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Hölder regularity for the general parabolic $p(x, t)$-Laplacian equations

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Abstract. In this paper we obtain the local Hölder regularity of the gradient of weak solutions for the general parabolic $p(x, t)$-Laplacian equations

$$u_t - \text{div } \mathcal{A}(\nabla u, x, t) = \text{div}\left(|\mathbf{f}|^{p(x,t)-2}\mathbf{f}\right),$$

provided $p(x, t)$, $\mathcal{A}$ and $\mathbf{f}$ satisfy some proper conditions. More precisely, we shall prove that

$$\nabla u \in C^{0;\alpha,\alpha/2}_{loc}(\Omega_T) \text{ for some } \alpha \in (0, 1).$$

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1. Introduction

In this paper we mainly consider the following general nonlinear parabolic problem

$$u_t - \text{div } \mathcal{A}(\nabla u, x, t) = \text{div}\left(|\mathbf{f}|^{p(x,t)-2}\mathbf{f}\right) \quad \text{in } \Omega_T =: \Omega \times (0, T], \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ for $n \geq 2$ and $\mathbf{f} = (f^1, \ldots, f^n)$ is a given vector field satisfying

$$1 \leq \frac{2n}{n+2} < \gamma_1 = \inf_{\Omega_T} p(x, t) \leq \sup_{\Omega_T} p(x, t) = \gamma_2 < \infty, \tag{1.2}$$

$$p(x, t) \in C^{0;\alpha_1,\alpha_1/2}_{loc}(\Omega_T) \quad \text{and} \quad f^i(x, t) \in C^{0;\alpha_2,\alpha_2/2}_{loc}(\Omega_T) \text{ for } 1 \leq i \leq n, \tag{1.3}$$

where the constants $\alpha_1, \alpha_2 \in (0, 1)$. We remark that the lower bound $2n/(n+2)$ on the exponent p is standard and unavoidable in the theory of the parabolic p-Laplacian operator. Moreover, the structural conditions on the function $\mathcal{A}(\xi, x, t)$ are given as follows

$$\begin{aligned} &[\mathcal{A}(\xi,x,t)-\mathcal{A}(\eta,x,t)]\cdot(\xi-\eta) \\ &\quad\geq \begin{cases} C_1|\xi-\eta|^{p(x,t)}, & p(x,t)\geq 2; \\ C_1\left(|\xi|+|\eta|\right)^{p(x,t)-2}|\xi-\eta|^2, & 1<p(x,t)<2, \end{cases} \end{aligned} \tag{1.4}$$

$$|\mathcal{A}(\xi,x,t)|\leq C_2\left(1+|\xi|^{p(x,t)-1}\right), \tag{1.5}$$

$$\mathcal{A}(\xi,x,t)\cdot\xi\geq C_3|\xi|^{p(x,t)}-C_4, \tag{1.6}$$

$$\begin{aligned} &|\mathcal{A}(\xi,x,t)-\mathcal{A}(\xi,y,s)| \\ &\quad\leq C_5\left(|x-y|+|t-s|^{\frac{1}{2}}\right)^{\alpha_3}\left|\log\left(1+|\xi|^2\right)\right|\left(1+|\xi|^{p(x,t)-1}\right) \end{aligned} \tag{1.7}$$

for all $\xi,\eta\in\mathbb{R}^n$, $(x,t),(y,s)\in\Omega_T$ and some positive constants $\alpha_3\in(0,1), C_j>0,\ j=1,2,3,4,5$. Especially when $\mathcal{A}(\xi,x,t)=(A\xi\cdot\xi)^{\frac{p(x,t)-2}{2}}A\xi$, (1.1) is reduced to the parabolic equations of $p(x,t)$-Laplacian type

$$u_t-\operatorname{div}\left((A\nabla u\cdot\nabla u)^{\frac{p(x,t)-2}{2}}A\nabla u\right)=\operatorname{div}\left(|\mathbf{f}|^{p(x,t)-2}\mathbf{f}\right)\quad\text{in }\Omega_T.$$

Many authors [7,8,15,21,25–27] have studied the regularity estimates for weak solutions of the quasilinear elliptic equations of p-Laplacian type. Moreover, there have been many investigations [2,9–11,22,28,29,31,32] on regularity estimates for the nonlinear elliptic equations of $p(x)$-Laplacian type. Different from the elliptic case, the quasilinear parabolic equation of p-Laplacian type is not homogeneous, which is one of the most difficulties. Many authors [3,6,20,23,24] have studied the regularity estimates of the gradient for the nonlinear parabolic equations of p-Laplacian type. Recently, Bögelein and Duzaar [4] have obtained a reverse Hölder inequality of the gradient for (1.1), which implies that

$$|\nabla u|^{p(x,t)(1+\epsilon)}\in L^1_{loc}(\Omega_T)\text{ for some }\ \epsilon>0.$$

Moreover, Xu and Chen [30] proved the interior Hölder continuity of weak solutions for

$$u_t-\operatorname{div}\left(|\nabla u|^{p(x,t)-2}\nabla u\right)=0$$

with $p(x,t)>2$. Furthermore, Bögelein and Duzaar [5] established the local Hölder continuity of the gradient of weak solutions for the parabolic $p(x,t)$-Laplacian system

$$u_t-\operatorname{div}\left(|\nabla u|^{p(x,t)-2}A\nabla u\right)=0.$$

In this paper we shall obtain the local Hölder regularity of the gradient for weak solutions of (1.1) with the assumptions (1.2)–(1.7). As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.1. Assume that $\mathbf{f}\in L^{p(x,t)}_{loc}(\Omega_T)$ (see Sect. 2.2). A function $u\in L^{p(x,t)}_{loc}(\Omega_T)\cap L^\infty_{loc}(0,T;L^2_{loc}(\Omega))$ with $|\nabla u|\in L^{p(x,t)}_{loc}(\Omega_T)$ is a local weak solution of (1.1) in Ω_T if for any compact subset $\mathcal{K}$ of Ω and for any subinterval $[t_1,t_2]$ of $(0,T]$ we have

$$\int_{\mathcal{K}} u\varphi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathcal{K}} \Big\{ -u\varphi_t + \mathcal{A}(\nabla u, x, t)\cdot\nabla\varphi \Big\} dxdt$$
$$= -\int_{t_1}^{t_2}\int_{\mathcal{K}} |\mathbf{f}|^{p(x,t)-2}\mathbf{f}\cdot\nabla\varphi \, dxdt$$

for any $\varphi \in C_0^{\infty}(\mathcal{K}\times[t_1,t_2])$.

Now let us state the main result of this work.

Theorem 1.2. *If u is a local weak solution of problem (1.1) in Ω_T under the assumptions (1.2)–(1.7), then we have*

$$\nabla u \in C_{loc}^{0;\alpha,\alpha/2}(\Omega_T)$$

for some $\alpha \in (0,1)$.

2. Proof of the main result

2.1. Geometric notation

1. A typical point in $\mathbb{R}^n \times \mathbb{R}$ is $z = (x,t)$.
2. $B_r = \{y \in \mathbb{R}^n : |y| < r\}$ is an open ball in $\mathbb{R}^n$ with center 0 and radius $r > 0$, and $B_r(x) = B_r + x$.
3. $Q_r = B_r \times (-r^2, r^2]$ is a centered parabolic cylinder, $Q_r(z) = Q_r + z$, and $\partial_p Q_r = \partial B_r \times [-r^2, r^2] \cup B_r \times \{-r^2\}$ is the parabolic boundary of Q_r.

2.2. $L^{p(x,t)}(\Omega_T)$ and $W^{1,p(x,t)}(\Omega_T)$ spaces

For the reader's convenience, we will give some definitions on the $L^{p(x,t)}(\Omega_T)$ and $W^{1,p(x,t)}(\Omega_T)$ spaces. We denote by $L^{p(x,t)}(\Omega_T)$ the variable exponent Lebesgue spaces

$$L^{p(x,t)}(\Omega_T) = \left\{ f : \Omega_T \to \mathbb{R} \mid f \text{ is measurable and} \int_{\Omega_T} |f|^{p(x,t)} dxdt < \infty \right\}$$

with the Luxemburg type norm

$$\|f\|_{L^{p(x,t)}(\Omega_T)} = \inf \left\{ \lambda > 0 \ : \ \int_{\Omega_T} \left|\frac{f}{\lambda}\right|^{p(x,t)} dxdt \le 1 \right\}.$$

Furthermore, we define

$$W^{1,p(x,t)}(\Omega_T) = \Big\{ u \in L^{p(x,t)}(\Omega_T) \ : \ |\nabla u| \in L^{p(x,t)}(\Omega_T) \Big\}$$

with the norm

$$\|u\|_{W^{1,p(x,t)}(\Omega_T)} = \|u\|_{L^{p(x,t)}(\Omega_T)} + \|\nabla u\|_{L^{p(x,t)}(\Omega_T)}.$$

By $W_0^{1,p(x,t)}(\Omega_T)$ we denote the closure of $C_0^{\infty}(\Omega_T)$ in $W^{1,p(x,t)}(\Omega_T)$. Actually, the $L^{p(x,t)}(\Omega_T)$, $W^{1,p(x,t)}(\Omega_T)$ and $W_0^{1,p(x,t)}(\Omega_T)$ spaces are Banach spaces. There have been many investigations (see for example [12–14,16,17,19]) on properties of such variable exponent Sobolev spaces.

2.3. Preliminary lemmas

In this subsection we shall give some important lemmas, which are very important to obtain the main result, Theorem 1.2. We first recall the following reverse Hölder inequality.

Lemma 2.1. (see [4], Theorem 2.2) *If u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.7), then there exist positive constants $\sigma_0, R_0 < 1, C$, depending on $n, \gamma_1, \gamma_2, \alpha_i (1 \le i \le 3), C_j (1 \le j \le 5)$, such that*

$$\fint_{Q_{R/2}} |\nabla u|^{p(x,t)(1+\sigma)}\, dxdt \le C \left(\fint_{Q_R} (|\nabla u| + |\mathbf{f}| + 1)^{p(x,t)}\, dxdt \right)^{1+d\sigma} + C \left(\fint_{Q_R} (1 + |\mathbf{f}|)^{p(x,t)(1+\sigma)}\, dxdt \right)$$

holds for any $R \le R_0$ and $\sigma \le \sigma_0$, where $Q_{R/2} \equiv Q_{R/2}(z_0)$ and $Q_R \equiv Q_R(z_0) \subset \Omega_T$ and

$$d = \max \left\{ \frac{2\gamma_1}{\gamma_1(n+2) - 2n}, \frac{\gamma_2}{2} \right\}.$$

For simplicity, from now on we shall denote

$$Q_r =: Q_r(z_0) \tag{2.1}$$

for any $r > 0$ with some $z_0 = (x_0, t_0) \in \Omega_T$.

We shall initially take R_1 small enough such that $0 < R_1 \le R_0$ and

$$|p(x,t) - p(y,s)| \le C_1 \left((2R_1)^{\alpha_1} + (2R_1^2)^{\alpha_1/2} \right) \le 2C_1(2R_1)^{\alpha_1} \le \frac{\sigma_0\gamma_1}{\sigma_0 + 2}$$

for any $(x,t), (y,s) \in Q_{R_1}$. Moreover, we define

$$p_1 =: p(x_m, t_m) = \min_{Q_{2R}} p(x,t) \quad \text{and} \quad p_2 =: p(x_M, t_M) = \max_{Q_{2R}} p(x,t) \tag{2.2}$$

for any $R \le R_1/2$ with $Q_{R_0} \subset \Omega_T$. Then we conclude that $p_1 \ge \gamma_1 > 1$,

$$p_2 \le p_2 - p_1 + p_1 \le p_1 + \frac{\sigma_0\gamma_1}{\sigma_0 + 2} \tag{2.3}$$

and

$$\begin{aligned} p_2 \le p_2(1 + \sigma_0/2) &\le \left(p_1 + \frac{\sigma_0\gamma_1}{\sigma_0 + 2} \right)(1 + \sigma_0/2) \\ &\le p_1(1 + \sigma_0) \le p(x,t)(1 + \sigma_0). \end{aligned} \tag{2.4}$$

Lemma 2.2. *If u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.7), then there exists a positive constant C, depending on $n, \gamma_1, \gamma_2, \alpha_i (1 \le i \le 3), C_j (1 \le j \le 5)$, the Hölder norms of $\{f^i\}$, $\|\nabla u\|_{L^{p(x,t)}_{loc}(\Omega_T)}$, such that*

$$\fint_{Q_{R/2}} |\nabla u|^{p(x,t)(1+\sigma_0)}\, dxdt \le CR^{-(n+2)d\sigma_0} \fint_{Q_R} (|\nabla u|^{p_2} + 1)\, dxdt \tag{2.5}$$

for any $R \le R_1/2$ with $Q_{R_0} \subset \Omega_T$.

Proof. From Lemma 2.1 and the fact that $\{f^i\} \in C^{0;\alpha_2,\alpha_2/2}$ we have

$$\fint_{Q_{R/2}} |\nabla u|^{p(x,t)(1+\sigma_0)}\,dxdt \le C\left\{\left(\fint_{Q_R} |\nabla u|^{p(x,t)}\ dxdt\right)^{1+d\sigma_0} + 1\right\}.$$

Since $|\nabla u| \in L^{p(x,t)}_{loc}(\Omega_T)$, for any $R \le R_1/2$ we deduce that

$$\begin{aligned}
&\fint_{Q_{R/2}} |\nabla u|^{p(x,t)(1+\sigma_0)}\,dxdt \\
&\quad\le C\left\{R^{-(n+2)d\sigma_0}\left(\int_{Q_R} |\nabla u|^{p(x,t)}\ dxdt\right)^{d\sigma_0}\cdot\left(\fint_{Q_R} |\nabla u|^{p(x,t)}\ dxdt\right) + 1\right\}\\
&\quad\le C\left(R^{-(n+2)d\sigma_0}\fint_{Q_R} |\nabla u|^{p(x,t)}\ dxdt + 1\right)\\
&\quad\le CR^{-(n+2)d\sigma_0}\fint_{Q_R}\left(|\nabla u|^{p(x,t)} + 1\right)\ dxdt\\
&\quad\le CR^{-(n+2)d\sigma_0}\fint_{Q_R}\left(|\nabla u|^{p_2} + 1\right)\ dxdt.
\end{aligned}$$

Thus, we finish the proof. □

Let us consider the following reference equation

$$\begin{cases} v_t - \operatorname{div}\mathcal{A}\left(\nabla v, x_M, t_M\right) = 0 & \text{in } Q_R,\\ v = u & \text{on } \partial_p Q_R\end{cases} \tag{2.6}$$

for any $R \le R_1/2$ with $Q_{R_0} \subset \Omega_T$, where (x_M, t_M) is defined in (2.2).

Lemma 2.3. *Assume that u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.7). If $v \in W^{1,p_2}(Q_R)$ is the weak solution of (2.6) in Q_R, then we have*

$$\fint_{Q_R} |\nabla v|^{p_2}\,dxdt \le C\left(\fint_{Q_R} |\nabla u|^{p_2}\,dxdt + 1\right).$$

Proof. Noting that both u and v are the weak solutions, we may as well select the test function $\varphi = v - u$, which is possible modulo Steklov averages since $v = u$ on $\partial_p Q_R$, and then a direct calculation shows the resulting expression as

$$I_1 + I_2 = I_3 + I_4 + I_5 + I_6 + I_7,$$

where

$$I_1 = \int_{B_R(x_0)} \frac{|v(x,t_0+R^2) - u(x,t_0+R^2)|^2}{2}dx,$$

$$I_2 = \int_{Q_R} \mathcal{A}(\nabla v, x_M, t_M)\cdot\nabla v\,dxdt, \qquad I_3 = \int_{Q_R} \mathcal{A}(\nabla v, x_M, t_M)\cdot\nabla u\,dxdt,$$

$$I_4 = -\int_{Q_R} \mathcal{A}(\nabla u, x, t)\cdot\nabla u\,dxdt, \qquad I_5 = \int_{Q_R} \mathcal{A}(\nabla u, x, t)\cdot\nabla v\,dxdt,$$

$$I_6 = \int_{Q_R} |\mathbf{f}|^{p_2-2}\mathbf{f}\cdot\nabla v\,dxdt, \qquad I_7 = -\int_{Q_R} |\mathbf{f}|^{p_2-2}\mathbf{f}\cdot\nabla u\,dxdt.$$

Estimate of I_2. (1.6) implies that

$$I_2 \geq C\left(\int_{Q_R} |\nabla v|^{p_2}\ dxdt - |Q_R|\right).$$

Estimate of I_3. From (1.5) and Young's inequality with τ we have

$$\begin{aligned} I_3 &\leq C\int_{Q_R}\left(1+|\nabla v|^{p_2-1}\right)|\nabla u|\ dxdt \\ &\leq \tau\int_{Q_R}|\nabla v|^{p_2}\ dxdt + C(\tau)\int_{Q_R}\left(1+|\nabla u|^{p_2}\right)\ dxdt. \end{aligned}$$

Estimate of $I_4 - I_7$. Similarly to the estimate of I_3, we have

$$\begin{aligned} I_4 &\leq C\int_{Q_R}\left(|\nabla u|^{p_2}+1\right)\ dxdt, \\ I_5 &\leq \tau\int_{Q_R}|\nabla v|^{p_2}\ dxdt + C(\tau)\int_{Q_R}\left(1+|\nabla u|^{p_2}\right)\ dxdt, \\ I_6 &\leq \tau\int_{Q_R}|\nabla v|^{p_2}\ dxdt + C(\tau)\int_{Q_R}|\mathbf{f}|^{p_2}\, dxdt, \\ I_7 &\leq C\int_{Q_R}|\nabla u|^{p_2}\ dxdt + C\int_{Q_R}|\mathbf{f}|^{p_2}\, dxdt. \end{aligned}$$

Combining the estimates of I_i $(1 \leq i \leq 7)$ and selecting a small enough constant $\tau > 0$, we deduce that

$$\begin{aligned} ⨍_{Q_R}|\nabla v|^{p_2}\, dxdt &\leq C\left(⨍_{Q_R}|\nabla u|^{p_2}\, dxdt + ⨍_{Q_R}|\mathbf{f}|^{p_2}\, dxdt + 1\right) \\ &\leq C\left(⨍_{Q_R}|\nabla u|^{p_2}\, dxdt + 1\right), \end{aligned}$$

since $\{f^i\} \in C^{0;\alpha_2,\alpha_2/2}$, and then finish the proof. □

Lemma 2.4. *Assume that* v *is the weak solution of (2.6) in* Q_R. *Then there exist positive constants* $R_2, \beta_1 < 1$ *such that*

$$⨍_{Q_\rho}|\nabla v|^{p_2}\, dxdt \leq C\rho^{-\tau} \tag{2.7}$$

and

$$⨍_{Q_\rho}\left|\nabla v - (\nabla v)_{Q_\rho}\right|^{p_2} dxdt \leq C\left(\frac{\rho}{R}\right)^{\beta_1}\left(⨍_{Q_R}|\nabla v|^{p_2}\, dxdt + 1\right) \tag{2.8}$$

for any $\tau > 0$ *and any* $\rho \leq R \leq R_2 \leq R_1/2$ *with* $Q_{R_0} \subset \Omega_T$, *where* C *depends on* $n, \gamma_1, \gamma_2, \alpha_i (1 \leq i \leq 3), C_j (1 \leq j \leq 5)$ *and* $\|\nabla u\|_{L^{p(x,t)}_{loc}(\Omega_T)}$.

Proof. We define

$$\widetilde{v}(x,t) = \begin{cases} \dfrac{v(rx, \lambda^{2-p_2} r^2 t)}{\lambda r} & \text{for } p_2 \geq 2, \\ \dfrac{v(\lambda^{(p_2-2)/2} rx, r^2 t)}{\lambda^{p_2/2} r} & \text{for } 2n/(n+2) < p_2 < 2, \end{cases} \tag{2.9}$$

$$\mathcal{A}_\lambda(\xi(x,t), x_M, t_M) = \begin{cases} \dfrac{\mathcal{A}(\lambda \xi(rx, \lambda^{2-p_2} r^2 t), x_M, t_M)}{\lambda^{p_2-1}} & \text{for } p_2 \geq 2, \\ \dfrac{\mathcal{A}(\lambda \xi(\lambda^{(p_2-2)/2} rx, r^2 t), x_M, t_M)}{\lambda^{p_2-1}} & \text{for } 2n/(n+2) < p_2 < 2, \end{cases} \tag{2.10}$$

and

$$Q_r^{(\lambda)} = \begin{cases} B_r \times \left(-\lambda^{2-p_2} r^2, \lambda^{2-p_2} r^2\right] & \text{for } p_2 \geq 2, \\ B_{\lambda^{\frac{p_2-2}{2}} r} \times \left(-r^2, r^2\right] & \text{for } 2n/(n+2) < p_2 < 2 \end{cases}$$

for some constant $\lambda > 1$. Obviously, $Q_r^{(\lambda)} \subset Q_r$. If $v(x,t)$ satisfies (2.6) in $Q_r \supset Q_r^{(\lambda)}$, then $\widetilde{v}(x,t)$ is a local weak solution of

$$\widetilde{v}_t - \operatorname{div} \mathcal{A}_\lambda (\nabla \widetilde{v}, x_M, t_M) = 0 \quad \text{in } Q_1^{(\lambda)}. \tag{2.11}$$

Especially when $\lambda = 2$ and $r = R/2$, v_λ satisfies (2.11) in $Q_2^{(2)}$ since $v(x,t)$ satisfies (2.6) in $Q_R^{(2)} \subset Q_R$. Then from Lemma 2.1 and Lemma 2.3 we deduce that

$$⨍_{Q_R^{(2)}} |\nabla v|^{p_2} \, dxdt \leq C ⨍_{Q_R} |\nabla v|^{p_2} \, dxdt \leq C,$$

which implies that

$$⨍_{Q_2^{(2)}} |\nabla \widetilde{v}|^{p_2} \, dxdt \leq C,$$

where C is independent of R. Then similarly to Lemma 2 and Lemma 11 in [23] there exist positive numbers $\widehat{r}_0, \beta_1 < 1$, such that (2.7) and (2.8) with $\widetilde{v}$ replacing v are true for any $\tau > 0$ and $\rho < r \leq \widehat{r}_0 < 1$. Finally, we can finish the proof by changing variables. □

Furthermore, we can obtain the following important result.

Lemma 2.5. *If u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.7), then there exists a small positive constant β, depending on $n, \gamma_1, \gamma_2, \alpha_i (1 \leq i \leq 3), C_j (1 \leq j \leq 5)$, such that*

$$⨍_{Q_R} |\nabla u - \nabla v|^{p_2} \, dxdt \leq C R^\beta ⨍_{Q_{2R}} (|\nabla u|^{p_2} + 1) \, dxdt$$

for any $R \leq R_2$ with $Q_{R_0} \subset \Omega_T$, where v is the weak solution of (2.6) and C depends on $n, \gamma_1, \gamma_2, \alpha_i (1 \leq i \leq 3), C_j (1 \leq j \leq 5)$, the Hölder norms of $\{f^i\}$ and $p(x,t)$, $\|\nabla u\|_{L^{p(x,t)}_{loc}(\Omega_T)}$.

Proof. Without loss of generality we may as well select the test function $\varphi = u - v$. Since div $\left(|\mathbf{f}(z_0)|^{p_2-2}\mathbf{f}(z_0)\right) = 0$, from the definitions of weak solutions, after a direct calculation we show the resulting expression as

$$I_1 + I_2 + I_3 = I_4 + I_5,$$

where

$$\begin{aligned}
I_1 &= -\fint_{Q_R} \left(\mathcal{A}(\nabla u, x, t) - \mathcal{A}(\nabla u, x_M, t_M)\right) \cdot \nabla(u-v)\, dxdt,\\
I_2 &= -\fint_{Q_R} \left(|\mathbf{f}|^{p(x,t)-2}\,\mathbf{f} - |\mathbf{f}(z_0)|^{p(x,t)-2}\mathbf{f}(z_0)\right) \cdot \nabla(u-v)\, dxdt,\\
I_3 &= -\fint_{Q_R} \left(|\mathbf{f}(z_0)|^{p(x,t)-2}\mathbf{f}(z_0) - |\mathbf{f}(z_0)|^{p_2-2}\mathbf{f}(z_0)\right) \cdot \nabla(u-v)\, dxdt\\
I_4 &= \fint_{Q_R} \left(\mathcal{A}(\nabla u, x_M, t_M) - \mathcal{A}(\nabla v, x_M, t_M)\right) \cdot \nabla(u-v)\, dxdt\\
I_5 &= \frac{d}{dt}\left\{\fint_{Q_R} \frac{|u-v|^2}{2} dxdt\right\}\\
&= \frac{1}{|Q_R|}\int_{B_R(x_0)} \frac{|u(x,t_0+R^2) - v(x,t_0+R^2)|^2}{2} dx \geq 0.
\end{aligned}$$

Estimate of I_1. From (1.7) and Hölder's inequality we obtain

$$\begin{aligned}
I_1 &\leq C\fint_{Q_R} R^{\alpha_3}\left|\log\left(1+|\nabla u|^2\right)\right|\left(1+|\nabla u|^{p_2-1}\right)|\nabla(u-v)|\, dxdt\\
&\leq CR^{\alpha_3}\fint_{Q_R}\left(1+|\nabla u|^{p_2-1+\delta}\right)|\nabla(u-v)|\, dxdt\\
&\leq CR^{\alpha_3}\left(\fint_{Q_R}|\nabla(u-v)|^{p_2}\, dxdt\right)^{\frac{1}{p_2}}\left(\fint_{Q_R}(|\nabla u|+1)^{\frac{(p_2-1+\delta)p_2}{p_2-1}}\, dxdt\right)^{\frac{p_2}{p_2-1}},
\end{aligned}$$

for any $\delta > 0$. Selecting proper $\delta \in (0, (\gamma_1 - 1)\sigma_0/2)$, we deduce from (2.4) that

$$\begin{aligned}
\frac{p_2(p_2-1+\delta)}{p_2-1} &= p_2\left(1+\frac{\delta}{p_2-1}\right)\\
&\leq p_2\left(1+\frac{\delta}{\gamma_1-1}\right) \leq p_2\left(1+\frac{\sigma_0}{2}\right) \leq p(x,t)(1+\sigma_0).
\end{aligned}$$

Therefore, from (2.5) we conclude that

$$\begin{aligned}
I_1 &\leq CR^{\alpha_3}\left(\fint_{Q_R}|\nabla(u-v)|^{p_2}\, dxdt\right)^{\frac{1}{p_2}}\left(\fint_{Q_R}(|\nabla u|+1)^{p(x,t)(1+\sigma_0)}\, dxdt\right)^{\frac{p_2-1}{p_2}}\\
&\leq CR^{\alpha_3}\left(\fint_{Q_R}|\nabla(u-v)|^{p_2}\, dxdt\right)^{\frac{1}{p_2}}\left(R^{-(n+2)d\sigma_0}\fint_{Q_{2R}}(|\nabla u|^{p_2}+1)\, dxdt\right)^{\frac{p_2-1}{p_2}}\\
&\leq CR^{\alpha_3-\frac{\gamma_1(n+2)d\sigma_0}{\gamma_1-1}}\left(\fint_{Q_R}|\nabla(u-v)|^{p_2}\, dxdt\right)^{\frac{1}{p_2}}\left(\fint_{Q_{2R}}(|\nabla u|^{p_2}+1)\, dxdt\right)^{\frac{p_2-1}{p_2}}.
\end{aligned}$$

Estimate of I_2. We divide it into two cases:

Case 1. $p(x,t) \geq 2$. Using the elementary inequality

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right| \leq C\left(|\xi| + |\eta|\right)^{p-2}|\xi - \eta|$$

for any $p \geq 2$, $\xi, \eta \in \mathbb{R}^n$ with $C = C(p)$, Hölder's inequality and the fact that $\{f^i\} \in C^{0;\alpha_2,\alpha_2/2}$, we have

$$\begin{aligned} I_2 &\leq \fint_{Q_R} \left||\mathbf{f}|^{p(x,t)-2}\mathbf{f} - |\mathbf{f}(z_0)|^{p(x,t)-2}\mathbf{f}(z_0)\right| |\nabla(u-v)|\ dxdt, \\ &\leq C\fint_{Q_R} \left(|\mathbf{f}(x)| + |\mathbf{f}(z_0)|\right)^{p(x,t)-2}|\mathbf{f}(x) - \mathbf{f}(z_0)|\,|\nabla(u-v)|\ dxdt, \\ &\leq CR^{\alpha_2}\fint_{Q_R}|\nabla(u-v)|\ dxdt \leq CR^{\alpha_2}\left(\fint_{Q_R}|\nabla(u-v)|^{p_2}\ dxdt\right)^{\frac{1}{p_2}}. \end{aligned}$$

Case 2. $1 < p(x,t) < 2$. Using the elementary inequality

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right| \leq C|\xi - \eta|^{p-1}$$

for any $p \in (1,2)$, $\xi, \eta \in \mathbb{R}^n$ with $C = C(p)$, Hölder's inequality and the fact that $\{f^i\} \in C^{0;\alpha_2,\alpha_2/2}$, we have

$$\begin{aligned} I_2 &\leq \fint_{Q_R}|\mathbf{f}(x) - \mathbf{f}(z_0)|^{p(x,t)-1}|\nabla(u-v)|\,dxdt \\ &\leq CR^{(\gamma_1-1)\alpha_2}\fint_{Q_R}|\nabla(u-v)|\ dxdt \\ &\leq CR^{(\gamma_1-1)\alpha_2}\left(\fint_{Q_R}|\nabla(u-v)|^{p_2}\ dxdt\right)^{\frac{1}{p_2}}. \end{aligned}$$

Estimate of I_3. From the mean value theorem, and the fact that $p(x,t) \in C^{0;\alpha_1,\alpha_1/2}$ and $\{f^i\} \in C^{0;\alpha_2,\alpha_2/2}$, we have

$$\begin{aligned} I_3 &\leq C\fint_{Q_R}|p_2 - p(x,t)|\left(1 + |\mathbf{f}(z_0)|^{p_2-1}\right)\ln\left(1 + |\mathbf{f}(z_0)|\right)|\nabla(u-v)|\ dxdt \\ &\leq CR^{\alpha_1}\fint_{Q_R}|\nabla(u-v)|\ dxdt \\ &\leq CR^{\alpha_1}\left(\fint_{Q_R}|\nabla(u-v)|^{p_2}\ dxdt\right)^{\frac{1}{p_2}}. \end{aligned}$$

Combing all the estimates of $I_i (1 \leq i \leq 3)$, we obtain

$$\begin{aligned} I_4 &\leq I_4 + I_5 = I_1 + I_2 + I_3 \\ &\leq CR^{\beta_2}\left(\fint_{Q_R}|\nabla(u-v)|^{p_2}\ dxdt\right)^{\frac{1}{p_2}}\left(\fint_{Q_{2R}}\left(|\nabla u|^{p_2} + 1\right)\ dxdt\right)^{\frac{p_2-1}{p_2}}, \end{aligned} \tag{2.12}$$

where

$$\beta_2 = \min\left\{\alpha_1, \alpha_2, (\gamma_1 - 1)\alpha_2, \alpha_3 - \frac{\gamma_1(n+2)d\sigma_0}{\gamma_1 - 1}\right\} > 0.$$

Here we have used the assumption that $\gamma_1(n+2)d\sigma_0/(\gamma_1-1) < \alpha_3$.

Estimate of I_4. We divide it into two cases:

Case 1. $p_2 \geq 2$. From (1.4) we have

$$I_4 \geq C \fint_{Q_R} |\nabla u - \nabla v|^{p_2} \; dxdt,$$

which implies that

$$\begin{aligned}\fint_{Q_R} |\nabla u - \nabla v|^{p_2} \; dxdt &\leq CR^{\frac{\beta_2 p_2}{p_2-1}} \fint_{Q_{2R}} \left(|\nabla u|^{p_2}+1\right) \; dxdt \\ &\leq CR^{\frac{\beta_2 \gamma_2}{\gamma_2-1}} \fint_{Q_{2R}} \left(|\nabla u|^{p_2}+1\right) \; dxdt.\end{aligned}$$

Case 2. $1 < p_2 < 2$. Using (1.4), we have

$$I_4 \geq C \fint_{Q_R} \left(|\nabla u| + |\nabla v|\right)^{p_2-2} |\nabla u - \nabla v|^2 \, dxdt. \tag{2.13}$$

Therefore, using Hölder's inequality, we have

$$\begin{aligned}&\fint_{Q_R} |\nabla u - \nabla v|^{p_2} \, dx \\ &= \fint_{Q_R} \left(|\nabla u| + |\nabla v|\right)^{\frac{p_2-2}{2}+\frac{2-p_2}{2}} |\nabla u - \nabla v| \, |\nabla u - \nabla v|^{p_2-1} \, dxdt \\ &\leq \left(\fint_{Q_R} \left(|\nabla u| + |\nabla v|\right)^{p_2-2} |\nabla u - \nabla v|^2 \, dxdt\right)^{\frac{1}{2}} \\ &\quad \times \left(\fint_{Q_R} \left(|\nabla u| + |\nabla v|\right)^{2-p_2} |\nabla u - \nabla v|^{2p_2-2} \, dxdt\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, it follows from (2.13) and Lemma 2.3 that

$$\begin{aligned}&\fint_{Q_R} |\nabla u - \nabla v|^{p_2} \, dxdt \\ &\leq CI_4^{1/2} \left(\fint_{Q_R} \left(|\nabla u| + |\nabla v|\right)^{2-p_2} \left(|\nabla u| + |\nabla v|\right)^{2p_2-2} \; dxdt\right)^{\frac{1}{2}} \\ &\leq CI_4^{1/2} \left(\fint_{Q_R} \left(|\nabla u|^{p_2} + |\nabla v|^{p_2}\right) \; dxdt\right)^{\frac{1}{2}} \\ &\leq CI_4^{1/2} \left(\fint_{Q_R} \left(1+|\nabla u|^{p_2}\right) dxdt\right)^{\frac{1}{2}} \leq CI_4^{1/2} \left(\fint_{Q_{2R}} \left(1+|\nabla u|^{p_2}\right) dxdt\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, in view of (2.12) we find that

$$\begin{aligned}&\fint_{Q_R} |\nabla u - \nabla v|^{p_2} \, dxdt \\ &\leq CR^{\beta_2/2} \left(\fint_{Q_R} |\nabla u - \nabla v|^{p_2} \, dxdt\right)^{\frac{1}{2p_2}} \left(\fint_{Q_{2R}} \left(1+|\nabla u|^{p_2}\right) dxdt\right)^{\frac{2p_2-1}{2p_2}},\end{aligned}$$

which implies that

$$\fint_{Q_R} |\nabla u - \nabla v|^{p_2}\, dxdt \le C R^{\frac{\beta_2 p_2}{2p_2-1}} \fint_{Q_{2R}} (|\nabla u|^{p_2}+1)\, dxdt$$
$$\le C R^{\frac{\beta_2 \gamma_2}{2\gamma_2-1}} \fint_{Q_{2R}} (|\nabla u|^{p_2}+1)\, dxdt.$$

Thus, combining the estimates of I_4 in Case 1 and Case 2, we obtain

$$\fint_{Q_R} |\nabla u - \nabla v|^{p_2}\, dxdt \le C R^{\frac{\beta_2 \gamma_2}{2\gamma_2-1}} \fint_{Q_{2R}} (|\nabla u|^{p_2}+1)\, dxdt,$$

which can finish the proof by choosing $\beta = \frac{\beta_2\gamma_2}{2\gamma_2-1}$. □

Lemma 2.6. (see Lemma 3.2 in [1]) *Let $\Phi : [0,a) \to \mathbb{R}$, $0 < a < 1$, be a positive bounded function such that $\Phi(s) \le 2\Phi(t)$ for any $s \le t$ and*

$$\Phi(\rho) \le C\left[\left(\frac{\rho}{R}\right)^n + \epsilon\right]\Phi(R) + CR^n$$

for any $0 < \rho \le R/8$. Then, for any $\theta \in (0,n)$ there exist positive constants c, ϵ_0, depending on C, θ, n, such that if $\epsilon \le \epsilon_0$, then

$$\Phi(\rho) \le c\left(\frac{\rho}{R}\right)^{n-\theta}\left[\Phi(R) + R^{n-\theta}\right]$$

for any $0 < \rho \le R/16$.

2.4. Final proof

Now we are ready to prove the main result, Theorem 1.2.

Proof. From Lemma 2.5, Hölder's inequality and (2.7) we have

$$\int_{Q_\rho} |\nabla u|^{p_2}\, dxdt$$
$$\le C\left(\int_{Q_\rho} |\nabla u - \nabla v|^{p_2}\, dxdt + \int_{Q_\rho} \left|\nabla v - (\nabla v)_{Q_\rho}\right|^{p_2} dxdt + \rho^{n+2}\left|(\nabla v)_{Q_\rho}\right|^{p_2}\right)$$
$$\le C\left(R^\beta \int_{Q_{R/4}} (|\nabla u|^{p_2}+1)\, dxdt + \left(\frac{\rho}{R}\right)^{n+2+\beta_1}\int_{Q_R} (|\nabla v|^{p_2}+1)\, dxdt + \rho^{n+2-\tau}\right).$$

for any $\rho \le R/8 \le R_2/8$, which implies that

$$\int_{Q_\rho} |\nabla u|^{p_2}\, dxdt \le C\left[\left(R^\beta + \left(\frac{\rho}{R}\right)^{n+2+\beta_1}\right)\int_{Q_R} |\nabla u|^{p_2}\, dxdt + R^{n+2-\tau}\right]$$
$$\le C\left[\left(R^\beta + \left(\frac{\rho}{R}\right)^{n+2-\tau}\right)\int_{Q_R} |\nabla u|^{p_2}\, dxdt + R^{n+2-\tau}\right].$$

Without loss of generality we may assume that $R^\beta \le R_2^\beta \le \epsilon_0$. Then from Lemma 2.6, for any $\theta \in (0, n+2-\tau)$ we obtain

$$\int_{Q_\rho} |\nabla u|^{p_2}\, dxdt \le C\left(\frac{\rho}{R}\right)^{n+2-\tau-\theta}\left[\int_{Q_R} |\nabla u|^{p_2}\, dxdt + R^{n+2-\tau-\theta}\right]$$

for $\rho \le R/16 \le R_2/16$. Let $\theta = \tau$. Then we have $\tau \in (0, (n+2)/2)$ and

$$\fint_{Q_\rho} |\nabla u|^{p_2}\, dxdt \le C\rho^{-2\tau} R^{2\tau-n-2}\left[\int_{Q_R} |\nabla u|^{p_2}\, dxdt + R^{n+2-2\tau}\right]$$

for any $\rho \le R/16 \le R_2/16$. Therefore, by choosing $R = R_2$ we deduce that

$$\fint_{Q_\rho} |\nabla u|^{p_2}\, dxdt \le C\rho^{-2\tau}. \tag{2.14}$$

From Young's inequality, (2.7), (2.8), (2.14) and Lemma 2.5, we have

$$\begin{aligned}
&\int_{Q_\rho} \left|\nabla u - (\nabla u)_{Q_\rho}\right|^{p_2} dxdt \\
&\le C \int_{Q_\rho} \left|\nabla u - (\nabla v)_{Q_\rho}\right|^{p_2} dxdt \\
&\le C\left[\rho^{n+2} \fint_{Q_\rho} \left|\nabla v - (\nabla v)_{Q_\rho}\right|^{p_2} dxdt + \int_{Q_\rho} |\nabla u - \nabla v|^{p_2}\, dxdt\right] \\
&\le C\left[\left(\frac{\rho}{R}\right)^{\beta_1} \rho^{n+2} \fint_{Q_{R/16}} (|\nabla v|^{p_2}+1)\, dxdt + R^{\beta} \int_{Q_{R/16}} (|\nabla u|^{p_2}+1)\, dxdt\right] \\
&\le C\left[\left(\frac{\rho}{R}\right)^{\beta_1} \rho^{n+2} R^{-\tau} + R^{n+2+\beta-2\tau}\right]
\end{aligned}$$

for any $0 < \rho \le R/32 \le R_2/32$ with $Q_{R_0} \subset \Omega_T$. Choose

$$\rho = \frac{R^{1+\mu}}{32} \quad \text{and} \quad \tau = \frac{\beta\beta_1}{2(n+2+2\beta_1)} < \beta,$$

where $\mu = \frac{\beta-\tau}{n+2+\beta_1} > 0$. Thus, we have

$$\begin{aligned}
\int_{Q_\rho} \left|\nabla u - (\nabla u)_{Q_\rho}\right|^{p_2} dxdt &\le CR^{n+2+\beta-2\tau} \le C\rho^{\frac{n+2+\beta-2\tau}{1+\mu}} \\
&\le C\rho^{n+2+\frac{\beta-\mu(n+2)-2\tau}{1+\mu}}.
\end{aligned}$$

Finally, from Hölder's inequality we have

$$\fint_{Q_\rho} \left|\nabla u - (\nabla u)_{Q_\rho}\right|^{\gamma_1} dxdt \le C\rho^{\frac{[\beta-\mu(n+2)-2\tau]\gamma_1}{(1+\mu)\gamma_2}} =: C\rho^{\alpha}$$

for any $0 < \rho \le R/32 \le R_2/32$ with $Q_{R_0} \subset \Omega_T$. It is easy to see that $\alpha > 0$ since

$$\beta - \mu(n+2) - 2\tau = \frac{\beta_1\beta - (n+2+2\beta_1)\tau}{n+2+\beta_1} = \frac{\beta_1\beta}{2(n+2+\beta_1)} > 0.$$

Then from Campanato's theorem (see [18], Theorem 1.3 of Chapter 3) we conclude that $\nabla u \in C^{0;\alpha,\alpha/2}(Q_{R_2/32})$. Thus, we can complete the proof of Theorem 1.2 by an elementary covering argument. □

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