Algebra Univers. 61 (2009) 449–453
DOI 10.1007/s00012-009-0028-9
Published online November 18, 2009

Algebra Universalis

MacNeille completion and profinite completion can coincide on finitely generated modal algebras

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Abstract. Following Bezhanishvili and Vosmaer, we confirm a conjecture of Yde Venema by piecing together results from various authors. Specifically, we show that if $\mathbb{A}$ is a residually finite, finitely generated modal algebra such that $\mathrm{HSP}(\mathbb{A})$ has equationally definable principal congruences, then the profinite completion of $\mathbb{A}$ is isomorphic to its MacNeille completion, and $\Diamond$ is smooth. Specific examples of such modal algebras are the free **K4**-algebra and the free **PDL**-algebra.

1. Introduction

In this paper we compare two mathematical constructions applied to modal algebras. The first is the MacNeille completion, which is an order-theoretic generalization of the construction of the reals from the rationals using Dedekind cuts [12]. It has been applied in logic to prove, for example, the completeness of predicate calculi [14]. The second is the profinite completion, which is a universal algebraic construction, transforming an algebra into a topological algebra endowed with a Stone (compact, Hausdorff, zero-dimensional) topology. This construction stems from Galois theory [15], but has more recently also been connected with lattice completions [3, 10, 19, 4].

This paper is a companion piece to [4]. In that paper, parallel versions of our Theorems 3.1 and 4.2 arise in a study of the connections between different completions of Heyting algebras, using Esakia duality. In light of the topological character of the profinite completion, in the present paper we will present topological algebra proofs instead. This establishes a strong connection with the body of work on canonicity [8] and MacNeille canonicity [17]. Another advantage of our present perspective is that we can show how the two main theorems pivot around an interaction between principal lattice filters and principal algebra congruences, which are in a one-to-one correspondence for Heyting algebras, but not for modal algebras. Finally, we will briefly mention some of the connections of our results to modal logic.

Presented by I. Hodkinson.
Received November 13, 2007; accepted in final form April 28, 2008.
2000 *Mathematics Subject Classification*: Primary: 06E25; Secondary: 06B23, 03B45, 22A30.
Key words and phrases: modal algebra, MacNeille completion, profinite completion.
This research was supported by VICI grant 639.073.501 of the Netherlands Organization for Scientific Research (NWO).

2. Completions and topologies

Let $\mathbb{B} = \langle B; \wedge, \vee, \neg, 0, 1\rangle$ be a Boolean algebra. Given $b \in B$ we write $b{\downarrow} = \{a \in B \mid a \leq b\}$ ($b{\uparrow}$ is defined dually). We say $S \subseteq B$ is *join-dense* in $\mathbb{B}$ iff, for every $a \in B$, $a = \bigvee(a{\downarrow} \cap S)$ (*meet-density* is defined dually). A *completion* of a lattice $\mathbb{B}$ is a pair $(m, \mathbb{C})$, where $m\colon \mathbb{B} \hookrightarrow \mathbb{C}$ is a lattice embedding into a complete lattice $\mathbb{C}$. Completions $(m, \mathbb{C})$ and $(k, \mathbb{D})$ of $\mathbb{B}$ are isomorphic if $gm = k$ for some lattice isomorphism $g\colon \mathbb{C} \to \mathbb{D}$. If $(m, \mathbb{C})$ is a completion of $\mathbb{B}$, let $\rho_{\mathbb{B}}$ be the topology on C generated by basis $\{[m(a), m(b)] \mid a, b \in B\}$ (where $[x, y] = \{z \in C \mid x \leq z \leq y\}$). By $\gamma^{\downarrow}_{\mathbb{B}}, \gamma^{\uparrow}_{\mathbb{B}}$ and $\gamma_{\mathbb{B}}$ we denote the Scott topology, the dual Scott topology, and the biScott topology on $\mathbb{B}$ respectively. Let $\operatorname{At}\mathbb{B}$ be the (possibly empty) set of atoms of $\mathbb{B}$, and let $\operatorname{At}_\omega \mathbb{B}$ be the set of all finite joins of atoms of $\mathbb{B}$. Then $\iota_{\mathbb{B}}$ is the topology generated by the basis $\{[a, \neg b] \mid a, b \in \operatorname{At}_\omega \mathbb{B}\}$. By [8, Section 2], $\iota_{\mathbb{B}} = \gamma_{\mathbb{B}}$ if $\mathbb{B}$ is complete and atomic.

The *MacNeille completion* [2] of a Boolean algebra $\mathbb{B}$ is defined up to isomorphism as a completion $(m, \mathbb{C})$ such that $m[B]$ is join-dense in $\mathbb{C}$ (by [5, Theorem V-27] $\mathbb{C}$ is then also a Boolean algebra). We denote the MacNeille completion of $\mathbb{B}$ by $\bar{\mathbb{B}}$. Alternatively [17, Theorem 4.5], $\bar{\mathbb{B}}$ can be characterized up to isomorphism as a completion $(m, \mathbb{C})$ of $\mathbb{B}$ such that $\langle C, \rho_{\mathbb{B}}\rangle$ is Hausdorff. If $f\colon \mathbb{B} \to \mathbb{C}$ is an order-preserving map between Boolean algebras, then $f^\circ\colon \bar{\mathbb{B}} \to \bar{\mathbb{C}}$, defined by $f^\circ\colon x \mapsto \bigvee\{f(a) \mid m_{\mathbb{B}}(a) \leq x\}$, is the *lower extension* of f. The *upper extension* $f^\bullet$ is defined dually. Alternatively [17, Section 5], f° is the (pointwise) largest $(\rho_{\mathbb{B}}, \gamma^{\downarrow}_{\mathbb{C}})$-continuous extension of f, and $f^\bullet$ is the smallest $(\rho_{\mathbb{B}}, \gamma^{\uparrow}_{\mathbb{C}})$-continuous extension of f. We say f is *smooth* if $f^\circ = f^\bullet$.

Given a modal algebra $\mathbb{A} = \langle A; \Diamond\rangle$, let $\Phi_{\mathbb{A}} := \{\theta \in \operatorname{Con}\mathbb{A} \mid \mathbb{A}/\theta \text{ is finite}\}$. We say $\mathbb{A}$ is *residually finite* if, for all $a, b \in \mathbb{A}$ with $a \neq b$, there exists $\theta \in \Phi_{\mathbb{A}}$ such that $a/\theta \neq b/\theta$. The inverse system $\langle\{\mathbb{A}/\theta\}_{\theta\in\Phi_{\mathbb{A}}}, f_{\theta\psi}\rangle$, where $f_{\theta\psi}\colon \mathbb{A}/\theta \twoheadrightarrow \mathbb{A}/\psi$ (for all $\theta, \psi \in \Phi_{\mathbb{A}}$ such that $\theta \subseteq \psi$) is defined by $f_{\theta\psi}\colon a/\theta \mapsto a/\psi$, has a projective limit

$$\hat{\mathbb{A}} = \big\{\alpha \in \textstyle\prod_{\Phi_{\mathbb{A}}} \mathbb{A}/\theta \mid \forall \theta, \psi \in \Phi_{\mathbb{A}} \text{ with } \theta \subseteq \psi, \text{ if } \alpha(\theta) = a/\theta \text{ then } \alpha(\psi) = a/\psi\big\}.$$

The map $\mu\colon \mathbb{A} \to \hat{\mathbb{A}}$, defined by $\mu\colon a \mapsto (a/\theta)_{\theta\in\Phi_{\mathbb{A}}}$, is a modal algebra homomorphism which is injective iff $\mathbb{A}$ is residually finite. We call $\hat{\mathbb{A}}$ the *profinite completion of* $\mathbb{A}$ [15]. Since $\hat{\mathbb{A}}$ is a complete lattice [10], it follows that $(\mu, \hat{\mathbb{A}})$ is a completion of $\mathbb{A}$ iff $\mathbb{A}$ is residually finite. If we define the discrete topology on each $\mathbb{A}/\theta$, $\hat{\mathbb{A}}$ inherits a topology $\tau_{\hat{\mathbb{A}}}$ as a closed subspace of the product $\prod_{\Phi_{\mathbb{A}}} \mathbb{A}/\theta$. Now $\langle \hat{A}, \tau_{\hat{\mathbb{A}}}\rangle$ is a Stone space [3, Section 2], and in particular $\widehat{\Diamond}\colon \hat{\mathbb{A}} \to \hat{\mathbb{A}}$ is $(\tau_{\hat{\mathbb{A}}}, \tau_{\hat{\mathbb{A}}})$-continuous [1].

Lemma 2.1. *If $\mathbb{A}$ is a Boolean algebra expansion, then $\tau_{\hat{\mathbb{A}}} = \iota_{\hat{\mathbb{A}}} = \gamma_{\hat{\mathbb{A}}}$.*

Proof. Since $\langle \hat{\mathbb{A}}, \tau_{\hat{\mathbb{A}}}\rangle$ is a compact Hausdorff topological lattice, it follows by [9, Corollary VII-2.3] that $\tau_{\hat{\mathbb{A}}} = \gamma_{\hat{\mathbb{A}}}$. Since $\hat{\mathbb{A}}$ is also a complete, atomic Boolean algebra [3, 19], we know that $\iota_{\hat{\mathbb{A}}} = \gamma_{\hat{\mathbb{A}}}$ [8, Section 2]. □

3. Comparing profinite completion and MacNeille completion

Theorem 3.1 (cf. [4, Theorem 4.12])**.** *Let $\mathbb{A}$ be a modal algebra. The following are equivalent:*

(1) *the profinite completion $(\mu, \hat{\mathbb{A}})$ is the MacNeille completion of $\mathbb{A}$, and $\Diamond$ is smooth;*
(2) *$\mathbb{A}$ is residually finite and, for every $\theta \in \Phi_{\mathbb{A}}$, $1/\theta$ is a principal lattice filter.*

Proof. If $(\mu, \hat{\mathbb{A}})$ is the MacNeille completion of $\mathbb{A}$, then $\mu\colon \mathbb{A} \to \hat{\mathbb{A}}$ must be injective, so that $\mathbb{A}$ is residually finite (see above). Let $\theta \in \Phi_{\mathbb{A}}$; then it follows from the definition of $\hat{\mathbb{A}}$ that the projection $\pi_\theta\colon \hat{\mathbb{A}} \twoheadrightarrow \mathbb{A}/\theta$ commutes with μ and the natural map $a \mapsto a/\theta$; i.e., $a/\theta = \pi_\theta\mu(a)$. By [3, Lemma 2.7], $\pi_\theta^{-1}(1/\theta)$ is a closed principal filter of $\hat{\mathbb{A}}$; say $\pi_\theta^{-1}(1/\theta) = \alpha{\uparrow}$. Because of the correspondence between modal filters and modal congruences [18, Theorem 29], $\alpha{\uparrow}$ completely characterizes $\mathbb{A}/\theta$ in the following sense: $\mathbb{A}/\theta \cong [0, \alpha]_{\hat{\mathbb{A}}}$ as a bounded lattice [7, Exercise 4.12]. This implies that $\alpha{\downarrow}$ is finite. Since $(\mu, \hat{\mathbb{A}})$ is the MacNeille completion of $\mathbb{A}$, $\mu[\mathbb{A}]$ is join-dense in $\hat{\mathbb{A}}$, so by finiteness of $\alpha{\downarrow}$, there must exist $a \in \mathbb{A}$ such that $\mu(a) = \alpha$. Now $b/\theta = 1/\theta$ iff $\mu(b) \in \pi_\theta^{-1}(1/\theta) = \alpha{\uparrow} = \mu(a){\uparrow}$ iff $b \geq a$, so $1/\theta = a{\uparrow}$ is a principal lattice filter.

Conversely, if $\mathbb{A}$ is residually finite, then $\mu\colon \mathbb{A} \to \hat{\mathbb{A}}$ is injective, so $(\mu, \hat{\mathbb{A}})$ is a completion of $\mathbb{A}$. To show that $(\mu, \hat{\mathbb{A}})$ is the MacNeille completion of $\mathbb{A}$, we will consider the different topologies on $\hat{\mathbb{A}}$. We first show that $\operatorname{At}\hat{\mathbb{A}} \subseteq \mu[\mathbb{A}]$. If $\alpha \in \operatorname{At}\hat{\mathbb{A}}$, there must be some $\theta \in \Phi_{\mathbb{A}}$ such that $\alpha(\theta) \in \operatorname{At}\mathbb{A}/\theta$. Because $1/\theta$ is a principal lattice filter $c{\uparrow}$, we know that $\mathbb{A}/\theta \cong [0, c]_{\mathbb{A}}$ as a bounded lattice, so there must be some $a \leq c$ with $a \in \operatorname{At}\mathbb{A}$ and $a/\theta = \alpha(\theta)$. But then $\mu(a) = \alpha$. It follows that $\operatorname{At}\hat{\mathbb{A}} \subseteq \mu[\mathbb{A}]$, whence $\iota \subseteq \rho$. Since ι is Hausdorff, so is ρ. Using [17, Theorem 4.5] we can thus conclude that, as far as the Boolean substructure of $\mathbb{A}$ is concerned, $(\mu, \hat{\mathbb{A}})$ is the MacNeille completion of $\mathbb{A}$. Now to show that $\Diamond$ is smooth, remember that $\widehat{\Diamond}\colon \hat{\mathbb{A}} \to \hat{\mathbb{A}}$ is (τ, τ)-continuous. Since $\tau = \iota = \gamma$ by Lemma 2.1 and $\iota \subseteq \rho$, it follows that $\widehat{\Diamond}$ is (ρ, γ)-continuous. But then since $\gamma^{\downarrow}, \gamma^{\uparrow} \subseteq \gamma$, it follows by [17, Proposition 5.9] that $\Diamond^{\bullet} \leq \widehat{\Diamond} \leq \Diamond^{\circ}$. Since also $\Diamond^{\circ} \leq \Diamond^{\bullet}$ [17, Proposition 5.6], it follows that $\Diamond$ is smooth. □

Note that the theorem above also admits a third equivalent condition, characterizing the dual space of $\mathbb{A}$. This perspective is further explored in [4].

4. Finitely generated modal algebras with EDPC

Having equationally definable principal congruences (EDPC) is a strong meta-logical property of varieties of algebras that coincides with, for example, the existence of a deduction theorem or of a master modality [6, 11] for the modal logic corresponding to a variety of modal algebras. Examples of such logics are logics of bounded depth, n-transitive logics such as **K4**, or regular test-free **PDL** with finitely many basic programs.

Lemma 4.1 (Proposition 3.4.3 of [11])**.** *Let $\mathcal{V}$ be a variety of modal algebras. $\mathcal{V}$ has EDPC iff every principal modal filter of an algebra in $\mathcal{V}$ is a principal lattice filter.*

Note that the hypotheses below strongly resemble those of [20, Theorem 4].

Theorem 4.2 (cf. [4, Corollary 4.5.3])**.** *If $\mathbb{A}$ is a residually finite, finitely generated modal algebra such that* $\mathrm{HSP}(\mathbb{A})$ *has equationally definable principal congruences, then the profinite completion $(\mu, \hat{\mathbb{A}})$ is the MacNeille completion of $\mathbb{A}$ and $\Diamond$ is smooth.*

Proof. Since $\mathbb{A}$ is a finitely generated algebra, every $\theta \in \Phi_{\mathbb{A}}$ is compact [16, Theorem 1]. As is remarked in [20], every compact congruence of a modal algebra is principal, so under our hypotheses, every $\theta \in \Phi_{\mathbb{A}}$ is principal. Now since $\mathrm{HSP}(\mathbb{A})$ has EDPC, Lemma 4.1 tells us that $1/\theta$ is a principal lattice filter for all $\theta \in \Phi_{\mathbb{A}}$, so by Theorem 3.1, $(\mu, \hat{\mathbb{A}})$ is the MacNeille completion of $\mathbb{A}$ and $\Diamond$ is smooth. □

Note that the EDPC clause in the theorem above is suppressed in the Heyting algebra case [4], because every variety of Heyting algebras has EDPC.

The conditions of Theorem 4.2 above are sufficient; what about necessity? From Theorem 3.1 we know that it is necessary that $\mathbb{A}$ is residually finite. Moreover, in light of [19, Section 3.3], we know that it is necessary for Theorem 4.2 that $\mathbb{A}$ is atomic. This helps us to find counterexamples to the theorem if we remove the requirements of being finitely generated or having EDPC. For instance, let $\mathbb{A}$ be the free algebra on one generator for the modal logic **T** (the logic of reflexive Kripke frames). Then [20, Corollary 7: Example 1] tells us that $\mathbb{A}$ is residually finite and finitely generated but not atomic. This shows us that being residually finite and finitely generated is not sufficient for the conclusion of Theorem 4.2. Alternatively, the free transitive modal algebra on ω generators is an example of a residually finite modal algebra $\mathbb{A}$, generating a variety with EDPC, such that $\mathbb{A}$ is not atomic. In summary, residual finiteness is necessary for Theorem 4.2, and if we remove either of the other two conditions, we can find non-atomic counterexamples.

Corollary 4.3. *Let $\mathbb{A}$ be a finitely generated free algebra for* **K4** *or* **PDL**. *Then the MacNeille completion and the profinite completion of $\mathbb{A}$ are the same and $\Diamond$ is smooth.*

Proof. Let **L** be either **K4** or **PDL** and let $\mathcal{V}$ be the variety corresponding to **L**. Since **L** has the finite model property, $\mathcal{V}$ is generated by its finite members. By [13, Theorem IV-14.4], this implies that $\mathbb{A}$, being a finitely generated free algebra for $\mathcal{V}$, is residually finite. Using the fact that **L** has a master modality, it follows that we can apply Theorem 4.2. □

Acknowledgements. The author would like to thank Yde Venema, who suggested that Theorem 4.2 might be true. Additionally, the author is grateful to the Editor and the Referee for their criticisms and suggestions.

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