Nonlinear Differ. Equ. Appl. 22 (2015), 629–650

1021-9722/15/040629-22
published online November 19, 2014
DOI 10.1007/s00030-014-0298-6

Nonlinear Differential Equations and Applications NoDEA

Dynamical solutions of singular parabolic equations modeling electrostatic MEMS

Qi Wang

Abstract. We study the electrostatic MEMS-device parabolic equation, $u_t - \Delta u = \dfrac{\lambda\rho(x)}{(1-u)^2}$ with Dirichlet boundary condition and a bounded domain Ω of $\mathbb{R}^N$. Here λ is positive parameter and ρ is a nonnegative continuous function. In this paper, we investigate the behavior of solutions for this problem. In particular, we show small initial value yields quenching behavior of the solutions. While large initial data leads global existence of the solutions.

Mathematics Subject Classification. 35B44, 35K10, 35K20, 35K58, 35K91.

Keywords. MEMS equation, Quench, Globally bounded.

1. Introduction

We consider the parabolic problem

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda\rho(x)}{(1-u)^2}, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\lambda > 0$, ρ is a continuous nonnegative function in Ω, $u_0(x)$ satisfies

$$u_0 \in L^1(\Omega), \quad 0 \le u_0 \le a < 1, \qquad u_0(x) = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

The associated stationary equation is

$$\begin{cases} -\Delta w = \dfrac{\lambda\rho(x)}{(1-w)^2}, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Problem (1.1) arises in the study of micro-electromechanical systems (MEMS) devices consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at +1. When a voltage-represented here by λ-is applied, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates a so-called "pull-in instability" which greatly affects the design of many devices. (see [13,22,23] for more details). The mathematical model lends to a nonlinear parabolic problem for the dynamic deflection of the elastic membrane which has been considered in [10,11]. The relative researches on the model are also discussed in [2–8,12,16–20,25,26] and the references therein.

We say the solution w to (1.3) is classical or regular, if $\|w\|_\infty < 1$. It is known in [9,10] that for any given ρ, there exists a critical value $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$, problem (1.3) has a unique stable classical solution w_λ and the solution to (1.1) is global with $u_0 = 0$. Moreover w_λ is the minimal solution and $\lambda \to w_\lambda$ is increasing. Here the minimal solution means that $w_\lambda \leq v$ for any solution v of (1.3). Furthermore from [3,5], we can see $1 \leq N \leq 7$,

$$\lambda_* := \inf\{\bar{\lambda} > 0 : \text{ for any } \lambda \in (\bar{\lambda}, \lambda^*), (1.3) \text{ has exactly two solutions}\} < \lambda^*. \tag{1.4}$$

In particular, if $N = 1$, then $\lambda_* = 0$. For $\lambda = \lambda^*$, it follows from [9,27] that problem (1.3) admits a unique weak solution $w^* := \lim\limits_{\lambda \to \lambda^*} w_\lambda$, called the extremal solution, in the sense that

$$-\int_\Omega w^* \Delta\psi \mathrm{d}x = \lambda \int_\Omega \frac{\rho\psi}{(1-w^*)^2} \mathrm{d}x,$$

for any $\psi \in C^2(\overline{\Omega}) \cap H_0^1(\Omega)$, where $w^* \in L^1(\Omega)$ and $\dfrac{\rho(x) dist(x, \partial\Omega)}{(1-w^*)^2} \in L^1(\Omega)$. Moreover, w^* is stable, which means the first eigenvalue μ_{1,λ^*} of the linearized operator $L_{w^*,\lambda^*} := -\Delta - \dfrac{2\lambda^*\rho}{(1-w^*)^3}$ is nonnegative. While for $\lambda > \lambda^*$, no solution of (1.3) exists, and the solution u of (1.1) with $u_0 \equiv 0$ reaches the value 1 in finite time T^*, called quenching time, i.e., the so called quenching or touchdown phenomenon occurs. More precisely $\|u(\cdot, t)\|_\infty < 1$ for $t \in [0, T^*)$ and $\lim\limits_{t \to (T^*)^-} \|u(\cdot, t)\|_\infty = 1$. We say the solution u to (1.1) quenches if it reaches 1 in the time $T^* \leq +\infty$ ($T^* = +\infty$ means u quenches as $t \to +\infty$).

For any given $u_0(x)$ satisfying (1.2), we say that $u \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times (0,T))$ is a solution (subsolution, supersolution respectively) of (1.1) in $\Omega \times (0,T)$, if u satisfies (1.1)($\leq, \geq$ respectively) with $u(x,t) = 0$ ($\leq, \geq$ respectively) on $\partial\Omega \times (0,T)$,

$$\sup_{t \in (0,T')} \|u(\cdot,t)\|_\infty < 1, \quad \forall\ 0 < T' < T,$$

and

$$\lim_{t \to 0} \|u(\cdot,t) - u_0\|_{L^1(\Omega)} = 0.$$

Define

$$\delta(x) = dist(x, \partial\Omega). \tag{1.5}$$

By a nonnegative weak solution of (1.1) in $\Omega \times (0,T)$ we mean $u \geq 0$, $u_0 < 1$, and there holds

$$u \in L^1(\Omega \times (0,T)), \quad \frac{\rho\delta}{(1-u)^2} \in L^1(\Omega \times (0,T)), \tag{1.6}$$

$$\int_0^T \int_\Omega \frac{\rho\psi}{(1-u)^2} \mathrm{d}x\mathrm{d}t = -\int_0^T \int_\Omega u(\psi_t + \Delta\psi)\mathrm{d}x\mathrm{d}t - \int_\Omega u_0\psi(\cdot,0)\mathrm{d}x, \tag{1.7}$$

for any $\psi \in C^2(\overline{\Omega} \times (0,T))$ such that $\psi(x,T) = 0$ and $\psi = 0$ on $\partial\Omega$.

It is known in [27] that if there exists a weak $H_0^1(\Omega)$ solution S of (1.3) with $\lambda < \lambda^*$, then the solution $u(x,t)$ of (1.1) is unique, global and

$$\lim_{t\to+\infty} \|u(\cdot,t) - w_\lambda\|_\infty = 0,$$

where w_λ is the minimal steady-state of (1.3), provided that $0 \leq u_0 \leq a < 1$(not necessarily smooth), $u_0 \leq S$ and $u_0 \neq S$. While for $\lambda > \lambda^*$, with any u_0, the solution quenches in finite time T^*.

Let $G(x,y,t)$, $x,y \in \Omega$, $t > 0$, be the Dirichlet Green function of the heat equation in $\Omega \times (0,+\infty)$. That is for any $x,y \in \Omega$,

$$\begin{cases} \partial_t G = \Delta_x G, & (x,t) \in \Omega \times (0,T), \\ G(x,y,t) = 0, & x \in \partial\Omega,\ t > 0, \\ \lim\limits_{t\to 0} G(x,y,t) = \delta_y, & \end{cases} \tag{1.8}$$

where δ_y is the delta mass at y. So $G(x,y,t) = G(y,x,t)$. By the maximum principle, $0 \leq G(x,y,t) \leq \dfrac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4t}}$. The parabolic MEMS equation (1.1) with nonzero initial data was addressed earlier in [15]. In [14], Hui also discussed problem (1.1) with general u_0. Some classic results are in the following.

Theorem 1.1. [14] *Let u_0 satisfy (1.2) for some constant $0 < a < 1$. Let ρ be a continuous nonnegative function in Ω. Then for any $\lambda > 0$, there exists $T > 0$ such that (1.1) has a solution which satisfies*

$$u(x,t) = \int_\Omega G(x,y,t)u_0(y)\mathrm{d}y + \lambda \int_0^t \int_\Omega \frac{\rho(y)G(x,y,t-s)}{(1-u(y,s))^2}\mathrm{d}y\mathrm{d}s, \quad \forall\ x \in \overline{\Omega},\ t \in (0,T). \tag{1.9}$$

Corollary 1.2. [14] *Let ρ be a continuous nonnegative function in Ω and u_0 satisfy (1.2) for some constant $0 < a < 1$. Suppose u is a bounded solution of (1.1) in $\Omega \times (0,T)$. Then u satisfies (1.9) in $\overline{\Omega} \times (0,T)$.*

The main purpose of this paper is to consider the effect of initial value u_0 on the quenching phenomenon of problem (1.1). More precisely, we will show the solution u with sufficiently large initial data must quench in finite time T^*, even if $\lambda < \lambda^*$. While u exists globally in time with small initial value, for $\lambda < \lambda^*$. Throughout this paper, we will use the notation

$$\rho_0 := \inf_{x\in\Omega} \rho(x). \tag{1.10}$$

We present our results (Theorems 2.1, 2.2, 3.3, 3.6) in the following sections.

2. The influence of initial data

In this section, we are going to investigate the influence of initial value on the behavior of the solutions. We first introduce an energy functional

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \lambda \int_\Omega \frac{\rho u}{1-u}\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \lambda \int_\Omega \rho \mathrm{d}x - \lambda \int_\Omega \frac{\rho}{1-u}\mathrm{d}x. \end{aligned} \tag{2.1}$$

Theorem 2.1. *Let $\lambda > 0, \rho$ be a continuous nonnegative function in Ω, such that $\int_\Omega \rho^{-\frac{1}{p-1}}\mathrm{d}x < +\infty$ for some $p > 1$, and u be a solution of (1.1) with initial value u_0. Suppose that*

$$E(0) \le -\frac{1}{2}\lambda c_p \int_\Omega \rho \mathrm{d}x,$$

where

$$c_p := \max_{0\le u\le 1}\left(u^{2p} - \frac{2u^2-u}{(1-u)^2}\right), \tag{2.2}$$

or suppose that

$$E(0) + \frac{1}{8}\lambda \|\rho\|_\infty |\Omega| \le -\frac{c_0}{4}$$

for some constant $c_0 > 0$. Then the solution u must quench in finite time T^*.

Proof. Define $G(t) = \int_\Omega u^2 \mathrm{d}x \le |\Omega|$, and let $E(t)$ be as in (2.1). Then we have

$$\begin{aligned} G'(t) &= 2\int_\Omega u u_t \mathrm{d}x = 2\int_\Omega u\Delta u \mathrm{d}x + 2\lambda \int_\Omega \frac{u\rho}{(1-u)^2}\mathrm{d}x \\ &= -2\int_\Omega |\nabla u|^2 \mathrm{d}x + 2\lambda \int_\Omega \frac{u\rho}{(1-u)^2}\mathrm{d}x \\ &= -4E(t) - 4\lambda \int_\Omega \frac{u\rho}{1-u}\mathrm{d}x + 2\lambda \int_\Omega \frac{u\rho}{(1-u)^2}\mathrm{d}x. \end{aligned} \tag{2.3}$$

Since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}E(t) &= \int_\Omega \nabla u \nabla u_t \mathrm{d}x - \lambda \int_\Omega \frac{\rho u_t}{(1-u)^2}\mathrm{d}x \\ &= -\int_\Omega \Delta u u_t \mathrm{d}x - \lambda \int_\Omega \frac{\rho u_t}{(1-u)^2}\mathrm{d}x \\ &= -\int_\Omega u_t^2 \mathrm{d}x \le 0, \end{aligned} \tag{2.4}$$

we have $E(t) \leq E(0) = \frac{1}{2}\int_\Omega |\nabla u_0|^2 \mathrm{d}x - \lambda \int_\Omega \frac{\rho u_0}{1-u_0}\mathrm{d}x$. Then we get

$$\begin{aligned} G'(t) &\geq -4E(0) + 2\lambda \int_\Omega \frac{\rho(2u^2-u)}{(1-u)^2}\mathrm{d}x \\ &\geq -4E(0) - \frac{1}{2}\lambda \int_\Omega \rho \mathrm{d}x \geq -4E(0) - \frac{1}{2}\|\rho\|_\infty |\Omega|, \end{aligned} \tag{2.5}$$

for any $t \geq 0$, as long as $u < 1$. If $E(0) + \frac{1}{8}\lambda\|\rho\|_\infty|\Omega| \leq -\frac{c_0}{4}$, then $G'(t) \geq c_0$ and $G(t) \geq G(0) + c_0 t \to +\infty$, as t tends to infinity. This is impossible, so $T^* < +\infty$.

On the other hand, if there exists $p > 1$, such that $\int_\Omega \rho^{-\frac{1}{p-1}}\mathrm{d}x < +\infty$, then we have

$$G'(t) \geq -4E(0) + 2\lambda \int_\Omega \rho u^{2p}\mathrm{d}x - 2\lambda c_p \int_\Omega \rho \mathrm{d}x, \tag{2.6}$$

where c_p is as in (2.2). The above identity and Hölder's inequality, along with $q := \frac{p}{p-1}$ yield

$$G'(t) \geq 2\lambda \left(\int_\Omega \rho^{-\frac{q}{p}}\mathrm{d}x\right)^{-\frac{p}{q}} G^p(t) - 4\left(E(0) + \frac{1}{2}\lambda c_p \int_\Omega \rho\mathrm{d}x\right), \tag{2.7}$$

for any $t \geq 0$, as long as $u < 1$. It follows from $E(0) \leq -\frac{1}{2}\lambda c_p \int_\Omega \rho \mathrm{d}x$ and the above inequality that

$$G'(t) \geq 2\lambda \left(\int_\Omega \rho^{-\frac{q}{p}}\right)^{-\frac{p}{q}} G^p(t), \quad G(0) \in (0, |\Omega|). \tag{2.8}$$

We now compare $E(t)$ with the solution $F(t)$ of

$$F'(t) = 2\lambda \left(\int_\Omega \rho^{-\frac{q}{p}}\right)^{-\frac{p}{q}} F^p(t), \quad F(0) = G(0) \in (0, |\Omega|). \tag{2.9}$$

Standard comparison principle yields that $E(t) \geq F(t)$ on their domains of existence. Therefore,

$$|\Omega| \geq G(t) \geq F(t). \tag{2.10}$$

It is easy to see from (2.9) that the quenching time T_1 for $F(t)$ is finite. Therefore, the solution u of (1.1) must quench in finite time $T^* \leq T_1$, and the proof is finished. □

Now let $\Omega = [-R, R]$ be a interval in $\mathbb{R}^1$. It is clear that $\frac{\pi}{4}\cos\frac{\pi}{2}x$ is the first normalized eigenfunction of $-\Delta$ in $C_c^2([-1,1])$ such that $\frac{\pi}{4}\int_0^1 \cos\frac{\pi}{2}x\mathrm{d}x = 1$. The corresponding first eigenvalue is $\lambda_0 := \frac{\pi^2}{4}$. Hence $\frac{\lambda_0}{R^2}$ is the first eigenvalue of $-\Delta$ in $C_c^2([-R,R])$ and $\frac{\pi}{4R}\cos\frac{\pi x}{2R}$ is the first eigenfunction of $-\Delta$ in

$C_c^2([-R, R])$ normalized such that $\frac{\pi}{4R}\int_{-R}^{R}\cos\frac{\pi x}{2R}\mathrm{d}x = 1$. Then we have the following theorem.

Theorem 2.2. *Let $\Omega = [-R, R]$ and $\lambda \in (0, \lambda^*)$. Assume that ρ is a continuous positive function in Ω, such that $\lambda\rho_0 R^2 > \frac{\pi}{2}\big(1 - \frac{2}{\pi}\big)^2$. If $u_0(x) = \frac{\mu\pi}{4R}\cos\frac{\pi x}{2R}$, where $\mu \in \big(2Rs_0, \frac{4R}{\pi}\big)$, $s_0 \in (\frac{1}{3}, 1)$ is the unique root of $\frac{\pi^2 s_0}{4R^2} = \frac{\lambda\rho_0}{(1-s_0)^2}$, then the solution u must quench in finite time T^*.*

Proof. We shall apply some similar ideas in [13] to prove this theorem. Denote $\frac{\pi^2}{4R^2}$ and $\frac{\pi}{4R}\cos\frac{\pi x}{2R}$ by λ_1 and $\varphi(x)$, then $u_0(x) = \mu\varphi(x)$. Since $\|u_0\|_\infty < 1$, $\mu < \frac{1}{\|\varphi\|_\infty} = \frac{4R}{\pi}$. Multiply (1.1) by φ and integrate over $[-R, R]$ to get

$$\int_{-R}^{R}\varphi u_t \mathrm{d}x = \int_{-R}^{R}\varphi\left(\Delta u + \frac{\lambda\rho}{(1-u)^2}\right)\mathrm{d}x. \tag{2.11}$$

Using Green's theorem, together with the lower bound ρ_0 of ρ, we get

$$\begin{aligned}\int_{-R}^{R}\varphi u_t \mathrm{d}x &\geq -\lambda_1\int_{-R}^{R}u\varphi\mathrm{d}x + \lambda\rho_0\int_{-R}^{R}\frac{\varphi}{(1-u)^2}\mathrm{d}x \\ &\geq -\lambda_1\int_{-R}^{R}u\varphi\mathrm{d}x + \frac{\lambda\rho_0}{(1-\int_{-R}^{R}u\varphi\mathrm{d}x)^2}.\end{aligned} \tag{2.12}$$

Here Jensen's inequality is applied in the second inequality. Next, we define $X(t) := \int_{-R}^{R}\varphi u\mathrm{d}x$, such that $X(t) \leq \int_{-R}^{R}\varphi\mathrm{d}x = 1$. Moreover $X(0) = \mu\int_{-R}^{R}\varphi^2\mathrm{d}x$. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}X + \lambda_1 X \geq \frac{\lambda\rho_0}{(1-X)^2}, \quad X(0) = \mu\int_{-R}^{R}\varphi^2\mathrm{d}x. \tag{2.13}$$

We then compare $X(t)$ with the solution $Y(t)$ of

$$\frac{\mathrm{d}}{\mathrm{d}t}Y + \lambda_1 Y = \frac{\lambda\rho_0}{(1-Y)^2}, \quad Y(0) = \mu\int_{-R}^{R}\varphi^2\mathrm{d}x. \tag{2.14}$$

Standard comparison principle gives that $X(t) \geq Y(t)$ on their domains of existence. Therefore, $1 \geq X(t) \geq Y(t)$. Next, we separate variables in (2.14) to determine t in terms of F, and it is easy to see from $1 = \int_{-R}^{R}\varphi\mathrm{d}x \leq$

$(\int_{-R}^{R}\varphi^2\mathrm{d}x)^{\frac{1}{2}}(2R)^{\frac{1}{2}}$ that the quenching time T_2 for F is given by

$$\begin{aligned} T_2 &= \int_{Y(0)}^{1}\left(-\lambda_1 s+\frac{\lambda\rho_0}{(1-s)^2}\right)^{-1}\mathrm{d}s=\int_{\mu\int_{-R}^{R}\varphi^2\mathrm{d}x}^{1}\left(-\lambda_1 s+\frac{\lambda\rho_0}{(1-s)^2}\right)^{-1}\mathrm{d}s \\ &\le \int_{\frac{\mu}{2R}}^{1}\left(-\lambda_1 s+\frac{\lambda\rho_0}{(1-s)^2}\right)^{-1}\mathrm{d}s. \end{aligned} \tag{2.15}$$

Note that T_2 is finite whenever the integral in (2.15) converges. Now define $g(s)=-\lambda_1 s+\frac{\lambda\rho_0}{(1-s)^2}$. Since $\lambda<\lambda^*$ and it is well known that $\lambda^*\le\dfrac{4\lambda_1}{27\rho_0}$ (see [9]), a simple calculation shows that there exists two zeros s_1, s_0, such that $0<s_1<\dfrac{1}{3}<s_0<1$ and $g(s)>0$ in $(s_0,1)$. Then T_2 is finite whenever $\mu>2Rs_0$. Hence, if T_2 is finite, then $X(t)\ge Y(t)$ implies that the quenching time T^* of (1.1) must also be finite. This completes the proof of Theorem 2.2. □

3. Sharp threshold behavior of initial value

In this section, we investigate the threshold behavior of solutions of problem (1.1) with $u_0(x)=\mu\phi(x)$, according to the value of μ. It is obvious that $0<\mu<\dfrac{1}{\|\phi\|_\infty}$. In this direction we establish the claims in Theorems 3.3 and 3.6 in the following.

We say the solution u of (1.1) is globally bounded, if u exists globally,

$$\sup_{t>0}\|u(\cdot,t)\|_\infty<1.$$

We prove in Sect. 3.1 the global existence and finite-time quenching in the case $\lambda<\lambda^*$. In Sect. 3.2, we discuss the case $\lambda=\lambda^*$.

We now recall a useful result. Let $G(x,y,t)$ be the Dirichlet Green function of the heat equation, as in (1.8). Then for any $a(x)$ and $f(x,t)$, $\int_\Omega G(x,y,t)a(y)\mathrm{d}y$ is a solution of

$$\begin{cases} u_t=\Delta u & \text{in } \Omega\times(0,+\infty),\\ u(x,t)=0 & \text{on } \partial\Omega,\\ u(x,0)=a(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

and $\int_0^t\int_\Omega G(x,y,t-s)f(y,s)\mathrm{d}y\mathrm{d}s$ is a solution of

$$\begin{cases} u_t-\Delta u=f(x,t) & \text{in } \Omega\times(0,T),\\ u(x,t)=0 & \text{on } \partial\Omega,\\ u(x,0)=0 & \text{in } \Omega, \end{cases} \tag{3.2}$$

for any $t\in[0,T)$.

3.1. The case: $\lambda < \lambda^*$

First, we note the following results.

Lemma 3.1. *Assume ρ is a continuous nonnegative function in Ω and u_0 satisfies (1.2), $\|u_0\|_\infty < 1$, and $\lambda < \lambda^*$. Suppose that the solution u of (1.1) does not quench in finite time. Then for any $0 \le v_0(x) \le u_0(x)$, $v_0(x) \not\equiv u_0(x)$, the solution $v(x,t)$ of problem (1.1) with $v(x,0) = v_0(x)$ is global bounded.*

Proof. Take now v_0 as in the statement of the lemma. Consider the function

$$V(x,t) := v(x,t) + \int_\Omega G(x,y,t)(u_0(y) - v_0(y))\mathrm{d}y.$$

It is a consequence of the maximum principle that $V(x,t) \ge v(x,t)$. The difference $u(x,t) - V(x,t)$ then satisfies

$$(u-V)_t - \Delta(u-V) \ge \frac{2\lambda\rho(x)}{(1-V)^3}(u-V) \quad \text{in } \Omega \times (0,T) \tag{3.3}$$

with initial data $u(x,0) - V(x,0) = v_0(x) \ge 0$ and zero boundary condition, where T is the maximal time for existence. The comparison principle and Hopf lemma yield that $u \ge V$ and

$$u(x,t) - v(x,t) \ge \int_\Omega G(x,y,t)(u_0(y) - v_0(y))\mathrm{d}y \ge c(t)\delta(x), \tag{3.4}$$

where $c(t)$ is a continuous function of t, and $\delta(x)$ is defined in (1.5). Fix now $\tau > 0$, and let $c_0 := c(\tau)$. Without loss of generality, take $u(x,\tau)$ and $v(x,\tau)$ instead of $u_0(x)$ and $v_0(x)$. Then

$$u_0(x) \ge v_0(x) + c_0\delta(x).$$

Define now

$$f(u) = \frac{1}{(1-u)^2}, \quad h(u) = \int_0^u \frac{1}{f(s)}\mathrm{d}s, \qquad 0 \le u < 1. \tag{3.5}$$

For any $\varepsilon \in (0,1)$, we also define

$$\tilde{f}(u) = \frac{1}{(1-u)^2} - K\varepsilon, \quad \tilde{h}(u) = \int_0^u \frac{1}{\tilde{f}(s)}\mathrm{d}s, \qquad 0 \le u < 1, \tag{3.6}$$

and $\Phi_\varepsilon(u) := \tilde{h}^{-1}(h(u))$. It is easy to check that $\Phi_\varepsilon(0) = 0$ and $0 < \Phi_\varepsilon(x) \le x$ for $x > 0$, and Φ_ε is increasing and concave with

$$\begin{aligned} &1 \ge \Phi'_\varepsilon(x) \ge \frac{(f(\Phi_\varepsilon(x)) - K\varepsilon)^+}{f(x)} \quad \text{for } x \ge 0, \\ &\lim_{\varepsilon\to 0} \Phi'_\varepsilon(x) \to 1 \quad \text{uniformly in } x. \end{aligned} \tag{3.7}$$

A direct calculation leads to

$$\lim_{s\to 1}\lim_{\varepsilon\to 0} \frac{u_0 - \Phi_\varepsilon(u_0)s}{\delta(x)} = 0 \quad \text{uniformly in } x, \tag{3.8}$$

which shows there exists $s \in (0,1)$ and $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$, we have

$$s\Phi_\varepsilon(u_0) \ge u_0 - c_0\delta(x) \ge v_0. \tag{3.9}$$

Setting $U_\varepsilon = \Phi_\varepsilon(u)$, we have U_ε is a super solution of the following problem

$$\begin{cases} U_t - \Delta U = \lambda\rho(x)\left(\dfrac{1}{(1-U)^2} - K\varepsilon\right)^+ & \text{in } \Omega\times(0,T), \\ U(x,t) = 0 & \text{on } \partial\Omega, \\ U(x,0) = \Phi_\varepsilon(u_0) & \text{in } \Omega, \end{cases} \tag{3.10}$$

where T is the maximal time for existence. Since $\Phi_\varepsilon(u) < u$ for any $\varepsilon > 0$ and u does not quench at a finite time, we have

$$0 < U_\varepsilon(x,t) < u(x,t) \leq 1.$$

Thus the solution U of (3.10) is global and $0 < U < 1$ by the maximum principle.

Fix now s, ε_1 as in (3.9). Consider the solution z of

$$\begin{cases} z_t - \Delta z = \lambda\rho(x)\left(\dfrac{1}{(1-z)^2} + \dfrac{Ks\varepsilon}{1-s}\right) & \text{in } \Omega\times(0,T), \\ z(x,t) = 0 & \text{on } \partial\Omega, \\ z(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{3.11}$$

Since $\lambda < \lambda^*$, there exists $0 < \varepsilon_2 \leq \varepsilon_1$ such that for any $\varepsilon \in (0,\varepsilon_2)$, z is a global solution and $z < 1$.

To complete the proof of Lemma 3.1, we set $\varepsilon \in (0,\varepsilon_2)$ and

$$Z(x,t) = sU(x,t) + (1-s)z(x,t).$$

Then $0 \leq Z(x,t) < 1$ satisfies

$$Z_t - \Delta Z \geq \frac{\lambda s\rho(x)}{(1-U)^2} + \frac{\lambda(1-s)\rho(x)}{(1-z)^2} \geq \frac{\lambda\rho(x)}{(1-Z)^2} \tag{3.12}$$

with initial data $Z(x,0) = s\Phi_\varepsilon(u_0(x)) \geq v_0(x)$ and zero boundary condition. Now the maximum principle gives that $v(x,t)$ is global and $v(x,t) < 1$, which finishes the proof of Lemma 3.1. □

Lemma 3.2. *Assume ρ is a continuous positive function in Ω, and assume u_0 satisfies (1.2), $\|u_0\|_\infty < 1$, and $\lambda < \lambda^*$. If u is a global solution of (1.1) with $u(x,0) = u_0(x)$, then there exists $0 < C < 1$ such that $\|u(\cdot,t)\varphi\|_{L^1} \leq C$, for all $t > 0$, where φ is the first normalized eigenfunction of $-\Delta$ in $H_0^1(\Omega)$, such that $\int_\Omega \varphi \mathrm{d}x = 1$, and C is independent of u_0 and t.*

Proof. We prove it by contradiction. First, observe that $\lim_{s\to1} \dfrac{1}{(1-s)^2} = +\infty$, so that there exists a constant $1 > M > 0$ such that

$$\frac{\lambda\rho_0}{(1-s)^2} - \lambda_1 s \geq \frac{\lambda\rho_0}{2(1-s)^2}, \quad \text{for } s \geq M, \tag{3.13}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. It follows from (1.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\varphi \mathrm{d}x + \lambda_1 \int_\Omega u\varphi \mathrm{d}x = \int_\Omega \frac{\lambda\rho\varphi}{(1-u)^2}\mathrm{d}x. \tag{3.14}$$

Indeed, applying $\int_\Omega \varphi \mathrm{d}x = 1$ and Jensen's inequality in (3.14), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\varphi \mathrm{d}x + \lambda_1 \int_\Omega u\varphi \mathrm{d}x \geq \frac{\lambda\rho_0}{(1-\int_\Omega u\varphi \mathrm{d}x)^2}. \tag{3.15}$$

If there exists $t_0 > 0$ such that $\int_\Omega u(\cdot, t_0)\varphi \mathrm{d}x > M$, where M is as in (3.13), then it follows from (3.13) and (3.15) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\varphi \mathrm{d}x\Big|_{t=t_0} \geq \frac{\lambda\rho_0}{2(1-\int_\Omega u\varphi \mathrm{d}x)^2}\Big|_{t=t_0}, \tag{3.16}$$

which implies that $\int_\Omega u\varphi \mathrm{d}x$ is increasing with respect to $t \geq t_0$. Hence for any $t \geq t_0$, we have $\int_\Omega u\varphi \mathrm{d}x > M$, and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\varphi \mathrm{d}x \geq \frac{\lambda\rho_0}{2(1-\int_\Omega u\varphi \mathrm{d}x)^2}, \tag{3.17}$$

for $t \geq t_0$, which is impossible, since $\int_0^1 (1-s)^2 \mathrm{d}s = \frac{1}{3}$. So we have

$$\int_\Omega u\varphi \mathrm{d}x \leq M < 1, \tag{3.18}$$

for all $t > 0$. □

Indeed from the above proof, we see

$$\lambda\rho_0 = 2\lambda_1 M(1-M)^2, \quad 0 < M < 1. \tag{3.19}$$

Let ξ be the first normalized eigenfunction of the following problem:

$$\begin{cases} -\Delta\xi = \lambda_0\xi & \text{in } B_1(0), \\ \xi = 0 & \text{on } \partial B_1(0), \end{cases} \tag{3.20}$$

such that $\int_{B_1(0)} \xi \mathrm{d}x = 1$.

Define now a cut-off smooth function $\eta(x) = \eta(|x|)$ such that

$$\begin{aligned} &0 \leq \eta \leq 1,\ \|\eta\|_\infty = 1,\ \eta'(1) < 0,\ \eta(1) \\ &\quad = 0,\ \eta > 0 \text{ in } B_1(0),\ \text{and } \int_{B_1(0)} \xi\eta \mathrm{d}x > \frac{1}{3}. \end{aligned} \tag{3.21}$$

Let N, λ satisfy

$$\begin{cases} N = 1,\ \text{and } \lambda \in (0, \lambda^*); \\ \text{or } 2 \leq N \leq 15,\ \text{and } \lambda \in (\lambda_*, \lambda^*), \lambda_* \text{ is as in (1.4)}. \end{cases} \tag{3.22}$$

Then we can introduce the following theorem.

Theorem 3.3. *Let $\Omega = B_R(0)$ and N, λ satisfy (3.22). Assume that ρ is a continuous positive function in Ω, and assume that η is as in (3.21), ξ, λ_0 are as in (3.20). Suppose that λ, ρ_0, R satisfy*

$$\lambda\rho_0 R^2 > 2\lambda_0 \left(1 - \int_{B_1(0)} \xi\eta \mathrm{d}x\right)^2 \int_{B_1(0)} \xi\eta \mathrm{d}x. \tag{3.23}$$

Then if $u_0(x) = \mu\eta(\frac{x}{R})$, there exists $\mu^* \in (0, 1)$ such that we have the following:

(1) *When $0 \le \mu < \mu^*$, the solution $u(x,t)$ of (1.1) is globally bounded and converges to its unique minimal steady-state w_λ.*
(2) *When $\mu = \mu^*$, the weak solution $u(x,t)$ of (1.1) does not quench in finite time.*
(3) *When $1 > \mu > \mu^*$, the solution $u(x,t)$ of (1.1) must quench in finite time.*

Proof. Denote $\eta(\frac{x}{R})$ by $\phi(x)$, and denote the solution of (1.1) with initial value $\mu\phi$ by u_μ for simplicity. It is obvious that $\|\phi\|_\infty = 1$ and $\mu < 1$. This theorem is in fact a direct consequence of Lemma 3.1 and the following claims:

(i) For $\mu > 0$ small enough, the solution u_μ is globally bounded.
(ii) For all $\mu < \mu^*$, the solution u_μ converges to w_λ in $L^\infty(B_R(0))$.
(iii) When $\mu = \mu^*$, (1.1) admits a global weak solution.
(iv) For $1 > \mu > \mu^*$, the solution u_μ will quench in a finite time.

Since $u_\mu \in C(B_R(0) \times (0,T))$, for any $a \in (0,1)$, there exists $\mu_1 > 0$ and $\tau_1 > 0$ such that for $\mu \in (0, \mu_1)$ and $0 \le t \le \tau_1$, $u(x,t) \le a < 1$. Then together with the regularity of the solution of (3.2) we reach that

$$\begin{aligned}\lambda \int_0^t \int_{B_R(0)} & G(x,y,t-s)\frac{\rho(y)}{(1-u_\mu(y,s))^2}\mathrm{d}y\mathrm{d}s \\ &\le \frac{\lambda^*\|\rho\|_\infty}{(1-a)^2}\int_0^t\int_{B_R(0)} G(x,y,t-s)\mathrm{d}y\mathrm{d}s \le \frac{\lambda^*\|\rho\|_\infty A(t)\delta(x)}{(1-a)^2},\end{aligned} \tag{3.24}$$

where $\lim_{t\to 0} A(t) = 0$. Note that Hopf lemma indicates that the minimal solution w_λ of (1.3) satisfies $w_\lambda(x) \ge c\delta(x)$, for some $c > 0$. Therefore we can find a $\tau_2 \in (0, \tau_1]$, such that for all $t \in [0, \tau_2]$,

$$\begin{aligned}\lambda \int_0^t \int_{B_R(0)} G(x,y,t-s)\frac{\rho(y)}{(1-u_\mu(y,s))^2}\mathrm{d}y\mathrm{d}s &\le \frac{\lambda^*\|\rho\|_\infty A(t)\delta(x)}{(1-a)^2} \\ &\le \frac{1}{2}w_\lambda.\end{aligned} \tag{3.25}$$

Fix now τ_2, then we have $\int_{B_R(0)} G(x,y,\tau_2)\phi(y)\mathrm{d}y \in C^1(\overline{B_R(0)})$, and then there exists $\mu_0 \in (0, \mu_1]$, such that

$$\mu_0 \int_{B_R(0)} G(x,y,\tau_2)\phi(y)\mathrm{d}y \le \mu_0 c(\tau_2)\delta(x) \le \frac{1}{2}w_\lambda. \tag{3.26}$$

Therefore the above two inequalities and Corollary 1.2 give that for all $0 \leq \mu \leq \mu_0$ and $0 \leq t \leq \tau_2$,

$$\begin{aligned} u_\mu(x,t) &= \mu \int_{B_R(0)} G(x,y,t)\phi(y)\mathrm{d}y \\ &\quad + \lambda \int_0^t \int_{B_R(0)} G(x,y,t-s)\frac{\rho(y)}{(1-u_\mu(y,s))^2}\mathrm{d}y\mathrm{d}s \\ &\leq \frac{1}{2}w_\lambda + \frac{1}{2}w_\lambda \leq w_\lambda. \end{aligned} \tag{3.27}$$

The comparison principle leads to $u_\mu(x,t) \leq w_\lambda(x)$, for all $x \in B_R(0)$ and $t \geq \tau_2$, which yields the claim (i).

From Lemma 3.2 we see that if u_μ is a global solution, then $\|u_\mu(\cdot,t)\varphi\|_{L^1(B_R(0))} \leq M$, for all $t \geq 0$. In particular, $\mu \int_{B_R(0)} \phi\varphi \mathrm{d}x \leq M$. It is clear that $\varphi(x) = \frac{1}{R^N}\xi\big(\frac{x}{R}\big)$. Since $\int_{B_R(0)} \phi\varphi\mathrm{d}x = \int_{B_1(0)} \eta\xi\mathrm{d}x > \frac{1}{3}$, the condition (3.23) gives $\int_{B_1(0)} \eta\xi\mathrm{d}x > M$. Then $\mu \leq \frac{M}{\int_\Omega \phi\varphi\mathrm{d}x} < 1$. This indicates that

$$\mu^* := \inf\{\mu > 0 : \text{the solution of (1.1) quench in finite time}\} < 1. \tag{3.28}$$

Hence the claim (iv) is correct. Furthermore,

$$\mu_* := \sup\{\mu > 0 : \text{the solution of (1.1) is globally bounded}\} \leq \mu^* < 1. \tag{3.29}$$

If there exists a $\tilde{\mu} > \mu_*$ such that the solution $u_{\tilde{\mu}}$ does not quench in a finite time, then by Lemma 3.1, we get for any $\mu \in (\mu_*, \tilde{\mu})$, the solution u_μ is globally bounded, which contradicts to the definition of μ_*. Hence $\mu_* = \mu^*$.

Next we show $\lim_{t\to+\infty} \|u_\mu(\cdot,t) - w_\lambda\|_\infty = 0$, for all $\mu < \mu_*$. Indeed it is sufficient to show there exists a sequence t_n, such that $t_n \to +\infty$, $\lim_{n\to+\infty} \|u_\mu(\cdot,t_n) - w_\lambda\|_\infty = 0$, since w_λ is stable. Assume by contradiction that there exists no subsequence t_n such that $u_\mu(x,t_n)$ converges to $w_\lambda(x)$. Since u is globally bounded, the ω-limit set of u contains a function w such that there exists a sequence $t_k \to +\infty$ and $u_\mu(\cdot,t_k;u_0) \to w$ in $C^1(B_R(0))$. By Proposition 2.2 in [21], we have w is steady state of (1.1). From assumption above, we have $w \not\equiv w_\lambda$, so $w > w_\lambda$. Then Hopf lemma yields $w(x) \geq w_\lambda(x) + c_0\delta(x)$, for some $c_0 > 0$. Therefore for large k we have $u_\mu(x,t_k) > w_\lambda(x) + c_1\delta(x)$, for some $c_1 > 0$. By possibly taking $u_\mu(x,t_k)$ instead of $u_0(x)$, we may suppose that $u_0(x) \in C^1(B_R(0))$ and $u_0(x) > w_\lambda(x) + c_1\delta(x)$.

Set now $v(x,t) = u_\mu(x,t) - w_\lambda(x)$, so $v(x,0) \geq c_1\delta(x)$. Hence there exists $c_2 > 0$ such that $\|v(\cdot,t)\|_{L^\infty(B_R(0))} \geq c_2$, for all $t > 0$. Using the parabolic regularity, we see

$$\left(\int_{B_R(0)} G\left(x,y,\frac{1}{3}\right)\delta^{-1}(x)\mathrm{d}y\right) \leq C. \tag{3.30}$$

Then we obtain

$$\begin{aligned}
&\frac{1}{R}\int_{B_R(0)} G(x,y,1)v(y,t)\mathrm{d}y \\
&\quad \le \left|\frac{1_{B_R(0)}G(x,y,1)v(y,t)\mathrm{d}y}{\delta(x)}\right| \\
&\quad = \left|\frac{1_{B_R(0)}G(x,y,\frac{1}{3})\big(1_{B_R(0)}G(y,z,\frac{2}{3})v(z,t)\mathrm{d}z\big)\mathrm{d}y}{\delta(x)}\right| \qquad (3.31)\\
&\quad \le \left(\int_{B_R(0)} G\left(x,y,\frac{1}{3}\right)\delta^{-1}(x)\mathrm{d}y\right)\left\|\int_{B_R(0)} G\left(y,z,\frac{2}{3}\right)v(z,t)\mathrm{d}z\right\|_{L^\infty(B_R(0))} \\
&\quad \le C\left\|\int_{B_R(0)} G\left(y,z,\frac{2}{3}\right)v(z,t)\mathrm{d}z\right\|_{L^\infty(B_R(0))}.
\end{aligned}$$

Using (3.30) again we find

$$\begin{aligned}
&\left\|\int_{B_R(0)} G\left(y,z,\frac{2}{3}\right)v(z,t)\mathrm{d}z\right\|_{L^\infty(B_R(0))} \\
&\quad = \left\|\int_{B_R(0)} G\left(y,z,\frac{1}{3}\right)\left(\int_{B_R(0)} G\left(z,\hat z,\frac{1}{3}\right)v(\hat z,t)\mathrm{d}\hat z\right)\mathrm{d}z\right\|_{L^\infty(B_R(0))} \\
&\quad \le C\int_{B_R(0)}\int_{B_R(0)} G\left(z,\hat z,\frac{1}{3}\right)\big|v(\hat z,t)\big|\mathrm{d}\hat z\mathrm{d}z \qquad (3.32)\\
&\quad = C\int_{B_R(0)} \big|v(\hat z,t)\big|\left(\int_{B_R(0)} G\left(\hat z,z,\frac{1}{3}\right)\mathrm{d}z\right)\mathrm{d}\hat z \\
&\quad \le C\int_{B_R(0)} \big|v(\hat z,t)\big|\delta(\hat z)\mathrm{d}\hat z\left\|\frac{1_{B_R(0)}G\left(\hat z,z,\frac{1}{3}\right)\mathrm{d}z}{\delta(\hat z)}\right\|_{L^\infty(B_R(0))} \\
&\quad \le C\|v(\cdot,t)\delta(\cdot)\|_{L^1(B_R(0))}.
\end{aligned}$$

We conclude

$$\frac{1}{R}\int_{B_R(0)} G(x,y,1)v(y,t)\mathrm{d}y \le C\|v(\cdot,t)\delta(\cdot)\|_{L^1(B_R(0))}. \qquad (3.33)$$

By the strong maximum principle, we find $v(x,t)>0$, which means $u_\mu > w_\lambda$. Therefore from direct calculations, letting $\tilde v(x,s)=v(x,s+t)$, we have

$$\left(e^{-\frac{2\lambda}{(1-\|u\|_{L^\infty(\Omega\times(0,T))})^3}s}\tilde v\right)_s - \Delta\left(e^{-\frac{2\lambda}{(1-\|u\|_{L^\infty(\Omega\times(0,T))})^3}s}\tilde v\right)\le 0, \qquad (3.34)$$

with positive initial value $v(x,t)$ and Dirichlet boundary condition. Then the standard comparison principle gives rise to

$$\int_\Omega G(x,y,1)v(y,t)\mathrm{d}y \ge e^{-\frac{2\lambda}{(1-\|u\|_{L^\infty(\Omega\times(0,T))})^3}} v(x,t+1), \qquad (3.35)$$

which together with (3.33) implies $\|v(\cdot,t)\delta(\cdot)\|_{L^1(B_R(0))} \geq cv(x,t+1)$, for some $c > 0$. This inequality shows that for all $t > 0$,

$$\|v(\cdot,t)\delta(\cdot)\|_{L^1(B_R(0))} \geq cc_2, \tag{3.36}$$

due to the arbitrary of x.

Consider the positive solution of

$$\begin{cases} f_t - \Delta f = \dfrac{2\lambda\rho}{(1-u_\mu)^3} f & \text{in } B_R(0)\times(0,+\infty), \\ f(x,t) = 0 & \text{on } \partial B_R(0), \\ f(x,0) = \delta(x) & \text{in } B_R(0), \end{cases} \tag{3.37}$$

then by letting $g(x,t) = f(x,T-t)$, for any $T \in (0,+\infty)$, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R(0)} (u_\mu - w_\lambda) g \mathrm{d}x &= \int_{B_R(0)} (u_\mu - w_\lambda) g_t \mathrm{d}x + \int_{B_R(0)} g u_t \mathrm{d}x \\ &= \int_{B_R(0)} (u_\mu - w_\lambda)\left(-\Delta g - \frac{2\lambda\rho}{(1-u_\mu)^3} g\right)\mathrm{d}x \\ &\quad + \int_{B_R(0)} \left(\Delta u_\mu + \frac{\lambda\rho}{(1-u_\mu)^2}\right) g\mathrm{d}x \\ &= \int_{B_R(0)} (-\Delta u_\mu + \Delta w_\lambda) g - \int_{B_R(0)} \frac{2\lambda\rho g}{(1-u_\mu)^3}(u_\mu - w_\lambda)\mathrm{d}x \\ &\quad + \int_{B_R(0)} \left(\Delta u_\mu + \frac{\lambda\rho}{(1-u_\mu)^2}\right) g\mathrm{d}x \\ &= \int_{B_R(0)} \lambda\rho\left(\frac{1}{(1-u_\mu)^2} - \frac{1}{(1-w_\lambda)^2} - \frac{2}{(1-u_\mu)^3}(u_\mu - w_\lambda)\right) g\mathrm{d}x \\ &\leq 0. \end{aligned} \tag{3.38}$$

Therefore by the regularity of $u_0 - w_\lambda$,

$$\begin{aligned} \int_{B_R(0)} f(x,T)\delta(x)\mathrm{d}x &= \int_{B_R(0)} g(x,0)\delta(x)\mathrm{d}x \\ &\geq C\int_{B_R(0)} g(x,0)(u_0(x) - w_\lambda(x))\mathrm{d}x \\ &\geq C\int_{B_R(0)} g(x,T)(u_\mu(x,T) - w_\lambda(x))\mathrm{d}x \\ &= C\int_{B_R(0)} g(x,T)v(x,T)\mathrm{d}x \geq C, \end{aligned} \tag{3.39}$$

where (3.36) is applied in the last inequality.

Choose now $\mu' \in (\mu, \mu_*)$, and let z be the solution of

$$\begin{cases} z_t - \Delta z = \dfrac{\lambda\rho}{(1-z)^2} & \text{in } B_R(0)\times(0,T), \\ z(x,t) = 0 & \text{on } \partial B_R(0), \\ z(x,0) = \mu'\phi & \text{in } B_R(0). \end{cases} \tag{3.40}$$

Since z is globally bounded, there exists a sequence $t_k \to +\infty$ such that $z(\cdot, t_k; u_0) \to W$ in $C^1(B_R(0))$, where W is a solution of (1.3). The direct calculations show that $z - u_\mu$ satisfies

$$\begin{cases} (z-u_\mu)_t - \Delta(z-u_\mu) = \dfrac{\frac{\lambda\rho}{(1-z)^2} - \frac{\lambda\rho}{(1-u_\mu)^2}}{z-u_\mu}(z-u_\mu) & \text{in } B_R(0)\times(0,T), \\ (z-u_\mu)(x,t) = 0 & \text{on } \partial B_R(0), \\ z(x,0) - u_\mu(x,0) = (\mu'-\mu)\phi > 0 & \text{in } B_R(0). \end{cases} \tag{3.41}$$

It follows from (3.21) that $(\mu'-\mu)\phi \geq c\delta(x)$, for some $c > 0$. Since the strong maximum principle yields $z > u_\mu$, applying the comparison principle, we obtain that $z - u_\mu \geq cf$, which implies $z \geq u_\mu + cf$. By (3.39), we now conclude that $W > w > w_\lambda$ are three solutions of (1.3), which contradicts to (3.22). This indicates there exists a sequence t_n, such that $t_n \to +\infty$, $\lim\limits_{n\to+\infty} \|u_\mu(\cdot,t_n) - w_\lambda\|_\infty = 0$. Then Theorem 4.1 and Theorem 4.2 in [24] imply $\lim\limits_{t\to+\infty} \|u_\mu(\cdot,t) - w_\lambda\|_\infty = 0$, which yields the claim (ii).

Consider now a nondecreasing sequence $\mu_n \to \mu^*$, $\mu_n < \mu^*$. Since u_{μ_n} is globally bounded, $\|u_{\mu_n}(\cdot,t)\delta(\cdot)\|_{L^1(B_R(0))} \leq c$, by Lemma 3.2. Applying the similar techniques in [1], we deduce that

$$\int_T^{T+1}\int_{B_R(0)} u_{\mu_n}(x,t)\delta(x)\mathrm{d}x\mathrm{d}t \leq c, \quad \int_T^{T+1}\int_{B_R(0)} u_{\mu_n}(x,t)\delta(x)\mathrm{d}x\mathrm{d}t \leq c. \tag{3.42}$$

Define $u_{\mu^*} := \lim\limits_{\mu_n\to\mu^*} u_{\mu_n}$. Then together with the definition of weak solution (1.7), we obtain that u_{μ^*} is a global weak solution of (1.1), by taking $\mu_n \to \mu^*$. This gives the claim (iii). □

3.2. The case: $\lambda = \lambda^*$

We now discuss the threshold behavior of (1.1) at $\lambda = \lambda^*$. For this critical case, there exists a unique steady-state w^* of (1.1) obtained as a pointwise limit of the minimal solution w_λ as $\lambda \to \lambda^*$. We begin with two lemmas.

Lemma 3.4. *Assume ρ is a continuous nonnegative function in Ω and u_0 satisfies (1.2), $\|u_0\|_\infty < 1$, and $\lambda = \lambda^*$. Let u be the solution of (1.1). Suppose that u does not quench in finite time. Then for any $0 \leq v_0(x) \leq u_0(x)$, $v_0(x) \not\equiv u_0(x)$, the solution $v(x,t)$ of problem (1.1) with $v(x,0) = v_0(x)$ does not quench in finite time.*

Proof. We prove Lemma 3.4 by adapting similar methods in [1]. Set

$$g(u) = \frac{1}{(1-u)^2}, \quad h(u) = \int_0^u \frac{1}{g(s)}\mathrm{d}s, \qquad 0 \leq u < 1. \tag{3.43}$$

For any $\varepsilon \in (0,1)$, we also set

$$\tilde{g}(u,t) = \frac{1}{(1-u)^2} - \varepsilon^{-2\lambda_1 t}, \quad \tilde{h}(u,t) = \int_0^u \frac{1}{\tilde{g}(s)}\mathrm{d}s, \qquad 0 \leq u < 1, \tag{3.44}$$

and define $\Phi_\varepsilon(u,t)$ as $\tilde{h}(\Phi_\varepsilon(u,t),t) = h(u)$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Let u be a solution which does not quench in finite time, and let $V_\varepsilon(x,t) = \Phi_\varepsilon(u,t) \le u$, then V_ε is a supersolution of

$$\begin{cases} V_t - \Delta V = \lambda^* \rho \left(\dfrac{1}{(1-V)^2} - \varepsilon e^{-2\lambda_1 t} \right) & \text{in } \Omega \times (0,T), \\ V(x,0) = \Phi_\varepsilon(u_0,0) & \text{in } \Omega, \\ V(x,t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.45}$$

The comparison principle gives $V \le V_\varepsilon$, and V does not quench in finite time. Hence V is a global solution. Take now v_0 as in the statement of the lemma. Similar to (3.4), we then have there exists $\tau > 0$, such that

$$u(x,\tau) - v(x,\tau) \ge \int_\Omega G(x,y,\tau)(u_0(y) - v_0(y))\mathrm{d}y \ge c_0 \delta(x). \tag{3.46}$$

Taking $u(x,\tau), v(x,\tau)$ instead of u_0, v_0, similar to (3.9) we get there exists $\varepsilon_1 > 0$, such that for all $\varepsilon \in (0,\varepsilon_1)$,

$$v_0 \le u_0 - c_0\delta(x) < \Phi_\varepsilon(u_0,0) - \frac{1}{2}c_0\delta(x) = V(x,0) - \frac{1}{2}c_0\delta(x). \tag{3.47}$$

Considering the first eigenfunction φ_1 of $-\Delta$ in $H_0^1(\Omega)$, and a smooth solution of the following equation

$$\begin{cases} -\Delta\chi = \lambda^*\rho & \text{in } \Omega, \\ \chi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.48}$$

Standard regularity theory for elliptic problems and Hopf lemma give $\varphi_1 \ge c\chi$, for some $c > 0$. Setting $g(x,t) = (\frac{2\varphi_1}{c} - \chi)e^{-2\lambda_1 t} \ge 0$, it is easy to observe that g is a subsolution of

$$\begin{cases} Z_t - \Delta Z = -\lambda^*\rho e^{-2\lambda_1 t} & \text{in } \Omega \times (0,T), \\ Z(x,0) = \dfrac{2\varphi_1}{c} & \text{in } \Omega, \\ Z(x,t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.49}$$

Then one can get $Z \ge g \ge 0$ by the maximum principle.

Set now $z(x,t) = V(x,t) - \varepsilon Z(x,t)$, for ε small enough, such that

$$\varepsilon Z(x,0) = \frac{2\varepsilon\varphi_1}{c} \le \frac{c_0}{2}\delta(x)$$

by the regularity of φ_1. Some calculations give

$$\begin{cases} z_t - \Delta z = \dfrac{\lambda^*\rho}{(1-V)^2} \ge \dfrac{\lambda^*\rho}{(1-z)^2} & \text{in } \Omega \times (0,T), \\ z(x,0) = V(x,0) - \varepsilon Z(x,0) \ge V(x,0) - \dfrac{c_0}{2}\delta(x) \ge v_0 & \text{in } \Omega, \\ z(x,t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.50}$$

Applying the comparison principle, we obtain $v \le z$, and v does not quench in finite time. □

Lemma 3.5. *Let $1 \leq N \leq 7$, $\lambda = \lambda^*$, and w^* be the extremal solution of* (1.3)*. Assume ρ is a continuous positive function in Ω, and u_0 satisfies* (1.2)*, $\|u_0\|_\infty < 1$. Suppose that $u_0 \geq w^*$ and $u_0 \not\equiv w^*$. Then the solution u of* (1.1) *must quench in finite time.*

Proof. It is known that $\|w^*\|_\infty < 1$, since $1 \leq N \leq 7$. We assume for contradiction that u does not quench in finite time. The same calculations as in (3.4) indicate there exists $t_0 > 0$, such that

$$u(x,t_0) - w^*(x) \geq \int_\Omega G(x,y,t_0)(u_0(y) - w^*(y))\mathrm{d}y \geq c_0\delta(x). \tag{3.51}$$

Taking $u(x,t_0)$ instead of u_0, then we get $u_0 \geq w^* + c_0\delta(x)$. Consider

$$\tilde{u}(x,t) = u(x,t) - w^*(x),$$

then $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_t - \Delta\tilde{u} = \lambda^*\rho\dfrac{\frac{1}{(1-u)^2} - \frac{1}{(1-w^*)^2}}{u - w^*}\tilde{u} \geq \dfrac{2\lambda^*\rho}{(1-w^*)^3}\tilde{u} & \text{in } \Omega\times(0,T), \\ \tilde{u}(x,0) = u_0 - w^* \geq c_0\delta(x) & \text{in } \Omega, \\ \tilde{u}(x,t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.52}$$

On the other hand, we denote by ψ the first eigenfunction of $(-\Delta - \dfrac{2\lambda^*\rho}{(1-w^*)^3})$ in $H_0^1(\Omega)$, the corresponding eigenvalue being 0. Choosing ψ such that $\|\psi\|_\infty = 1$, then there exists $C > 0$ such that $\psi \leq C\delta(x)$. We now conclude that $\tilde{u} \geq C_1\psi$, for some C_1, by the maximum principle.

We introduce the function

$$F(t) = \int_\Omega u\psi\mathrm{d}x. \tag{3.53}$$

Then we have

$$\begin{aligned} F'(t) &= \int_\Omega u_t\psi\mathrm{d}x = \int_\Omega \psi\left(\Delta u + \frac{\lambda^*\rho}{(1-u)^2}\right)\mathrm{d}x \\ &= \int_\Omega \left(u\Delta\psi + \frac{\lambda^*\rho}{(1-u)^2}\psi\right)\mathrm{d}x \\ &= \lambda^*\int_\Omega \rho\psi\left(\frac{1}{(1-u)^2} - \frac{2}{(1-w^*)^3}u\right)\mathrm{d}x. \end{aligned} \tag{3.54}$$

Since (1.3) gives $\int_\Omega \frac{\rho\psi}{(1-w^*)^2}\mathrm{d}x = \int_\Omega \frac{2\rho w^*\psi}{(1-w^*)^3}\mathrm{d}x$, it follows that

$$\begin{aligned} F'(t) &= \lambda^*\int_\Omega \rho\psi\left(\frac{1}{(1-u)^2} - \frac{2}{(1-w^*)^3}\tilde{u} - \frac{2}{(1-w^*)^3}w^*\right)\mathrm{d}x \\ &= \lambda^*\int_\Omega \rho\psi\left(\frac{1}{(1-u)^2} - \frac{2}{(1-w^*)^3}\tilde{u} - \frac{1}{(1-w^*)^2}\right)\mathrm{d}x \\ &= \lambda^*\int_\Omega \rho\psi\int_{w^*}^u\left(\frac{2}{(1-s)^3} - \frac{2}{(1-w^*)^3}\right)\mathrm{d}s\mathrm{d}x \\ &= \lambda^*\int_\Omega \rho\psi\int_{w^*}^u\left(\int_{w^*}^s \frac{6}{(1-\sigma)^4}\mathrm{d}\sigma\right)\mathrm{d}s\mathrm{d}x \geq 0. \end{aligned} \tag{3.55}$$

It is clear that there exists two subsets $\Omega_1 \subset \Omega_2 \subset \Omega$, such that $|\Omega_1|, |\Omega_2| \neq 0$, $1 > w_{\lambda^*}|_{\Omega_2} \geq K$, $\psi|_{\Omega_2} \geq C_2$, and $w^*|_{\Omega_1} \in [K, \ \min\{K + \frac{C_1C_2}{2}, \frac{K+1}{2}\}]$. Then we derive from $\tilde{u} \geq C_1\psi$ that

$$1 \geq u|_{\Omega_1} = (\tilde{u} + w^*)|_{\Omega_1} \geq (C_1\psi + w^*)|_{\Omega_1} \geq C_1C_2 + K, \tag{3.56}$$

and

$$\begin{aligned} F'(t) &\geq 6\lambda^*\rho_0 C_2 \int_{\Omega_1} \int_{w^*}^{u} \left(\int_{w^*}^{s} \mathrm{d}\sigma \right) \mathrm{d}s\mathrm{d}x \\ &\geq 6\lambda^*\rho_0 C_2 \int_{\Omega_1} \int_{\min\{K+\frac{C_1C_2}{2}, \frac{K+1}{2}\}}^{C_1C_2+K} \left(\int_{\min\{K+\frac{C_1C_2}{2}, \frac{K+1}{2}\}}^{s} \mathrm{d}\sigma \right) \mathrm{d}s\mathrm{d}x \\ &\geq C > 0. \end{aligned} \tag{3.57}$$

Therefore

$$F(t) \geq F(0) + Ct \to +\infty,$$

as $t \to +\infty$, which contradicts to Lemma 3.2, so we are done. □

Now we shall establish the following theorem.

Theorem 3.6. *Suppose $\Omega = B_R(0)$, $1 \leq N \leq 7$, $\lambda = \lambda^*$ and ρ is a continuous positive function in Ω. Let $u_0(x) = \mu\eta(\frac{x}{R})$, where η is as in (3.21). Then there exits $\mu^*, \mu_* \in (0, 1)$ such that*

- (i) *For $0 \leq \mu < \mu_*$, the solution u of (1.1) is globally bounded and converges as $t \to +\infty$ to its unique minimal steady-state w^*.*
- (ii) *For $\mu = \mu^*$, there exists a global weak solution u^* of (1.1).*
- (iii) *For $\mu_* \leq \mu \leq \mu^*$, the solution u of (1.1) is global and*

$$\lim_{t \to +\infty} \int_{B_R(0)} |u^*(x,t) - w^*(x)|\delta(x)\mathrm{d}x = 0.$$

- (iv) *For $1 > \mu > \mu^*$, the solution of (1.1) must quench in finite time.*

Proof. Denote $\eta(\frac{x}{R})$ by $\phi(x)$, and denote the solution of (1.1) with initial value $\mu\phi$ by u_μ for simplicity. It is obvious that $\|\phi\|_\infty = 1$ and $\mu < 1$. We define μ^*, μ_* as in the proof of Theorem 3.3. Of course $\mu_* \leq \mu^*$.

Note that the conclusion of the case where $\mu > \mu^*$ can be derived from the proof of Theorem 3.3 similarly, then we need only to show the statement (i), (ii), (iii).

For $\mu < \mu^*$, we deduce from Lemma 3.4 that the solution u_μ is global. Then by a similar method in the proof of Theorem 3.3, the weak solution u^* corresponding to μ^* is obtained as the limit of the solution u_μ for $\mu \to \mu^*$.

It is clear that the extremal solution w^* of (1.3) is regular, since $1 \leq N \leq 7$. Then a similar proof as in Theorem 3.3, yields u_μ is globally bounded and converges to w^* as $t \to +\infty$, in the case where $\mu \leq \mu_*$. We now show that

$\lim_{t\to+\infty}\int_{B_R(0)} |u^*(x,t)-w^*(x)|\delta(x)\mathrm{d}x = 0$. This result will follow for $\mu_* \le \mu < \mu^*$. Let v be the unique classical solution of

$$\begin{cases} v_t - \Delta v = \dfrac{\lambda^*\rho}{(1-v)^2} & \text{in } B_R(0)\times(0,T),\\ v(x,0)=0 & \text{in } B_R(0),\\ v(x,t)=0 & \text{on } \partial B_R(0). \end{cases} \tag{3.58}$$

Standard comparison principle gives $w^* \ge v$ and $u^* \ge v$. Using Theorem 2.2 in [10], we get

$$\lim_{t\to+\infty} \|v(\cdot,t)-w^*\|_\infty = 0.$$

Consequently, there exists $s_n > 0$ such that for all $t > s_n$,

$$0 \le w^*(x) - v(x,t) \le \frac{1}{n}. \tag{3.59}$$

Suppose for contradiction that there exists $C>0$ and $t_n > s_n$ such that

$$\int_{B_R(0)} |u^*(x,t_n)-w^*(x)|\delta(x)\mathrm{d}x > C. \tag{3.60}$$

Since $(u^*-w^*)^- = (w^*-u^*)^+ \le w^* - v$,

$$\|(u^*(x,t_n)-w^*(x))^-\|_\infty \le \|w^*(x)-v(x,t_n)\|_\infty \le \frac{1}{n}. \tag{3.61}$$

Let $U(x,t) = u^*(x,t) - w^*(x)$. Then U satisfies

$$\begin{cases} U_t - \Delta U = \lambda^*\rho\left(\dfrac{1}{(1-u*)^2} - \dfrac{1}{(1-w^*)^2}\right) \ge \dfrac{2\lambda^*\rho}{(1-w^*)^3}U & \text{in } B_R(0)\times(0,T),\\ U(x,0) = \mu^*\phi - w^* & \text{in } B_R(0),\\ U(x,t)=0 & \text{on } \partial B_R(0). \end{cases} \tag{3.62}$$

By (3.60) and (3.61),

$$\int_{B_R(0)} U^-(x,t_n)\delta(x)\mathrm{d}x < \frac{C}{2}, \tag{3.63}$$

$$\int_{B_R(0)} U^+(x,t_n)\delta(x)\mathrm{d}x \ge \frac{C}{2}. \tag{3.64}$$

This implies $U^+ \ge 0$ and $U^+ \not\equiv 0$. Letting $c_0 := \|\dfrac{2}{(1-w^*)^3}\|_{L^\infty(B_R(0)\times(0,+\infty))}$, we have

$$U_t - \Delta U \ge -\lambda^* c_0 \rho U^-, \quad \text{in } B_R(0)\times(t_n,T) \tag{3.65}$$

with $U(x,t_n) \ge U^+(x,t_n) - \dfrac{1}{n}$. Fix $\tau > 0$, we now claim that there exists $n > 0$ such that $U(x,\tau+t_n) > 0$. Indeed, consider z_1, z_2 the solutions of

$$\begin{cases} (z_1)_t - \Delta z_1 = \lambda^* c_0 \rho z_1 & \text{in } B_R(0)\times(0,T),\\ z_1(x,0) = \dfrac{1}{n} & \text{in } B_R(0),\\ z_1(x,t)=0 & \text{on } \partial B_R(0), \end{cases} \tag{3.66}$$

and

$$\begin{cases} (z_2)_t - \Delta z_2 = 0 & \text{in } B_R(0) \times (0,T), \\ z_2(x,0) = U^+(x,t_n) & \text{in } B_R(0), \\ z_2(x,t) = 0 & \text{on } \partial B_R(0). \end{cases} \tag{3.67}$$

Clearly, there exists $c_1, c_2 > 0$ such that $z_1(x,\tau) \le \frac{c_1}{n}\delta(x)$, $z_2(x,\tau) \ge c_2\delta(x)$. Then we see for n large enough, there holds $z_1(x,\tau) \le z_2(x,\tau)$. Applying the comparison principle, we obtain $U(x,\tau+t_n) > (z_2 - z_1)(x,\tau) > 0$, for large n. That means $u^*(x,\tau+t_n) > w^*(x)$, and there exists a function $w^* < p(x) \le u^*(x,\tau+t_n)$. Applying Lemma 3.4, we get the solution of

$$\begin{cases} \tilde{u}_t - \Delta\tilde{u} = \dfrac{\lambda^*\rho}{(1-\tilde{u})^2} & \text{in } B_R(0) \times (0,T), \\ \tilde{u}(x,0) = p(x) & \text{in } B_R(0), \\ \tilde{u}(x,t) = 0 & \text{on } \partial B_R(0), \end{cases} \tag{3.68}$$

must quench in finite time. However, it is easy to see $\tilde{u} \le u^*$, which means $\tilde{u}$ is a global solution. So we get a contradiction. Therefore $\lim_{t\to+\infty}\int_{B_R(0)} |u^*(x,t) - w^*(x)|\delta(x)\mathrm{d}x = 0$. This completes the proof of this theorem. □

Acknowledgements

The author thanks the referee for his/her valuable suggestions that improve this paper. Thank also Professor Lou B. for his useful seminar on the blow-up theory.

References

[1] Brezis, H., Cazenave, T., Martel, Y., Ramiandrisoa, A.: Blow-up for $u_t - \Delta u = g(u)$ revisited. Adv. Differ. Equ. **1**, 73–90 (1996)

[2] Driscoll, T.A., Pelesko, J.A.: The effect of the small-aspect-ratio approximation on canonical electrostatic MEMS models. J. Eng. Math. **53**, 239–252 (2005)

[3] Esposito, P.: Compactness of a nonlinear eigenvalue problem with a singular nonlinearity. Commun. Contemp. Math. **10**(2), 17–45 (2008)

[4] Esposito, P., Ghoussoub, N.: Uniqueness of solutions for an elliptic equation modeling MEMS. Methods Appl. Anal. **15**(3), 341–353 (2008)

[5] Esposito, P., Ghoussoub, N., Guo, Y.J.: Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity. CommUN. Pure Appl. Math. **60**(12), 1731–1768 (2007)

[6] Esposito, P., Ghoussoub, N., Guo, Y.J.: Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS. Courant Lecture Notes in Mathematics, vol. 20. Courant Institute of Mathematical Sciences/American Mathematical Society, New York/Providence (2010)

[7] Flores, G., Mercado, G.A., Pelesko, J.A.: Dynamics and touchdown in electrostatic MEMS. In: Proceedings of ASME DETC03, pp. 1–8, IEEE Computer Society (2003)

[8] Ghoussoub, N., Guo, Y.J.: Estimates for the quenching time of a parabolic equation modeling electrostatic MEMS. Methods Appl. Anal. **15**(3), 361–376 (2008)

[9] Ghoussoub, N., Guo, Y.J.: On the partial differential equations of electrostatic MEMS devices: stationary case. SIAM J. Math. Anal. **38**, 1423–1449 (2007)

[10] Ghoussoub, N., Guo, Y.J.: On the partial differential equations of electrostatic MEMS devices II: dynamic case. Nonlinear Differ. Equ. Appl. **15**, 115–145 (2008)

[11] Guo, Y.J.: On the partial differential equations of electrostatic MEMS devices III: refined touchdown behavior. J. Differ. Equ. **244**, 2277–2309 (2008)

[12] Guo, Y.J.: Dynamical solutions of singular wave equations modeling electrostatic MEMS. SIAM J. Appl. Dyn. Syst. **9**(4), 1135–1163 (2010)

[13] Guo, Y.J., Pan, Z.G., Ward, M.J.: Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties. SIAM J. Appl. Math. **66**(1), 309–338 (2005)

[14] Hui, K.M.: Global and touchdown behaviour of the generalized MEMS device equation. Adv. Math. Sci. Appl. **19**(1), 347–370 (2009)

[15] Kavallaris, N.I., Miyasita, T., Suzuki, T.: Touchdown and related problems in electrostatic MEMS device equation. Nonlinear Differ. Equ. Appl. **15**(3), 363–385 (2008)

[16] Lega, J., Lindsay, A.E.: Multiple quenching solutions of a fourth order parabolic PDE with a singular nonlinearity modeling a MEMS capacitor. SIAM J. Appl. Math. **72**(3), 935–958 (2012)

[17] Lin, F.H., Yang, Y.S.: Nonlinear non-local elliptic equation modelling electrostatic actuation. Proc. R. Soc. Lond. Ser. A **463**, 1323–1337 (2007)

[18] Lindsay, A.E., Ward, M.J.: Asymptotics of nonlinear eigenvalue problems modeling a MEMS capacitor: Part I: Fold point asymptotics. Methods Appl. Anal. **15**, 297–326 (2008)

[19] Liu, Z., Wang, X.: On a parabolic equation in MEMS with fringing field. Archiv der Mathematik **98**(4), 373–381 (2012)

[20] Luo, X.: On the quenching behavior of the MEMS with fringing field. arXiv:1402.0066 (2014, preprint)

[21] Ni, W.-M.: The Mathematics of Diffusion. Society for Industrial and Applied Mathematics, Philadelphia (2011)

[22] Pelesko, J.A., Berstein, D.H.: Modeling MEMS and NEMS. Chapman & Hall/CRC, Boca Raton (2002)

[23] Pelesko, J.A.: Mathematical modeling of electrostatic MEMS with tailored dielectric properties. SIAM J. Appl. Math. **62**(3), 888–908 (2002)

[24] Sattinger, D.H.: Monotone methods in nonlinear elliptic and parabolic boundary value problems. Indiana Univ. Math. J. **21**, 979–1000 (1972)

[25] Wang, Q.: Estimates for the quenching time of a MEMS equation with fringing field. J. Math. Anal. Appl. **405**, 135–147 (2013)

[26] Wei, J.C., Ye, D.: On MEMS equation with fringing field. Proc. Am. Math. Soc. **138**(5), 1693–1699 (2010)

[27] Ye, D., Zhou, F.: On a general family of nonautonomous elliptic and parabolic equations. Calc. Var. **37**, 259–274 (2010)

Qi Wang
Department of Mathematics
Tongji University
Shanghai 200092
China
e-mail: wangqi0168@gmail.com

Received: 1 July 2014.
Accepted: 3 November 2014.