Results Math (2018) 73:63

1422-6383/18/020001-13
published online April 13, 2018
https://doi.org/10.1007/s00025-018-0824-7

Results in Mathematics

Examples of Minimal G-structures

Kamil Niedziałomski

Abstract. Let M be an oriented Riemannian manifold and $SO(M)$ its oriented orthonormal frame bundle. Assume there exists a reduction $P \subset SO(M)$ of the structure group $SO(\dim M)$ to a subgroup G. We say that a G-structure M is minimal if P is a minimal submanifold of $SO(M)$, where we equip $SO(M)$ in the natural Riemannian metric. We give non-trivial examples of minimal G-structures for $G = U(\dim M/2)$ and $G = U((\dim M - 1)/2) \times 1$ having some special features—locally conformally Kähler and α-Kenmotsu manifolds, respectively.

Mathematics Subject Classification. 53C10, 53C25, 53C43.

Keywords. G-structure, intrinsic torsion, minimal submanifold, locally conformally Kähler manifold, α-Kenmotsu manifold.

1. Introduction

Existence of a geometric structure compatible with a Riemannian metric on an oriented manifold is equivalent to reduction of the structure group of oriented orthonormal frame bundle $SO(M)$ to certain subgroup $G \subset SO(n)$, $n = \dim M$. For example, almost Hermitian structure is defined by the unitary group $U(m) \subset SO(2m)$, almost contact structure by $U(m)\times 1 \subset SO(2m+1)$ or an almost quaternion–Hermitian structure by $Sp(m)Sp(1) \subset SO(4m)$. Considering additionally the Levi-Civita connection ∇ we may ask if this connection is compatible with the given reduction. The failure is measured by the intrinsic torsion. In particular, if the intrinsic torsion vanishes, the holonomy algebra is contained in the Lie algebra $\mathfrak{g}$ of the structure group.

We may classify intrinsic torsion with respect to the action of G obtaining irreducible components, often called Grey–Hervella classes. Another approach was initiated by Wood [11,12] and, in general case, by Martin-Cabrera and

Gonzalez-Davilla [3] by studying harmonicity of induced section of certain homogeneous associated bundle. Namely, the reduction of the structure group gives a subbundle $P \subset SO(M)$, which defines the unique section σ_P of the bundle $N = SO(M)/G = SO(M) \times_{SO(n)} (SO(n)/G)$. If σ_P is a harmonic section we call G-structure harmonic. In [8] the author studied properties of the intrinsic torsion by considering extrinsic geometry of a reduction P inside $SO(M)$ (The Riemannian metric on $SO(M)$ is induced from Riemannian metric on M and Killing form on $SO(n)$). If P is a minimal submanifold in $SO(M)$ we call G-structure minimal. In [8] the author gave necessary and sufficient condition for minimality of a G-structure (being the condition on the intrinsic torsion). However, this condition is quite complicated and therefore it is not easy to give nontrivial examples of minimal G-structures.

This note shows that results developed in [8] are not trivially satisfied, i.e., we provide non-trivial examples of minimal G-structures. We concentrate on the cases for $G = U(m)$ and $G = U(m) \times 1$. The dimension of the space of all possible intrinsic torsions is, in general, quite big. Thus, it is convenient to restrict attention to certain subclass. Here, we focus on locally conformally Kähler (for $G = U(m)$) and α-Kenmotsu manifolds (for $G = U(m) \times 1$). The advantage of such restriction, is that condition for minimality of a G-structure takes relatively simple form in each case, since it depends only on a (closed) one form, called the Lee form, and a single function, respectively. On the other hand the considered classes of manifolds are still large enough to find nontrivial examples. These include Hopf manifolds (then the Lee form is parallel) and Kenmotsu manifolds (then $\alpha = 1$) of constant sectional curvature.

We begin by recalling basic information about the intrinsic torsion, harmonicity and minimality of G-structures. Then we compute the minimality condition for above mentioned structures. We conclude providing appropriate examples.

2. Minimal G-structures via the Intrinsic Torsion

All the information in this section can be found in [3,8]. Let (M, g) be an oriented Riemannian manifold. Consider an oriented orthonormal frame bundle $SO(M)$. Let ∇ denote the Levi-Civita connection of g. It induces the horizontal distribution $\mathcal{H} \subset TSO(M)$. Any vector $X \in TM$ has the unique lift X^h_p to $\mathcal{H}_p$, $p \in SO(M)$. Vertical distribution $\mathcal{V} = \ker \pi_*$, where $\pi : SO(M) \to M$ is a natural projection, is pointwise, isomorphic to the Lie algebra $\mathfrak{so}(n)$ of the structure group $SO(n)$. Denote by A^* the fundamental vertical vector field induced by an element $A \in \mathfrak{so}(n)$. The Riemannian metric on $SO(M)$ is given as follows:

$$g_{SO(M)}(X^h, Y^h) = g(X, Y),$$
$$g_{SO(M)}(X^h, A^*) = 0,$$

$$g_{SO(M)}(A^*, B^*) = -\mathrm{tr}(AB),$$

where $X \in TM$, $A \in \mathfrak{so}(n)$. Define a structure on M by restricting the structure group $SO(n)$ to a subgroup G such that on the level of Lie algebras, the following decomposition

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^{\perp} \tag{1}$$

is $\mathrm{ad}(G)$-invariant ($\mathfrak{g}^{\perp}$ denotes the orthogonal complement with respect to the Killing form).

We say that a G-structure M is *minimal* if the induced subbundle P with the structure group G is minimal as a submanifold of $SO(M)$. Let us now define the intrinsic torsion and formulate the condition of harmonicity and minimality using this notion. Let ω be the connection form of the horizontal distribution $\mathcal{H}$ (induced by ∇). By the invariance of the splitting (1) the decomposition

$$\omega = \omega_{\mathfrak{g}} + \omega_{\mathfrak{g}^{\perp}}$$

defines a connection $\omega_{\mathfrak{g}}$ on P. Denote the horizontal distribution induced by $\omega_{\mathfrak{g}}$ by $\mathcal{H}'$ and the associated horizontal lift of $X \in TM$ by $X^{h'}$. Define the *intrinsic torsion* by the formula

$$\xi_X = -\,\omega_{\mathfrak{g}^{\perp}}\left(X_p^{h'}\right), \quad X \in T_xM, \quad \pi(p) = x.$$

By $\mathrm{ad}(G)$-invariance of $\omega_{\mathfrak{g}^{\perp}}$ and the horizontal lift, it follows that ξ_X is defined up to the adjoint action, thus is an element of the adjoint bundle $\mathfrak{g}_P^{\perp} = P \times_{\mathrm{ad}(G)} \mathfrak{g}^{\perp}$. Thus we may treat ξ_X as an endomorphism $\xi_X : TM \to TM$. It follows by above considerations that

$$\xi_X = \nabla_X^G - \nabla_X,$$

where ∇^G is a metric connection on M induced by $\omega_{\mathfrak{g}}$.

The reduction $P \subset SO(M)$ defines the unique section σ_P of the associated bundle $N = SO(M) \times_{SO(n)} (SO(n)/G)$,

$$P \ni p \mapsto [[p, eG]] \in N,$$

where $e \in SO(n)$ is the identity element. We define a Riemannian metric on N by inducing from the Riemannian metric g on M and the Killing form restricted to $\mathfrak{g}^{\perp}$. We say that a G-structure M is *harmonic* if a section $\sigma_P : M \to N$ is a harmonic section. Denote by $\mathbf{v}W$ the vertical component in TN of a vector $W \in TN$. Then [3] $\mathbf{v}\sigma_{P*}(X) = -\,\xi_X$, thus harmonicity is coded in the intrinsic torsion. Moreover, we say that a G-structure is a *harmonic map*, if the unique section σ_P is a harmonic map. Notice that notions of harmonicity and harmonicity as a map of a G-structure are different. Harmonic section do not need to be a harmonic map. In the former case we consider variations of the energy functional of the norm of the differential σ_* among sections, whereas in the later case among all maps from M to N (compare Proposition 1 below).

Let us state results obtained in [3,8] concerning harmonicity and minimality of G-structures. For any endomorphism $T : TM \to TM$ let

$$R_T(X) = \sum_j R(e_j, T(e_j))X, \quad X \in TM. \tag{2}$$

Proposition 1 ([3])**.** *A G-structure M is harmonic if and only if the following condition holds*

$$\sum_j (\nabla_{e_j}\xi)_{e_j} = 0, \tag{3}$$

where (e_j) is a g-orthonormal basis. Moreover, a G-structure M is a harmonic map if it is a harmonic G-structure and

$$\sum_j R_{\xi_{e_j}}(e_j) = 0.$$

Consider a Riemannian metric $\tilde{g}$ on M defined by

$$\tilde{g}(X,Y) = g(X,Y) + \sum_j g(\xi_X e_j, \xi_Y e_j), \quad X, Y \in TM, \tag{4}$$

where (e_j) is any g-orthonormal basis.

Proposition 2 ([8])**.** *A G-structure M is minimal if and only if the following condition holds*

$$\sum_j (\nabla_{\tilde{e}_j}\xi)_{\tilde{e}_j} + \xi_{R_{\xi_{\tilde{e}_j}}(\tilde{e}_j)} = 0, \tag{5}$$

where $(\tilde{e}_j)$ is any $\tilde{g}$-orthonormal basis. Alternatively, if and only if the section $\sigma_P : M \to N$ is a harmonic map, where we consider the Riemannian metric $\tilde{g}$ instead of g on M.

Remark 1. Recall that condition for harmonicity of a map $\sigma_P : (M, \tilde{g}) \to N$ is of the following form

$$\sum_j (\nabla_{\tilde{e}_j}\xi)_{\tilde{e}_j} = \sum_j \xi_{S_{\tilde{e}_j}\tilde{e}_j} \quad \text{and} \quad \sum_j R_{\xi_{\tilde{e}_j}}(\tilde{e}_j) = -\sum_j S_{\tilde{e}_j}\tilde{e}_j,$$

where S is the difference of the Levi-Civita connection $\tilde{\nabla}$ of the metric $\tilde{g}$ and the Levi-Civita connection ∇ of the metric g [8].

Notice that in [8] the author considered intrinsic torsion differing by the sign form the intrinsic torsion considered in this article and by other authors.

3. Examples of Minimal G-structures

In this section we will compute the condition (5) for $G = U(n)$ and $G = U(n) \times 1$ assuming that considered structures satisfy some additional properties. Let us begin by explaining our choices.

In general, the space of all possible intrinsic torsions $T^*M \otimes \mathfrak{g}_P^\perp$ (with the notation from the previous section) is large, so we restrict our approach to certain submodules. In some cases, the intrinsic torsion is given by concrete formula depending on a vector field or a function. This happens for locally conformally Kähler structures in the case $G = U(m)$ and α-Kenmotsu manifolds in the case $G = U(n) \times 1$. Then the intrinsic torsion, hence the condition of minimality, depends on a (closed) one form θ, called the Lee form, or a single function α, respectively. In these cases it is possible to find appropriate examples of manifolds satisfying condition of minimality.

3.1. Locally Conformally Kähler Structures

Let (M, g, J) be a Hermitian manifold, i.e., $J^2 = -\operatorname{id}_{TM}$, J is integrable and g-invariant,

$$g(JX, JY) = g(X, Y), \quad X, Y \in TM.$$

Assume that M is locally conformally Kähler (LcK, for short) [7,10]. Then, there exists closed one-form θ, called the Lee form, such that

$$d\Omega = \theta \wedge \Omega,$$

where Ω is the Kähler form, $\Omega(X, Y) = g(X, JY)$, $X, Y \in TM$. Moreover [7],

$$(\nabla_X J)Y = \frac{1}{2}\left(\theta(JY)X - \theta(Y)JX - g(X, JY)\theta^\sharp + g(X, Y)J\theta^\sharp\right). \tag{6}$$

Structure (M, g, J) induces the subbundle $U(M)$ of oriented orthonormal frame bundle $SO(M)$ with the structure group $G = U(n)$, $n = \frac{1}{2}\dim M$. On the level of Lie algebras we have the following splitting

$$\mathfrak{so}(2n) = \mathfrak{u}(n) \oplus \mathfrak{u}(n)^\perp,$$

where $\mathfrak{u}(n)^\perp$ is an orthogonal complement of $\mathfrak{u}(n)$ with respect to the Killing form on $\mathfrak{so}(2n)$,

$$\mathfrak{u}(n) = \{A \in \mathfrak{so}(2n) \mid AJ = JA\},$$
$$\mathfrak{u}(n)^\perp = \{A \in \mathfrak{so}(2n) \mid AJ = -JA\},$$

where J is considered here as a block matrix $\left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. The projection $\operatorname{pr}_{\mathfrak{u}(n)^\perp}$: $\mathfrak{so}(2n) \to \mathfrak{u}(n)^\perp$ respecting above decomposition is given by

$$\operatorname{pr}_{\mathfrak{u}(n)^\perp}(A) = \frac{1}{2}\left(A + JAJ\right).$$

Thus the intrinsic torsion ξ_X is given by the formula [2]

$$\xi_X = -\frac{1}{2}J(\nabla_X J).$$

which, by (6) implies

$$\xi_X Y = -\frac{1}{4}\left(\theta(Y)X + \theta(JY)JX - g(X, Y)\theta^\sharp - g(X, JY)J\theta^\sharp\right).$$

In further considerations, we will use the notion of $*$-Ricci tensor, which is defined as follows

$$\mathrm{Ric}^*(X) = \sum_j R(JX, Je_j)e_j, \quad X \in TM,$$

where (e_j) is any orthonormal basis.

Remark 2. Our definition of $*$-Ricci tensor differs slightly from the one considered, for example, in [3]. We have $\mathrm{Ric}^*(X,Y) = \overline{\mathrm{Ric}}^*(Y,X)$, where $\overline{\mathrm{Ric}}^*$ is the $*$-Ricci tensor in [3] and here we consider these tensors as $(0,2)$-tensors. Notice that $*$-Ricci is not, in general, symmetric.

We will compute the condition of minimality of a G-structure induced by LcK manifold. First of all, let us derive the formula for the Riemannian metric $\tilde{g}$. Denote by (e_j) any g-orthonormal basis. Then

$$\begin{aligned}\tilde{g}(X,Y) &= g(X,Y) + \sum_j g(\xi_X e_j, \xi_Y e_j) \\ &= g(X,Y) + \frac{1}{4}\left(g(X,Y)|\theta^\sharp|^2 - \theta(X)\theta(Y) - \theta(JX)\theta(JY)\right) \\ &= \left(1 + \frac{1}{4}|\theta^\sharp|^2\right) g(X,Y) - \frac{1}{4}\left(\theta(X)\theta(Y) + \theta(JX)\theta(JY)\right).\end{aligned}$$

Denote by $\mathcal{D}$ the J-invariant distribution spanned by the vector fields $\theta^\sharp$, $J\theta^\sharp$. Let $\mathcal{D}^\perp$ be the orthogonal complement of D in TM with respect to g. Notice that $X \in D^\perp$ if and only if $\theta(X) = \theta(JX) = 0$, which implies that for $X \in \mathcal{D}^\perp$ we have $\tilde{g}(X,Y) = \left(1 + \frac{1}{4}|\theta^\sharp|^2\right) g(X,Y)$. Thus by dimensional reasons, orthogonal complement to D with respect to $\tilde{g}$ is just $D^\perp$. Hence, there should be no confusion in writing $D^\perp$. Moreover, if $X \in \mathcal{D}$, then $\tilde{g}(X,Y) = g(X,Y)$. For a g-orthonormal basis (e_j) such that $e_{2n-1} = \frac{1}{|\theta^\sharp|}\theta^\sharp$ and $e_{2n} = \frac{1}{|\theta^\sharp|}J\theta^\sharp$, we define a related $\tilde{g}$-orthonormal basis by

$$\begin{aligned}\tilde{e}_1 &= \frac{1}{\sqrt{1 + \frac{1}{4}|\theta^\sharp|^2}} e_1, \ldots, \quad \tilde{e}_{2n-2} = \frac{1}{\sqrt{1 + \frac{1}{4}|\theta^\sharp|^2}} e_{2n-2}, \\ \tilde{e}_{2n-1} &= e_{2n-1}, \quad \tilde{e}_{2n} = e_{2n}.\end{aligned}$$

Before computing minimality condition, let us introduce one useful notation. For a vector $X \in TM$ put

$$X' = \sum_j g(X, \tilde{e}_j)\tilde{e}_j.$$

Let us collect properties of the assignment $X \mapsto X'$ in the Proposition below.

Proposition 3. *The following conditions hold:*

1. $X' = \frac{1}{1 + \frac{1}{4}|\theta^\sharp|^2} X$ *for* $X \in \mathcal{D}^\perp$,
2. $X' = X$ *for* $X \in \mathcal{D}$,

3. *in general,* $X' = \frac{1}{1+\frac{1}{4}|\theta^\sharp|^2}\left(X + \frac{1}{4}\left(g(X,\theta^\sharp)\theta^\sharp + g(X,J\theta^\sharp)J\theta^\sharp\right)\right)$, $X \in TM$,
4. $(JX)' = JX'$ *and* $\theta(X') = \theta(X)$ *for any* $X \in TM$.
5. $g(X',Y) = g(X,Y')$ *for any* $X,Y \in TM$.

After lengthy computations we get

$$\sum_j g((\nabla_{\tilde{e}_j}\xi)_{\tilde{e}_j}Y,Z) = -\frac{1}{4}((\nabla_{Z'}\theta)Y - (\nabla_{Y'}\theta)Z - (\nabla_{JZ'}\theta)JY + (\nabla_{JY'}\theta)JZ$$
$$- \theta((\nabla_{JZ'}J)Y) + \theta((\nabla_{JY'}J)Z)$$
$$+ \theta(JY)g(\mathrm{div}'J,Z) - \theta(JZ)g(\mathrm{div}'J,Y)),$$

where the divergence $\mathrm{div}'J$ equals $\mathrm{div}'J = \sum_j(\nabla_{\tilde{e}_j}J)\tilde{e}_j$. By (6) and Proposition 3 the assignment

$$(X,Y) \mapsto 2\theta((\nabla_{JX'}J)Y) = \theta(JX)\theta(JY) + \theta(X)\theta(Y) - g(X',Y)|\theta^\sharp|^2$$

is symmetric with respect to X and Y. Moreover,

$$2\mathrm{div}'J = \sum_j\left(\theta(J\tilde{e}_j)\tilde{e}_j - \theta(\tilde{e}_j)J\tilde{e}_j + |\tilde{e}_j|^2 J\theta^\sharp\right)$$
$$= -J\theta^\sharp - J\theta^\sharp + \left(\sum_j |\tilde{e}_j|^2\right)J\theta^\sharp$$
$$= \frac{2n-2}{1+\frac{1}{4}|\theta^\sharp|^2}J\theta^\sharp.$$

Thus the bilinear map $(X,Y) \mapsto \theta(JX)g(\mathrm{div}'J,Y)$ is also symmetric.

For any $X \in TM$,

$$R_{\xi_X}(X) = \sum_j R(e_j,\xi_X e_j)X = -\frac{1}{2}\left(R(\theta^\sharp,X)X - R(J\theta^\sharp,JX)X)\right).$$

Hence

$$\sum_j R_{\xi_{\tilde{e}_j}}(\tilde{e}_j) = -\frac{1}{2}\frac{1}{1+\frac{1}{4}|\theta^\sharp|^2}\left(\mathrm{Ric}(\theta^\sharp) - \mathrm{Ric}^*(\theta^\sharp)\right). \tag{7}$$

By (7) we get

$$\sum_j g(\xi_{R_{\xi_{\tilde{e}_j}}(\tilde{e}_j)}Y,Z) = \frac{1}{8}\frac{1}{1+\frac{1}{4}|\theta^\sharp|^2}(\theta(Y)g(\mathcal{R},Z) - \theta(Z)g(\mathcal{R},Y)$$
$$- \theta(JY)\Omega(\mathcal{R},Z) + \theta(JZ)\Omega(\mathcal{R},Y)),$$

where, to simplify notation, we put

$$\mathcal{R} = \mathrm{Ric}(\theta^\sharp) - \mathrm{Ric}^*(\theta^\sharp).$$

Concluding, by Proposition 2, a $U(n)$-structure on locally conformally Kähler manifold (M,g,J) with the Lee form θ is minimal $U(n)$ if and only if the following condition holds

$$\begin{aligned}0 = &(\nabla_{Z'}\theta)Y - (\nabla_{Y'}\theta)Z - (\nabla_{JZ'}\theta)JY + (\nabla_{JY'}\theta)JZ\\ &-\frac{1}{2}\frac{1}{1+\frac{1}{4}|\theta^\sharp|^2}(\theta(Y)g(\mathcal{R},Z) - \theta(Z)g(\mathcal{R},Y)\\ &-\theta(JY)\Omega(\mathcal{R},Z) + \theta(JZ)\Omega(\mathcal{R},Y))\end{aligned} \tag{8}$$

for all $Y, Z \in TM$.

To check validity of the condition (8), we may restrict to certain vectors Y, Z. Indeed, since the right hand side is skew-symmetric with respect to Y and Z, by linearity, we have the following four possibilities:

(i) $Y, Z \in \mathcal{D}^\perp$, (ii) $Y \in \mathcal{D}^\perp, Z = \theta^\sharp$,

(iii) $Y \in \mathcal{D}^\perp, Z = J\theta^\sharp$, (iv) $Y = \theta^\sharp, Z = J\theta^\sharp$.

Now we will use the fact that $d\theta = \mathrm{Alt}(\nabla\theta) = 0$. In the cases (i) and (iv), (8) is trivially satisfied. Finally, the cases (ii) and (iii) lead to the same condition

$$\theta(\nabla_Y\theta^\sharp) + \theta(\nabla_{JY}J\theta^\sharp) + 2g(\mathcal{R}, Y) = 0, \quad Y \in \mathcal{D}^\perp. \tag{9}$$

Thus we have proved the following result.

Theorem 1. *A $U(n)$-structure on a LcK manifold (M, g, J) is minimal if and only if (9) holds.*

Example 1. Let (M, g_0) be the Euclidean space $\mathbb{R}^{2n}$ with the canonical complex structure J. Consider the coordinates $(x_1, y_1, \ldots, x_n, y_n)$ with $J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$. Let f be arbitrary smooth function depending only on x_1, y_1 and consider the conformal deformation $g = e^{-2f}g_0$. We will compute the condition of minimality of (M, g, J). The Lee form equals $\theta = df$. Therefore

$$\mathcal{D}^\perp = \mathrm{span}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \mid i = 2, 3, \ldots, n\right\}.$$

Thus $\nabla^0_X\theta^\sharp = \nabla^0_{JX}J\theta^\sharp = 0$ for $X \in \mathcal{D}^\perp$. Hence, by the formula for the Levi-Civita connections of conformally related metrics we get

$$\begin{aligned}\nabla_X\theta^\sharp &= \nabla^0_X\theta^\sharp + |\theta^\sharp|^2X = |\theta^\sharp|^2X,\\ \nabla_{JX}J\theta^\sharp &= \nabla^0_{JX}J\theta^\sharp = 0,\end{aligned}$$

for $X \in \mathcal{D}^\perp$. Consequently, condition (9) simplifies to

$$g(\mathcal{R}, Y) = 0, \quad Y \in \mathcal{D}^\perp.$$

Recall that the curvature tensor R is given by the formula

$$\begin{aligned}-g(R(X,Y)Z,W) = &L(X,Z)g(Y,W) + L(Y,W)g(X,Z)\\ &- L(X,W)g(Y,Z) - L(Y,Z)g(X,W)\\ &+ e^{4f}|df|^2(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)),\end{aligned}$$

where $L(X,Y) = (\nabla^0_X df)Y + df(X)df(Y)$ and hessian is computed with respect to the Levi-Civita connection ∇^0 of the Euclidean metric g_0 [9]. Simple calculations lead to the equality

$$g(\mathcal{R},Y) = (2n-3)L(\theta^\sharp,Y) - L(J\theta^\sharp,JY), \quad Y \in \mathcal{D}^\perp.$$

Since f depends only on x_1 and y_1, it follows that both $L(\theta^\sharp,Y)$ and $L(J\theta^\sharp,JY)$ vanish. Thus (9) holds.

Theorem 2. *Assume (M,g,J) is with parallel Lee form θ. If the $U(n)$-structure on locally conformally Kähler manifold (M,g,J) is a harmonic map, then it is a minimal. In particular, the $U(n)$-structure on any Hopf manifold is minimal.*

Before we will prove the above theorem let us recall the notion of Hopf manifold [10]. Consider the complex space $\mathbb{C}^n \backslash \{0\}$ without the origin and denote by Δ_λ, where λ is nonzero complex number such that $|\lambda| \neq 1$, the cyclic group generated by the transformation $z \mapsto \lambda z$, $z \in \mathbb{C}^n$. The Hopf manifold is a quotient $(\mathbb{C}^n \backslash \{0\})/\Delta_\lambda$ equipped with the Hermitian metric induced from the Hermitian metric

$$h = \frac{1}{\sum_j z_j \bar{z}_j} \sum_j dz_j \otimes d\bar{z}_j$$

on $\mathbb{C}^n \backslash \{0\}$. It can be shown that Hopf manifold is diffeomorphic to the product $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$.

Proof (of Theorem 2). By (6) and the fact that $\theta^\sharp$ is parallel we have

$$\nabla_X J\theta^\sharp = (\nabla_X J)\theta^\sharp = \begin{cases} -\frac{1}{2}|\theta^\sharp|^2 JX & \text{for } \ X \in \mathcal{D}^\perp \\ 0 & \text{for } \ X \in \mathcal{D} \end{cases}$$

and $\xi_{\theta^\sharp} = \xi_{J\theta^\sharp} = 0$. Put, for simplicity, $c = \frac{1}{1+\frac{1}{4}|\theta^\sharp|^2}$. Then

$$\sum_j (\nabla_{\tilde{e}_j}\xi)_{\tilde{e}_j} = c\sum_j (\nabla_{e_j}\xi)_{e_j} + \frac{1-c}{|\theta^\sharp|^2}\left((\nabla_{\theta^\sharp}\xi)_{\theta^\sharp} + (\nabla_{J\theta^\sharp}\xi)_{J\theta^\sharp}\right) = c\sum_j (\nabla_{e_j}\xi)_{e_j}.$$

Moreover, by (7), we have

$$\sum_j R_{\xi_{\tilde{e}_j}}(\tilde{e}_j) = c\sum_j R_{\xi_{e_j}}(e_j).$$

Since, by assumption, a $U(n)$-structure on M is a harmonic map, then (see Proposition 1) $\sum_j (\nabla_{e_j}\xi)_{e_j} = 0$ and $\sum_j R_{\xi_{e_j}}(e_j) = 0$. Thus, by above considerations, $\sum_j (\nabla_{\tilde{e}_j}\xi)_{\tilde{e}_j} = 0$ and $\sum_j R_{\xi_{\tilde{e}_j}}(\tilde{e}_j) = 0$. In particular, by Proposition 2, a $U(n)$-structure on M is minimal. Thus we have proved the first part of the theorem. The second part follows by the fact that Hopf manifolds are examples of LcK manifolds, which define $U(n)$-structures being harmonic maps [3]. □

Remark 3. The above theorem can be derived directly from Theorem 1 and the characterization of $U(n)$-structures on locally conformally Kähler manifolds which are harmonic maps [3]. The mentioned condition is $g(\mathcal{R}, X) = 0$ for all $X \in TM$ [3, Theorem 4.11(iii)]. In particular, $g(\mathcal{R}, X) = 0$ for $X \in \mathcal{D}^\perp$, which is equivalent to (9) since we assume that $\theta^\sharp$ is parallel.

3.2. α-Kenmotsu Manifolds

Let $(M, g, \varphi, \eta, \zeta)$ be an almost contact metric structure (of dimension $2n+1$), i.e., the Riemannian metric g, endomorphism $\varphi : TM \to TM$, one-form η and a vector field ζ satisfy the following conditions

$$\varphi^2 = -\mathrm{Id}_{TM} + \eta \otimes \zeta, \qquad \eta = \zeta^\flat,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \qquad |\zeta|^2 = 1.$$

Then φ defines almost complex structure on the distribution $\mathcal{E}$ orthogonal to ζ. We call ζ the Reeb field. Such conditions imply reduction of the structure group of the oriented orthonormal frame bundle to $G = U(n) \times 1 \subset SO(2n+1)$. On the level of Lie algebras we have

$$\mathfrak{so}(2n+1) = \mathfrak{u}(n) \oplus \mathfrak{u}(n)^\perp,$$

where $\mathfrak{u}(n)^\perp$ is the orthogonal complement of $\mathfrak{u}(n)$ in $\mathfrak{so}(2n+1)$ with respect to the Killing form. The projection $\mathrm{pr}_{\mathfrak{u}(n)^\perp} : \mathfrak{so}(2n+1) \mapsto \mathfrak{u}(n)^\perp$ respecting the above decomposition is given by

$$\mathrm{pr}_{\mathfrak{m}}(A) = \frac{1}{2}\left(A + \varphi A \varphi + \eta A \otimes \zeta + \eta \otimes A\zeta\right).$$

Thus the intrinsic torsion equals [4]

$$\xi_X Y = \frac{1}{2}(\nabla_X \varphi)\varphi Y + \frac{1}{2}(\nabla_X \eta) Y \,\zeta - \eta(Y)\nabla_X \zeta. \tag{10}$$

Recall the following identities [1]

$$(\nabla_X \eta) Y = g(Y, \nabla_X \zeta) = \eta((\nabla_X \varphi)\varphi Y). \tag{11}$$

We focus on α-Kenmotsu manifolds, i.e., almost contact metric structures satisfying the following condition [1]

$$(\nabla_X \varphi) Y = \alpha(g(\varphi X, Y)\zeta - \eta(Y)\varphi X). \tag{12}$$

Here α is a smooth function on M. Notice that, originally, α-Kenmotsu manifold was defined for constant α [5] and for $\alpha = 1$ we obtain Kenmotsu manifold [6].

Comparing (12) and (10) we have

$$(\nabla_X \eta) Y = \alpha(g(X, Y) - \eta(X)\eta(Y)),$$
$$\nabla_X \zeta = \alpha(X - \eta(X)\zeta).$$

Hence, using (11) we get the formula for the intrinsic torsion on α-Kenmotsu manifold

$$\xi_X Y = \alpha(g(X, Y)\zeta - \eta(Y)X), \quad X, Y \in TM.$$

In particular,

$$\xi_\zeta = 0, \quad \xi_X \zeta = -\,\alpha \mathrm{pr} X.$$

By a simple computation we get

$$\tilde{g}(X,Y) = (1+2\alpha^2) g(X,Y) - 2\alpha^2 \eta(X)\eta(Y).$$

For a g-orthonormal basis (e_j) such that $e_{2n+1} = \zeta$, we define a related $\tilde{g}$-orthonormal basis $(\tilde{e}_j)$ by

$$\tilde{e}_1 = \frac{1}{\sqrt{1+2\alpha^2}} e_1, \ldots, \quad \tilde{e}_{2n} = \frac{1}{\sqrt{1+2\alpha^2}} e_{2n}, \quad \tilde{e}_{2n+1} = \zeta.$$

Analogously as in the Hermitian case, for a vector $X \in TM$ put

$$X' = \sum_j g(X, \tilde{e}_j)\tilde{e}_j.$$

Then $X' = \frac{1}{1+2\alpha^2} X + \frac{2\alpha^2}{1+2\alpha^2}\eta(X)\zeta$, which implies

$$X' = \frac{1}{1+2\alpha^2} X \quad \text{for } X \in \mathcal{E} \quad \text{and} \quad \zeta' = \zeta.$$

Now we may turn to computing the condition of minimality of considered $U(n) \times 1$-structure. We have $R_{\xi_X}(X) = -2\alpha R(\zeta, X)X$, thus

$$\sum_j R_{\xi_{\tilde{e}_j}}(\tilde{e}_j) = -\frac{2\alpha}{1+2\alpha^2} \mathrm{Ric}(\zeta).$$

Hence,

$$\sum_j g(\xi_{R_{\xi_{\tilde{e}_j}}(\tilde{e}_j)} Y, Z) = \frac{2\alpha^2}{1+2\alpha^2}(\eta(Y)\mathrm{Ric}(\zeta, Z) - \eta(Z)\mathrm{Ric}(\zeta, Y)).$$

Moreover,

$$\begin{aligned}(\nabla_X \xi)_X Y &= (X\alpha)(g(X,Y)\zeta - \eta(Y)X) + \alpha(g(X,Y)\nabla_X\zeta - (\nabla_X \eta)Y \cdot X)\\ &= (X\alpha)(g(X,Y)\zeta - \eta(Y)X) + \alpha^2 \eta(X)(\eta(Y)X - g(X,Y)\zeta),\end{aligned}$$

which implies

$$\sum_j g((\nabla_{\tilde{e}_j}\xi)_{\tilde{e}_j} Y, Z) = \eta(Z) Y'\alpha - \eta(Y) Z'\alpha.$$

Thus we have obtained the following observation. A $U(n) \times 1$-structure on α-Kenmotsu manifold is minimal if and only if for any $Y, Z \in TM$ the following condition holds

$$0 = \eta(Z)Y'\alpha - \eta(Y)Z'\alpha + \frac{2\alpha^2}{1+2\alpha^2}(\eta(Y)\mathrm{Ric}(\zeta, Z) - \eta(Z)\mathrm{Ric}(\zeta, Y)). \quad (13)$$

Let us rewrite the above condition by splitting into following two cases (by skew-symmetry in Y and Z). For $Y, Z \in \mathcal{E}$ (13) holds trivially, whereas for $Y \in \mathcal{E}$ and $Z = \zeta$ we obtain

$$0 = Y\alpha - 2\alpha^2 \mathrm{Ric}(\zeta, Y).$$

Concluding we may state the following result.

Theorem 3. *A* $U(n) \times 1$*-structure on* α*-Kenmotsu manifold is minimal if and only if*

$$Y\alpha = 2\alpha^2 \mathrm{Ric}(\zeta, Y), \quad Y \in \mathcal{E}.$$

Corollary 1. *A* $U(n) \times 1$*-structures on any* α*-Kenmotsu manifold with* α *constant and such that* $\mathrm{Ric}(\zeta, Y) = 0$ *for* $Y \in \mathcal{E}$ *is minimal.*

Proof. Follows directly by Theorem 3 and equality $Y\alpha = 0$. □

Let us finish by giving one example.

Example 2. Consider the hyperbolic space $H^{2n+1} = \{(x_1, \ldots, x_{2n+1}) \mid x_1 > 0\}$, where the Riemannian metric g is of the form

$$g = \frac{1}{c^2 x_1^2} \sum_j dx_j^2$$

for some non-zero constant c. One can show that H^{2n+1} is of constant sectional curvature $-c^2$ and induces α-Kenmotsu structure, with $\zeta = cx_1 \frac{\partial}{\partial x_1}$ and $\alpha = -c$ [1,4]. Since H^{2n+1} is a space form it follows that $\mathrm{Ric}(\zeta, Y) = 0$ for Y orthogonal to ζ. Thus by Corollary 1 the described $U(n) \times 1$-structure on H^{2n+1} is minimal.

Acknowledgements

I wish to thank anonymous referee for many suggestions that led to improvement of the paper.

References

[1] Chinea, D., Gonzalez-Davila, J.C.: A classification of almost contact metric manifolds. Ann. Mat. Pura Appl. (4) **156**, 15–36 (1990)

[2] Falcitelli, M., Farinola, A., Salamon, S.M.: Almost-Hermitian geometry. Differ. Geom. Appl. **4**, 259–282 (1994)

[3] Gonzalez-Davila, J.C., Martin Cabrera, F.: Harmonic G-structures. Math. Proc. Camb. Philos. Soc. **146**(2), 435–459 (2009)

[4] Gonzalez-Davila, J.C., Martin Cabrera, F.: Harmonic almost contact structures via the intrinsic torsion. Isr. J. Math. **181**, 145–187 (2011)

[5] Janssens, D., Vanhecke, L.: Almost contact structures and curvature tensors. Kodai Math. J. **4**(1), 1–27 (1981)

[6] Kenmotsu, K.: A class of almost contact Riemannian manifolds. Tohoku Math. J. (2) **24**, 93–103 (1972)

[7] Moroianu, A.: Compact lcK manifolds with parallel vector fields. Complex Manifolds **2**(1), 26–33 (2015)

[8] Niedziałomski, K.: Geometry of G-structures via the intrinsic torsion. SIGMA **12**, 107 (2016)

[9] Tricerri, F., Vanhecke, L.: Curvature tensors on almost Hermitian manifolds. Trans. Am. Math. Soc. **267**, 365–398 (1981)

[10] Vaisman, I.: On locally conformal almost Kähler manifolds. Isr. J. Math. **24**(3–4), 338–351 (1976)

[11] Wood, C.M.: Harmonic almost-complex structures. Compos. Math. **99**, 183–212 (1995)

[12] Wood, C.M.: Harmonic sections of homogeneous fibre bundles. Differ. Geom. Appl. **19**, 193–210 (2003)

Kamil Niedziałomski
Department of Mathematics and Computer Science
University of Łódź
Banacha 22
90-238 Lodz
Poland
e-mail: kamiln@math.uni.lodz.pl

Received: September 22, 2016.
Accepted: April 7, 2018.