ldots, 5$, in an obvious way.

Finally, we define $\overline{L}$, $\overline{\psi}$, f by (3.21), (3.22), (3.24), where we may take as $\mathcal{A}$ the set of A-valued random variables. In particular, the cost (4.3) can be written as in (3.3).

It is now easy to check that all conditions of Corollary 3.3, and Corollary 3.6 with $\mathcal{C}_0 = \mathbb{R}^d$ are satisfied. Therefore, our results follow as a consequence of these corollaries. □

Many other results could be stated, which involve various possible combinations of the data moduli types. That is, some of the maps b, σ, H, K, L, ψ may have moduli of power type, some moduli with suitable concavity properties, and some even arbitrary moduli (provided a suitable quantity is finite). If conditions (2.39), (2.41) (as mentioned in Theorem 4.1) hold for p sufficiently large, one can establish generalized semiconcavity results for the value function.

Remark 4.2. (Removing the global Lipschitz hypotheses on g, L and ψ, local results) Of course, as noted in Subsect. 3.2, in order to ensure the validity of the conclusion of Theroem 4.1 about semiconcavity of value function on bounded subsets of $\mathbb{R}^d$, instead of assuming that L and ψ be globally Lipschitz in state variable, uniformly in time and control variables, we can assume that solutions of (4.1) departing from points x^0 belonging to any bounded $K \subset \mathbb{R}^d$ remain uniformly (as $x^0 \in K$) bounded for subsequent times $t \in [s, T]$ almost surely. The local Lipschitz continuity of g, L, ψ (which follows by their ω-semiconcavity or $C^{1,\omega}$-regularity under reasonable assumptions such as local boundedness), enables us to prove the ω-semiconcavity of V on any bounded set $K \subset I\!R^d$.

Remark 4.3. (Regularity results for solutions of HJB PIDEs) Under same assumptions on b, σ, H, K, L, ψ as in Theorem 4.1, or as in following remarks, it follows by results in [25], that V is the unique viscosity solution of (1.1) with polynomial growth in x, therefore such viscosity solution of (1.1) enjoys the regularity properties prescribed by that theorem, or following remarks, respectively.

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