



Large Time Existence of Special Strong Solutions to MHD Equations in Cylindrical Domains

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Abstract. We investigate the problem of the existence of regular solutions to the three-dimensional MHD equations in cylindrical domains with perfectly conducting boundaries and under the Navier boundary conditions for the velocity field. We show that if the initial and external data do not change too rapidly along the axis of the cylinder, then there exists a unique regular solution for any finite time.

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1. Introduction

In this work we examine the existence and regularity of solutions to the magnetohydrodynamics equations (MHD) in $3d$ cylindrical domains. The governing system of equations reads

$$\begin{aligned} \mathbf{v}_{,t} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \Delta \mathbf{v} + \nabla \left(p + \frac{1}{2} |\mathbf{H}|^2 \right) - (\mathbf{H} \cdot \nabla) \mathbf{H} &= \mathbf{f} && \text{in } \Omega^T := \Omega \times (t_0, T), \\ \mathbf{H}_{,t} + (\mathbf{v} \cdot \nabla) \mathbf{H} - (\mathbf{H} \cdot \nabla) \mathbf{v} - \nu_\kappa \Delta \mathbf{H} &= \mathbf{0} && \text{in } \Omega^T, \\ \operatorname{div} \mathbf{v} &= 0, \quad \operatorname{div} \mathbf{H} = 0 && \text{in } \Omega^T, \\ \operatorname{rot} \mathbf{v} \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{v} \cdot \mathbf{n} = 0 && \text{on } \partial\Omega^T := \partial\Omega \times (t_0, T), \\ \operatorname{rot} \mathbf{H} \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{H} \cdot \mathbf{n} = 0 && \text{on } \partial\Omega^T, \\ \mathbf{v}|_{t=t_0} &= \mathbf{v}_{t_0}, \quad \mathbf{H}|_{t=t_0} = \mathbf{H}_{t_0} && \text{in } \Omega \times \{t = t_0\}, \end{aligned} \tag{1.1}$$

where $\Omega = \Omega' \times (-a, a)$, $a > 0$ and $\Omega' \subset \mathbb{R}^2$ is a bounded subset with the \mathcal{C}^2 -boundary. Here, the unknowns are:

- the velocity field $\mathbf{v}: \Omega \rightarrow \mathbb{R}^3$, $\mathbf{v} = \mathbf{v}(x, t) = (v_1(x_1, x_2, x_3, t), v_2(x_1, x_2, x_3, t), v_3(x_1, x_2, x_3, t))$,
- the magnetic field $\mathbf{H}: \Omega \rightarrow \mathbb{R}^3$, $\mathbf{H} = \mathbf{H}(x, t) = (H_1(x_1, x_2, x_3, t), H_2(x_1, x_2, x_3, t), H_3(x_1, x_2, x_3, t))$,
- the pressure $p: \Omega \rightarrow \mathbb{R}$, $p = p(x, t) = p(x_1, x_2, x_3, t)$.

The external force $\mathbf{f}: \Omega \rightarrow \mathbb{R}^3$, $\mathbf{f} = \mathbf{f}(x, t)$, $\mathbf{f} = (f_1(x_1, x_2, x_3, t), f_2(x_1, x_2, x_3, t), f_3(x_1, x_2, x_3, t))$, the viscosity coefficients $\nu, \nu_\kappa > 0$ and the initial conditions $\mathbf{v}_{t_0}, \mathbf{H}_{t_0}$ are given.

System (1.1) describes the motion of a viscous, incompressible and resistive fluid filling a region Ω (see e.g. [1, 2]). This motion under the presence of the magnetic fields generates electric field and electric currents, thereby evoking forces which alter the magnetic field and the fluid motion itself. Clearly, (1.1) is a combination of the Navier-Stokes equations (NSE) and Maxwell's equations. Since the problem of regularity of weak solutions to the $3d$ NSE for arbitrary smooth data is still open, we cannot get any better results for (1.1) than there are for the ordinary NSE. Therefore, to obtain a new result for (1.1)

we need to make further assumptions. Two basic strategies are: either take the initial data small enough or require some conditional regularity of \mathbf{v} (see e.g. [3]).

In the following paper we are interested in proving the existence of strong solutions (see Definition 2.1) to system (1.1) without any assumptions on the magnitude of the L_2 -norms of the initial and external data. Instead, we assume that the L_2 -norms of the derivatives of the initial and external data along the axis of the cylinder Ω are small. More precisely, we prove the following:

Theorem 1. *Suppose that $\mathbf{v}_{t_0}, \mathbf{H}_{t_0} \in H^1(\Omega)$, $\operatorname{div} \mathbf{v}_{t_0} = \operatorname{div} \mathbf{H}_{t_0} = 0$, $\mathbf{f} \in L_2(\Omega^T)$, $\mathbf{f}_{,x_3} \in L_2(t_0, T; L_{\frac{6}{5}}(\Omega))$. Let us introduce*

$$\begin{aligned} E^2(T) &:= \|\mathbf{v}_{t_0}\|_{H^1(\Omega)}^2 + \|\mathbf{H}_{t_0}\|_{H^1(\Omega)}^2 + \|\mathbf{f}\|_{L_2(\Omega^T)}^2, \\ d^2(T) &:= \|\mathbf{v}_{t_0}\|_{L_2(\Omega)}^2 + \|\mathbf{H}_{t_0}\|_{L_2(\Omega)}^2 + \|\mathbf{f}\|_{L_2(t_0, T; L_{\frac{6}{5}}(\Omega))}^2, \\ \delta^2(T) &:= \|\mathbf{v}_{t_0, x_3}\|_{L_2(\Omega)}^2 + \|\mathbf{H}_{t_0, x_3}\|_{L_2(\Omega)}^2 + \|\mathbf{f}_{,x_3}\|_{L_2(t_0, T; L_{\frac{6}{5}}(\Omega))}^2. \end{aligned}$$

Fix $T > t_0$. If $\delta^2(T)$ is sufficiently small then there exists a unique strong solution to (1.1). This unique solution satisfies the following exponential growth estimate for all $t \in [t_0, T]$

$$\|\mathbf{v}\|_{V_2^1(\Omega^t)}^2 + \|\mathbf{H}\|_{V_2^1(\Omega^t)}^2 \leq cE^2(T)e^{c(1+d^2(T))^2},$$

where a generic constant c depends on ν , ν_κ and Ω but not on T .

Before we briefly discuss the proof, we should clarify that all constants in the sequel are denoted by c and they may vary from line to line. They may depend on ν , ν_κ and Ω but neither on t nor on the initial and external data. The definition of the space $V_2^k(\Omega^T)$, $k \in \mathbb{N}$, is given in Sect. 2. The proof of the above theorem is based on the refined energy estimates and a fixed point principle. We derive these estimates by utilizing the multiplicative Sobolev inequality (see e.g. [4, Remark 2.1], [5, Remark (iii)]), which distinguishes the differentiation with respect to x_3 from the whole gradient (see Lemma 3.7). To exploit this inequality we introduce an auxiliary problem for $(\mathbf{v}_{,x_3}, \mathbf{H}_{,x_3})$ (see Lemma 4.2). The first energy estimate for the solutions to this problem reads

$$\|\mathbf{v}_{,x_3}\|_{V_2^0(\Omega^t)}^2 + \|\mathbf{H}_{,x_3}\|_{V_2^0(\Omega^t)}^2 \leq c \exp\left(c\|\Delta \mathbf{v}\|_{L_2(\Omega^t)}^2 + c\|\Delta \mathbf{H}\|_{L_2(\Omega^t)}^2\right) \delta^2(T), \quad \forall t \leq T.$$

Next, we test (1.1)_{1,2} with $-\Delta \mathbf{v}$ and $-\Delta \mathbf{H}$, respectively, and use the multiplicative Sobolev inequality to estimate the non-linear terms (see Lemma 6.1). First we obtain

$$\begin{aligned} &\left(\|\mathbf{v}(t)\|_{H^1(\Omega)}^2 + \|\mathbf{H}(t)\|_{H^1(\Omega)}^2\right) + \|\Delta \mathbf{v}\|_{L_2(\Omega^t)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega^t)}^2 \leq c\varphi\left(\|\Delta \mathbf{v}\|_{L_2(\Omega^t)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega^t)}^2\right) \delta^{\frac{4}{3}}(T) \\ &+ cE^2(T) + c \sup_{t_0 \leq t \leq T} \left(\|\mathbf{v}(t)\|_{H^1(\Omega)}^2 + \|\mathbf{H}(t)\|_{H^1(\Omega)}^2\right)^3 \delta^{\frac{4}{3}}(T) + \int_{t_0}^t \left(\|\mathbf{v}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{H}(\tau)\|_{H^1(\Omega)}^2\right) g(\tau) d\tau, \end{aligned}$$

where φ is a positive, increasing function of exponential type and $g(t)$ is a function such that $\int_{t_0}^T g(t) dt$ is bounded only by the data. Using the continuity argument (see e.g. [6, Ch. 1, §1.3]) it is clear that for sufficiently small $\delta^2(T)$ we can eliminate φ from the right-hand side. Next, we use the Gronwall inequality, basic energy estimates and the continuity argument again. In the end we can conclude the estimate for \mathbf{v} and \mathbf{H} (see Lemma 6.1). Finally, the existence of solutions to (1.1) follows from the Leray–Schauder fixed point theorem (see Lemma 7.1).

There is a slight problem with the above idea, namely the multiplicative Sobolev inequality cannot be utilized directly for \mathbf{v} and \mathbf{H} because of the boundary conditions (1.1)_{4,5}. They do not imply that $\mathbf{H}' = (H_1, H_2, 0)$ and $\mathbf{v}' = (v_1, v_2, 0)$ vanish on the bottom and the top of the cylinder, therefore we have to subtract from \mathbf{v}' and \mathbf{H}' their mean values along the x_3 -variable. More precisely, we introduce (cf. Sect. 3)

$$\bar{\mathbf{v}} = \left(\int_{-a}^a v_1(x_1, x_2, s) \, ds, \int_{-a}^a v_2(x_1, x_2, s) \, ds, 0 \right),$$

$$\bar{\mathbf{H}} = \left(\int_{-a}^a H_1(x_1, x_2, s) \, ds, \int_{-a}^a H_2(x_1, x_2, s) \, ds, 0 \right).$$

Now we easily observe that the functions $\mathbf{v} - \bar{\mathbf{v}}$ and $\mathbf{H} - \bar{\mathbf{H}}$ satisfy the assumptions of the multiplicative Sobolev inequality (see Remark 3.8), whereas the functions $\bar{\mathbf{v}}, \bar{\mathbf{H}}$ are two-dimensional, i.e. they depend merely on x_1 and x_2 . The mean value operator that we have just introduced, was successfully used in e.g. [4] or more recently in [7].

Let us shortly discuss the consequences of the smallness of $\delta^2(T)$. Let $\mathbf{u} \in \{\mathbf{v}, \mathbf{H}\}$. By the definition

$$\|\mathbf{u}_{t_0}\|_{H^1(\Omega)}^2 = \|\mathbf{u}_{t_0}\|_{L_2(\Omega)}^2 + \|\mathbf{u}_{t_0, x_1}\|_{L_2(\Omega)}^2 + \|\mathbf{u}_{t_0, x_2}\|_{L_2(\Omega)}^2 + \|\mathbf{u}_{t_0, x_3}\|_{L_2(\Omega)}^2.$$

In (3.2) we shall see that conditions (1.1)_{4,5} imply that $v_3 = H_3 = 0$ on the top and the bottom of the cylinder. Thus, we have the Poincaré inequality for u_3 with respect to x_3 and we easily get

$$\begin{aligned} \|\mathbf{v}_{t_0}\|_{H^1(\Omega)}^2 + \|\mathbf{H}_{t_0}\|_{H^1(\Omega)}^2 &\leq \|\mathbf{v}'_{t_0}\|_{L_2(\Omega)}^2 + \|\mathbf{H}'_{t_0}\|_{L_2(\Omega)}^2 + \|\mathbf{v}'_{t_0, x'}\|_{L_2(\Omega)}^2 + \|\mathbf{H}'_{t_0, x'}\|_{L_2(\Omega)}^2 \\ &\quad + (1 + 2a)\delta^2(T), \end{aligned}$$

where $x' = (x_1, x_2)$, $\mathbf{v}' = (v_1, v_2)$, $\mathbf{H}' = (H_1, H_2)$. This means that the third components of the initial velocity and magnetic fields must be close to zero. Thus, the initial flow is close to two-dimensional flow and in light of Lemma 4.2 it remains so for any time T . Note that this limitation is a direct consequence of the boundary conditions for \mathbf{v} and \mathbf{H} on the top and the bottom of the cylinder. It would not hold if Ω was e.g. a cylinder periodic with respect to x_3 . Then, $\|v_3(t_0)\|_{L_2(\Omega)}$ and $\|H_3(t_0)\|_{L_2(\Omega)}$ could be arbitrarily large.

For further discussion and references of MHD equations in periodic, cylindrical domains we refer the reader to e.g. [8] and [9]. Cylindrical domains play an important role in studying plasma physics (see [10] and [11]) and to some extent are used as approximations of a torus (see [12]).

We end this Introduction remarking that to the best of our knowledge results like Theorem 1 have not appeared in the literature. However, in case of the whole space and slightly more general system (i.e. with the fractional Laplacian) a result similar to ours follows from [13][Theorems 1.2 and 1.4]. Furthermore, if we assume $\mathbf{v} = \mathbf{0}$ on the boundary, then (1.1) was studied in [14–16] and [17], where the existence of regular solutions was proved either for short time or for any time but under the assumption of the smallness of the initial data. For a detailed summary of various results related to MHD equations we refer the reader to the Introduction in [18].

2. Notation

The Boundary of the Domain

The boundary $S := \partial\Omega$ is a union of three sets:

$$\begin{aligned} S_L &= \{x \in \mathbb{R}^3 : \varphi(x_1, x_2) = c_\varphi, -a < x_3 < a\} && \text{(the lateral surface),} \\ S_B &= \{x \in \mathbb{R}^3 : \varphi(x_1, x_2) < c_\varphi, x_3 = -a\} && \text{(the bottom),} \\ S_T &= \{x \in \mathbb{R}^3 : \varphi(x_1, x_2) < c_\varphi, x_3 = a\} && \text{(the top),} \end{aligned}$$

where $\varphi(x_1, x_2) = c_\varphi$ is a closed \mathcal{C}^2 -curve in the plane $x_3 = \text{const}$.

Shorthand Notation

Some formulas appear often in the sequel. To shorten the notation we use

$$\begin{aligned} p_H &:= p + \frac{1}{2} |\mathbf{H}|^2, \\ d^2(T) &\text{--- a constant from the energy estimate, see Theorem 1,} \\ \delta^2(T) &\text{--- the smallness assumption from Theorem 1,} \\ E^2(T) &\text{--- a sum of some norms of the initial and the external data from Theorem 1,} \\ \Gamma^T &:= \Gamma \times (t_0, T), \quad \text{where } \Gamma \in \{\Omega, S, S_L, S_B, S_T\}, \\ \mathbb{N} &:= \{0, 1, 2, \dots\}. \end{aligned}$$

Function Spaces

Throughout the paper we use the standard Lebesgue $L_p(\Omega)$ and Sobolev $W_2^k(\Omega) = H^k(\Omega)$ spaces. We also need $V_2^k(\Omega^T)$, which is defined as follows:

$$V_2^k(\Omega^T) = \left\{ \mathbf{u}: \|\mathbf{u}\|_{V_2^k(\Omega^T)} \equiv \operatorname{ess\,sup}_{t_0 \leq t \leq T} \|\mathbf{u}\|_{H^k(\Omega)} + \left(\int_{t_0}^T \|\mathbf{u}(t)\|_{H^{k+1}(\Omega)}^2 \, dt \right)^{\frac{1}{2}} < \infty \right\}, \quad k \in \mathbb{N}.$$

Weak and Strong Solutions

Definition 2.1. By a *weak solution* to problem (1.1) we mean a pair of functions $(\mathbf{v}, \mathbf{H}) \in V_2^0(\Omega^T) \times V_2^0(\Omega^T)$ such that $\operatorname{div} \mathbf{v} = \operatorname{div} \mathbf{H} = 0$ and satisfying

$$\begin{aligned} & \int_{\Omega^T} (-\mathbf{v} \cdot \varphi_{,t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \varphi + \nu \operatorname{rot} \mathbf{v} \cdot \operatorname{rot} \varphi + (\mathbf{H} \cdot \nabla) \varphi \cdot \mathbf{H}) \, dx \, dt \\ & \quad + \int_{\Omega} \mathbf{v} \cdot \varphi|_{t=T} \, dx - \int_{\Omega} \mathbf{v} \cdot \varphi|_{t=t_0} \, dx = \int_{\Omega^T} \mathbf{f} \cdot \varphi \, dx \, dt, \\ & \int_{\Omega^T} (-\mathbf{H} \cdot \psi_{,t} - (\mathbf{v} \cdot \nabla) \psi \cdot \mathbf{H} + (\mathbf{H} \cdot \nabla) \psi \cdot \mathbf{v} + \nu_{\kappa} \operatorname{rot} \mathbf{H} \cdot \operatorname{rot} \psi) \, dx \, dt \\ & \quad + \int_{\Omega} \mathbf{H} \cdot \psi|_{t=T} \, dx - \int_{\Omega} \mathbf{H} \cdot \psi|_{t=t_0} \, dx = 0, \end{aligned}$$

where $\varphi, \psi \in H^1(\Omega^T)$, $\varphi \cdot \mathbf{n} = 0$, $\psi \cdot \mathbf{n} = 0$, $\operatorname{div} \varphi = \operatorname{div} \psi = 0$. If the pair $(\mathbf{v}, \mathbf{H}) \in V_2^1(\Omega^T) \times V_2^1(\Omega^T)$ satisfies the above integral identities then we call (\mathbf{v}, \mathbf{H}) a *strong solution*.

3. Auxiliary Remarks and Tools

Boundary Conditions

Since Ω is a cylinder, we can easily determine the unit outward normal vector \mathbf{n} and the unit tangent vectors $\boldsymbol{\tau}_i$, $i = 1, 2$. We have:

$$\begin{aligned} \text{on } S_L \quad \mathbf{n} &= \frac{1}{|\nabla \varphi|} [\varphi_{,x_1}, \varphi_{,x_2}, 0] & \boldsymbol{\tau}_1 &= \frac{1}{|\nabla \varphi|} [-\varphi_{,x_2}, \varphi_{,x_1}, 0] & \boldsymbol{\tau}_2 &= [0, 0, 1], \\ \text{on } S_B \quad \mathbf{n} &= [0, 0, -1] & \boldsymbol{\tau}_1 &= [1, 0, 0] & \boldsymbol{\tau}_2 &= [0, 1, 0], \\ \text{on } S_T \quad \mathbf{n} &= [0, 0, 1] & \boldsymbol{\tau}_1 &= [1, 0, 0] & \boldsymbol{\tau}_2 &= [0, 1, 0]. \end{aligned} \tag{3.1}$$

Using the above formulas in (1.1)_{4,5} we immediately get the following identities on S_T and S_B

$$\begin{aligned} v_3 &= H_3 = 0, & v_{3,x_i} &= H_{3,x_i} = 0 \quad i \in \{1, 2\}, \\ v_{i,x_3} &= H_{i,x_3} = 0 \quad i \in \{1, 2\}, & v_{3,x_3x_3} &= H_{3,x_3x_3} = 0, \\ (v_{1,x_2} - v_{2,x_1})_{,x_3} &= (H_{1,x_2} - H_{2,x_1})_{,x_3} = 0. \end{aligned} \quad (3.2)$$

On S_L we only obtain

$$v_{1,x_2} - v_{2,x_1} = H_{1,x_2} - H_{2,x_1} = 0. \quad (3.3)$$

Let us observe that the discussed boundary conditions suggest the following *integration by parts* formula: Let \mathbf{u} and \mathbf{w} belong to $H^1(\Omega)$ and satisfy $\operatorname{rot} \mathbf{u} \times \mathbf{n} = \operatorname{rot} \mathbf{w} \times \mathbf{n} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{n} = \mathbf{w} \cdot \mathbf{n} = 0$. Then

$$\begin{aligned} \int_{\Omega} \operatorname{rot} \mathbf{u} \cdot \mathbf{w} \, dx &= \int_{\Omega} \operatorname{rot} \mathbf{w} \cdot \mathbf{u} \, dx + \int_S \mathbf{u} \times \mathbf{n} \cdot \mathbf{w} \, dS \\ &= \int_{\Omega} \operatorname{rot} \mathbf{w} \cdot \mathbf{u} \, dx - \int_S \mathbf{w} \times \mathbf{n} \cdot \mathbf{u} \, dS. \end{aligned} \quad (3.4)$$

Finally, let us note that there are other possible choices for the boundary conditions for the magnetic field. For an insightful discussions we refer the interested reader into [19–21] and [22].

Divergence-Free Magnetic Field

At first sight it might seem that system (1.1) is overdetermined since it contains four equations for \mathbf{H} while \mathbf{H} has only three unknown components. This is perfectly fine and consistent with the MHD theory (see e.g. [23, Ch.2]). However, it may appear that condition (1.1)₅ violates the equation $\operatorname{div} \mathbf{H} = 0$. This issue was partly addressed in [15, Lemma 2.1]. In Lemma 3.12 we show that if $\operatorname{div} \mathbf{H}_{t_0} = 0$ and \mathbf{H} satisfies (1.1)₅, then $\operatorname{div} \mathbf{H} = 0$ for all $t \geq t_0$.

Mean Value Operator

The proof of Theorem 1 is based on the multiplicative Sobolev inequality, which requires that a given function \mathbf{u} satisfies: $u_3|_{S_T} = u_3|_{S_B} = 0$ and $\int_{\Omega} u_j(x_1, x_2, x_3) \, dx_3 = 0$ for $j = 1, 2$. Since we do not know whether $\int_{\Omega} v_j \, dx_3$ and $\int_{\Omega} H_j \, dx_3$ are equal to zero for $j = 1, 2$, so we define the mean value for $\mathbf{v}' = (v_1, v_2)$ and $\mathbf{H}' = (H_1, H_2)$ with respect to the x_3 -variable as follows: if $z: \mathbb{R}^3 \rightarrow \mathbb{R}$ then we write

$$\bar{z}(x_1, x_2) = \frac{1}{2a} \int_{-a}^a z(x_1, x_2, s) \, ds \quad (3.5)$$

and

$$\bar{\mathbf{v}}(x_1, x_2) = (\bar{v}_1(x_1, x_2), \bar{v}_2(x_1, x_2), 0), \quad \bar{\mathbf{H}}(x_1, x_2) = (\bar{H}_1(x_1, x_2), \bar{H}_2(x_1, x_2), 0).$$

Such defined functions have several properties, which we summarize in lemmas below.

Lemma 3.1. *Let $\mathbf{z}: \Omega \rightarrow \mathbb{R}^3$, $z_3|_{S_B} = z_3|_{S_T} = 0$. Then $\bar{\mathbf{z}} = (\bar{z}_1, \bar{z}_2, 0)$ has the following properties:*

$$\begin{aligned} (1) \quad & \partial_{x_j}^k \bar{\mathbf{z}} = \overline{\partial_{x_j}^k \mathbf{z}}, \quad j \in \{1, 2\}, \quad k \in \mathbb{N}, \\ (2) \quad & \int_{-a}^a (\mathbf{z} - \bar{\mathbf{z}})' \, dx_3 = \mathbf{0}, \\ (3) \quad & \int_{-a}^a (\mathbf{z} - \bar{\mathbf{z}})'_{,x_j} \, dx_3 = \mathbf{0}, \quad j \in \{1, 2\}, \\ (4) \quad & \operatorname{rot} \bar{\mathbf{z}} = [0, 0, \bar{z}_{1,x_2} - \bar{z}_{2,x_1}], \\ (5) \quad & \Delta \bar{\mathbf{z}} = \overline{\Delta \mathbf{z}}, \end{aligned} \quad (3.6)$$

where $'$ means restriction to the two first coordinates, i.e. $\bar{\mathbf{z}}' = (\bar{z}_1, \bar{z}_2)$. If $\operatorname{div} \mathbf{z} = 0$, then

$$(6) \quad \operatorname{div} \bar{\mathbf{z}} = 0.$$

Finally, let $\operatorname{rot} \mathbf{z} \times \mathbf{n} = \mathbf{0}$, $\mathbf{z} \cdot \mathbf{n} = 0$ on S . Then

$$(7) \quad \begin{aligned} \bar{\mathbf{z}} \cdot \mathbf{n} &= 0 \\ \operatorname{rot} \bar{\mathbf{z}} \times \mathbf{n} &= \mathbf{0} \end{aligned} \quad \text{on } S.$$

Proof. Properties (1) and (2) follow immediately from (3.5). Property (3) is a direct consequence of (1) and (2). The computation of $\operatorname{rot} \bar{\mathbf{z}}$ in (4) is straightforward. For the justification of (5) we refer the reader to [4, Lemma 1.1].

To prove (6) we note

$$\operatorname{div} \bar{\mathbf{z}} \stackrel{(1)}{=} \frac{1}{2a} \int_{-a}^a (z_{1,x_1}(x_1, x_2, s) + z_{2,x_2}(x_1, x_2, s)) \, ds = -\frac{1}{2a} \int_{-a}^a z_{3,x_3}(x_1, x_2, s) \, ds = 0.$$

Finally, we deduce (7) from (3.1) and (1.1)_{4,5}. This completes the proof. \square

Lemma 3.2. Let $\mathbf{a}, \mathbf{b}: \Omega \rightarrow \mathbb{R}^3$ belong to $H^1(\Omega)$, $\operatorname{div} \mathbf{a} = \operatorname{div} \mathbf{b} = 0$. Suppose that \mathbf{a} and \mathbf{b} satisfy the Navier boundary conditions (i.e. (1.1)_{4,5}). If $\bar{\mathbf{a}} = [\bar{a}_1, \bar{a}_2, 0]$ and $\bar{\mathbf{b}} = [\bar{b}_1, \bar{b}_2, 0]$, then

$$\int_{\Omega} \operatorname{rot}(\mathbf{a} - \bar{\mathbf{a}}) \cdot \operatorname{rot} \bar{\mathbf{b}} \, dx = 0$$

and

$$\int_{\Omega} \Delta(\mathbf{a} - \bar{\mathbf{a}}) \cdot \Delta \bar{\mathbf{b}} \, dx = 0.$$

Proof. Consider the first integral. We see that in light of (3.6)₄ we only need to examine

$$\int_{-a}^a \operatorname{rot}(\mathbf{a} - \bar{\mathbf{a}})_3 \, dx_3 = \int_{-a}^a (z_{1,x_2} - \bar{z}_{1,x_2} - (z_{2,x_1} - \bar{z}_{2,x_1})) \, dx_3.$$

From (3.6)₁ it follows that the above integral vanishes.

To investigate the second integral we use (3.6)₄ twice. Since $\operatorname{div} \bar{\mathbf{b}} = 0$ (see Lemma 3.1) we conclude that the third component of $\Delta \bar{\mathbf{b}}$ is zero and the first two components do not depend on x_3 . Thus, we need to examine

$$\int_{-a}^a \Delta(\mathbf{a} - \bar{\mathbf{a}})' \, dx_3 = \int_{-a}^a (\partial_{x_1 x_1}^2 + \partial_{x_2 x_2}^2) (\mathbf{a} - \bar{\mathbf{a}})' \, dx_3 + \int_{-a}^a \partial_{x_3 x_3}^2 (\mathbf{a} - \bar{\mathbf{a}})' \, dx_3.$$

The first integral on the right-hand side vanishes due to (3.6)₁. The second integral is equal to

$$\mathbf{a}'_{,x_3}(x_1, x_2, a) - \mathbf{a}'_{,x_3}(x_1, x_2, -a).$$

The boundary conditions for \mathbf{a} imply $\mathbf{a}'_{,x_3} = \mathbf{0}$ (we saw it in (3.2)₂). This concludes the proof. \square

Lemma 3.3. Suppose that $\mathbf{z} \in H^1(\Omega)$, $z_3|_{S_B} = z_3|_{S_T} = 0$. Then

$$\begin{aligned} \|\mathbf{z}\|_{L_2(\Omega)}^2 &= \|\mathbf{z} - \bar{\mathbf{z}}\|_{L_2(\Omega)}^2 + \|\bar{\mathbf{z}}\|_{L_2(\Omega)}^2, \\ \|\mathbf{z}\|_{H^1(\Omega)}^2 &= \|\mathbf{z} - \bar{\mathbf{z}}\|_{H^1(\Omega)}^2 + \|\bar{\mathbf{z}}\|_{H^1(\Omega)}^2, \\ \|\mathbf{z}\|_{H^2(\Omega)}^2 &= \|\mathbf{z} - \bar{\mathbf{z}}\|_{H^2(\Omega)}^2 + \|\bar{\mathbf{z}}\|_{H^2(\Omega)}^2. \end{aligned}$$

Proof. We have

$$\begin{aligned} |\mathbf{z}|^2 &= (\mathbf{z} - \bar{\mathbf{z}} + \bar{\mathbf{z}}) \cdot (\mathbf{z} - \bar{\mathbf{z}} + \bar{\mathbf{z}}), \\ |\nabla \mathbf{z}|^2 &= (\nabla(\mathbf{z} - \bar{\mathbf{z}}) + \nabla \bar{\mathbf{z}}) \cdot (\nabla(\mathbf{z} - \bar{\mathbf{z}}) + \nabla \bar{\mathbf{z}}), \\ |\Delta \mathbf{z}|^2 &= (\Delta(\mathbf{z} - \bar{\mathbf{z}}) + \Delta \bar{\mathbf{z}}) \cdot (\Delta(\mathbf{z} - \bar{\mathbf{z}}) + \Delta \bar{\mathbf{z}}). \end{aligned}$$

From Lemma 3.1 it follows that the integrals

$$\int_{\Omega} (\mathbf{z} - \bar{\mathbf{z}}) \cdot \bar{\mathbf{z}} \, dx \quad \text{and} \quad \int_{\Omega} \nabla (\mathbf{z} - \bar{\mathbf{z}}) \cdot \nabla \bar{\mathbf{z}} \, dx$$

vanish because $\bar{\mathbf{z}}$ and $\nabla \bar{\mathbf{z}}$ do not depend on x_3 . The integrals containing Δ were analyzed in Lemma 3.2. This concludes the proof. \square

Remark 3.4. From the above Lemma we instantly deduce that

$$\|\bar{\mathbf{z}}\|_{H^k(\Omega)} \leq \|\mathbf{z}\|_{H^k(\Omega)} \quad \text{and} \quad \|\mathbf{z} - \bar{\mathbf{z}}\|_{H^k(\Omega)} \leq \|\mathbf{z}\|_{H^k(\Omega)}$$

for $k = 0, 1, 2$.

The subsequent lemma will be very useful for estimating the integrals originating from the non-linear terms.

Lemma 3.5. *Let $\mathbf{a}, \mathbf{b}, \mathbf{w}: \Omega \rightarrow \mathbb{R}^3$. Then*

$$\int_{\Omega} (\mathbf{a} \cdot \nabla) \mathbf{b} \cdot \bar{\mathbf{w}} \, dx = \int_{\Omega} (\bar{\mathbf{a}} \cdot \nabla) \bar{\mathbf{b}} \cdot \bar{\mathbf{w}} \, dx + \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot \bar{\mathbf{w}} \, dx \quad (3.7a)$$

and

$$\begin{aligned} \int_{\Omega} (\mathbf{a} \cdot \nabla) \mathbf{b} \cdot (\mathbf{w} - \bar{\mathbf{w}}) \, dx &= \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) \bar{\mathbf{b}} \cdot (\mathbf{w} - \bar{\mathbf{w}}) \, dx \\ &+ \int_{\Omega} (\bar{\mathbf{a}} \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot (\mathbf{w} - \bar{\mathbf{w}}) \, dx + \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot (\mathbf{w} - \bar{\mathbf{w}}) \, dx. \end{aligned} \quad (3.7b)$$

Proof. We have

$$\begin{aligned} (\mathbf{a} \cdot \nabla) \mathbf{b} &= ((\mathbf{a} - \bar{\mathbf{a}} + \bar{\mathbf{a}}) \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}} + \bar{\mathbf{b}}) \\ &= ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) + ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) \bar{\mathbf{b}} + (\bar{\mathbf{a}} \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) + (\bar{\mathbf{a}} \cdot \nabla) \bar{\mathbf{b}}. \end{aligned}$$

Multiplying the above formula by $\bar{\mathbf{w}}$ and integrating over Ω results in four integrals on the right-hand side. We denote these integrals by J_k . If $\mathbf{c} = \bar{\mathbf{w}}$ then

$$J_2 = \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) \bar{\mathbf{b}} \cdot \bar{\mathbf{w}} \, dx = \int_{\Omega'} \left(\int_{-a}^a (\mathbf{a} - \bar{\mathbf{a}}) \, dx_3 \cdot \nabla \right) \bar{\mathbf{b}} \cdot \bar{\mathbf{w}} \, dx' = 0,$$

which follows from Lemma 3.1. Similarly $J_3 = 0$ and (3.7a) is proved.

We now set $\mathbf{c} = \mathbf{w} - \bar{\mathbf{w}}$. Then, Lemma 3.1 implies that $J_4 = 0$ which justifies (3.7b). This ends the proof. \square

Remark 3.6. From the above Lemma it follows that

$$\begin{aligned} \int_{\Omega} (\mathbf{a} \cdot \nabla) \mathbf{b} \cdot \Delta \mathbf{w} \, dx &= \int_{\Omega} (\mathbf{a} \cdot \nabla) \mathbf{b} \cdot \Delta \bar{\mathbf{w}} \, dx + \int_{\Omega} (\mathbf{a} \cdot \nabla) \mathbf{b} \cdot \Delta (\mathbf{w} - \bar{\mathbf{w}}) \, dx \\ &= \int_{\Omega} (\bar{\mathbf{a}} \cdot \nabla) \bar{\mathbf{b}} \cdot \Delta \bar{\mathbf{w}} \, dx + \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot \Delta \bar{\mathbf{w}} \, dx \\ &+ \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) \bar{\mathbf{b}} \cdot \Delta (\mathbf{w} - \bar{\mathbf{w}}) \, dx + \int_{\Omega} (\bar{\mathbf{a}} \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot \Delta (\mathbf{w} - \bar{\mathbf{w}}) \, dx \\ &+ \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot \Delta (\mathbf{w} - \bar{\mathbf{w}}) \, dx. \end{aligned}$$

Thus

$$\begin{aligned} \int_{\Omega} (\mathbf{a} \cdot \nabla) \mathbf{b} \cdot \Delta \mathbf{w} \, dx &= \int_{\Omega} (\bar{\mathbf{a}} \cdot \nabla) \bar{\mathbf{b}} \cdot \Delta \bar{\mathbf{w}} \, dx + \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot \Delta \mathbf{w} \, dx \\ &+ \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) \bar{\mathbf{b}} \cdot \Delta (\mathbf{w} - \bar{\mathbf{w}}) \, dx + \int_{\Omega} (\bar{\mathbf{a}} \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot \Delta (\mathbf{w} - \bar{\mathbf{w}}) \, dx. \end{aligned}$$

In light of the above Remark we see that every integral containing the non-linear term generates four new integrals. As we shall see these new integrals will be much easier to handle.

Auxiliary Tools

Lemma 3.7. *Suppose that $\mathbf{u} \in H^1(\Omega)$, $u_3|_{S_B} = u_3|_{S_T} = 0$ and $\int_{\Omega} u_j(x_1, x_2, x_3) dx_3 = 0$ for $j = 1, 2$. Then*

$$\|\mathbf{u}\|_{L_6(\Omega)} \leq c \|\mathbf{u}_{,x_3}\|_{L_2(\Omega)}^{\frac{1}{3}} \|\nabla \mathbf{u}\|_{L_2(\Omega)}^{\frac{2}{3}}.$$

Proof. In cylindrical domains we have the following Sobolev inequality (see [4, Remark 2.1])

$$\|\mathbf{u}\|_{L_6(\Omega)} \leq c \left(\|\mathbf{u}\|_{L_2(\Omega)} + \|\mathbf{u}_{,x_3}\| \right)^{\frac{1}{3}} \left(\|\mathbf{u}\|_{L_2(\Omega)} + \|\mathbf{u}_{,x_1}\|_{L_2(\Omega)} + \|\mathbf{u}_{,x_2}\|_{L_2(\Omega)} \right)^{\frac{2}{3}}.$$

Using the Poincaré inequality with respect to x_3 we conclude the proof. \square

Remark 3.8. Lemma 3.7 is valid for $\mathbf{u} = \mathbf{w} - \bar{\mathbf{w}}$, where $\mathbf{w} \in \{\mathbf{v}, \mathbf{H}\}$. Thus

$$\|\mathbf{u} - \bar{\mathbf{u}}\|_{L_6(\Omega)} \leq c \|\mathbf{u}_{,x_3}\|_{L_2(\Omega)}^{\frac{1}{3}} \|\nabla (\mathbf{u} - \bar{\mathbf{u}})\|_{L_2(\Omega)}^{\frac{2}{3}}.$$

Moreover, if \mathbf{w} satisfies $\text{rot } \mathbf{w} \times \mathbf{n} = \mathbf{0}$, $\mathbf{w} \cdot \mathbf{n} = 0$ on the boundary, then we also have

$$\|\nabla' (\mathbf{u} - \bar{\mathbf{u}})\|_{L_p(\Omega)} \leq c \|\nabla' \mathbf{u}_{,x_3}\|_{L_2(\Omega)}^{\frac{1}{3}} \|\nabla^2 (\mathbf{u} - \bar{\mathbf{u}})\|_{L_2(\Omega)}^{\frac{2}{3}},$$

where $2 \leq p \leq 6$. Indeed, let $\mathbf{g} = \mathbf{u}_{,x'}$. Then

$$\nabla' (\mathbf{u} - \bar{\mathbf{u}}) = \mathbf{u}_{,x'} - \overline{\mathbf{u}_{,x'}} = \mathbf{g} - \bar{\mathbf{g}},$$

where the first equality follows from (3.6)₁. From (3.6)₂ we infer that

$$\int_{-a}^a (\mathbf{g} - \bar{\mathbf{g}})' dx_3 = \mathbf{0}.$$

Since $g_3 - \bar{g}_3 = u_{3,x'} - \bar{u}_{3,x'} = 0$ (see (3.2) and Lemma 3.1), all assumption from Lemma 3.7 are satisfied and the Remark is proved.

Lemma 3.9. (see Theorem 1.1 in [24]) *Suppose that \mathbf{z} is a solution to the following problem*

$$\begin{aligned} \text{rot } \mathbf{z} &= \boldsymbol{\alpha}, \\ \text{div } \mathbf{z} &= \beta, \\ \mathbf{z} \times \mathbf{n} &= \mathbf{0} \quad \text{or} \quad \mathbf{z} \cdot \mathbf{n} = 0. \end{aligned}$$

Then

$$\|\mathbf{z}\|_{H^{k+1}(\Omega)} \leq c \left(\|\boldsymbol{\alpha}\|_{H^k(\Omega)} + \|\beta\|_{H^k(\Omega)} \right), \quad k \in \mathbb{N}.$$

The direct consequence of the above Lemma is the following useful inequality:

Lemma 3.10. *Suppose that $(\mathbf{w} - \bar{\mathbf{w}}) \in H^2(\Omega)$ and $\text{rot } \mathbf{w} \times \mathbf{n} = \mathbf{0}$, $\mathbf{w} \cdot \mathbf{n} = 0$. Then*

$$\|\mathbf{w} - \bar{\mathbf{w}}\|_{H^2(\Omega)} \leq c \|\Delta (\mathbf{w} - \bar{\mathbf{w}})\|_{L_2(\Omega)}.$$

Proof. We set $\mathbf{z} = \mathbf{w} - \bar{\mathbf{w}}$ in Lemma 3.9. Clearly $\mathbf{z} \cdot \mathbf{n} = 0$ on S (see Lemma 3.1), thus

$$\|\mathbf{w} - \bar{\mathbf{w}}\|_{H^2(\Omega)} \leq c \|\text{rot } (\mathbf{w} - \bar{\mathbf{w}})\|_{H^1(\Omega)}.$$

Next we set $\mathbf{z} = \text{rot } (\mathbf{w} - \bar{\mathbf{w}})$. Then $\mathbf{z} \times \mathbf{n} = \mathbf{0}$ (see Lemma 3.1). Then, by Lemma 3.9 and the above inequality we get

$$\|\mathbf{w} - \bar{\mathbf{w}}\|_{H^2(\Omega)} \leq c \|\text{rot } (\mathbf{w} - \bar{\mathbf{w}})\|_{H^1(\Omega)} \leq c \|\text{rot rot } (\mathbf{w} - \bar{\mathbf{w}})\|_{L_2(\Omega)},$$

and since $\text{div } (\mathbf{w} - \bar{\mathbf{w}}) = 0$ we conclude the proof. \square

Lemma 3.11. (Gagliardo–Nirenberg interpolation inequality, see [25]) *Suppose that $u: \Omega \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^n$. Fix $1 \leq q, r \leq \infty$ and $m \in \mathbb{N}$. Suppose that a real number $\alpha \geq 0$ and $j \in \mathbb{N}$ are such that*

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n} \right) \alpha + \frac{1-\alpha}{q}, \quad \text{where} \quad \frac{j}{m} \leq \alpha \leq 1.$$

Then

$$\|D^j u\|_{L_p(\Omega)} \leq c_1 \|D^m u\|_{L_r(\Omega)}^\alpha \|u\|_{L_q(\Omega)}^{1-\alpha} + c_2 \|u\|_{L_s(\Omega)},$$

where $s > 0$ is arbitrary and the constants c_1 and c_2 do not depend on u .

Finally, we justify why (1.1)₅ does not violate the condition $\operatorname{div} \mathbf{H} = 0$.

Lemma 3.12. *Suppose that \mathbf{H} satisfy (1.1)₅ and $\operatorname{div} \mathbf{H}_0 = 0$. Then $\operatorname{div} \mathbf{H} = 0$ for all $t \geq 0$.*

Proof. Let $h := \operatorname{div} \mathbf{H}$. We take div of both sides in (1.1)₂ and we get

$$\begin{aligned} h_{,t} - \nu_\kappa \Delta h &= 0 && \text{in } \Omega^T, \\ h|_{t=t_0} &= 0 && \text{on } \Omega \times \{t = t_0\}. \end{aligned}$$

From (3.2) we immediately deduce that

$$\nabla h \cdot \mathbf{n}|_{S_B} = \nabla h \cdot \mathbf{n}|_{S_T} = 0.$$

We only need to find the boundary condition for h on S_L . Using the vector identities

$$-\Delta = \operatorname{rot} \operatorname{rot} - \nabla \operatorname{div},$$

$$\operatorname{rot}(\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B},$$

we rewrite (1.1)₂ in the form

$$\mathbf{H}_{,t} + \operatorname{rot}(\mathbf{H} \times \mathbf{v}) + \nu_\kappa \operatorname{rot} \operatorname{rot} \mathbf{H} - \nu_\kappa \nabla \operatorname{div} \mathbf{H} = \mathbf{0}.$$

Multiplying this equation by \mathbf{n} and projecting the result onto S_L yields

$$\nabla h \cdot \mathbf{n}|_{S_L} = \frac{1}{\nu_\kappa} \operatorname{rot}(\mathbf{H} \times \mathbf{v}) \cdot \mathbf{n}|_{S_L} + \operatorname{rot} \operatorname{rot} \mathbf{H} \cdot \mathbf{n}|_{S_L}. \quad (3.8)$$

Using the identity

$$\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \operatorname{rot} \mathbf{A} - \mathbf{A} \cdot \operatorname{rot} \mathbf{B}$$

we see that

$$-\frac{1}{\nu_\kappa} \operatorname{rot}(\mathbf{H} \times \mathbf{v}) \cdot \mathbf{n}|_{S_L} = \frac{1}{\nu_\kappa} \operatorname{div}(\mathbf{n} \times (\mathbf{H} \times \mathbf{v}))|_{S_L} - \frac{1}{\nu_\kappa} \mathbf{H} \times \mathbf{v} \cdot \operatorname{rot} \mathbf{n}|_{S_L}.$$

Now

$$\mathbf{n} \times (\mathbf{H} \times \mathbf{v}) = (\mathbf{n} \cdot \mathbf{v}) \mathbf{H} - (\mathbf{n} \cdot \mathbf{H}) \mathbf{v} = \mathbf{0}$$

because of (1.1)_{4,5}. From (3.1)₁ we get

$$\operatorname{rot} \mathbf{n}|_{S_L} = \nabla \left(\frac{1}{|\nabla \varphi|} \right) \times [\varphi_{,x_1}, \varphi_{,x_2}, 0]|_{S_L} = \left[0, 0, \partial_{x_1} \left(\frac{1}{|\nabla \varphi|} \right) \varphi_{,x_2} - \partial_{x_2} \left(\frac{1}{|\nabla \varphi|} \right) \varphi_{,x_1} \right]|_{S_L}$$

and

$$\begin{aligned} \mathbf{H} \cdot \mathbf{n}|_{S_L} &= H_1 n_1 + H_2 n_2|_{S_L} = 0, \\ \mathbf{v} \cdot \mathbf{n}|_{S_L} &= v_1 n_1 + v_2 n_2|_{S_L} = 0. \end{aligned}$$

Thus

$$(\mathbf{H} \times \mathbf{v})_3 = H_1 v_2 - H_2 v_1 = -H_2 \frac{n_2}{n_1} \left(-v_1 \frac{n_1}{n_2} \right) - H_2 v_1 = 0.$$

Hence

$$\frac{1}{\nu_\kappa} \mathbf{H} \times \mathbf{v} \cdot \operatorname{rot} \mathbf{n}|_{S_L} = 0$$

and (3.8) becomes

$$\nabla h \cdot \mathbf{n}|_{S_L} = \operatorname{rot} \operatorname{rot} \mathbf{H} \cdot \mathbf{n}|_{S_L}.$$

Using (3.1) we obtain

$$\begin{aligned} \operatorname{rot} \operatorname{rot} \mathbf{H} \cdot \mathbf{n}|_{S_L} &= \frac{1}{|\nabla \varphi|} [(H_{2,x_1x_2} - H_{1,x_2x_2} - H_{1,x_3x_3} + H_{3,x_1x_3}) \varphi_{,x_1} \\ &\quad + (H_{3,x_2x_3} - H_{2,x_3x_3} - H_{2,x_1x_1} + H_{1,x_1x_2}) \varphi_{,x_2}]. \end{aligned}$$

Since $\mathbf{e}_3 = [0, 0, 1]$ is tangent to S_L we add to the above equality $\partial_{x_3} (\operatorname{rot} \mathbf{H} \times \mathbf{n}) = 0$

$$\operatorname{rot} \operatorname{rot} \mathbf{H} \cdot \mathbf{n}|_{S_L} = \frac{1}{|\nabla \varphi|} [(H_{2,x_1x_2} - H_{1,x_2x_2}) \varphi_{,x_1} + (H_{1,x_1x_2} - H_{2,x_1x_1}) \varphi_{,x_2}].$$

Using (3.3) we eventually get

$$\operatorname{rot} \operatorname{rot} \mathbf{H} \cdot \mathbf{n}|_{S_L} = 0,$$

thus

$$\nabla h \cdot \mathbf{n}|_{S_L} = 0.$$

Summarizing, we get

$$\begin{aligned} h_{,t} - \nu_\kappa \Delta h &= 0 && \text{in } \Omega^T, \\ \nabla h \cdot \mathbf{n}|_S &= 0 && \text{on } S^T, \\ h|_{t=t_0} &= 0 && \text{on } \Omega \times \{t = t_0\}. \end{aligned}$$

Multiplying the first equation by h , integrating with respect to Ω and t we easily see that $h \equiv 0$ for all $t \in (t_0, T)$ a.e. This concludes the proof. \square

Existence and Estimates of Solutions to Evolutionary Systems

Lemma 3.13. *Let us consider the Stokes problem*

$$\begin{aligned} \mathbf{v}_{,t} - \nu \Delta \mathbf{v} + \nabla p &= \mathbf{F} && \text{in } \Omega^T, \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega^T, \\ \mathbf{v} \cdot \mathbf{n} &= 0 && \text{on } S^T, \\ \operatorname{rot} \mathbf{v} \times \mathbf{n} &= \mathbf{0} && \text{on } S^T, \\ \mathbf{v}|_{t=t_0} &= \mathbf{v}_{t_0} && \text{on } \Omega \times \{t = t_0\}. \end{aligned} \tag{3.9}$$

If $\mathbf{F} \in L_s(\Omega^T)$ and $\mathbf{v}_{t_0} \in W_s^{2-\frac{2}{s}}(\Omega)$, where $1 < s < \infty$, then there exist a unique solution such that $\mathbf{v} \in W_s^{2,1}(\Omega^T)$, $\nabla p \in L_s(\Omega^T)$ and

$$\|\mathbf{v}\|_{W_s^{2,1}(\Omega^T)} + \|\nabla p\|_{L_s(\Omega^T)} \leq c \left(\|\mathbf{F}\|_{L_s(\Omega^T)} + \|\mathbf{v}_{t_0}\|_{W_s^{2-\frac{2}{s}}(\Omega)} \right).$$

Sketch of the proof. An identical problem but with slip boundary conditions with friction was considered in [26]. In cylindrical domains there is a tight relation between the Navier and slip boundary conditions, see e.g. [27, Lemma 6.5]. If we were to follow the details from [26], we would have to make sure that the problem for p has proper boundary conditions. Applying div to both sides of (3.9)₁ yields

$$\Delta p = \operatorname{div} \mathbf{F}.$$

To derive the boundary conditions we multiply (3.9)₁ by \mathbf{n} and project the result onto S . From (3.9)_{3,4} and (3.2) we immediately obtain

$$-\nu \Delta \mathbf{v} \cdot \mathbf{n} + \nabla p \cdot \mathbf{n} = \mathbf{F} \cdot \mathbf{n} \quad \text{on } S.$$

The first term on the left-hand side is zero. Indeed, we have

$$-\Delta \mathbf{v} \cdot \mathbf{n} = \operatorname{rot} \operatorname{rot} \mathbf{v} \cdot \mathbf{n} = 0$$

because the tangential component of $\operatorname{rot} \mathbf{v}$ is zero and therefore by the Stokes' theorem the normal component of $\operatorname{rot} \operatorname{rot} \mathbf{v}$ is zero, too. Summarizing, we have

$$\begin{aligned} \Delta p &= \operatorname{div} \mathbf{F} && \text{in } \Omega, \\ \nabla p \cdot \mathbf{n} &= \mathbf{F} \cdot \mathbf{n} && \text{on } S, \end{aligned} \quad (3.10)$$

which is a simpler version of Problem (2.29) from [26] (in [26] condition (3.10) reads $\nabla p \cdot \mathbf{n} = \mathbf{F} \cdot \mathbf{n} + b$). Therefore, the whole proof of the main Theorem in [26] can be safely repeated. \square

Lemma 3.14. *Consider the following initial-boundary value problem*

$$\begin{aligned} \mathbf{H}_{,t} - \nu_{\kappa} \Delta \mathbf{H} &= \mathbf{G} && \text{in } \Omega^T, \\ \mathbf{H} \cdot \mathbf{n} &= 0 && \text{on } S^T, \\ \operatorname{rot} \mathbf{H} \times \mathbf{n} &= \mathbf{0} && \text{on } S^T, \\ \mathbf{H}|_{t=t_0} &= \mathbf{H}_{t_0} && \text{on } \Omega \times \{t = t_0\}. \end{aligned} \quad (3.11)$$

Assume that $\mathbf{G} \in L_p(\Omega^T)$, $\mathbf{H}_{t_0} \in W_p^{2-\frac{2}{p}}(\Omega)$, $1 < p < \infty$. Then, there exist a unique solution \mathbf{H} such that $\mathbf{H} \in W_p^{2,1}(\Omega^T)$ and

$$\|\mathbf{H}\|_{W_p^{2,1}(\Omega^T)} \leq c \left(\|\mathbf{G}\|_{L_p(\Omega^T)} + \|\mathbf{H}_{t_0}\|_{W_p^{2-\frac{2}{p}}(\Omega)} \right).$$

Sketch of the proof. Since the boundary is not smooth we proceed as follows. First we reflect \mathbf{H} outside Ω with respect to x_3 . This is possible due to (3.11)_{2,3}, which imply (3.2). In that way we obtain (3.11) in $\bar{\Omega} := \Omega' \times (-3a, 3a)$.

In the second step we introduce a partition of unity $\sum_{l=0}^N \zeta_l(x_3) = 1$ on Ω , i.e. $\Omega \subset \bigcup_{l=0}^N \operatorname{supp} \zeta_l$. For a given l two situations may occur: either $\operatorname{supp} \zeta_l \cap S = \emptyset$ or $\operatorname{supp} \zeta_l \cap S \neq \emptyset$. In the first situation we get a model problem in the whole space. In the second situation after straightening up the boundary we obtain the model problem in the half-space. For further investigation of these model problems we refer the reader to e.g. [28]. \square

4. Basic Energy Estimates

In this Section we establish energy estimates for (\mathbf{v}, \mathbf{H}) and $(\mathbf{v}_{,x_3}, \mathbf{H}_{,x_3})$.

Lemma 4.1. *Suppose that $\mathbf{v}_{t_0}, \mathbf{H}_{t_0} \in L_2(\Omega)$, $\operatorname{div} \mathbf{v}_{t_0} = \operatorname{div} \mathbf{H}_{t_0} = 0$, and $\mathbf{f} \in L_2(t_0, T; L_{\frac{6}{5}}(\Omega))$. Then, for any $t \in (t_0, T)$ we have $\mathbf{v}, \mathbf{H} \in V_2^0(\Omega^t)$ and*

$$\|\mathbf{v}\|_{V_2^0(\Omega^t)}^2 + \|\mathbf{H}\|_{V_2^0(\Omega^t)}^2 \leq cd^2(T). \quad (4.1)$$

Proof. We multiply (1.1)_{1,2} by \mathbf{v} and \mathbf{H} , respectively, integrate over Ω and add to each other

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\mathbf{v}\|_{L_2(\Omega)}^2 + \|\mathbf{H}\|_{L_2(\Omega)}^2 \right) + \nu \|\operatorname{rot} \mathbf{v}\|_{L_2(\Omega)}^2 + \nu_{\kappa} \|\operatorname{rot} \mathbf{H}\|_{L_2(\Omega)}^2 \\ &= \int_{\Omega} (\mathbf{H} \cdot \nabla) \mathbf{H} \cdot \mathbf{v} \, dx + \int_{\Omega} (\mathbf{H} \cdot \nabla) \mathbf{v} \cdot \mathbf{H} \, dx + \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx. \end{aligned}$$

The sum of the two first integrals on the right-hand side is equal to zero. To the last integral we apply the Hölder and Cauchy inequalities. To get H^1 -norms we use Lemma 3.9. We have

$$\frac{d}{dt} \left(\|\mathbf{v}\|_{L_2(\Omega)}^2 + \|\mathbf{H}\|_{L_2(\Omega)}^2 \right) + c \|\mathbf{v}\|_{H^1(\Omega)}^2 + c \|\mathbf{H}\|_{H^1(\Omega)}^2 \leq c \|\mathbf{f}\|_{L_{\frac{6}{5}}(\Omega)}^2.$$

Integration with respect to time concludes the proof. \square

Lemma 4.2. *Suppose that $\operatorname{div} \mathbf{v}_{t_0} = \operatorname{div} \mathbf{H}_{t_0} = 0$, $\nabla \mathbf{v}, \nabla \mathbf{H} \in L_2(t_0, t; L_3(\Omega))$ for any $t_0 \leq t \leq T$ and $\delta(T) < \infty$. Then $\mathbf{v}_{,x_3}, \mathbf{H}_{,x_3} \in V_2^0(\Omega^t)$ and*

$$\|\mathbf{v}_{,x_3}\|_{V_2^0(\Omega^t)}^2 + \|\mathbf{H}_{,x_3}\|_{V_2^0(\Omega^t)}^2 \leq c e^{c(\|\Delta \mathbf{v}\|_{L_2(\Omega^t)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega^t)}^2)} \delta^2(T).$$

To make the proof of the above Lemma more readable we introduce a temporary notation

$$\begin{aligned} \mathbf{w} &:= \mathbf{v}_{,x_3}, \quad \mathbf{K} := \mathbf{H}_{,x_3}, \\ q_H &:= \partial_{x_3} \left(p + \frac{1}{2} |\mathbf{H}|^2 \right). \end{aligned}$$

Differentiating (1.1)_{1,2,3} with respect to x_3 and using (3.2) we get the following systems for \mathbf{w} and \mathbf{K} :

$$\begin{aligned} \mathbf{w}_{,t} - \nu \Delta \mathbf{w} + \nabla q_H &= -(\mathbf{w} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{w} + (\mathbf{K} \cdot \nabla) \mathbf{H} + (\mathbf{H} \cdot \nabla) \mathbf{K} + \mathbf{f}_{,x_3} && \text{in } \Omega^T, \\ \operatorname{div} \mathbf{w} &= 0 && \text{in } \Omega^T, \\ \operatorname{rot} \mathbf{w} \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{w} \cdot \mathbf{n} = 0 && \text{on } S_L^T, \\ \mathbf{w}' &= \mathbf{0}, \quad w_{3,x_3} = 0 && \text{on } S_B^T \text{ and } S_T^T, \\ \mathbf{w}|_{t=t_0} &= \mathbf{w}_{t_0} && \text{on } \Omega \times (t_0, T) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \mathbf{K}_{,t} - \nu_\kappa \Delta \mathbf{K} &= (\mathbf{K} \cdot \nabla) \mathbf{v} + (\mathbf{H} \cdot \nabla) \mathbf{w} - (\mathbf{w} \cdot \nabla) \mathbf{H} - (\mathbf{v} \cdot \nabla) \mathbf{K} && \text{in } \Omega^T, \\ \operatorname{div} \mathbf{K} &= 0 && \text{in } \Omega^T, \\ \operatorname{rot} \mathbf{K} \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{K} \cdot \mathbf{n} = 0 && \text{on } S_L^T, \\ \mathbf{K}' &= \mathbf{0}, \quad K_{3,x_3} = 0 && \text{on } S_B^T \text{ and } S_T^T, \\ \mathbf{K}|_{t=t_0} &= \mathbf{K}_{t_0} && \text{on } \Omega \times (t_0, T). \end{aligned} \tag{4.3}$$

Proof of Lemma 4.2. We multiply (4.2)₁ and (4.3)₁ by \mathbf{w} and \mathbf{K} , respectively, integrate over Ω and add to each other

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|\mathbf{w}\|_{L_2(\Omega)}^2 + \|\mathbf{K}\|_{L_2(\Omega)}^2 \right) + \nu \|\operatorname{rot} \mathbf{w}\|_{L_2(\Omega)}^2 + \nu_\kappa \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \\ &= \int_\Omega [(-(\mathbf{w} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{w} + (\mathbf{K} \cdot \nabla) \mathbf{H} + (\mathbf{H} \cdot \nabla) \mathbf{K} + \mathbf{f}_{,x_3}) \cdot \mathbf{w} \\ &\quad + ((\mathbf{K} \cdot \nabla) \mathbf{v} + (\mathbf{H} \cdot \nabla) \mathbf{w} - (\mathbf{w} \cdot \nabla) \mathbf{H} - (\mathbf{v} \cdot \nabla) \mathbf{K}) \cdot \mathbf{K}] \, dx. \end{aligned}$$

On the right-hand side we have nine terms under the integral sign. The second and the ninth terms vanish due to (4.2)₂ and (4.3)₂. The fourth and the seventh terms have opposite sign. Hence

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|\mathbf{w}\|_{L_2(\Omega)}^2 + \|\mathbf{K}\|_{L_2(\Omega)}^2 \right) + \nu \|\operatorname{rot} \mathbf{w}\|_{L_2(\Omega)}^2 + \nu_\kappa \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \\ &= \int_\Omega [(-(\mathbf{w} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} + (\mathbf{K} \cdot \nabla) \mathbf{H} \cdot \mathbf{w} + \mathbf{f}_{,x_3} \cdot \mathbf{w}) + ((\mathbf{K} \cdot \nabla) \mathbf{v} \cdot \mathbf{K} - (\mathbf{w} \cdot \nabla) \mathbf{H} \cdot \mathbf{K})] \, dx. \end{aligned}$$

Next, we apply Lemma 3.9 on the left-hand side and the Hölder, Young and Sobolev inequalities on the right-hand side

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|\mathbf{w}\|_{L_2(\Omega)}^2 + \|\mathbf{K}\|_{L_2(\Omega)}^2 \right) + c \left(\|\mathbf{w}\|_{H^1(\Omega)}^2 + \|\mathbf{K}\|_{H^1(\Omega)}^2 \right) \leq c \epsilon_1 \left(\|\mathbf{w}\|_{H^1(\Omega)}^2 + \|\mathbf{K}\|_{H^1(\Omega)}^2 \right) \\ &\quad + \frac{c}{\epsilon_1} \left(\|\mathbf{w}\|_{L_2(\Omega)}^2 + \|\mathbf{K}\|_{L_2(\Omega)}^2 \right) \left(\|\nabla \mathbf{v}\|_{L_3(\Omega)}^2 + \|\nabla \mathbf{H}\|_{L_3(\Omega)}^2 \right) + c \epsilon_2 \|\mathbf{w}\|_{H^1(\Omega)}^2 + \frac{c}{\epsilon_2} \|\mathbf{f}_{,x_3}\|_{L_{\frac{6}{5}}(\Omega)}^2. \end{aligned}$$

Choosing ϵ_1 and ϵ_2 small enough we get

$$\begin{aligned} & \frac{d}{dt} \left(\|\mathbf{w}\|_{L_2(\Omega)}^2 + \|\mathbf{K}\|_{L_2(\Omega)}^2 \right) + c \left(\|\mathbf{w}\|_{H^1(\Omega)}^2 + \|\mathbf{K}\|_{H^1(\Omega)}^2 \right) \\ & \leq c \left(\|\mathbf{w}\|_{L_2(\Omega)}^2 + \|\mathbf{K}\|_{L_2(\Omega)}^2 \right) \left(\|\nabla \mathbf{v}\|_{L_3(\Omega)}^2 + \|\nabla \mathbf{H}\|_{L_3(\Omega)}^2 \right) + c \|\mathbf{f}_{,x_3}\|_{L_{\frac{6}{5}}(\Omega)}^2. \end{aligned} \quad (4.4)$$

The Gronwall inequality yields

$$\|\mathbf{w}(t)\|_{L_2(\Omega)}^2 + \|\mathbf{K}(t)\|_{L_2(\Omega)}^2 \leq c \exp \left(c \|\nabla \mathbf{v}\|_{L_2(t_0, t; L_3(\Omega))}^2 + c \|\nabla \mathbf{H}\|_{L_2(t_0, t; L_3(\Omega))}^2 \right) \delta^2(T).$$

Integrating (4.4) with respect to time and using the above inequality along with the Sobolev inequality ends the proof. \square

5. Auxiliary Estimates

We start this section with a simple observation that follows from the Gagliardo–Nirenberg inequality.

Remark 5.1. If we set $p = 4, r = q = 2, j = 0$ and $m = 1$ in Lemma 3.11, then

$$\begin{aligned} \|\bar{z}\|_{L_4(\Omega')} & \leq c \|\nabla \bar{z}\|_{L_2(\Omega')}^{\frac{1}{2}} \|\bar{z}\|_{L_2(\Omega')}^{\frac{1}{2}} + c \|\bar{z}\|_{L_2(\Omega')}, \\ \|\nabla \bar{z}\|_{L_4(\Omega')} & \leq c \|D^2 \bar{z}\|_{L_2(\Omega')}^{\frac{1}{2}} \|\nabla \bar{z}\|_{L_2(\Omega')}^{\frac{1}{2}} + c \|\nabla \bar{z}\|_{L_2(\Omega')}. \end{aligned}$$

If additionally $\bar{z}|_{\partial\Omega'} = 0$ or $\int_{\Omega'} \bar{z} \, dx' = 0$, then (see e.g. [29, Ch. 2, §2, Thm. 2.2 and Rem. 2.1])

$$\|\bar{z}\|_{L_4(\Omega')} \leq c \|\nabla \bar{z}\|_{L_2(\Omega')}^{\frac{1}{2}} \|\bar{z}\|_{L_2(\Omega')}^{\frac{1}{2}}.$$

Below we prove a couple of results related to integrals corresponding to non-linear terms. We saw in Remark 3.6 that every integral involving a non-linear term splits into four integrals. These four integrals are estimated separately in a series of Lemmas 5.3–5.6.

Lemma 5.2. *Suppose that $\operatorname{div} \mathbf{w} = 0$ and $\operatorname{rot} \mathbf{w} \times \mathbf{n} = \mathbf{0}$. If $\bar{\mathbf{w}} = (\bar{w}_1, \bar{w}_2, 0)$, then*

$$\int_{\Omega} (\bar{\mathbf{w}} \cdot \nabla) \bar{\mathbf{w}} \cdot \Delta \bar{\mathbf{w}} \, dx = 0.$$

Proof. First we note that the given integral does not depend on x_3 . Next, we use that $-\Delta \bar{\mathbf{w}} = \operatorname{rot} \operatorname{rot} \bar{\mathbf{w}}$ and integrate by parts

$$\begin{aligned} - \int_{\Omega} (\bar{\mathbf{w}} \cdot \nabla) \bar{\mathbf{w}} \cdot \Delta \bar{\mathbf{w}} \, dx & = 2a \int_{\Omega'} (\bar{\mathbf{w}} \cdot \nabla') \bar{\mathbf{w}} \cdot \operatorname{rot} \operatorname{rot} \bar{\mathbf{w}} \, dx' \\ & = 2a \int_{\Omega'} \operatorname{rot} ((\bar{\mathbf{w}} \cdot \nabla') \bar{\mathbf{w}}) \cdot \operatorname{rot} \bar{\mathbf{w}} \, dx' - \int_{S'} \operatorname{rot} \bar{\mathbf{w}} \times \mathbf{n} \cdot (\bar{\mathbf{w}} \cdot \nabla') \bar{\mathbf{w}} \, dS'. \end{aligned}$$

The boundary integral vanishes due to Lemma 3.1. Moreover,

$$\begin{aligned} 2a \int_{\Omega'} \operatorname{rot} ((\bar{\mathbf{w}} \cdot \nabla') \bar{\mathbf{w}}) \cdot \operatorname{rot} \bar{\mathbf{w}} \, dx' & = 2a \int_{\Omega'} (\bar{w}_{1,x_2} \bar{w}_{1,x_1} + \bar{w}_1 \bar{w}_{1,x_1 x_2} + \bar{w}_{2,x_2} \bar{w}_{1,x_2} + \bar{w}_2 \bar{w}_{1,x_2 x_2} \\ & \quad - \bar{w}_{1,x_1} \bar{w}_{2,x_1} - \bar{w}_1 \bar{w}_{2,x_1 x_1} - \bar{w}_{2,x_1} \bar{w}_{2,x_2} - \bar{w}_2 \bar{w}_{2,x_2 x_1}) (\bar{w}_{1,x_2} - \bar{w}_{2,x_1}) \, dx'. \end{aligned}$$

Since $\operatorname{div} \bar{\mathbf{w}} = 0$ we infer that the above integral is equal to

$$2a \int_{\Omega'} \left(w_1 (\bar{w}_{1,x_2} - \bar{w}_{2,x_1})_{,x_1} + w_2 (\bar{w}_{1,x_2} - \bar{w}_{2,x_1})_{,x_2} \right) (\bar{w}_{1,x_2} - \bar{w}_{2,x_1}).$$

Integrating by parts and using $\operatorname{div} \bar{\mathbf{w}} = 0$ and $\bar{\mathbf{w}} \cdot \mathbf{n} = 0$ we conclude the proof. \square

Lemma 5.3. *Suppose that $\mathbf{a} \in H^1(\Omega)$, $\mathbf{b}, \mathbf{w} \in H^2(\Omega)$ and $\operatorname{div} \mathbf{b} = 0$. Then*

$$\begin{aligned} \int_{\Omega} (\bar{\mathbf{a}} \cdot \nabla) \bar{\mathbf{b}} \cdot \Delta \bar{\mathbf{w}} \, dx &\leq \epsilon \|\Delta \mathbf{w}\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \left(\|\mathbf{a}\|_{L_2(\Omega)} \|\nabla \mathbf{a}\|_{L_2(\Omega)} \|\nabla \mathbf{b}\|_{L_2(\Omega)} \|\Delta \mathbf{b}\|_{L_2(\Omega)} + \|\mathbf{a}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{b}\|_{L_2(\Omega)}^2 \right) \\ &\quad + \frac{c}{\epsilon} \left(\|\mathbf{a}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{b}\|_{L_2(\Omega)} \|\Delta \mathbf{b}\|_{L_2(\Omega)} + \|\nabla \mathbf{b}\|_{L_2(\Omega)}^2 \|\mathbf{a}\|_{L_2(\Omega)} \|\nabla \mathbf{a}\|_{L_2(\Omega)} \right). \end{aligned}$$

Proof. In light of the Hölder and Young inequalities

$$\int_{\Omega} (\bar{\mathbf{a}} \cdot \nabla) \bar{\mathbf{b}} \cdot \Delta \bar{\mathbf{w}} \, dx \leq \epsilon \|\Delta \bar{\mathbf{w}}\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\bar{\mathbf{a}}\|_{L_4(\Omega)}^2 \|\nabla \bar{\mathbf{b}}\|_{L_4(\Omega)}^2.$$

To estimate the L_4 -norms we use Remark 5.1 and the Young inequality

$$\begin{aligned} \|\bar{\mathbf{a}}\|_{L_4(\Omega)}^2 \|\nabla \bar{\mathbf{b}}\|_{L_4(\Omega)}^2 &= \left(\|\bar{\mathbf{a}}\|_{L_2(\Omega)}^{\frac{1}{2}} \|\nabla \bar{\mathbf{a}}\|_{L_2(\Omega)}^{\frac{1}{2}} + \|\bar{\mathbf{a}}\|_{L_2(\Omega)} \right)^2 \left(\|\nabla \bar{\mathbf{b}}\|_{L_2(\Omega)}^{\frac{1}{2}} \|\nabla^2 \bar{\mathbf{b}}\|_{L_2(\Omega)}^{\frac{1}{2}} + \|\nabla \bar{\mathbf{b}}\|_{L_2(\Omega)} \right)^2 \\ &\leq c \|\bar{\mathbf{a}}\|_{L_2(\Omega)} \|\nabla \bar{\mathbf{a}}\|_{L_2(\Omega)} \|\nabla \bar{\mathbf{b}}\|_{L_2(\Omega)} \|\nabla^2 \bar{\mathbf{b}}\|_{L_2(\Omega)} + c \|\bar{\mathbf{a}}\|_{L_2(\Omega)}^2 \|\nabla \bar{\mathbf{b}}\|_{L_2(\Omega)}^2 \\ &\quad + c \|\bar{\mathbf{a}}\|_{L_2(\Omega)}^2 \|\nabla \bar{\mathbf{b}}\|_{L_2(\Omega)} \|\nabla^2 \bar{\mathbf{b}}\|_{L_2(\Omega)} + c \|\nabla \bar{\mathbf{b}}\|_{L_2(\Omega)}^2 \|\bar{\mathbf{a}}\|_{L_2(\Omega)} \|\nabla \bar{\mathbf{a}}\|_{L_2(\Omega)}. \end{aligned}$$

Using $\operatorname{div} \bar{\mathbf{b}} = 0$ we get $\bar{b}_{i,x_j x_i} = -\bar{b}_{j,x_j x_i}$ for $i, j \in \{1, 2\}$. To conclude the proof we use Remark 3.4. \square

Lemma 5.4. *Suppose that $\mathbf{a}, \mathbf{b} \in \{\mathbf{v}, \mathbf{H}\}$. Then*

$$\begin{aligned} \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot \Delta \mathbf{w} \, dx &\leq \epsilon \left(\|\Delta \mathbf{w}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{b}\|_{L_2(\Omega)}^2 \right) \\ &\quad + \frac{c}{\epsilon} \|\mathbf{a}_{,x_3}\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla \mathbf{a}\|_{L_2(\Omega)}^{\frac{8}{3}} \|\nabla \mathbf{b}\|_{L_2(\Omega)}^2. \end{aligned}$$

Proof. Using the Hölder and the Young inequalities we get

$$\int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot \Delta \mathbf{w} \, dx \leq \epsilon \|\Delta \mathbf{w}\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\mathbf{a} - \bar{\mathbf{a}}\|_{L_6(\Omega)}^2 \|\nabla (\mathbf{b} - \bar{\mathbf{b}})\|_{L_3(\Omega)}^2.$$

By the interpolation inequality between L_p -spaces

$$\|\mathbf{a} - \bar{\mathbf{a}}\|_{L_6(\Omega)}^2 \|\nabla (\mathbf{b} - \bar{\mathbf{b}})\|_{L_3(\Omega)}^2 \leq \|\mathbf{a} - \bar{\mathbf{a}}\|_{L_6(\Omega)}^2 \|\nabla (\mathbf{b} - \bar{\mathbf{b}})\|_{L_2(\Omega)} \|\nabla (\mathbf{b} - \bar{\mathbf{b}})\|_{L_6(\Omega)}.$$

Remark 3.8, the embedding theorem and the Cauchy inequality yield

$$\begin{aligned} &\|\mathbf{a} - \bar{\mathbf{a}}\|_{L_6(\Omega)}^2 \|\nabla (\mathbf{b} - \bar{\mathbf{b}})\|_{L_2(\Omega)} \|\nabla (\mathbf{b} - \bar{\mathbf{b}})\|_{L_6(\Omega)} \\ &\leq \epsilon \|\Delta (\mathbf{b} - \bar{\mathbf{b}})\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\mathbf{a}_{,x_3}\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla (\mathbf{a} - \bar{\mathbf{a}})\|_{L_2(\Omega)}^{\frac{8}{3}} \|\nabla (\mathbf{b} - \bar{\mathbf{b}})\|_{L_2(\Omega)}^2. \end{aligned}$$

We conclude the proof using Lemma 3.10 and Remark 3.4. \square

Lemma 5.5. *Suppose that $\mathbf{a}, \mathbf{b} \in \{\mathbf{v}, \mathbf{H}\}$. Then*

$$\begin{aligned} \int_{\Omega} ((\mathbf{a} - \bar{\mathbf{a}}) \cdot \nabla) \bar{\mathbf{b}} \cdot \Delta (\mathbf{w} - \bar{\mathbf{w}}) \, dx &\leq \epsilon \left(\|\Delta \mathbf{w}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{b}\|_{L_2(\Omega)}^2 \right) \\ &\quad + \frac{c}{\epsilon} \|\mathbf{a}_{,x_3}\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla \mathbf{a}\|_{L_2(\Omega)}^{\frac{8}{3}} \|\nabla \mathbf{b}\|_{L_2(\Omega)}^2. \end{aligned}$$

Proof. The proof is identical to the proof of Lemma 5.4. \square

Lemma 5.6. *Suppose that $\mathbf{a}, \mathbf{b} \in \{\mathbf{v}, \mathbf{H}\}$ and \mathbf{b} satisfies $\operatorname{rot} \mathbf{b} \times \mathbf{n} = \mathbf{0}$, $\mathbf{b} \cdot \mathbf{n} = 0$ on S_B and on S_T . Then*

$$\begin{aligned} \int_{\Omega} (\bar{\mathbf{a}} \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot \Delta (\mathbf{w} - \bar{\mathbf{w}}) \, dx &\leq \epsilon \left(\|\Delta \mathbf{w}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{b}\|_{L_2(\Omega)}^2 \right) \\ &\quad + \frac{c}{\epsilon} \|\mathbf{a}\|_{H^1(\Omega)}^6 \|\nabla \mathbf{b}_{,x_3}\|_{L_2(\Omega)}^2. \end{aligned}$$

Proof. Since $\bar{a}_3 = 0$ we get from the application of the Hölder and the Young inequalities

$$\int_{\Omega} (\bar{\mathbf{a}} \cdot \nabla) (\mathbf{b} - \bar{\mathbf{b}}) \cdot \Delta (\mathbf{w} - \bar{\mathbf{w}}) \, dx \leq \epsilon \|\Delta (\mathbf{w} - \bar{\mathbf{w}})\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\bar{\mathbf{a}}\|_{L_6(\Omega)}^2 \|\nabla' (\mathbf{b} - \bar{\mathbf{b}})\|_{L_3(\Omega)}^2,$$

where $\nabla' = [\partial_{x_1}, \partial_{x_2}]$. By Remark 3.8 we get

$$\|\nabla' (\mathbf{b} - \bar{\mathbf{b}})\|_{L_3(\Omega)}^2 \leq c \|\nabla \mathbf{b}_{,x_3}\|_{L_2(\Omega)}^{\frac{2}{3}} \|\nabla^2 (\mathbf{b} - \bar{\mathbf{b}})\|_{L_2(\Omega)}^{\frac{4}{3}}.$$

Application of the Young inequality along with Lemma 3.10 and Remark 3.4 ends the proof. \square

6. Higher Order Estimates

Lemma 6.1. *Suppose that $\mathbf{v}_{t_0}, \mathbf{H}_{t_0} \in H^1(\Omega)$, $\operatorname{div} \mathbf{v}_{t_0} = \operatorname{div} \mathbf{H}_{t_0} = 0$, and $\mathbf{f} \in L_2(\Omega^T)$. If $\delta^2(T)$ is sufficiently small, then*

$$\|\mathbf{v}\|_{V_2^1(\Omega^t)}^2 + \|\mathbf{H}\|_{V_2^1(\Omega^t)}^2 \leq cE^2(T)e^{c(1+d^2(T))} \quad (6.1)$$

for any $t \in (t_0, T)$.

Proof. We multiply (1.1)_{1,2} by $-\Delta \mathbf{v}$ and $-\Delta \mathbf{H}$ respectively, integrate over Ω and add to each other

$$\begin{aligned} & - \int_{\Omega} \mathbf{v}_{,t} \cdot \Delta \mathbf{v} \, dx - \int_{\Omega} \mathbf{H}_{,t} \cdot \Delta \mathbf{H} \, dx + \nu \int_{\Omega} \Delta \mathbf{v} \cdot \Delta \mathbf{v} \, dx + \nu_{\kappa} \int_{\Omega} \Delta \mathbf{H} \cdot \Delta \mathbf{H} \, dx \\ & = \int_{\Omega} \nabla \left(p + \frac{1}{2} |\mathbf{H}|^2 \right) \cdot \Delta \mathbf{v} \, dx + \int_{\Omega} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \Delta \mathbf{v} \, dx - \int_{\Omega} (\mathbf{H} \cdot \nabla) \mathbf{H} \cdot \Delta \mathbf{v} \, dx \\ & \quad + \int_{\Omega} (\mathbf{v} \cdot \nabla) \mathbf{H} \cdot \Delta \mathbf{H} \, dx - \int_{\Omega} (\mathbf{H} \cdot \nabla) \mathbf{v} \cdot \Delta \mathbf{H} \, dx - \int_{\Omega} \mathbf{f} \cdot \Delta \mathbf{v} \, dx =: \sum_{k=1}^6 J_k. \end{aligned} \quad (6.2)$$

In the two first integrals of the left-hand side we integrate by parts using (3.4). Thus, the left-hand side of (6.2) is estimated from below by

$$\frac{1}{2} \frac{d}{dt} \left(\|\operatorname{rot} \mathbf{v}\|_{L_2(\Omega)}^2 + \|\operatorname{rot} \mathbf{H}\|_{L_2(\Omega)}^2 \right) + c \left(\|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 \right).$$

On the right-hand side in (6.2) we have 6 integrals, which we denote by J_k . In light of (3.4) and Lemma 3.1 we see that $J_1 = 0$. To estimate integrals J_2 – J_5 we first use Remark 3.6

$$\begin{aligned} J_2 &= \int_{\Omega} (\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{v}} \cdot \Delta \bar{\mathbf{v}} \, dx + \int_{\Omega} ((\mathbf{v} - \bar{\mathbf{v}}) \cdot \nabla) (\mathbf{v} - \bar{\mathbf{v}}) \cdot \Delta \mathbf{v} \, dx \\ & \quad + \int_{\Omega} ((\mathbf{v} - \bar{\mathbf{v}}) \cdot \nabla) \bar{\mathbf{v}} \cdot \Delta (\mathbf{v} - \bar{\mathbf{v}}) \, dx + \int_{\Omega} (\bar{\mathbf{v}} \cdot \nabla) (\mathbf{v} - \bar{\mathbf{v}}) \cdot \Delta (\mathbf{v} - \bar{\mathbf{v}}) \, dx =: \sum_{m=1}^4 J_{2m}, \\ J_3 &= - \int_{\Omega} (\bar{\mathbf{H}} \cdot \nabla) \bar{\mathbf{H}} \cdot \Delta \bar{\mathbf{v}} \, dx - \int_{\Omega} ((\mathbf{H} - \bar{\mathbf{H}}) \cdot \nabla) (\mathbf{H} - \bar{\mathbf{H}}) \cdot \Delta \mathbf{v} \, dx \\ & \quad - \int_{\Omega} ((\mathbf{H} - \bar{\mathbf{H}}) \cdot \nabla) \bar{\mathbf{H}} \cdot \Delta (\mathbf{v} - \bar{\mathbf{v}}) \, dx - \int_{\Omega} (\bar{\mathbf{H}} \cdot \nabla) (\mathbf{H} - \bar{\mathbf{H}}) \cdot \Delta (\mathbf{v} - \bar{\mathbf{v}}) \, dx =: - \sum_{m=1}^4 J_{3m}, \\ J_4 &= \int_{\Omega} (\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{H}} \cdot \Delta \bar{\mathbf{H}} \, dx + \int_{\Omega} ((\mathbf{v} - \bar{\mathbf{v}}) \cdot \nabla) (\mathbf{H} - \bar{\mathbf{H}}) \cdot \Delta \mathbf{H} \, dx \\ & \quad + \int_{\Omega} ((\mathbf{v} - \bar{\mathbf{v}}) \cdot \nabla) \bar{\mathbf{H}} \cdot \Delta (\mathbf{H} - \bar{\mathbf{H}}) \, dx + \int_{\Omega} (\bar{\mathbf{v}} \cdot \nabla) (\mathbf{H} - \bar{\mathbf{H}}) \cdot \Delta (\mathbf{H} - \bar{\mathbf{H}}) \, dx =: \sum_{m=1}^4 J_{4m} \end{aligned}$$

and

$$\begin{aligned} J_5 = & - \int_{\Omega} (\bar{\mathbf{H}} \cdot \nabla) \bar{\mathbf{v}} \cdot \Delta \bar{\mathbf{H}} \, dx - \int_{\Omega} ((\mathbf{H} - \bar{\mathbf{H}}) \cdot \nabla) (\mathbf{v} - \bar{\mathbf{v}}) \cdot \Delta \mathbf{H} \, dx \\ & - \int_{\Omega} ((\mathbf{H} - \bar{\mathbf{H}}) \cdot \nabla) \bar{\mathbf{v}} \cdot \Delta (\mathbf{H} - \bar{\mathbf{H}}) \, dx - \int_{\Omega} (\bar{\mathbf{H}} \cdot \nabla) (\mathbf{v} - \bar{\mathbf{v}}) \cdot \Delta (\mathbf{H} - \bar{\mathbf{H}}) \, dx =: - \sum_{m=1}^4 J_{5m}. \end{aligned}$$

We have obtained 16 new integrals, which we estimate using various lemmas from Sect. 5. Using Lemma 5.2 we see that $J_{21} = 0$.

Next, the Hölder and Cauchy inequalities along with the following lemmas yield:

- Lemma 5.3:

$$\begin{aligned} -J_{31} \leq & \epsilon \left(\|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 \right) + \frac{c}{\epsilon} \left(\|\mathbf{H}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{H}\|_{L_2(\Omega)}^4 \right. \\ & \left. + \|\mathbf{H}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 + \|\mathbf{H}\|_{L_2(\Omega)}^4 \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 + \|\mathbf{H}\|_{L_2(\Omega)} \|\nabla \mathbf{H}\|_{L_2(\Omega)}^3 \right), \\ J_{41} \leq & \epsilon \left(\|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 \right) + \frac{c}{\epsilon} \left(\|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 \|\mathbf{v}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{v}\|_{L_2(\Omega)}^2 \right. \\ & \left. + \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 \|\mathbf{v}\|_{L_2(\Omega)}^2 + \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 \|\mathbf{v}\|_{L_2(\Omega)}^4 + \|\mathbf{v}\|_{L_2(\Omega)} \|\nabla \mathbf{v}\|_{L_2(\Omega)} \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 \right) \end{aligned}$$

and

$$\begin{aligned} -J_{51} \leq & \epsilon \left(\|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 \right) + \frac{c}{\epsilon} \left(\|\mathbf{H}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{v}\|_{L_2(\Omega)}^2 \right. \\ & \left. + \|\mathbf{H}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{v}\|_{L_2(\Omega)}^2 + \|\mathbf{H}\|_{L_2(\Omega)}^4 \|\nabla \mathbf{v}\|_{L_2(\Omega)}^2 + \|\mathbf{H}\|_{L_2(\Omega)} \|\nabla \mathbf{H}\|_{L_2(\Omega)} \|\nabla \mathbf{v}\|_{L_2(\Omega)}^2 \right). \end{aligned}$$

- Lemma 5.4:

$$\begin{aligned} J_{22} \leq & \epsilon \|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\mathbf{v}, x_3\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla \mathbf{v}\|_{L_2(\Omega)}^{\frac{14}{3}}, \\ -J_{32} \leq & \epsilon \left(\|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 \right) + \frac{c}{\epsilon} \|\mathbf{H}, x_3\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla \mathbf{H}\|_{L_2(\Omega)}^{\frac{14}{3}}, \\ J_{42} \leq & \epsilon \|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\mathbf{v}, x_3\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla \mathbf{v}\|_{L_2(\Omega)}^{\frac{8}{3}} \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 \end{aligned}$$

and

$$-J_{52} \leq \epsilon \left(\|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 \right) + \frac{c}{\epsilon} \|\mathbf{H}, x_3\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla \mathbf{H}\|_{L_2(\Omega)}^{\frac{8}{3}} \|\nabla \mathbf{v}\|_{L_2(\Omega)}^2$$

- Lemma 5.5:

$$\begin{aligned} J_{23} \leq & \epsilon \|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\mathbf{v}, x_3\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla \mathbf{v}\|_{L_2(\Omega)}^{\frac{14}{3}}, \\ -J_{33} \leq & \epsilon \left(\|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 \right) + \frac{c}{\epsilon} \|\mathbf{H}, x_3\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla \mathbf{H}\|_{L_2(\Omega)}^{\frac{14}{3}}, \\ J_{43} \leq & \epsilon \|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\mathbf{v}, x_3\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla \mathbf{v}\|_{L_2(\Omega)}^{\frac{8}{3}} \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 \end{aligned}$$

and

$$-J_{53} \leq \epsilon \left(\|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 \right) + \frac{c}{\epsilon} \|\mathbf{H}, x_3\|_{L_2(\Omega)}^{\frac{4}{3}} \|\nabla \mathbf{H}\|_{L_2(\Omega)}^{\frac{8}{3}} \|\nabla \mathbf{v}\|_{L_2(\Omega)}^2$$

- Lemma 5.6:

$$\begin{aligned} J_{24} \leq & \epsilon \|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\mathbf{v}\|_{H^1(\Omega)}^6 \|\nabla \mathbf{v}, x_3\|_{L_2(\Omega)}^2, \\ -J_{34} \leq & \epsilon \left(\|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 \right) + \frac{c}{\epsilon} \|\mathbf{H}\|_{H^1(\Omega)}^6 \|\nabla \mathbf{H}, x_3\|_{L_2(\Omega)}^2, \\ J_{44} \leq & \epsilon \|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\mathbf{v}\|_{H^1(\Omega)}^6 \|\nabla \mathbf{H}, x_3\|_{L_2(\Omega)}^2, \\ -J_{54} \leq & \epsilon \left(\|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 \right) + \frac{c}{\epsilon} \|\mathbf{H}\|_{H^1(\Omega)}^6 \|\nabla \mathbf{v}, x_3\|_{L_2(\Omega)}^2. \end{aligned}$$

Finally, the estimate for the last term on the right-hand side in (6.2) is straightforward

$$\int_{\Omega} \mathbf{f} \cdot \Delta \mathbf{v} \, dx \leq \epsilon \|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \frac{c}{\epsilon} \|\mathbf{f}\|_{L_2(\Omega)}^2.$$

In the end we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\operatorname{rot} \mathbf{v}\|_{L_2(\Omega)}^2 + \|\operatorname{rot} \mathbf{H}\|_{L_2(\Omega)}^2 \right) + c \left(\|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 \right) \\ & \leq 16\epsilon \left(\|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 \right) + \frac{c}{\epsilon} \left(\|\nabla \mathbf{v}\|_{L_2(\Omega)}^2 + \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 \right) \\ & \quad \cdot \left(\|\mathbf{H}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 + \|\mathbf{H}\|_{L_2(\Omega)} \|\nabla \mathbf{H}\|_{L_2(\Omega)} + \|\mathbf{v}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{v}\|_{L_2(\Omega)}^2 + \|\mathbf{v}\|_{L_2(\Omega)} \|\nabla \mathbf{v}\|_{L_2(\Omega)} \right. \\ & \quad \left. + \|\mathbf{v}\|_{L_2(\Omega)}^4 + \|\mathbf{v}\|_{L_2(\Omega)}^2 + \|\mathbf{H}\|_{L_2(\Omega)}^4 + \|\mathbf{H}\|_{L_2(\Omega)}^2 \right) \\ & \quad + \frac{c}{\epsilon} \left(\|\nabla \mathbf{v}\|_{L_2(\Omega)}^2 + \|\nabla \mathbf{H}\|_{L_2(\Omega)}^2 \right) \left(\|\mathbf{H}_{,x_3}\|_{L_2(\Omega)}^{\frac{4}{3}} + \|\mathbf{v}_{,x_3}\|_{L_2(\Omega)}^{\frac{4}{3}} \right) \left(\|\nabla \mathbf{H}\|_{L_2(\Omega)}^{\frac{8}{3}} + \|\nabla \mathbf{v}\|_{L_2(\Omega)}^{\frac{8}{3}} \right) \\ & \quad + \frac{c}{\epsilon} \left(\|\nabla \mathbf{H}_{,x_3}\|_{L_2(\Omega)}^2 + \|\nabla \mathbf{v}_{,x_3}\|_{L_2(\Omega)}^2 \right) \left(\|\mathbf{v}\|_{H^1(\Omega)}^6 + \|\mathbf{H}\|_{H^1(\Omega)}^6 \right) + \frac{c}{\epsilon} \|\mathbf{f}\|_{L_2(\Omega)}^2. \end{aligned}$$

Now we chose $\epsilon = \frac{c}{32}$, multiply the result by 2, apply Lemma 3.9 on the left-hand side and estimate some L_2 -norms by H^1 -norms and use the Young inequality on the right-hand side. We obtain

$$\begin{aligned} & \frac{d}{dt} \left(\|\mathbf{v}(t)\|_{H^1(\Omega)}^2 + \|\mathbf{H}(t)\|_{H^1(\Omega)}^2 \right) + c \left(\|\Delta \mathbf{v}\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega)}^2 \right) \\ & \leq c \left(\|\mathbf{v}\|_{H^1(\Omega)}^2 + \|\mathbf{H}\|_{H^1(\Omega)}^2 \right) \left(\|\mathbf{H}\|_{L_2(\Omega)}^2 \|\mathbf{H}\|_{H^1(\Omega)}^2 + \|\mathbf{H}\|_{H^1(\Omega)}^2 + \|\mathbf{v}\|_{L_2(\Omega)}^2 \|\mathbf{v}\|_{H^1(\Omega)}^2 + \|\mathbf{v}\|_{H^1(\Omega)}^2 \right) \\ & \quad + c \left(\|\mathbf{v}\|_{H^1(\Omega)}^2 + \|\mathbf{H}\|_{H^1(\Omega)}^2 \right)^3 \left(\|\mathbf{H}_{,x_3}\|_{L_2(\Omega)}^{\frac{4}{3}} + \|\mathbf{v}_{,x_3}\|_{L_2(\Omega)}^{\frac{4}{3}} + \|\mathbf{H}_{,x_3}\|_{H^1(\Omega)}^2 + \|\mathbf{v}_{,x_3}\|_{H^1(\Omega)}^2 \right) + c \|\mathbf{f}\|_{L_2(\Omega)}^2. \end{aligned} \tag{6.3}$$

Let

$$y(t) := \left(\|\mathbf{v}(t)\|_{H^1(\Omega)}^2 + \|\mathbf{H}(t)\|_{H^1(\Omega)}^2 \right).$$

We integrate (6.3) with respect to t and get

$$\begin{aligned} & y(t) + c \int_{t_0}^t \left(\|\Delta \mathbf{v}(\tau)\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{H}(\tau)\|_{L_2(\Omega)}^2 \right) \, d\tau \\ & \leq c \int_{t_0}^t y(\tau) \left(\|\mathbf{H}(\tau)\|_{L_2(\Omega)}^2 \|\mathbf{H}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{H}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{v}(\tau)\|_{L_2(\Omega)}^2 \|\mathbf{v}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{v}(\tau)\|_{H^1(\Omega)}^2 \right) \, d\tau \\ & \quad + c \sup_{t_0 \leq \tau \leq t} y^3(\tau) \int_{t_0}^t \left(\|\mathbf{H}_{,x_3}(\tau)\|_{L_2(\Omega)}^{\frac{4}{3}} + \|\mathbf{v}_{,x_3}(\tau)\|_{L_2(\Omega)}^{\frac{4}{3}} + \|\mathbf{H}_{,x_3}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{v}_{,x_3}(\tau)\|_{H^1(\Omega)}^2 \right) \, d\tau \\ & \quad + cE^2(T). \end{aligned} \tag{6.4}$$

To estimate the second integral on the right-hand side we utilize Lemma 4.2. Since $\delta^2(T) < \delta^{\frac{4}{3}}(T)$ we have by the Cauchy inequality

$$\begin{aligned} & c \sup_{t_0 \leq \tau \leq t} y^3(\tau) \int_{t_0}^t \left(\|\mathbf{H}_{,x_3}(\tau)\|_{L_2(\Omega)}^{\frac{4}{3}} + \|\mathbf{v}_{,x_3}(\tau)\|_{L_2(\Omega)}^{\frac{4}{3}} + \|\mathbf{H}_{,x_3}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{v}_{,x_3}(\tau)\|_{H^1(\Omega)}^2 \right) \, d\tau \\ & \leq c \sup_{t_0 \leq \tau \leq t} y^6(\tau) \delta^{\frac{4}{3}}(T) + c e^{c(\|\Delta \mathbf{v}\|_{L_2(\Omega^t)}^2 + \|\Delta \mathbf{H}\|_{L_2(\Omega^t)}^2)} \delta^{\frac{4}{3}}(T). \end{aligned} \tag{6.5}$$

Using the above estimate in (6.4) and taking $\delta(T)$ small enough yields

$$\begin{aligned} y(t) &\leq c \int_{t_0}^t y(\tau) \left(\|\mathbf{H}(\tau)\|_{L_2(\Omega)}^2 \|\mathbf{H}(\tau)\|_{H^1(\Omega)}^2 \right. \\ &\quad \left. + \|\mathbf{H}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{v}(\tau)\|_{L_2(\Omega)}^2 \|\mathbf{v}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{v}(\tau)\|_{H^1(\Omega)}^2 \right) d\tau \\ &\quad + c \sup_{t_0 \leq \tau \leq t} y^6(\tau) \delta^{\frac{4}{3}}(T) + cE^2(T). \end{aligned}$$

By the Gronwall inequality

$$\begin{aligned} y(t) &\leq \left(c \sup_{t_0 \leq \tau \leq t} y^6(\tau) \delta^{\frac{4}{3}}(T) + cE^2(T) \right) \\ &\quad \cdot \exp \left(c \int_{t_0}^t \left(\|\mathbf{H}(\tau)\|_{L_2(\Omega)}^2 \|\mathbf{H}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{H}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{v}(\tau)\|_{L_2(\Omega)}^2 \|\mathbf{v}(\tau)\|_{H^1(\Omega)}^2 + \|\mathbf{v}(\tau)\|_{H^1(\Omega)}^2 \right) d\tau \right) \end{aligned}$$

In light of Lemma 4.1 the above integral is bounded by $c(1 + d^2(T))^2$. Therefore, if $\delta(T)$ is small enough, then

$$y(t) \leq \sup_{t_0 \leq \tau \leq t} y(\tau) \leq cE^2(T) e^{c(1+d^2(T))^2}.$$

We use the above estimate along with (6.5) in (6.4). By Lemma 4.1 and the smallness of $\delta(T)$ we obtain

$$\begin{aligned} \sup_{t_0 \leq \tau \leq t} y(\tau) + c \int_{t_0}^t \left(\|\Delta \mathbf{v}(\tau)\|_{L_2(\Omega)}^2 + \|\Delta \mathbf{H}(\tau)\|_{L_2(\Omega)}^2 \right) d\tau &\leq cE^2(T) e^{c(1+d^2(T))^2} (1 + d^2(T))^2 + cE^2(T) \\ &\leq cE^2(T) e^{c(1+d^2(T))^2}. \end{aligned}$$

This completes the proof. □

Remark 6.2. Using (6.1) in Lemmas 3.13 and 3.14 we obtain the following estimate

$$\|\mathbf{v}\|_{W_s^{2,1}(\Omega^T)} + \|\mathbf{H}\|_{W_s^{2,1}(\Omega^T)} \leq cE^8(T) e^{c(1+d^2(T))^2},$$

where $1 < s < +\infty$.

7. Proof of Theorem 1

The key tool for proving the existence of solutions to problem (1.1) is the Leray–Schauder fixed point theorem, which we recall below following [30, §3]:

Theorem 2. *Let B be a Banach space. Assume that $\Phi: [0, 1] \times B \rightarrow B$ is mapping with the following properties:*

- (i) *The mapping $\Phi(\lambda, \cdot): B \rightarrow B$ is completely continuous for any fixed $\lambda \in [0, 1]$.*
- (ii) *For any bounded $X \subset B$ the family of mappings $\Phi(\cdot, x): [0, 1] \rightarrow B$, $x \in X$, is uniformly equicontinuous.*
- (iii) *There exists a bounded subset $X \subset B$ such that any fixed point of $\Phi(\lambda, \cdot)$ in B for $0 \leq \lambda \leq 1$ is contained in X .*
- (iv) *The mapping $\Phi(0, \cdot)$ has exactly one fixed point in B .*

Then, $\Phi(1, \cdot)$ has at least one fixed point in B .

First we rewrite (1.1)

$$\begin{aligned}
 \mathbf{v}_{,t} - \nu \Delta \mathbf{v} + \nabla p &= -\lambda \left((\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{v}} - (\bar{\mathbf{H}} \cdot \nabla) \bar{\mathbf{H}} + \nabla \frac{1}{2} |\bar{\mathbf{H}}|^2 \right) + \mathbf{f} && \text{in } \Omega^T, \\
 \mathbf{H}_{,t} - \nu_\kappa \Delta \mathbf{H} &= -\lambda \left(-(\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{H}} + (\bar{\mathbf{H}} \cdot \nabla) \bar{\mathbf{v}} \right) && \text{in } \Omega^T, \\
 \operatorname{div} \mathbf{v} &= 0, \quad \operatorname{div} \mathbf{H} = 0 && \text{in } \Omega^T, \\
 \operatorname{rot} \mathbf{v} \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{v} \cdot \mathbf{n} = 0 && \text{on } S^T, \\
 \operatorname{rot} \mathbf{H} \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{H} \cdot \mathbf{n} = 0 && \text{on } S^T, \\
 \mathbf{v}|_{t=t_0} &= \mathbf{v}_{t_0}, \quad \mathbf{H}|_{t=t_0} = \mathbf{H}_{t_0} && \text{on } \Omega \times \{t = t_0\},
 \end{aligned} \tag{7.1}$$

where $\lambda \in [0, 1]$ and $\bar{\mathbf{v}}, \bar{\mathbf{H}}$ are considered as given. We introduce the space

$$\mathfrak{M}(\Omega^T) := \left\{ \mathbf{u}: \|\mathbf{u}\|_{L^{\frac{20}{3}}(\Omega^T)} < \infty, \|\nabla \mathbf{u}\|_{L^{\frac{20}{7}}(\Omega^T)} < \infty, \operatorname{div} \mathbf{u} = 0 \right\}.$$

We see that (7.1) determines the mapping

$$\begin{aligned}
 \Phi: \mathfrak{M}(\Omega^T) \times \mathfrak{M}(\Omega^T) \times [0, 1] &\rightarrow \mathfrak{M}(\Omega^T) \times \mathfrak{M}(\Omega^T), \\
 \Phi(\bar{\mathbf{v}}, \bar{\mathbf{H}}, \lambda) &= (\mathbf{v}, \mathbf{H}).
 \end{aligned}$$

Lemma 7.1. *Under the assumptions of Theorem 1 there exists a strong solution to problem (1.1).*

Proof. We have to check all assumptions from Theorem 2.

Assumption (i). To check this assumption we will show that $\Phi(\lambda, \cdot): B \rightarrow B$ is compact and continuous. Indeed, assume that $\bar{\mathbf{u}}, \bar{\mathbf{w}} \in \mathfrak{M}(\Omega^T)$, where $\bar{\mathbf{u}}, \bar{\mathbf{w}} \in \{\bar{\mathbf{v}}, \bar{\mathbf{H}}\}$. Then

$$\|(\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{w}}\|_{L_2(\Omega^T)} \leq \|\bar{\mathbf{u}}\|_{L^{\frac{20}{3}}(\Omega^T)} \|\nabla \bar{\mathbf{w}}\|_{L^{\frac{20}{7}}(\Omega^T)} \leq \|\bar{\mathbf{u}}\|_{\mathfrak{M}(\Omega^T)} \|\bar{\mathbf{w}}\|_{\mathfrak{M}(\Omega^T)}.$$

In light of Lemmas 3.13 and 3.14 we see that the solution to (7.1) belongs to $W_2^{2,1}(\Omega^T)$. Since the embeddings

$$\begin{aligned}
 W_2^{2,1}(\Omega^T) &\hookrightarrow L^{\frac{20}{3}}(\Omega^T), \\
 W_2^{2,1}(\Omega^T) &\hookrightarrow L^{\frac{20}{7}}(\Omega^T)
 \end{aligned} \tag{7.2}$$

are compact we have justified that Φ is a compact mapping.

To prove the continuity of Φ we introduce

$$\begin{aligned}
 \mathbf{v}_{,t}^k - \nu \Delta \mathbf{v}^k + \nabla p^k &= -\lambda \left((\bar{\mathbf{v}}^k \cdot \nabla) \bar{\mathbf{v}}^k - (\bar{\mathbf{H}}^k \cdot \nabla) \bar{\mathbf{H}}^k + \nabla \frac{1}{2} |\bar{\mathbf{H}}^k|^2 \right) + \mathbf{f} && \text{in } \Omega^T, \\
 \mathbf{H}_{,t}^k - \nu_\kappa \Delta \mathbf{H}^k &= -\lambda \left(-(\bar{\mathbf{v}}^k \cdot \nabla) \bar{\mathbf{H}}^k + (\bar{\mathbf{H}}^k \cdot \nabla) \bar{\mathbf{v}}^k \right) && \text{in } \Omega^T, \\
 \operatorname{div} \mathbf{v}^k &= 0, \quad \operatorname{div} \mathbf{H}^k = 0 && \text{in } \Omega^T, \\
 \operatorname{rot} \mathbf{v}^k \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{v}^k \cdot \mathbf{n} = 0 && \text{on } S^T, \\
 \operatorname{rot} \mathbf{H}^k \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{H}^k \cdot \mathbf{n} = 0 && \text{on } S^T, \\
 \mathbf{v}^k|_{t=t_0} &= \mathbf{v}_{t_0}^k, \quad \mathbf{H}^k|_{t=t_0} = \mathbf{H}_{t_0}^k && \text{on } \Omega \times \{t = t_0\},
 \end{aligned}$$

where $k \in \{1, 2\}$. Let $\mathbf{V} = \mathbf{v}^1 - \mathbf{v}^2$, $\mathbf{h} = \mathbf{H}^1 - \mathbf{H}^2$, $P = p^1 - p^2$. Then (\mathbf{V}, \mathbf{h}) is a solution to the problem

$$\begin{aligned} \mathbf{V}_{,t} - \nu \Delta \mathbf{V} + \nabla P &= -\lambda \left((\bar{\mathbf{V}} \cdot \nabla) \bar{\mathbf{v}}_1 + (\bar{\mathbf{v}}_2 \cdot \nabla) \bar{\mathbf{V}} \right. \\ &\quad \left. - (\bar{\mathbf{h}} \cdot \nabla) \bar{\mathbf{H}}_1 - (\bar{\mathbf{H}}_2 \cdot \nabla) \bar{\mathbf{h}} + \bar{\mathbf{h}} \cdot (\nabla \bar{\mathbf{H}}_1) + \bar{\mathbf{H}}_2 \cdot (\nabla \bar{\mathbf{h}}) \right) \quad \text{in } \Omega^T, \\ \mathbf{H}_{,t} - \nu_\kappa \Delta \mathbf{H} &= -\lambda \left(-(\bar{\mathbf{V}} \cdot \nabla) \bar{\mathbf{H}}_1 - (\bar{\mathbf{v}}_2 \cdot \nabla) \bar{\mathbf{h}} + (\bar{\mathbf{h}} \cdot \nabla) \bar{\mathbf{v}}_1 + (\bar{\mathbf{H}}_2 \cdot \nabla) \bar{\mathbf{V}} \right) \quad \text{in } \Omega^T, \\ \operatorname{div} \mathbf{V} &= 0, \quad \operatorname{div} \mathbf{h} = 0 \quad \text{in } \Omega^T, \\ \operatorname{rot} \mathbf{V} \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{V} \cdot \mathbf{n} = 0 \quad \text{on } S^T, \\ \operatorname{rot} \mathbf{h} \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{h} \cdot \mathbf{n} = 0 \quad \text{on } S^T, \\ \mathbf{V}|_{t=t_0} &= \mathbf{V}_{t_0}, \quad \mathbf{h}|_{t=t_0} = \mathbf{h}_{t_0} \quad \text{on } \Omega \times \{t = t_0\}. \end{aligned}$$

From (7.2), Lemmas 3.13, 3.14 and the Hölder inequality we immediately deduce

$$\begin{aligned} \|\mathbf{V}\|_{\mathfrak{M}(\Omega^T)} + \|\mathbf{h}\|_{\mathfrak{M}(\Omega^T)} &\leq c \left(\|\mathbf{V}\|_{W_2^{2,1}(\Omega^T)} + \|\mathbf{h}\|_{W_2^{2,1}(\Omega^T)} \right) \\ &\leq c \left(\|\bar{\mathbf{V}}\|_{L_{\frac{20}{3}}(\Omega^T)} + \|\nabla \bar{\mathbf{V}}\|_{L_{\frac{20}{7}}(\Omega^T)} + \|\bar{\mathbf{h}}\|_{L_{\frac{20}{3}}(\Omega^T)} + \|\nabla \bar{\mathbf{h}}\|_{L_{\frac{20}{7}}(\Omega^T)} \right) \\ &\quad \cdot \left(\|\nabla \bar{\mathbf{v}}_1\|_{L_{\frac{20}{7}}(\Omega^T)} + \|\bar{\mathbf{v}}_2\|_{L_{\frac{20}{3}}(\Omega^T)} + \|\nabla \bar{\mathbf{H}}_1\|_{L_{\frac{20}{7}}(\Omega^T)} + \|\bar{\mathbf{H}}_2\|_{L_{\frac{20}{3}}(\Omega^T)} \right). \end{aligned}$$

In view of Remark 6.2 and (7.2) we can estimate all terms in the second bracket in terms of data only. This shows that

$$\|\mathbf{V}\|_{\mathfrak{M}(\Omega^T)} + \|\mathbf{h}\|_{\mathfrak{M}(\Omega^T)} \leq c(T, \Omega, \text{data}) \left(\|\bar{\mathbf{V}}\|_{\mathfrak{M}(\Omega^T)} + \|\bar{\mathbf{h}}\|_{\mathfrak{M}(\Omega^T)} \right),$$

which justifies the continuity of Φ .

Assumption (ii). In light of definition of Φ we easily see that this assumption is met.

Assumption (iii). This follows immediately from Lemma 6.1.

Assumption (iv). To check this assumption we take two different solutions $(\mathbf{v}^1, \mathbf{H}^1)$ and $(\mathbf{v}^2, \mathbf{H}^2)$ to (7.1) with $\lambda = 0$. Then, the pair (\mathbf{V}, \mathbf{h}) , where $\mathbf{V} = \mathbf{v}^1 - \mathbf{v}^2$, $\mathbf{h} = \mathbf{H}^1 - \mathbf{H}^2$ is a solution to the problem

$$\begin{aligned} \mathbf{V}_{,t} - \nu \Delta \mathbf{V} + \nabla P &= \mathbf{0} \quad \text{in } \Omega^T, \\ \mathbf{h}_{,t} - \nu_\kappa \Delta \mathbf{h} &= \mathbf{0} \quad \text{in } \Omega^T, \\ \operatorname{div} \mathbf{V} &= 0, \quad \operatorname{div} \mathbf{h} = 0 \quad \text{in } \Omega^T, \\ \operatorname{rot} \mathbf{V} \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{V} \cdot \mathbf{n} = 0 \quad \text{on } S^T, \\ \operatorname{rot} \mathbf{h} \times \mathbf{n} &= \mathbf{0}, \quad \mathbf{h} \cdot \mathbf{n} = 0 \quad \text{on } S^T, \\ \mathbf{V}|_{t=t_0} &= \mathbf{0}, \quad \mathbf{h}|_{t=t_0} = \mathbf{0} \quad \text{on } \Omega \times \{t = t_0\}. \end{aligned} \tag{7.3}$$

Multiplying (7.3)_{1,2} by \mathbf{V} , \mathbf{h} , respectively, integrating over Ω yields

$$\frac{1}{2} \frac{d}{dt} \left(\|\mathbf{V}\|_{L_2(\Omega)}^2 + \|\mathbf{h}\|_{L_2(\Omega)}^2 \right) + \nu \|\operatorname{rot} \mathbf{V}\|_{L_2(\Omega)}^2 + \nu_\kappa \|\operatorname{rot} \mathbf{h}\|_{L_2(\Omega)}^2 = 0.$$

Integrating with respect to t gives

$$\|\mathbf{V}(t)\|_{L_2(\Omega)}^2 + \|\mathbf{h}(t)\|_{L_2(\Omega)}^2 \leq 0,$$

which proves that $\mathbf{V} = \mathbf{h} = \mathbf{0}$ a.e. Thus, assumption (iv) is met. □

Proof of Theorem 1. The existence of strong solutions follows from Lemma 7.1, whereas their estimate from Lemma 6.1. The uniqueness of solutions is straightforward. □

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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References

- [1] Landau, L.D., Lifshitz, E.M.: Course of theoretical physics. Vol. 8. Pergamon International Library of Science, Technology, Engineering and Social Studies, Pergamon Press, Oxford. Electrodynamics of continuous media, Translated from the second Russian edition by Sykes, J.B., Bell, J.S., Kearsley, M.J.: Second Russian edition revised by Lifshits and L. P. Pitaevskii (1984)
- [2] Ströhmer, G.: About an initial-boundary value problem from magnetohydrodynamics. *Math. Z.* **209**(3), 345–362 (1992)
- [3] Cao, C., Wu, J.: Two regularity criteria for the 3D MHD equations. *J. Differ. Equ.* **248**(9), 2263–2274 (2010)
- [4] Temam, R., Ziane, M.: Navier–Stokes equations in three-dimensional thin domains with various boundary conditions. *Adv. Differ. Equ.* **1**(4), 499–546 (1996)
- [5] Cho, Y.-K.: Multiplicative Sobolev inequalities of the Ladyzhenskaya type. *Math. Inequal. Appl.* **14**(2), 335–341 (2011)
- [6] Tao, T.: Nonlinear dispersive equations, vol. 106 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI. Local and global analysis (2006)
- [7] Chemin, J.-Y., Gallagher, I., Paicu, M.: Global regularity for some classes of large solutions to the Navier–Stokes equations. *Ann. Math. (2)* **173**(2), 983–1012 (2011)
- [8] Montgomery, D., Turner, L., Vahala, G.: Three-dimensional magnetohydrodynamic turbulence in cylindrical geometry. *Phys. Fluids* **21**(5), 757–764 (1978)
- [9] Montgomery, D., Phillips, L., Theobald, M.L.: Helical, dissipative, magnetohydrodynamic states with flow. *Phys. Rev. A* **40**, 1515–1523 (1989)
- [10] Knaepen, B., Debliquy, O., Carati, D.: Modelling of MHD Turbulence, pp. 247–262. Springer, Dordrecht (2007)
- [11] Stefani, F., Gailitis, A., Gerbeth, G.: Magnetohydrodynamic experiments on cosmic magnetic fields. *ZAMM Z. Angew. Math. Mech.* **88**(12), 930–954 (2008)
- [12] Dobler, W., Frick, P., Stepanov, R.: Screw dynamo in a time-dependent pipe flow. *Phys. Rev. E* **67**, 056309 (2003)
- [13] Yamazaki, K.: On the global well-posedness of N -dimensional generalized MHD system in anisotropic spaces. *Adv. Differ. Equ.* **19**(3–4), 201–224 (2014)
- [14] Duvaut, G., Lions, J.-L.: Inéquations en thermoélasticité et magnétohydrodynamique. *Arch. Rational Mech. Anal.* **46**, 241–279 (1972)
- [15] Sermange, M., Temam, R.: Some mathematical questions related to the MHD equations. *Commun. Pure Appl. Math.* **36**(5), 635–664 (1983)
- [16] Yoshida, Z., Giga, Y.: On the Ohm–Navier–Stokes system in magnetohydrodynamics. *J. Math. Phys.* **24**(12), 2860–2864 (1983)
- [17] Akiyama, T.: On the existence of L^p solutions of the magnetohydrodynamic equations in a bounded domain. *Nonlinear Anal.* **54**(6), 1165–1174 (2003)
- [18] Nowakowski, B., Zajęczkowski, W.M.: Stability of two-dimensional magnetohydrodynamic motions in the periodic case. *Math. Methods Appl. Sci.* **39**(1), 44–61 (2016)
- [19] Ströhmer, G.: An existence result for partially regular weak solutions of certain abstract evolution equations, with an application to magnetohydrodynamics. *Math. Z.* **213**(3), 373–385 (1993)
- [20] Ströhmer, G., Zajęczkowski, W.: Existence and stability theorems for abstract parabolic equations, and some of their applications. In: Singularities and Differential Equations (Warsaw, 1993), vol. 33 of Banach Center Publ., pp. 369–382. Polish Acad. Sci., Warsaw (1996)
- [21] Ströhmer, G.: About the equations of non-stationary magneto-hydrodynamics with non-conducting boundaries. *Nonlinear Anal. Ser. A Theory Methods* **39**(5), 629–647 (2000)
- [22] Ströhmer, G.: About the equations of non-stationary magneto-hydrodynamics. *Nonlinear Anal.* **52**(4), 1249–1273 (2003)
- [23] Davidson, P.A.: An Introduction to Magnetohydrodynamics, Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge (2001)

- [24] Solonnikov, V.: Overdetermined elliptic boundary-value problems. *J. Math. Sci. (N. Y.)* **1**(4), 477–512 (1973). Translation from: *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **38**, 153–231 (1973). **(Russian)**
- [25] Nirenberg, L.: On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa* **3**(13), 115–162 (1959)
- [26] Zajączkowski, W.M.: Nonstationary Stokes system in cylindrical domains under boundary slip conditions. *J. Math. Fluid Mech.* **19**(1), 1–16 (2017)
- [27] Nowakowski, B.: Large time existence of strong solutions to micropolar equations in cylindrical domains. *Nonlinear Anal. Real World Appl.* **14**(1), 635–660 (2013)
- [28] Solonnikov, V.: On boundary value problems for linear parabolic systems of differential equations of general form. *Proc Steklov Inst. Math.* **83**, 1–184 (1965). Translation from: *Trudy Mat. Inst. Steklov* **83**, 3–163 (1965). **(Russian)**
- [29] Ladyženskaja, O., Solonnikov, V., Ural'ceva, N.: Linear and quasilinear equations of parabolic type. Translated from the Russian by S. Smith. *Translations of Mathematical Monographs*, vol. 23. American Mathematical Society, Providence (1967)
- [30] Dafermos, C., Hsiao, L.: Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity. *Nonlinear Anal. Theory Methods Appl.* **6**, 435–454 (1982)

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