Nonlinear Differ. Equ. Appl. 19 (2012), 79–95

1021-9722/12/010079-17
published online June 7, 2011
DOI 10.1007/s00030-011-0119-0

Nonlinear Differential Equations and Applications NoDEA

Spherical twists, stationary paths and harmonic maps from generalised annuli into spheres

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Abstract. Let $\mathbf{X} \subset \mathbb{R}^n$ be a *generalised* annulus and consider the Dirichlet energy functional

$$\mathbb{E}[u; \mathbf{X}] := \frac{1}{2} \int_{\mathbf{X}} |\nabla u(x)|^2 \, dx,$$

on the space of admissible maps

$$\mathcal{A}_\varphi(\mathbf{X}) = \left\{ u \in W^{1,2}(\mathbf{X}, \mathbb{S}^{n-1}) : u|_{\partial \mathbf{X}} = \varphi \right\}.$$

Here $\varphi \in \mathbf{C}(\partial \mathbf{X}, \mathbb{S}^{n-1})$ is *fixed* and $\mathcal{A}_\varphi(\mathbf{X})$ is non-empty. In this paper we introduce a class of maps referred to as *spherical* twists and examine them in connection with the Euler–Lagrange equation associated with $\mathbb{E}[\cdot, \mathbf{X}]$ on $\mathcal{A}_\varphi(\mathbf{X})$ [the so-called harmonic map equation on $\mathbf{X}$]. The main result here is an interesting discrepancy between *even* and *odd* dimensions. Indeed for even n subject to a compatibility condition on φ the latter system admits *infinitely* many smooth solutions modulo isometries whereas for odd n this number reduces to *one* or *none*. We discuss qualitative features of the solutions in view of their novel and *explicit* representation through the exponential map of the compact Lie group $\mathbf{SO}(n)$.

Mathematics Subject Classification (2000). 58E20, 22CXX, 35RXX.

1. Introduction

Let $\mathbf{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$ with $0 < a < b < \infty$ and consider the Dirichlet energy functional

$$\mathbb{E}[u; \mathbf{X}] := \frac{1}{2} \int_{\mathbf{X}} |\nabla u(x)|^2 \, dx, \tag{1.1}$$

on the space of admissible maps

$$\mathcal{A}_\varphi(\mathbf{X}) = \Big\{ u \in W^{1,2}(\mathbf{X}, \mathbb{S}^{n-1}) : u|_{\partial \mathbf{X}} = \varphi \Big\}. \tag{1.2}$$

Here $\mathbb{S}^{n-1}$ represent the Euclidean unit sphere and as customary we have set

$$W^{1,2}(\mathbf{X}, \mathbb{S}^{n-1}) = \Big\{ u \in W^{1,2}(\mathbf{X}, \mathbb{R}^n) : u(x) \in \mathbb{S}^{n-1} \text{ for } \mathcal{L}^n\text{-a.e. } x \in \mathbf{X} \Big\}.$$

Moreover $\varphi \in \mathbf{C}(\partial \mathbf{X}, \mathbb{S}^{n-1})$ is *fixed* while the space $\mathcal{A}_\varphi(\mathbf{X})$ is non-empty. In view of $\partial \mathbf{X} = \partial \mathbf{X}_a \cup \partial \mathbf{X}_b := a\mathbb{S}^{n-1} \cup b\mathbb{S}^{n-1}$ it is convenient to set

$$\begin{cases} \varphi_a = \varphi|_{\partial \mathbf{X}_a} \circ \delta_a, \\ \varphi_b = \varphi|_{\partial \mathbf{X}_b} \circ \delta_b, \end{cases}$$

where δ_a, δ_b are space dilatations by factors a and b respectively. As a result we speak of $\varphi_a, \varphi_b \in \mathbf{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1})$. The Euler–Lagrange equation associated with $\mathbb{E}[\cdot, \mathbf{X}]$ on $\mathcal{A}_\varphi(\mathbf{X})$ takes the form[1]

$$\begin{cases} \Delta u + |\nabla u|^2 u = 0 & \text{in } \mathbf{X}, \\ |u| = 1 & \text{in } \mathbf{X}, \\ u = \varphi & \text{on } \partial \mathbf{X}, \end{cases}$$

which is the well-known harmonic map equation on $\mathbf{X}$ and into $\mathbb{S}^{n-1}$. Motivated by the significance of *Dehn* twists in the study of mapping class groups of surfaces (see, e.g., [2]) and the interesting role played by *generalised* twists in the multiple solution problems of nonlinear elasticity (cf., e.g., [12]) in this article we introduce their $\mathbb{S}^{n-1}$-valued counterparts, the *spherical* twists, and out of pure curiosity examine them in connection with the above system of Euler–Lagrange equations. Indeed a spherical twist by definition is a map $u \in \mathcal{A}_\varphi(\mathbf{X})$ in the form

$$u : x = r\theta \mapsto \mathbf{Q}(r)\theta,$$

where $x \in \mathbf{X}, r = |x|, \theta = x/|x|$ and $\mathbf{Q} \in W^{1,2}([a,b], \mathbf{SO}(n))$. It is evident that subject to this assumption φ must take the form

$$\begin{cases} \varphi_a(\theta) = \mathbf{R}_a \theta, \\ \varphi_b(\theta) = \mathbf{R}_b \theta, \end{cases}$$

for $\theta \in \mathbb{S}^{n-1}$ where $\mathbf{R}_a, \mathbf{R}_b \in \mathbf{SO}(n)$ (in fact $\mathbf{Q}(a) = \mathbf{R}_a$ and $\mathbf{Q}(b) = \mathbf{R}_b$). Now by restricting the energy to the space of *spherical* twists we have that

$$\begin{aligned} \mathbb{E}[\mathbf{Q}(r)\theta, \mathbf{X}] &= \frac{1}{2} \int_a^b \int_{\mathbb{S}^{n-1}} \left\{ \frac{1}{r^2} \Big[(n-1) + r^2 |\dot{\mathbf{Q}}\theta|^2 \Big] \right\} r^{n-1} \, d\mathcal{H}^{n-1}(\theta) dr \\ &= \frac{1}{2} \omega_n \int_a^b \left\{ n(n-1) \frac{1}{r^2} + |\dot{\mathbf{Q}}|^2 \right\} r^{n-1} \, dr. \end{aligned}$$

[1] For a comprehensive treatment of harmonic maps and some fundamental results cf. [5]. Also [6,10,11] for regularity, [9] for the role of domain topology and the monographs [3,8] and the references therein.

As this last expression has no explicit θ dependence a natural starting point is to analyse the resulting Euler–Lagrange equation, a *second* order differential equation on the compact Lie group $\mathbf{SO}(n)$ and integrate the latter to classify all *extremals* of this restricted energy. Interestingly with the aid of the exponential map of $\mathbf{SO}(n)$ these take the form (see Theorems 2.1 and 2.2)

$$\mathbf{Q} : [a, b] \ni r \mapsto e^{\beta(r)\mathbf{A}}\mathbf{Q}_\circ \in \mathbf{SO}(n)$$

with $\mathbf{A} \in \mathbb{M}_{n\times n}$ *skew*-symmetric, $\mathbf{Q}_\circ \in \mathbf{SO}(n)$ and $\beta = \beta(|x|)$ the *fundamental* solution of $-\Delta$ on $\mathbb{R}^n$. The next step is to extract from within this class those *spherical* twists that grant solutions to the original harmonic map equation on $\mathbf{X}$ and this requires a careful analysis of the *full* versus the restricted Euler–Lagrange equations. The result points at a discrepancy between even and odd dimensions. Indeed subject to a compatibility condition between $\mathbf{R}_a$ and $\mathbf{R}_b$ (cf. (3.1) in Remark 3.1) for even n the latter system of equations admits *infinitely* many solutions all in the form

$$u(x) = u(r\theta) = \mathbf{R}_a\mathbf{P}_u e^{g(r)\mathcal{J}_n}\mathbf{P}_u^t\theta = \mathbf{R}_a\mathbf{P}_u\mathcal{R}_n[g](r)\mathbf{P}_u^t\theta,$$

where $\mathcal{J}_n = diag(\mathcal{J}, \mathcal{J}, \dots, \mathcal{J})$ and $\mathcal{R}_n[g](r) = diag(\mathcal{R}[g](r), \mathcal{R}[g](r), \dots, \mathcal{R}[g](r))$ with the 2×2 *skew*-symmetric matrix $\mathcal{J}$ and the *rotation* (by angle g) matrix $\mathcal{R}[g]$ as in (4.1) and (4.2) while $\mathbf{P}_u \in \mathbf{SO}(n)$ is suitably related to $\mathbf{R}_a$ and $\mathbf{R}_b$. Furthermore the *rotation* angle g is related to $\beta = \beta(|x|)$ and depending on n can be expressed as:
[**1**] $(n = 2)$

$$g(r) = \frac{\log r/a}{\log b/a}(\eta + 2\pi m).$$

[**2**] $(n \geq 4)$

$$g(r) = \frac{(r/a)^{2-n} - 1}{(b/a)^{2-n} - 1}(\eta + 2\pi m).$$

Here $\eta \in \mathbb{R}$ is as in Remark 3.1 while $m \in \mathbb{Z}$. In sharp contrast for odd n the number of such solutions severely reduces to *one*, i.e.,

$$u(x) = u(r\theta) = \mathbf{R}\theta,$$

when $\mathbf{R} = \mathbf{R}_a = \mathbf{R}_b$ and *none* otherwise (cf. Theorems 3.1 and 3.2). As $\mathbb{E}[\cdot, \mathbf{X}]$ attains its infimum on $\mathcal{A}_\phi$ it follows in particular that here the energy minimizer does not have the rotational *symmetry* one intuitively expects, i.e., is *not* a spherical twists (cf. [12] for further results).

Finally it is well-known that $\Delta u + |\nabla u|^2 u = 0$ for liftings $u = e^{i\phi}$ is equivalent to $\Delta\phi = 0$. The result here gives a generalisation of this to all *even* dimensions. This observation seems to have gone unnoticed before.

2. Spherical twists on annuli

Let $\mathbf{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$ and for $x \in \overline{\mathbf{X}}$ put $r = |x|$ and $\theta = x/|x|$. Then a *continuous* map u on $\overline{\mathbf{X}}$ into $\mathbb{S}^{n-1}$ in the form

$$u : x \mapsto \mathbf{Q}(r)\theta,$$

with $\mathbf{Q} \in \mathbf{C}([a,b], \mathbf{SO}(n))$ is referred to as a *spherical twist* on $\mathbf{X}$; $\mathbf{Q}$ is the twist *path* and when $\mathbf{Q}(a) = \mathbf{Q}(b)$ the twist loop.[2]

Proposition 2.1. *Suppose that u is a spherical twist on $\mathbf{X}$. Then $u \in \mathcal{A}_\varphi(\mathbf{X})$ provided that the following hold.*

[1] $\mathbf{Q}(a) = \mathbf{R}_a$,

[2] $\mathbf{Q}(b) = \mathbf{R}_b$,

[3] $\mathbf{Q} \in W^{1,2}([a,b], \mathbf{SO}(n))$.

Proof. Evidently for u as described $u \in \mathcal{A}_\varphi(\mathbf{X})$ if and only if the following hold:

[a] $||u||_{1,2} < \infty$,

[b] $u = \varphi$ on $\partial\mathbf{X}$.

Now anticipating on [**a**] a straight-forward *differentiation* gives

$$\nabla u = \frac{1}{r}\left(\mathbf{Q} + (r\dot{\mathbf{Q}} - \mathbf{Q})\theta \otimes \theta\right) \tag{2.1}$$

with $x = r\theta \in \mathbf{X}$ and $\dot{\mathbf{Q}} := d\mathbf{Q}/dr$. Therefore

$$\begin{aligned} |\nabla u|^2 = tr\Big\{[\nabla u][\nabla u]^t\Big\} &= \frac{1}{r^2} tr\Big\{\mathbf{I}_n + \mathbf{Q}\theta \otimes (r\dot{\mathbf{Q}} - \mathbf{Q})\theta + (r\dot{\mathbf{Q}} - \mathbf{Q})\theta \otimes \mathbf{Q}\theta \\ &\quad + \Big[(r\dot{\mathbf{Q}} - \mathbf{Q})\theta \otimes \theta\Big]\Big[\theta \otimes (r\dot{\mathbf{Q}} - \mathbf{Q})\theta\Big]\Big\} \\ &= \frac{1}{r^2} tr\Big\{\mathbf{I}_n - \mathbf{Q}\theta \otimes \mathbf{Q}\theta + r^2 \dot{\mathbf{Q}}\theta \otimes \dot{\mathbf{Q}}\theta\Big\} \\ &= \frac{1}{r^2}[n - \langle \mathbf{Q}\theta, \mathbf{Q}\theta\rangle + r^2 \langle \dot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta\rangle] \\ &= \frac{1}{r^2}[(n-1) + r^2 |\dot{\mathbf{Q}}\theta|^2] \end{aligned} \tag{2.2}$$

where in concluding the *last* identity we have used $\langle \mathbf{Q}\theta, \mathbf{Q}\theta\rangle = 1$ for all $\theta \in \mathbb{S}^{n-1}$. Thus recalling that $|u|^2 = 1$ in $\mathbf{X}$ we can write

$$\begin{aligned} \int_{\mathbf{X}} |u|^2 + |\nabla u|^2 &= \int_a^b \int_{\mathbb{S}^{n-1}} \left(1 + \frac{1}{r^2}\Big[(n-1) + r^2|\dot{\mathbf{Q}}\theta|^2\Big]\right) r^{n-1}\, d\mathcal{H}^{n-1}(\theta) dr \\ &= \int_a^b \omega_n \left[n + \frac{1}{r^2} n(n-1) + |\dot{\mathbf{Q}}|^2\right] r^{n-1}\, dr \end{aligned}$$

and so [**a**] results from [**3**]. Finally [**b**] $\iff$ ([**1**], [**2**]) and the proof is complete. □

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[2] In view of $\mathbf{Q}(r) \in \mathbf{SO}(n)$ on $[a,b]$ we have $|u(x)|^2 = |\mathbf{Q}(r)\theta|^2 = |\theta|^2$ and so u is $\mathbb{S}^{n-1}$-valued and thus *well*-defined.

Proposition 2.2. *Let u be a spherical twist with twist path $\mathbf{Q} \in \mathbf{C}^2(]a,b[, \mathbf{SO}(n))$. Then*

$$\Delta u = \frac{1}{r^2}[(n-1)(r\dot{\mathbf{Q}} - \mathbf{Q}) + r^2\ddot{\mathbf{Q}}]\theta \tag{2.3}$$

and subsequently

$$\Delta u + |\nabla u|^2 u = \left[\ddot{\mathbf{Q}} + \frac{n-1}{r}\dot{\mathbf{Q}} + |\dot{\mathbf{Q}}\theta|^2\mathbf{Q}\right]\theta \tag{2.4}$$

in $\mathbf{X}$.

Proof. Indeed (2.3) follows by a further differentiation of (2.1) and (2.4) follows upon substitution from (2.2) and (2.3). We abbreviate the details. □

It is plain that energy of a spherical twist with the aid of (2.2) in Proposition 2.1 can be described by the integral

$$\begin{aligned}
\mathbb{E}[u;\mathbf{X}] &= \frac{1}{2}\int_a^b \int_{\mathbb{S}^{n-1}} \left\{\frac{1}{r^2}\left[(n-1) + r^2|\dot{\mathbf{Q}}\theta|^2\right]\right\} r^{n-1}\, d\mathcal{H}^{n-1}(\theta)dr \\
&= \frac{1}{2}\omega_n \int_a^b \left\{n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2\right\} r^{n-1}\, dr.
\end{aligned}$$

Upon denoting the integral on the right by $e[\mathbf{Q}]$ in what follows we proceed by computing the first variation of this energy on the space of admissible paths on the pointed space $(\mathbf{SO}(n), \mathbf{I}_n)$, that is,[3]

$$\mathcal{E} = \mathcal{E}[a,b] := \left\{\begin{array}{l} \mathbf{Q} \in W^{1,2}([a,b], \mathbf{SO}(n)) \\ \mathbf{Q}(a) = \mathbf{I}_n \\ \mathbf{Q}(b) = \mathbf{R} \end{array}\right\}.$$

Proposition 2.3. (Stationary paths) *The Euler–Lagrange equation associated with* e$[\cdot]$ *on* $\mathcal{E}$ *takes the form*

$$\frac{d}{dr}\left\{\left[r^{n-1}\frac{d}{dr}\mathbf{Q}\right]\mathbf{Q}^t\right\} = 0 \tag{2.5}$$

on $]a,b[$.

Proof. First *fix* $\mathbf{Q}$ as described and for $\varepsilon \in \mathbb{R}$ put $\mathbf{Q}_\varepsilon = \mathbf{Q} + \varepsilon(\mathbf{F} - \mathbf{F}^t)\mathbf{Q}$ where $\mathbf{F} \in \mathbf{C}_0^\infty(]a,b[, \mathbb{M}_{n\times n})$ is arbitrary. Then

$$\mathbf{Q}_\varepsilon\mathbf{Q}_\varepsilon^t = [\mathbf{Q} + \varepsilon(\mathbf{F} - \mathbf{F}^t)\mathbf{Q}][\mathbf{Q}^t - \varepsilon\mathbf{Q}^t(\mathbf{F} - \mathbf{F}^t)] = \mathbf{I}_n - \varepsilon^2(\mathbf{F} - \mathbf{F}^t)^2$$

[3] In view of the *trivial* identity $\Delta(\mathbf{R}u) + |\nabla(\mathbf{R}u)|^2\mathbf{R}u = \mathbf{R}(\Delta u + |\nabla u|^2 u)$ (here $\mathbf{R} \in \mathbf{O}(n)$ is *fixed*) in what follows we assume without loss of generality that $\mathbf{R}_a = \mathbf{I}_n$ while $\mathbf{R}_b = \mathbf{R}$.

and so to the first order in ε the perturbation $\mathbf{Q}_\varepsilon$ takes values on $\mathbf{SO}(n)$. Now with a slight abuse of notation we can write

$$\begin{aligned}\frac{1}{\omega_n}\frac{d}{d\varepsilon}e[\mathbf{Q}_\varepsilon]\Big|_{\varepsilon=0} &= \int_a^b \langle \dot{\mathbf{Q}}, (\dot{\mathbf{F}}-\dot{\mathbf{F}}^t)\mathbf{Q} + (\mathbf{F}-\mathbf{F}^t)\dot{\mathbf{Q}}\rangle r^{n-1}\,dr \\ &= \int_a^b (\langle \dot{\mathbf{Q}}, (\dot{\mathbf{F}}-\dot{\mathbf{F}}^t)\mathbf{Q}\rangle + \langle \dot{\mathbf{Q}}, (\mathbf{F}-\mathbf{F}^t)\dot{\mathbf{Q}}\rangle) r^{n-1}\,dr \\ &= \int_a^b \langle \dot{\mathbf{Q}}\mathbf{Q}^t, (\dot{\mathbf{F}}-\dot{\mathbf{F}}^t)\rangle r^{n-1}\,dr \\ &= \int_a^b -\langle \frac{d}{dr}[r^{n-1}\dot{\mathbf{Q}}\mathbf{Q}^t], (\mathbf{F}-\mathbf{F}^t)\rangle\,dr = 0.\end{aligned}$$

Note that in concluding the *last* line we have used the integration by parts formula together with the *boundary* conditions $\mathbf{F}(a) = \mathbf{F}(b) = 0$. The conclusion now follows in view of $\dot{\mathbf{Q}}\mathbf{Q}^t$ being *skew-symmetric*. □

Remark 2.1. For the sake of convenience in what follows we often assume the orthogonal matrix $\mathbf{R}$ (see the definition of $\mathcal{E}[a.b]$ preceding Proposition 2.3) to have been expressed in *block* diagonal forms (cf. the Appendix for notation), specifically,

[**1**] $(n = 2k)$

$$\mathbf{R} = \mathfrak{P}_{\mathbf{R}}\mathfrak{D}_{\mathbf{R}}\mathfrak{P}_{\mathbf{R}}^t = \mathfrak{P}_{\mathbf{R}}diag(\mathcal{R}[\eta_1], \mathcal{R}[\eta_2], \ldots, \mathcal{R}[\eta_k])\mathfrak{P}_{\mathbf{R}}^t.$$

[**2**] $(n = 2k+1)$

$$\mathbf{R} = \mathfrak{P}_{\mathbf{R}}\mathfrak{D}_{\mathbf{R}}\mathfrak{P}_{\mathbf{R}}^t = \mathfrak{P}_{\mathbf{R}}diag(\mathcal{R}[\eta_1], \mathcal{R}[\eta_2], \ldots, \mathcal{R}[\eta_k], 1)\mathfrak{P}_{\mathbf{R}}^t.$$

The sequences $(e^{\pm i\eta_j})_{j=1}^k$ in [**1**] and $(1, e^{\pm i\eta_j})_{j=1}^k$ in [**2**] consist of *eigen*-values of $\mathbf{R}$ (here $\eta_1, \ldots, \eta_k \in [0, \pi]$) while $\mathfrak{P}_{\mathbf{R}} \in \mathbf{O}(n)$. (Note that there is *no* uniqueness associated with the choices of $\mathfrak{D}_{\mathbf{R}}$ and $\mathfrak{P}_{\mathbf{R}}$ *yet* in what follows we pick one such pair and assume them *fixed* throughout.)

Theorem 2.1. (Stationary paths) *The general solution to* (2.5) *is given by the matrix exponential*

$$\mathbf{Q}(r) = e^{\beta(r)\mathbf{A}}\,\mathbf{Q}_\circ. \tag{2.6}$$

Here $\mathbf{Q}_\circ \in \mathbf{SO}(n), \mathbf{A}$ *is skew-symmetric and*

$$\beta(r) = \begin{cases} \log r & n = 2, \\ \frac{1}{2-n}r^{2-n} & n \geq 3. \end{cases}$$

Moreover subject to $\mathbf{Q}(a) = \mathbf{I}_n$ *and* $\mathbf{Q}(b) = \mathbf{R}$, *depending on the dimension* n *being even or odd the following hold.*

[**a**] $(n = 2k)$

$$\mathbf{A} = \mathfrak{P}_{\mathbf{A}}\mathfrak{D}_{\mathbf{A}}\mathfrak{P}_{\mathbf{A}}^t = \mathfrak{P}_{\mathbf{R}}\mathbf{P}diag(\zeta_1\mathcal{J}, \zeta_2\mathcal{J}, \ldots, \zeta_k\mathcal{J})\mathbf{P}^t\mathfrak{P}_{\mathbf{R}}^t,$$

[**b**] $(n = 2k+1)$

$$\mathbf{A} = \mathfrak{P}_{\mathbf{A}}\mathfrak{D}_{\mathbf{A}}\mathfrak{P}_{\mathbf{A}}^t = \mathfrak{P}_{\mathbf{R}}\mathbf{P}diag(\zeta_1\mathcal{J}, \zeta_2\mathcal{J}, \ldots, \zeta_k\mathcal{J}, 0)\mathbf{P}^t\mathfrak{P}_{\mathbf{R}}^t.$$

In either of the cases [**a**] *or* [**b**] *the sequence* $(\zeta_j)_{j=1}^k \subset \mathbb{R}$ *must satisfy the set of conditions*

$$\zeta_j = \frac{1}{s}\Big[\eta_j + 2\pi m_j\Big], \tag{2.7}$$

for all $1 \le j \le k$ *where* $m_j \in \mathbb{Z}$, $s = \beta(b) - \beta(a)$ *and* $\mathbf{P} \in \mathbf{C}[\mathfrak{D}_\mathbf{R}]$ *the centraliser of* $\mathfrak{D}_\mathbf{R}$ *in* $\mathbf{O}(n)$. *Finally for each such* $\mathbf{A}$ *the choice of* $\mathbf{Q}_\circ$ *is unique; in fact one precisely has* $\mathbf{Q}_\circ = e^{-\beta(a)\mathbf{A}}$.[4]

Proof. Since $\mathbf{Q}$ is a solution to (2.5), integrating once, there exits a constant and *skew-symmetric* matrix $\mathbf{A}$ such that

$$\frac{d}{dr}\mathbf{Q} = \frac{1}{r^{n-1}}\mathbf{A}\mathbf{Q}.$$

Integrating again gives (2.6). Note that here we have absorbed a *constant* resulting from integrating r^{1-n} into the special orthogonal matrix $\mathbf{Q}_\circ$. Next enforcing the boundary conditions $\mathbf{Q}(a) = \mathbf{I}_n$ and $\mathbf{Q}(b) = \mathbf{R}$ we obtain

$$\mathbf{R} = e^{[\beta(b)-\beta(a)]\mathbf{A}}.$$

Thus with $s = \beta(b) - \beta(a)$ it remains to characterise all *skew*-symmetric matrices $\mathbf{A}$ for which

$$e^{s\mathbf{A}} = \mathbf{R}. \tag{2.8}$$

In order to solve this equation for $\mathbf{A}$ consider expressing $\mathbf{A}$ in block *diagonal* form as described in Proposition 4.1. Denoting the *spectrum* of $\mathbf{A}$ by $\sigma(\mathbf{A}) = (\pm i\zeta_j)_{j=1}^k$ in [**a**] and $\sigma(\mathbf{A}) = (0, \pm i\zeta_j)_{j=1}^k$ in [**b**], (2.8) and the *spectral* mapping theorem (see, e.g., [4]) lead to the identities:
[**a**] $(n = 2k)$

$$e^{s\sigma(\mathbf{A})} = (e^{\pm is\zeta_j})_{j=1}^k = (e^{\pm i\eta_j})_{j=1}^k = \sigma(\mathbf{R}).$$

[**b**] $(n = 2k+1)$

$$e^{s\sigma(\mathbf{A})} = (1, e^{\pm is\zeta_j})_{j=1}^k = (1, e^{\pm i\eta_j})_{j=1}^k = \sigma(\mathbf{R}).$$

Thus up to re-labeling and a possible re-naming upon *sign* differences in either of the cases [**a**] or [**b**] we have

$$e^{s\mathbf{A}} = \mathbf{R} \implies \left\{\begin{array}{l} e^{is\zeta_1} = e^{i\eta_1}, \\ e^{is\zeta_2} = e^{i\eta_2}, \\ . \\ . \\ . \\ e^{is\zeta_k} = e^{i\eta_k}, \end{array}\right\} \iff \left\{\begin{array}{l} s\zeta_1 = \eta_1 + 2\pi m_1, \\ s\zeta_2 = \eta_2 + 2\pi m_2, \\ . \\ . \\ . \\ s\zeta_k = \eta_k + 2\pi m_k. \end{array}\right\}$$

[4] Recall that in a group $\mathbf{G}$ the *centraliser* of an element $g \in \mathbf{G}$ denoted $\mathbf{C}[g]$ is the subgroup consisting of all elements in $\mathbf{G}$ commuting with g, i.e., $\mathbf{C}[g] = \{h \in \mathbf{G} : g = hgh^{-1}\}$.

[**a**] $(n = 2k)$ Here without loss of generality using the above set of identities we can write

$$\begin{aligned} e^{s\mathbf{A}} &= e^{s\mathfrak{P}_{\mathbf{A}}\mathfrak{D}_{\mathbf{A}}\mathfrak{P}^t_{\mathbf{A}}} \\ &= e^{s\mathfrak{P}_{\mathbf{A}} diag(\zeta_1\mathcal{J},\zeta_2\mathcal{J},\ldots,\zeta_k\mathcal{J})\mathfrak{P}^t_{\mathbf{A}}} \\ &= \mathfrak{P}_{\mathbf{A}} diag(\mathcal{R}[s\zeta_1],\mathcal{R}[s\zeta_2],\ldots,\mathcal{R}[s\zeta_k])\mathfrak{P}^t_{\mathbf{A}} \\ &= \mathfrak{P}_{\mathbf{A}} diag(\mathcal{R}[\eta_1],\mathcal{R}[\eta_2],\ldots,\mathcal{R}[\eta_k])\mathfrak{P}^t_{\mathbf{A}} \\ &= \mathfrak{P}_{\mathbf{R}} diag(\mathcal{R}[\eta_1],\mathcal{R}[\eta_2],\ldots,\mathcal{R}[\eta_k])\mathfrak{P}^t_{\mathbf{R}} \\ &= \mathfrak{P}_{\mathbf{R}}\mathfrak{D}_{\mathbf{R}}\mathfrak{P}^t_{\mathbf{R}} \\ &= \mathbf{R}. \end{aligned}$$

As a result the above chain of equalities enforces the following

$$\begin{aligned} \mathfrak{P}^t_{\mathbf{R}}\mathfrak{P}_{\mathbf{A}} \in \mathbf{C}[\mathfrak{D}_{\mathbf{R}}] &\iff \mathfrak{D}_{\mathbf{R}} = [\mathfrak{P}^t_{\mathbf{R}}\mathfrak{P}_{\mathbf{A}}]\mathfrak{D}_{\mathbf{R}}[\mathfrak{P}^t_{\mathbf{R}}\mathfrak{P}_{\mathbf{A}}]^t \\ &\iff \mathfrak{P}_{\mathbf{R}}\mathfrak{D}_{\mathbf{R}}\mathfrak{P}^t_{\mathbf{R}} = \mathfrak{P}_{\mathbf{A}}\mathfrak{D}_{\mathbf{R}}\mathfrak{P}^t_{\mathbf{A}}. \end{aligned}$$

[**b**] $(n = 2k + 1)$ Again without loss of generality using (2.7) we can write

$$\begin{aligned} e^{s\mathbf{A}} &= e^{s\mathfrak{P}_{\mathbf{A}}\mathfrak{D}_{\mathbf{A}}\mathfrak{P}^t_{\mathbf{A}}} \\ &= e^{s\mathfrak{P}_{\mathbf{A}} diag(\zeta_1\mathcal{J},\zeta_2\mathcal{J},\ldots,\zeta_k\mathcal{J},0)\mathfrak{P}^t_{\mathbf{A}}} \\ &= \mathfrak{P}_{\mathbf{A}} diag(\mathcal{R}[s\zeta_1],\mathcal{R}[s\zeta_2],\ldots,\mathcal{R}[s\zeta_k],1)\mathfrak{P}^t_{\mathbf{A}} \\ &= \mathfrak{P}_{\mathbf{A}} diag(\mathcal{R}[\eta_1],\mathcal{R}[\eta_2],\ldots,\mathcal{R}[\eta_k],1)\mathfrak{P}^t_{\mathbf{A}} \\ &= \mathfrak{P}_{\mathbf{R}} diag(\mathcal{R}[\eta_1],\mathcal{R}[\eta_2],\ldots,\mathcal{R}[\eta_k],1)\mathfrak{P}^t_{\mathbf{R}} \\ &= \mathfrak{P}_{\mathbf{R}}\mathfrak{D}_{\mathbf{R}}\mathfrak{P}^t_{\mathbf{R}} \\ &= \mathbf{R}. \end{aligned}$$

Therefore the argument can be completed as in the previous case. Plainly once $\mathbf{A}$ has been *fixed* as described $\mathbf{Q}_\circ$ can be *uniquely* expressed as the value of $e^{-\beta(a)\mathbf{A}}$. □

Remark 2.2. Note that apart from a *scaling* factor the function β is the fundamental solution for the Laplace operator on $\mathbb{R}^n$ (see, e.g., [7], p. 51). Indeed, by utilising (2.5) this can be justified since here

$$\begin{aligned} [\Delta_x\beta]\mathbf{A} &= \frac{1}{r^{n-1}}\frac{d}{dr}\left\{r^{n-1}\frac{d\beta}{dr}\right\}\mathbf{A} \\ &= \frac{1}{r^{n-1}}\frac{d}{dr}\left\{\left[r^{n-1}\frac{d\beta}{dr}\mathbf{A}\mathbf{Q}\right]\mathbf{Q}^t\right\} \\ &= \frac{1}{r^{n-1}}\frac{d}{dr}\left\{\left[r^{n-1}\frac{d}{dr}\mathbf{Q}\right]\mathbf{Q}^t\right\} = 0 \end{aligned}$$

with $\mathbf{Q}(r) = e^{\beta(r)\mathbf{A}}\mathbf{Q}_\circ$.

Theorem 2.2. *The solution* $\mathbf{Q}$ *described in Theorem* 2.1 *can be alternatively expressed in the following form.*
[**a**] $(n = 2k)$

$$\mathbf{Q} = \mathbf{Q}(r; a, b, \mathbf{m}) = \mathfrak{P}_{\mathbf{R}}\mathbf{P} diag(\mathcal{R}[g_1](r), \mathcal{R}[g_2](r), \ldots, \mathcal{R}[g_k](r))\mathbf{P}^t\mathfrak{P}^t_{\mathbf{R}},$$

[**b**] $(n = 2k + 1)$

$$\mathbf{Q} = \mathbf{Q}(r; a, b, \mathbf{m}) = \mathfrak{P}_{\mathbf{R}} \mathbf{P} diag(\mathcal{R}[g_1](r), \mathcal{R}[g_2](r), \dots, \mathcal{R}[g_k](r), 1) \mathbf{P}^t \mathfrak{P}^t_{\mathbf{R}}.$$

In either of the cases [**a**] *and* [**b**] *above we have* $\mathbf{P} \in \mathbf{C}[\mathfrak{D}_{\mathbf{R}}]$ *and* $\mathbf{m} = (m_1, \dots, m_k)$ *with* $m_j \in \mathbb{Z}$ *for all* $1 \le j \le k$ *while*

$$g_j(r) = \frac{\beta(r) - \beta(a)}{\beta(b) - \beta(a)} \Big[\eta_j + 2\pi m_j \Big].$$

Proof. Let $\mathbf{Q}$ denote the solution as described in Theorem 2.1. Then substituting for $\mathbf{Q}_\circ$ we have that

$$\mathbf{Q}(r) = e^{\beta(r)\mathbf{A}}\, \mathbf{Q}_\circ = e^{\beta(r)\mathbf{A}} e^{-\beta(a)\mathbf{A}} = e^{[\beta(r) - \beta(a)]\mathbf{A}}.$$

Now suppose that $(m_j)_{j=1}^k$ is an arbitrary sequence of *integers*. Then referring to Theorem 2.1 and using the block diagonal form of $\mathbf{A}$ whilst observing the identity

$$\zeta_j = s^{-1}(\eta_j + 2\pi m_j)$$

we obtain the following expressions for the solution $\mathbf{Q}$.
[**a**] $(n = 2k)$

$$\begin{aligned} \mathbf{Q} &= \mathbf{Q}(r; a, b, \mathbf{m}) \\ &= e^{\beta(r)\mathbf{A}}\, \mathbf{Q}_\circ \\ &= e^{[\beta(r)-\beta(a)]\mathfrak{P}_{\mathbf{R}} \mathbf{P} diag(\zeta_1 \mathcal{J}, \zeta_2 \mathcal{J}, \dots, \zeta_k \mathcal{J}) \mathbf{P}^t \mathfrak{P}^t_{\mathbf{R}}} \\ &= \mathfrak{P}_{\mathbf{R}} \mathbf{P} diag(\mathcal{R}[g_1](r), \mathcal{R}[g_2](r), \dots, \mathcal{R}[g_k](r)) \mathbf{P}^t \mathfrak{P}^t_{\mathbf{R}}, \end{aligned}$$

[**b**] $(n = 2k + 1)$

$$\begin{aligned} \mathbf{Q} &= \mathbf{Q}(r; a, b, \mathbf{m}) \\ &= e^{\beta(r)\mathbf{A}}\, \mathbf{Q}_\circ \\ &= e^{[\beta(r)-\beta(a)]\mathfrak{P}_{\mathbf{R}} \mathbf{P} diag(\zeta_1 \mathcal{J}, \zeta_2 \mathcal{J}, \dots, \zeta_k \mathcal{J}, 0) \mathbf{P}^t \mathfrak{P}^t_{\mathbf{R}}} \\ &= \mathfrak{P}_{\mathbf{R}} \mathbf{P} diag(\mathcal{R}[g_1](r), \mathcal{R}[g_2](r), \dots, \mathcal{R}[g_k](r), 1) \mathbf{P}^t \mathfrak{P}^t_{\mathbf{R}}. \end{aligned}$$

In either of the cases [**1**] and [**2**] above we have set

$$\begin{aligned} g_j(r) &:= [\beta(r) - \beta(a)]\zeta_j \\ &= \frac{\beta(r) - \beta(a)}{\beta(b) - \beta(a)} (\eta_j + 2\pi m_j) \end{aligned}$$

for all $r \in [a, b]$ and $1 \le j \le k$. This completes the proof. □

Note that by referring to the definition of the function β given in Theorem 2.1 we can alternatively express the twist *angles* g_j in the following more suggestive form.
[**1**] $(n = 2)$ As $k = 1$ setting $g := g_1$ for $m \in \mathbb{Z}$ and $r \in [a, b]$ we have that

$$g(r) = \frac{\log r/a}{\log b/a} (\eta + 2\pi m). \tag{2.9}$$

[**2**] ($n \geq 3$) For the sequence of integers $(m_j)_{j=1}^k$ and $r \in [a,b]$ we have that

$$g_j(r) = \frac{(r/a)^{2-n} - 1}{(b/a)^{2-n} - 1}(\eta_j + 2\pi m_j). \tag{2.10}$$

We end the section by giving explicit expressions for the energies of the solutions to the Euler–Lagrange equation associated with $e[\cdot]$ on $\mathcal{E}$ from Theorem 2.1. To this end we now proceed by considering the cases corresponding to $n = 2$ and $n \geq 3$ separately.
[**1**] ($n = 2$) Here we have that

$$e[\mathbf{Q}] = \frac{\pi}{2}\int_a^b \left\{\frac{2}{r^2} + |\dot{\mathbf{Q}}(r)|^2\right\} r\,dr = \frac{\pi}{2}\int_a^b (2 + |\mathbf{A}|^2)\frac{dr}{r} = \pi(1+\zeta^2)\log\frac{b}{a}. \tag{2.11}$$

where $\pm i\zeta$ denote the eigen-values of the *skew*-symmetric matrix $\mathbf{A}$.
[**2**] ($n \geq 3$) Here, again, we have that

$$\begin{aligned} e[\mathbf{Q}] &= \frac{\omega_n}{2}\int_a^b \left\{n(n-1)\frac{1}{r^2} + \frac{1}{r^{2(n-1)}}|\mathbf{AQ}|^2\right\} r^{n-1}\,dr \\ &= n\frac{\omega_n}{2}\left[\frac{n-1}{n-2}(b^{n-2} - a^{n-2}) + \frac{1}{n}\int_a^b \frac{1}{r^{n-1}}|\mathbf{A}|^2\,dr\right] \\ &= n\frac{\omega_n}{2}\left[(n-1) + \frac{2}{n}\frac{1}{(ab)^{n-2}}\sum_{j=1}^k \zeta_j^2\right]\frac{b^{n-2} - a^{n-2}}{n-2}. \end{aligned} \tag{2.12}$$

where depending on n being *even* ($n = 2k$) or *odd* ($n = 2k+1$) the quantities $\pm i\zeta_1, \ldots, \pm i\zeta_k$ or $\pm i\zeta_1, \ldots, \pm i\zeta_k, 0$ denote the eigen-values of the *skew*-symmetric matrix $\mathbf{A}$.

Alternatively using Theorem 2.2 we can re-write the energy $e[\mathbf{Q}]$ in both [**1**] and [**2**] above in the forms:
[**1**] ($n = 2$) with $\mathbf{Q} = \mathbf{Q}(r; a, b, m)$ we have $e[\mathbf{Q}] = \pi s[1 + (\eta + 2\pi m)^2 s^{-2}]$ where $s = \beta(b) - \beta(a) = \log b/a$,
[**2**] ($n \geq 3$) with $\mathbf{Q} = \mathbf{Q}(r; a, b, \mathbf{m})$ we have $e[\mathbf{Q}] = \omega_n s/2[n(n-1)(ab)^{n-2} + 2\sum_{1\leq j\leq k}(\eta_j + 2\pi m_j)^2 s^{-2}]$ where $s = \beta(b) - \beta(a) = (a^{2-n} - b^{2-n})/(n-2)$.

3. Harmonic twists as solutions to the harmonic map equation

We begin this section by introducing the notion of a *harmonic* twists, that is, a *twice* continuously differentiable *spherical* twist that is a harmonic map.

Definition 3.1. (*Harmonic twist*)
Let $\mathbf{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$. A *harmonic* twist u on $\mathbf{X}$ is a *spherical* twist on $\mathbf{X}$ that satisfies the following:

[1] $u \in \mathbf{C}(\overline{\mathbf{X}}, \mathbb{S}^{n-1})$,
[2] $u \in \mathbf{C}^2(\mathbf{X}, \mathbb{S}^{n-1})$,
[3] $\Delta u + |\nabla u|^2 u = 0$ *in* $\mathbf{X}$.

Here we aim to extract from amongst solutions in Theorem 2.1 those that constitute the twist *path* of a *harmonic* twist. Before confronting this however we find it helpful to discuss a condition on the matrix $\mathbf{R}$ that will indeed turn to be both *necessary* and *sufficient* for the existence of such *harmonic* twists (cf. Remark 3.1 below).

Remark 3.1. As seen the Euler–Lagrange equation (2.5) admits *infinitely* many solutions (cf. Theorem 2.2). The situation is completely different for harmonic twists. Indeed it will become clear that here solvability and multiplicity depend crucially on a structural property of $\mathbf{R}$. In fact a necessary and sufficient condition for this can be formulated depending on the dimension being *even* or *odd* as follows.
[**1**] ($n = 2k$) It must be that $\eta_1 = \eta_2 = \cdots = \eta_k := \eta$ (with $\eta \in [0, \pi]$) and hence

$$\begin{aligned} \mathbf{R} &= \mathfrak{P}_{\mathbf{R}} \mathfrak{D}_{\mathbf{R}} \mathfrak{P}^t_{\mathbf{R}} \\ &= \mathfrak{P}_{\mathbf{R}} diag(\mathcal{R}[\eta_1], \mathcal{R}[\eta_2], \ldots, \mathcal{R}[\eta_k]) \mathfrak{P}^t_{\mathbf{R}} \\ &= \mathfrak{P}_{\mathbf{R}} diag(\mathcal{R}[\eta], \mathcal{R}[\eta], \ldots, \mathcal{R}[\eta]) \mathfrak{P}^t_{\mathbf{R}}. \end{aligned} \tag{3.1}$$

[**2**] ($n = 2k + 1$) It must be that $\eta_1 = \eta_2 = \cdots = \eta_k = 0$ and hence

$$\mathbf{R} = \mathbf{I}_n \tag{3.2}$$

(See Remark 2.1 for notation.)

Theorem 3.1. *Let u be the spherical twist on $\mathbf{X}$ with twist path $\mathbf{Q}$ described in Theorem 2.1. Then u is a harmonic twist if and only if the following conditions hold.*
[**1**] ($n = 2k$) $\mathbf{R}$ *must be as in* (3.1) *while*

$$\begin{aligned} \mathbf{A} &= \mathfrak{P}_{\mathbf{R}} \mathbf{P} diag(\zeta_1 \mathcal{J}, \zeta_2 \mathcal{J}, \ldots, \zeta_k \mathcal{J}) \mathbf{P}^t \mathfrak{P}^t_{\mathbf{R}} \\ &= \mathfrak{P}_{\mathbf{R}} \mathbf{P} diag(\zeta \mathcal{J}, \zeta \mathcal{J}, \ldots, \zeta \mathcal{J}) \mathbf{P}^t \mathfrak{P}^t_{\mathbf{R}}. \end{aligned} \tag{3.3}$$

Here $\mathbf{P} \in \mathbf{C}[\mathfrak{D}_{\mathbf{R}}]$ and the sequence $(\zeta_j)_{j=1}^k \subset \mathbb{R}$ is so that $\zeta_1 = \zeta_2 = \cdots = \zeta_k =: \zeta$ where

$$\zeta = \frac{1}{s}\Big[\eta + 2\pi m\Big].$$

As before, $s = \beta(b) - \beta(a)$ and $m \in \mathbb{Z}$.
[**2**] ($n = 2k + 1$) $\mathbf{R}$ *must be as in* (3.2) *while*

$$\mathbf{A} = \mathbf{0}. \tag{3.4}$$

Proof. Let u be a *spherical* twist on $\mathbf{X}$ with twist path $\mathbf{Q}$ as in Theorem 2.1. We show in order for u to be a *harmonic* twist the *skew-symmetric* matrix $\mathbf{A}$

has to be *further* restricted as described in [**1**] and [**2**] above. Indeed to begin note that a straight-forward differentiation gives

$$\dot{\mathbf{Q}} = \frac{1}{r^{n-1}}\mathbf{A}\mathbf{Q}$$
$$\ddot{\mathbf{Q}} = \frac{1-n}{r^n}\mathbf{A}\mathbf{Q} + \frac{1}{r^{2(n-1)}}\mathbf{A}^2\mathbf{Q}.$$

Therefore in light of Proposition 2.2 upon substituting for these quantities we can write

$$\begin{aligned}\Delta u + |\nabla u|^2 u &= \left(\ddot{\mathbf{Q}} + \frac{n-1}{r}\dot{\mathbf{Q}} + |\dot{\mathbf{Q}}\theta|^2\mathbf{Q}\right)\theta \\ &= \left(\frac{1-n}{r^n}\mathbf{A}\mathbf{Q} + \frac{1}{r^{2(n-1)}}\mathbf{A}^2\mathbf{Q} + \frac{n-1}{r^n}\mathbf{A}\mathbf{Q} + \frac{1}{r^{2(n-1)}}|\mathbf{A}\mathbf{Q}\theta|^2\mathbf{Q}\right)\theta \\ &= \frac{1}{r^{2(n-1)}}(\mathbf{A}^2 + |\mathbf{A}\mathbf{Q}\theta|^2\mathbf{I}_n)\mathbf{Q}\theta = 0.\end{aligned}$$

Setting $\omega = \mathbf{Q}\theta$ it is then evident that the above is *equivalent* to the identity

$$[\mathbf{A}^2 + |\mathbf{A}\omega|^2\mathbf{I}_n]\omega = 0,$$

for all $\omega \in \mathbb{S}^{n-1}$. Hence an application of Proposition 4.3 to this gives $\mathbf{A}^2 = -s\mathbf{I}_n$ for some $s \geq 0$. Now in order to proceed further we consider the cases of *even* and *odd* dimensions separately.

[**1**] $(n = 2k)$

$$\begin{aligned}\mathbf{A}^2 = -s\mathbf{I}_n &\iff [\mathfrak{P}_{\mathbf{R}}\mathbf{P}diag(\zeta_1\mathcal{J}, \zeta_2, \mathcal{J}, \ldots, \zeta_k\mathcal{J})\mathbf{P}^t\mathfrak{P}^t_{\mathbf{R}}]^2 = -s\mathbf{I}_n \\ &\iff \mathfrak{P}_{\mathbf{R}}\mathbf{P}diag(\zeta_1^2\mathbf{I}_2, \zeta_2^2\mathbf{I}_2, \ldots, \zeta_k^2\mathbf{I}_2)\mathbf{P}^t\mathfrak{P}^t_{\mathbf{R}} = s\mathbf{I}_n \\ &\iff diag(\zeta_1^2\mathbf{I}_2, \zeta_2^2\mathbf{I}_2, \ldots, \zeta_k^2\mathbf{I}_2) = s\mathbf{I}_n \\ &\iff \zeta_1^2 = \zeta_2^2 = \cdots = \zeta_k^2 = s. \end{aligned} \tag{3.5}$$

As a result for $1 \leq j, j' \leq k$ we have that either $\zeta_j = \zeta_{j'}$ or $\zeta_j = -\zeta_{j'}$. We now describe the implication of each of these *two* identities separately. Indeed using (2.7) we can write

$$\begin{aligned}\zeta_j = \zeta_{j'} &\iff \eta_j + 2\pi m_j = \eta_{j'} + 2\pi m_{j'} \\ &\iff \eta_j - \eta_{j'} = -2\pi(m_j - m_{j'}) \\ &\iff m_j = m_{j'} \\ &\iff \eta_j = \eta_{j'},\end{aligned}$$

as $\eta_j - \eta_{j'} \in [-\pi, \pi]$. On the other hand

$$\begin{aligned}\zeta_j = -\zeta_{j'} &\iff \eta_j + 2\pi m_j = -(\eta_{j'} + 2\pi m_{j'}) \\ &\iff \eta_j + \eta_{j'} = -2\pi(m_j + m_{j'}) \\ &\iff \eta_j = \eta_{j'} \in \{0, \pi\},\end{aligned}$$

as $\eta_j + \eta_{j'} \in [0, 2\pi]$ and so

$$\zeta_j = -\zeta_{j'} \iff \begin{cases} \eta_j = \eta_{j'} = 0, \\ m_j = -m_{j'}, \\ \text{or}, \\ \eta_j = \eta_{j'} = \pi, \\ m_j = -(m_{j'} + 1). \end{cases}$$

Hence, *summarising*, in either of these cases we have that $\eta_1 = \eta_2 = \cdots = \eta_k := \eta$ with $\eta \in [0, \pi]$. As a consequence depending on η we have the following *three* distinct possibilities.

Case 1. ($\eta = 0$)
Here $m_j \in \{\pm m\}$ for all $1 \le j \le k$ with $m \in \mathbb{Z}$ and so $|\zeta_1| = |\zeta_2| = \cdots = |\zeta_k| = |\zeta|$ with

$$\zeta = 2\pi s^{-1} m.$$

Evidently $\eta = 0 \iff \mathbf{R} = \mathbf{I}_n$. Therefore here $\mathbf{C}[\mathfrak{D}_\mathbf{R}] = \mathbf{O}(n)$. In particular as $\mathbf{P} \in \mathbf{O}(n)$ in (3.3) is arbitrary we can arrange without any loss of generality that $\zeta_1 = \zeta_2 = \cdots = \zeta_k = \zeta$.

Case 2. ($\eta \in]0, \pi[$)
Here $m_1 = m_2 = \cdots = m_k =: m$ with $m \in \mathbb{Z}$ and so $\zeta_1 = \zeta_2 = \cdots = \zeta_k = \zeta$ with

$$\zeta = s^{-1}(\eta + 2\pi m).$$

Evidently $\eta \in]0, \pi[\iff \mathbf{R} \notin \{\pm \mathbf{I}_n\}$ and therefore here $\mathbf{C}[\mathfrak{D}_\mathbf{R}] \subsetneq \mathbf{O}(n)$.

Case 3. ($\eta = \pi$)
Here $m_j \in \{m, -(m+1)\}$ for all $1 \le j \le k$ with $m \in \mathbb{Z}$ and so $|\zeta_1| = |\zeta_2| = \cdots = |\zeta_k| = |\zeta|$ with

$$\zeta = s^{-1}(\pi + 2\pi m).$$

Evidently $\eta = \pi \iff \mathbf{R} = -\mathbf{I}_n$. Therefore as in [**1a**], $\mathbf{C}[\mathfrak{D}_\mathbf{R}] = \mathbf{O}(n)$. Again as $\mathbf{P} \in \mathbf{O}(n)$ in (3.3) is arbitrary we can arrange without any loss of generality that $\zeta_1 = \zeta_2 = \cdots = \zeta_k = \zeta$.

[**2**] ($n = 2k + 1$)

$$\begin{aligned} \mathbf{A}^2 = -s\mathbf{I}_n &\implies 0 = (\det \mathbf{A})^2 = \det \mathbf{A}^2 \\ &\implies s = 0 \\ &\implies \mathbf{A} = 0. \end{aligned}$$

(Note that in *odd* dimensions any *skew*-symmetric matrix has zero determinant.) The proof is thus complete. □

Theorem 3.2. *Let u be the spherical twist on $\mathbf{X}$ with twist path $\mathbf{Q} = \mathbf{Q}(r; a.b, \mathbf{m})$ as given in Theorem 2.2. Then u is a harmonic twist if and only if the following conditions hold.*
[**1**] ($n = 2k$) $\mathbf{R}$ *must be as in (3.1) and then*

$$\begin{aligned} \mathbf{Q}(r) &= \mathfrak{P}_\mathbf{R} \mathbf{P} diag(\mathcal{R}[g_1](r), \mathcal{R}[g_2](r), \ldots, \mathcal{R}[g_k](r)) \mathbf{P}^t \mathfrak{P}^t_\mathbf{R} \\ &= \mathfrak{P}_\mathbf{R} \mathbf{P} diag(\mathcal{R}[g](r), \mathcal{R}[g](r), \ldots, \mathcal{R}[g](r)) \mathbf{P}^t \mathfrak{P}^t_\mathbf{R}. \end{aligned}$$

Here $\mathbf{P} \in \mathbf{C}[\mathfrak{D}_{\mathbf{R}}]$ *and the sequence* $(g_j)_{j=1}^k$ *is such that* $g_1 = g_2 = \cdots = g_k =: g$ *where depending on* $n = 2$ *or* $n \geq 4$ *we have that*
[**1a**] $(n = 2)$

$$g(r) = \frac{\log r/a}{\log b/a}(\eta + 2\pi m).$$

[**1b**] $(n \geq 4)$

$$g(r) = \frac{(r/a)^{2-n} - 1}{(b/a)^{2-n} - 1}(\eta + 2\pi m).$$

[**2**] $(n = 2k + 1)$ $\mathbf{R}$ *must be as in* (3.2) *and then*

$$\mathbf{Q}(r) = \mathbf{I}_n,$$

i.e., the twist path $\mathbf{Q}$ *is the constant path at* $\mathbf{I}_n$*.*

Proof. This follows at once from Theorem 3.1 by substituting for $\mathbf{A}$ from (3.3) or (3.4) into (2.6) and evaluating the corresponding *exponential* term as in Theorem 2.2. □

4. Appendix

Recall from linear algebra that *all* eigen-values of a [real] *skew*-symmetric matrix have zero *real* parts. Hence they *either* appear as *purely* imaginary conjugate pairs *or* zero. In particular when n is *odd* there is necessarily a zero eigen-value. Thus distinguishing between the cases when n is *even* and *odd* respectively we can bring every *skew-symmetric* matrix to a *block* diagonal form. In what follows we set

$$\mathcal{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4.1}$$

Proposition 4.1. *Let* $\mathbf{A} \in \mathbb{M}_{n\times n}$ *be skew-symmetric. There exist* $\mathbf{P} \in \mathbf{O}(n)$ *and* $(\zeta_j)_{j=1}^k \subset \mathbb{R}$ *such that the following hold.*
[**1**] $(n = 2k)$

$$\mathbf{A} = \mathbf{P} diag(\zeta_1\mathcal{J}, \zeta_2\mathcal{J}, \ldots, \zeta_k\mathcal{J})\mathbf{P}^t,$$

[**2**] $(n = 2k + 1)$

$$\mathbf{A} = \mathbf{P} diag(\zeta_1\mathcal{J}, \zeta_2\mathcal{J}, \ldots, \zeta_k\mathcal{J}, 0)\mathbf{P}^t.$$

Proof. Indeed, here, $\mathbf{A}$ is *normal* (i.e., it commutes with its *transpose* $\mathbf{A}^t = -\mathbf{A}$) and so the conclusion follows from the the well-known *spectral* theorem.[5] □

[5]Note that the choices of $\mathbf{P}$ and $(\zeta_j)_{j=1}^k$ are in general *non-unique*. Indeed it is a *trivial* matter to see that by suitably adjusting $\mathbf{P}$ one can replace *any* ζ_j with $-\zeta_j$.

With the aid of the above representation evaluating the exponential function for skew-symmetric matrices becomes remarkably convenient. In what follows we set

$$\mathcal{R}[s] := \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}. \tag{4.2}$$

Proposition 4.2. *Let* $\mathbf{A} \in \mathbb{M}_{n\times n}$ *be skew-symmetric. Then using the notation in Proposition 4.1 we have that*
[**1**] $(n = 2k)$

$$e^{s\mathbf{A}} = \mathbf{P}diag(\mathcal{R}[s\zeta_1], \mathcal{R}[s\zeta_2], \dots, \mathcal{R}[s\zeta_k])\mathbf{P}^t,$$

[**2**] $(n = 2k + 1)$

$$e^{s\mathbf{A}} = \mathbf{P}diag(\mathcal{R}[s\zeta_1], \mathcal{R}[s\zeta_2], \dots, \mathcal{R}[s\zeta_k], 1)\mathbf{P}^t.$$

Proof. A straight-forward calculation gives

$$e^{s\mathcal{J}} = \sum_{n=0}^{\infty} \frac{1}{n!} s^n \mathcal{J}^n = \mathcal{R}[s].$$

The conclusion now follows by noting that for any block *diagonal* matrix $\mathbf{D}$ (as, e.g., in Proposition 4.1) we can write

$$e^{\mathbf{A}} = e^{\mathbf{PDP}^t} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^t.$$

□

Proposition 4.3. *Let* $\mathbf{A} \in \mathbb{M}_{n\times n}$ *be skew-symmetric. Then the following are equivalent.*
[**1**] $\mathbf{A}^2 = -s\mathbf{I}_n$ *for some* $s \geq 0$.
[**2**] $[\mathbf{A}^2 + |\mathbf{A}\omega|^2\mathbf{I}_n]\omega = 0$ *for all* $\omega \in \mathbb{S}^{n-1}$.

Proof. The implication ([**1**] $\implies$ [**2**]) follows by direct verification. Now for the *reverse* implication consider re-writting [**2**] in the form

$$\mathbf{A}^2\omega = -|\mathbf{A}\omega|^2\omega.$$

Then for any $\omega \in \mathbb{S}^{n-1}$ the quantity $-|\mathbf{A}\omega|^2$ is the associated eigen-value. However since $\mathbf{A}^2$ has at most n distinct eigen-values it follows from the continuity of $\omega \mapsto |\mathbf{A}\omega|^2$ that the latter must be *constant* (say s) and this gives [**1**]. □

Similar to the case of *skew*-symmetric matrices we can bring any *orthogonal* matrix to a block diagonal form. Below we specialise to the case of the *special* orthogonal group.[6]

[6]Note that the exponential map acts between the Lie algebra of *skew*-symmetric matrices in $\mathbb{M}_{n\times n}$ onto its corresponding Lie group $\mathbf{SO}(n)$.

Proposition 4.4. *Let* $\mathbf{R} \in \mathbf{SO}(n)$. *There exist* $\mathbf{P} \in \mathbf{O}(n)$ *and* $(\eta_j)_{j=1}^k \subset \mathbb{R}$ *such that the following hold.*
[**1**] $(n = 2k)$

$$\begin{aligned}\mathbf{R} &= \mathbf{P} diag(\mathcal{R}[\eta_1], \mathcal{R}[\eta_2], \ldots, \mathcal{R}[\eta_k])\mathbf{P}^t \\ &= \mathbf{P} e^{diag(\eta_1 \mathcal{J}, \eta_2 \mathcal{J}, \ldots, \eta_k \mathcal{J})} \mathbf{P}^t\end{aligned}$$

[**2**] $(n = 2k + 1)$

$$\begin{aligned}\mathbf{R} &= \mathbf{P} diag(\mathcal{R}[\eta_1], \mathcal{R}[\eta_2], \ldots, \mathcal{R}[\eta_k], 1)\mathbf{P}^t \\ &= \mathbf{P} e^{diag(\eta_1 \mathcal{J}, \zeta_2 \mathcal{J}, \ldots, \eta_k \mathcal{J}, 0)} \mathbf{P}^t.\end{aligned}$$

Proof. Again, **R**, here, is normal and so the conclusion follows from the *spectral* theorem. □

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Received: 4 October 2010.
Accepted: 7 May 2011.