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Means of iterates

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Dedicated to Professor Karol Baron on his 70th birthday.

Abstract. We determine continuous bijections f, acting on a real interval into itself, whose k-fold iterate is the quasi-arithmetic mean of all its subsequent iterates from f^0 up to f^n (where $0 \leqslant k \leqslant n$). Namely, we prove that if at most one of the numbers k, n is odd, then such functions consist of at most three affine pieces.

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1. Introduction

Consider integers $n \geqslant 2$, $0 \leqslant k \leqslant n$ and a non-trivial interval $I \subset \mathbb{R}$. The following problem arises when studying polynomial-like iterative equations, means of functions or convergence of means of iterates: Determine all continuous functions $F\colon I \to I$ with the k-fold iterate F^k being a mean of all its subsequent iterates up to F^n; more precisely, find all self-mappings $F \in C(I)$ such that

$$F^k(x) = M(x, F(x), F^2(x), \dots, F^n(x)) \tag{1}$$

for every $x \in I$, where $M\colon I^{n+1} \to I$ is a mean and $C(I)$ stands for the family of all continuous real functions acting on I.

In this paper we concentrate on Eq. (1) in the case where M is a quasi-arithmetic mean, i.e., M being of the form

$$M(x_0, \dots, x_n) = \varphi^{-1}\left(\frac{1}{n+1}\sum_{i=0}^{n}\varphi(x_i)\right) \tag{2}$$

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for all $x_0, \dots, x_n \in I$, where φ is a continuous bijection form I onto an interval $J \subset \mathbb{R}$. Observe that it is enough to consider Eq. (1) with M being the arithmetic mean, which follows from the following remark.

Remark 1.1. Assume that $J \subset \mathbb{R}$ is an interval, $\varphi \colon I \to J$ is a bijection onto J and $F \colon I \to I$. Then F satisfies (1) with M given by (2) on I if and only if $f = \varphi \circ F \circ \varphi^{-1}$ satisfies

$$f^k = \frac{1}{n+1} \sum_{i=0}^{n} f^i \tag{3}$$

on J. Moreover, f is a (continuous) bijection if and only if F is a (continuous) bijection.

It is easy to prove that any self-mapping satisfying (3) is injective (see, e.g. [2, Lemma 2.1]). Therefore, any continuous surjective solution to Eq. (3) is automatically a continuous bijection; in particular it is strictly monotone.

The paper is organized as follows. In Sect. 2 we collect basic properties of polynomial-like iterative equations and preliminary information on linear homogeneous recurrence relations. In Sect. 3 we solve Eq. (3) in the cases where $k = 0$ or $k = n$. The case where $0 < k < n$ is treated in Sect. 4. The final section contains examples and remarks on the considered equation as well as some further problems.

2. Polynomial-like iterative equations and recurrence relations

Equation (3) is a very special case of an iterative equation belonging to an interesting and widely studied class of functional equations, called polynomial-like iterative equations, of the form

$$\sum_{i=0}^{n} a_i f^i(x) = 0, \tag{4}$$

where f^k stands for the k-fold iterate of a self-mapping unknown function f (defined on an interval $I \subset \mathbb{R}$) and $a_0, a_1, \dots, a_n$ are given real coefficients. It turns out that continuous solutions to (4) deeply depend on the roots of its characteristic equation

$$\sum_{i=0}^{n} a_i r^i = 0 \tag{5}$$

which is obtained by assuming that f has the form $f(x) = rx$.

In general, it is very difficult to find all continuous functions satisfying Eq. (4). These difficulties follow from the non-linearity of the operator $f \mapsto f^n$. Up to now the complete solution is known only in the case where $n = 2$ and $I = \mathbb{R}$ [12]. The problem still remains open even for $n = 3$ [7]. A partial solution in the case where $n = 3$ and $I = \mathbb{R}$ was obtained in [16] and some results in the case where $n \geqslant 3$ and $I \neq \mathbb{R}$ can be found in [3,6,10,11,13,15,17].

It is known [9] that if a polynomial $\sum_{i=0}^{m} b_i r^i$ divides a polynomial $\sum_{i=0}^{n} a_i r^i$ and a function f satisfies $\sum_{i=0}^{m} b_i f^i(x) = 0$ then it satisfies also (4). One of the methods for finding solutions to (4) is based on a reverse reasoning, i.e., if a continuous function f satisfies Eq. (4), then we want to find a divisor of the polynomial $\sum_{i=0}^{n} a_i r^i$ such that f is a solution to the corresponding equation of lower order. The first such results on the whole real line were obtained in [8] in the case where all roots of the characteristic equation are real and satisfy some special conditions. Further research in this direction was done in [1,2,14,18] and some of them will be crucial tools in this paper.

Equation (5) can be considered as the characteristic equation of the recurrence relation

$$\sum_{i=0}^{n} a_i x_{m+i} = 0 \tag{6}$$

which might be obtained by choosing $x_0 \in I$ arbitrarily and putting $x_m = f(x_{m-1})$ for every $m \in \mathbb{N}$. It is easy to see that if $a_0 \neq 0$ and a function f satisfies (4), then it is injective (see, e.g., [2, Lemma 2.1.]). This observation implies that every continuous solution to (4) is strictly monotone. It means that the sequence $(x_m)_{m\in\mathbb{N}_0}$ given by $x_0 \in I$ and $x_m = f(x_{m-1})$ for $m \in \mathbb{N}$ is either monotone (in the case of increasing f) or anti-monotone (in the case of decreasing f). By *anti-monotone* we mean that the expression $(-1)^m(x_m - x_{m-1})$ does not change its sign when m runs through $\mathbb{N}_0$. In the case where f is bijective we can consider the dual equation

$$\sum_{i=0}^{n} a_i f^{n-i}(x) = 0. \tag{7}$$

Putting $f^{-n}(x)$ in place of x we see that f satisfies (4) if and only if f^{-1} satisfies (7). We can also extend the sequence $(x_m)_{m\in\mathbb{N}_0}$ to the whole $\mathbb{Z}$ by setting $x_{-m} = f^{-1}(x_{-m+1})$ for $m \in \mathbb{N}$. Then relation (6) is satisfied for $m \in \mathbb{Z}$.

For the theory of linear recurrence relations we refer the reader, for instance, to [4, §3.2]. We shall recall only the most significant theorem in this matter. In order to do this and simplify the writing we introduce the following notation: For a given polynomial $c_n r^n + \cdots + c_1 r + c_0$ we denote by $\mathcal{R}(c_n,\ldots,c_0)$ the collection $\{(r_1,k_1),\ldots,(r_p,k_p)\}$ of all pairs of its pairwise distinct (complex) roots $r_1,\ldots,r_p$ and their multiplicities $k_1,\ldots,k_p$, respectively. Here and throughout the paper by a polynomial we mean a polynomial with real coefficients.

Theorem 2.1. *Assume that*

$$\begin{aligned}\mathcal{R}(a_n,\ldots,a_0) = \{&(\lambda_1,l_1),\ldots,(\lambda_p,l_p),\\ &(\mu_1,m_1),(\overline{\mu_1},m_1),\ldots,(\mu_q,m_q),(\overline{\mu_q},m_q)\}.\end{aligned}$$

Then a real-valued sequence $(x_j)_{j\in\mathbb{N}_0}$ *satisfies* (6) *if and only if it is given by*

$$x_j = \sum_{k=1}^{p} A_k(j)\lambda_k^j + \sum_{k=1}^{q} \big(B_k(j)\cos j\phi_k + C_k(j)\sin j\phi_k\big)|\mu_k|^j$$

for every $j \in \mathbb{N}_0$*, where* A_k *is a polynomial whose degree equals at most* $l_k - 1$ *for* $k = 1,\ldots,p$ *and* B_k, C_k *are polynomials whose degrees equal at most* $m_k - 1$*, with* ϕ_k *being an argument of* μ_k*, for* $k = 1,\ldots,q$*.*

3. Cases $k = 0$ and $k = n$

In the case where $k = 0$ Eq. (3) takes the form

$$x = \frac{1}{n}\sum_{i=1}^{n} f^i(x). \tag{8}$$

Its characteristic equation is of the form

$$\sum_{i=1}^{n} r^i - n = 0 \tag{9}$$

which, after multiplication by $r-1$, can be written as $r^{n+1} - (n+1)r + n = 0$. Therefore, by the previous remarks, if a function f satisfies (8), then it also satisfies the equation $f^{n+1}(x) - (n+1)f(x) + nx = 0$. Now, applying [2, Theorem 5.7], we obtain the following result.

Theorem 3.1. *If* $f \in C(I)$ *satisfies* (8)*, then:*

(i) $f(x) = x$ *in the case where* n *is an odd number or in the case where* n *is an even number and* $I \neq \mathbb{R}$*;*

(ii) $f(x) = x$ *or* $f(x) = r_0x + c$*, where* c *is a constant and* r_0 *stands for the negative root of Eq.* (9)*, in the case where* n *is an even number and* $I = \mathbb{R}$*.*

Now we shall consider the case where $k = n$. Then Eq. (3) takes the form

$$f^n(x) = \frac{1}{n}\sum_{i=0}^{n-1} f^i(x) \tag{10}$$

and its characteristic equation is of the form

$$nr^n - \sum_{i=0}^{n-1} r^i = 0. \tag{11}$$

We will need the following lemma which is an elaboration of [12, Theorem 9]; the crucial point is that an unknown function acts on a subinterval of the real line.

Lemma 3.2. *Assume that $f \in C(I)$ satisfies*

$$f^2(x) - (1+\rho)f(x) + \rho x = 0. \tag{12}$$

If either

(i) $\rho \in (-1, 0)$

or

(ii) $\rho \in (-\infty, -1)$ *and f is a surjection,*

then $f(x) = x$ or $f(x) = \rho x + c$, where c is a constant.

Proof. (i) In the case where f is increasing we can apply [2, Theorem 4.1 and Remark 4.4] to conclude that $f(x) = x$. Therefore, assume f is decreasing.

Fix $x \in I$ and put $x_0 = x$ and $x_m = f(x_{m-1})$ for $m \in \mathbb{N}$. It is easy to see that the sequence $(x_m)_{m\in\mathbb{N}_0}$ satisfies the relation

$$x_{m+2} - (1+\rho)x_{m+1} + \rho x_m = 0 \quad \text{for } m \in \mathbb{N}_0.$$

By Theorem 2.1 we have $x_m = A + B\rho^m$ for some constants A and B (depending on x). Consequently, there exists a finite limit $\lim_{m\to\infty} f^m(x)$.

Fix $x, y \in I$. Since f is decreasing, the sequence $(f^m(x) - f^m(y))_{m\in\mathbb{N}_0}$ is anti-monotone. Moreover, it is convergent, so it must converge to zero. It means that the limit $\lim_{m\to\infty} f^m(x)$ does not depend on x.

We can rewrite Eq. (12) as $f^2(x) - \rho f(x) = f(x) - \rho x$. Putting $f(x)$ in place of x we obtain

$$f^3(x) - \rho f^2(x) = f^2(x) - \rho f(x) = f(x) - \rho x$$

and, by a simple induction, we obtain

$$f^{m+1}(x) - \rho f^m(x) = f(x) - \rho x \quad \text{for all } m \in \mathbb{N} \text{ and } x \in I.$$

Passing with m to ∞ gives

$$f(x) = \rho x + (1-\rho)\lim_{m\to\infty} f^m(x) \quad \text{for every } x \in I.$$

Setting $c = (1-\rho)\lim_{m\to\infty} f^m(x)$ ends the proof of assertion (i).

(ii) It is enough to apply assertion (i) to the dual equation to (12). □

Theorem 3.3. *If $f \in C(I)$ satisfies (10), then:*

(i) *$f(x) = x$ in the case where n is an odd number;*

(ii) *$f(x) = x$ or $f(x) = r_0 x + c$, where c is a constant and r_0 stands for the negative root of Eq. (11), in the case where n is an even number.*

Proof. Multiplying (11) by $r - 1$ gives

$$nr^{n+1} - (n+1)r^n + 1 = 0. \tag{13}$$

Clearly, Eqs. (11) and (13) have the same roots, wherein the multiplicity of root 1 is greater by 1 in the second equation. Define a function $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(r) = nr^{n+1} - (n+1)r^n + 1.$$

Since $g(1) = g'(1) = 0$ and $g''(1) \neq 0$, Eq. (13) has a double root 1.

In the case of odd n, g has the only extremum at 1, so it is the only real root of g. In the case of even n, g has extrema at 0 and 1, so it also has a negative root r_0. Since $g(0)g(-1) < 0$, we have $r_0 > -1$ and since $g'(r_0) > 0$, it follows that r_0 is a single root.

We shall show that if n is even and if z is a non-real root of (13), then $|z| > -r_0$. Suppose, for an indirect proof, the contrary. If $|z| = -r_0$, then we would have

$$1 = |n(-z)^{n+1} + (n+1)z^n| \leqslant n|z|^{n+1} + (n+1)|z|^n = -\,nr_0^{n+1} + (n+1)r_0^n = 1,$$

which means that $n(-z)^{n+1}$ and $(n+1)z^n$ are linearly dependent over $\mathbb{R}$, and consequently $z \in \mathbb{R}$; a contradiction. If $|z| < -r_0$, then we would have

$$1 = |n(-z)^{n+1} + (n+1)z^n| \leqslant n|z|^{n+1} + (n+1)|z|^n < -nr_0^{n+1} + (n+1)r_0^n = 1,$$

which is a contradiction again.

By [2, Theorem 3.6] Eq. (10) is equivalent to the equation $f(x) - x = 0$ in the case of odd n, which proves assertion (i), and to the equation $f^2(x) - (r_0+1)f(x) + r_0 x = 0$ in the case of even n. Now assertion (ii) follows from assertion (i) of Lemma 3.2. □

4. Case $0 < k < n$

In order to find continuous solutions to Eq. (3) in the case where $0 < k < n$ we need information on the roots of its characteristic equation which is of the form

$$(n+1)r^k = \sum_{i=0}^{n} r^i. \tag{14}$$

We start with a general observation on complex roots of Eq. (14).

Lemma 4.1. *All complex roots of Eq. (14) are in modulus less than $2n+1$.*

Proof. Assume that $z \in \mathbb{C}$ is a root of Eq. (14). Without loss of generality we can assume that $|z| > 1$. Then

$$|z|^n = \left|(n+1)z^k - \sum_{i=0}^{n-1} z^i\right| \leqslant (n+1)|z|^k + \sum_{i=0}^{n-1} |z|^i,$$

and hence

$$|z| \leqslant (n+1)|z|^{k-n+1} + \sum_{i=0}^{n-1} |z|^{i-n+1} < (n+1) + n = 2n+1.$$

This completes the proof. □

To obtain more precise information on real roots of Eq. (14) define functions $g, G\colon \mathbb{R} \to \mathbb{R}$ by

$$g(r) = r^{n-k+1} - (k+1)r + k, \quad G(r) = r^{n+1} - (n+1)r^k(r-1) - 1.$$

It is easy to see that

$$G(1) = g(1) = 0, \quad g(0) > 0 \quad \text{and} \quad G(0) < 0. \tag{15}$$

Moreover, the set of all real roots of Eq. (14) coincides with the set of all solutions to the equation

$$G(r) = 0 \tag{16}$$

and

$$G'(r) = (n+1)r^{k-1}g(r).$$

Note also that in the case where $n-k$ is odd the function g is strictly convex with the global minimum at the point

$$r_{\min} = \left(\frac{k+1}{n-k+1}\right)^{1/(n-k)}$$

and, by (15), we have

$$g(r_{\min}) < 0 \quad \text{if and only if} \quad r_{\min} \neq 1. \tag{17}$$

However, in the case where $n-k$ is even the function $g|_{(0,+\infty)}$ is strictly convex with the global minimum at the point $r_{\min}$ and (17) holds, whereas the function $g|_{(-\infty,0)}$ is strictly concave with the global maximum at the point $r_{\max} = -r_{\min}$ and, by (15), we have

$$g(r_{\max}) > 0. \tag{18}$$

The described properties of functions g and G will play a key role in the next lemma which will allow us to locate real roots of Eq. (14).

Lemma 4.2. *Assume that $0 < k < n$.*

(i) *If k is an odd number and $n = 2k$, then Eq. (14) has one real root: $r_1 = 1$.*

(ii) *If k is an odd number and n is an even number with $n < 2k$, then Eq. (14) has two real roots: $r_1 = 1$ and $r_2 \in (1, 2n+1)$.*

(iii) *If k is an odd number and n is an even number with $n > 2k$, then Eq. (14) has two real roots: $r_1 = 1$ and $r_2 \in (0, 1)$.*

(iv) *If k and n are odd numbers with $n < 2k$, then Eq. (14) has three real roots: $r_1 = 1$, $r_2 \in (1, 2n+1)$ and $r_3 \in (-2n-1, -1)$.*

(v) *If k and n are odd numbers with $n > 2k$, then Eq. (14) has three real roots: $r_1 = 1$, $r_2 \in (0, 1)$ and $r_3 \in (-2n-1, -1)$.*

(vi) *If k is an even number and n is an odd number with $n < 2k$, then Eq. (14) has three real roots: $r_1 = 1$, $r_2 \in (1, 2n+1)$ and $r_3 \in (-1, 0)$.*

(vii) *If k is an even number and n is an odd number with $n > 2k$, then Eq. (14) has three real roots: $r_1 = 1$, $r_2 \in (0, 1)$ and $r_3 \in (-1, 0)$.*

(viii) *If k is an even number and $n = 2k$, then Eq. (14) has three real roots: $r_1 = 1$, $r_2 \in (-1, 0)$ and $r_3 \in (-2n - 1, -1)$.*

(ix) *If k and n are even numbers with $n < 2k$, then Eq. (14) has four real roots: $r_1 = 1$, $r_2 \in (1, 2n + 1)$, $r_3 \in (-1, 0)$ and $r_4 \in (-2n - 1, -1)$.*

(x) *If k and n are even numbers with $n > 2k$, then Eq. (14) has four real roots: $r_1 = 1$, $r_2 \in (0, 1)$, $r_3 \in (-1, 0)$ and $r_4 \in (-2n - 1, -1)$.*

Moreover, in all the above cases r_1, r_2, r_3, r_4 are single roots, except the case $n = 2k$ in which r_1 is a double root.

Proof. We first observe that if k is odd, then for every $r \in \mathbb{R}\backslash\{0\}$ we have

$$G'(r) > 0 \iff g(r) > 0.$$

(i) The assumption $n = 2k$ implies $r_{\min} = 1$. It yields $g(r) > g(r_{\min}) = 0$ for every $r \neq r_{\min}$. Consequently, the function G is strictly increasing, which jointly with (15) shows that $r_{\min}$ is the unique real root of Eq. (14).

(ii) The assumption $n < 2k$ implies $r_{\min} > 1$. We conclude that there exists exactly one $t_0 \in \mathbb{R}\backslash\{1\}$ such that $g(t_0) = 0$ and, moreover, $t_0 > r_{\min}$. Consequently, the function G has exactly one local maximum at the point $r_1 = 1$ and exactly one local minimum at the point t_0. This jointly with (15) shows that Eq. (14) has two real roots: $r_1 = 1$ and $r_2 \in (t_0, \infty)$. By Lemma 4.1 we have $r_2 < 2n + 1$.

(iii) The reasoning is similar as in (ii).

(iv) The assumption $n < 2k$ implies $r_{\min} > 1$ and $r_{\max} < -1$. We conclude that there exists exactly one $t_0 \in (0, 1) \cup (1, \infty)$ such that $g(t_0) = 0$ and, moreover, $t_0 \in (r_{\min}, \infty)$. Further, there exists exactly one $u_0 \in (-\infty, 0)$ such that $g(u_0) = 0$ and, moreover, $u_0 \in (-\infty, r_{\max})$. Consequently, the function G has exactly one local maximum at the point $r_1 = 1$ and exactly two local minimums at points t_0 and u_0. This jointly with (15) shows that Eq. (14) has three real roots: $r_1 = 1$, $r_2 \in (t_0, \infty)$ and $r_3 \in (-\infty, u_0)$. By Lemma 4.1 we have $r_2 < 2n + 1$ and $r_3 > -2n - 1$.

(v) The reasoning is similar as in (iv).

Now observe that if k is even, then for every $r \in (0, \infty)$ we have

$$G'(r) > 0 \iff g(r) > 0$$

and for every $r \in (-\infty, 0)$ we have

$$G'(r) > 0 \iff g(r) < 0.$$

(vi) The assumption $n < 2k$ implies $r_{\min} > 1$. We conclude that there exists exactly one $t_0 \in \mathbb{R}\backslash\{1\}$ such that $g(t_0) = 0$ and, moreover, $t_0 > r_{\min}$. Consequently, the function G has exactly one local maximum at the point $r_1 = 1$ and exactly two local minimums at points 0 and t_0. This jointly with

(15) shows that Eq. (14) has three real roots: $r_1 = 1$, $r_2 \in (t_0, \infty)$ and $r_3 \in (-\infty, 0)$. By Lemma 4.1 we have $r_2 < 2n+1$ and since

$$G(-1) > 0 \tag{19}$$

in this case, we have $r_3 > -1$.

(vii) The reasoning is similar as in (vi).

(viii) The assumption $n = 2k$ implies $r_{\min} = 1$ and $r_{\max} = -1$. We conclude that $g(r) > g(r_{\min}) = 0$ for every $r \in (0, r_{\min}) \cup (r_{\min}, \infty)$. Furthermore, there exists exactly one $y_0 \in (-\infty, 0)$ such that $g(y_0) = 0$ and, moreover, $y_0 \in (-\infty, r_{\max})$. Consequently, the function G has exactly one local maximum at the point y_0 and exactly one local minimum at point 0. This jointly with (15) and (19) shows that Eq. (14) has three real roots: $r_1 = 1$, $r_2 \in (-1, 0)$ and $r_3 \in (-\infty, -1)$. By Lemma 4.1 we have $r_3 > -2n - 1$.

(ix) The assumption $n < 2k$ implies $r_{\min} > 1$ and $r_{\max} < -1$. We conclude that there exists exactly one $t_0 \in (0, 1) \cup (1, \infty)$ such that $g(t_0) = 0$ and, moreover, $t_0 \in (r_{\min}, \infty)$. Furthermore, there exists exactly one $u_0 \in (-\infty, 0)$ such that $g(u_0) = 0$ and, moreover, $u_0 \in (-\infty, r_{\max})$. Consequently, the function G has exactly two local maximums at points $r_1 = 1$ and u_0 and exactly two local minimums at points 0 and t_0. This jointly with (15) and (19) shows that Eq. (14) has four real roots: $r_1 = 1$, $r_2 \in (t_0, \infty)$, $r_3 \in (-1, 0)$ and $r_4 \in (-\infty, -1)$. By Lemma 4.1 we have $r_2 < 2n + 1$ and $r_4 > -2n - 1$.

(x) The reasoning is similar as in (ix).

To prove the 'moreover' part note that equality $g'(r) = 0$ can hold only for $r \in \{r_{\min}, r_{\max}\}$. Thus $G''(r_i) = (n+1)r_i^{k-1}g'(r_i) \neq 0$ for every $i \in \{2, 3, 4\}$ and for $i = 1$ in the case where $n \neq 2k$. If $n = 2k$, then $G''(r_1) = 0$ and $G'''(r_1) = (2k+1)(k+1)k \neq 0$. □

Now we want to obtain some information on the location of non-real roots of Eq. (14). It will be more convenient for us to consider (16) instead of (14).

Lemma 4.3. *Assume that $0 < k < n$ and at least one of the numbers k and n is odd. Then for a non-real root z and a real root r_0 of Eq. (14) we have $|z| \neq |r_0|$.*

Proof. Let $r_0 \in (0, \infty)$; in this case the parity of k or n does not play any role. Suppose, for a contradiction, $|z| = r_0$. Then

$$(n+1)r_0^k = (n+1)|z|^k = \left|\sum_{i=0}^{n} z^i\right| < \sum_{i=0}^{n} |z|^i = \sum_{i=0}^{n} r_0^i,$$

which is impossible.

Now assume $r_0 \in (-\infty, 0)$ and as before, suppose that $|z| = -r_0$. By Lemma 4.2 we need to consider only the case where n is odd.

In the case where both k and n are odd, we have

$$\begin{aligned} r_0^{n+1} &= |z|^{n+1} = \left|(n+1)z^{k+1} - (n+1)z^k + 1\right| \\ &< (n+1)|z|^{k+1} + (n+1)|z|^k + 1 = (n+1)r_0^{k+1} - (n+1)r_0^k + 1; \end{aligned}$$

a contradiction. Similarly, in the case where k is even and n is odd, we have

$$\begin{aligned} 1 &= \left|z^{n+1} - (n+1)z^{k+1} + (n+1)z^k\right| < |z|^{n+1} + (n+1)|z|^{k+1} + (n+1)|z|^k \\ &= r_0^{n+1} - (n+1)r_0^{k+1} + (n+1)r_0^k; \end{aligned}$$

a contradiction. □

The last lemma we need is basically Theorems 8 and 10 (iii) from [12]; the only difference is that in our lemma the unknown function acts on a subinterval of the real line.

Lemma 4.4. *Let $\rho \in (0,\infty)$ and assume a surjection $f \in C(I)$ satisfies (12).*

(i) *If $\rho = 1$, then $f(x) = x + c$, where c is a constant.*

(ii) *If $\rho \in (0,1) \cup (1,\infty)$, then*

$$f(x) = \begin{cases} \rho(x-a) + a & \text{for } x \leqslant a, \\ x & \text{for } x \in (a,b), \\ \rho(x-b) + b & \text{for } x \geqslant b, \end{cases} \tag{20}$$

where $a, b \in \operatorname{cl} I$ with $a \leqslant b$.

Proof. Since $f(x) = \frac{1}{1+\rho}\left(f^2(x) + \rho x\right)$, it follows that f is strictly increasing.

(i) If $\rho = 1$, then the monotonicity of f and a simple induction applied to (12) imply

$$0 < \frac{f^m(x) - f^m(y)}{x-y} = 1 + m\left(\frac{f(x)-f(y)}{x-y} - 1\right)$$

for all $m \in \mathbb{Z}$ and $x, y \in I$ with $x \neq y$. Therefore, passing with m in turn to ∞ and $-\infty$ we conclude that $\frac{f(x)-f(y)}{x-y} = 1$ for all $x, y \in I$.

(ii) Assume that $\rho \in (0,1)$; the case when $\rho \in (1,\infty)$ can be proved similarly, or else can be deduced from the dual case. Applying induction to (12) with $\rho \neq 1$ we obtain

$$f^m(x) = \frac{1}{\rho-1}(\rho x - f(x)) + \frac{\rho^m}{\rho-1}(f(x) - x)$$

for all $m \in \mathbb{Z}$ and $x \in I$. If $f(x) \neq x$ for some $x \in I$, then passing with m to ∞ we obtain

$$\lim_{m\to\infty} f^m(x) = \frac{1}{\rho-1}(\rho x - f(x)),$$

which means that $f(x) = \rho(x-a) + a$ with some $a \in \operatorname{cl} I$. In consequence (20) holds. □

Note that if $a = \inf I$ and $b = \sup I$, then (20) represents the identity solution, while if $a = b$, then (20) represents an affine solution.

Theorem 4.5. *Assume* $0 < k < n$*. If* $f \in C(I)$ *is a surjection satisfying* (3)*, then:*

(i) $f(x) = x + c$*, where* c *is a constant, in the case where* k *is an odd number and* $n = 2k$*;*

(ii)
$$f(x) = \begin{cases} r_0(x-a)+a & \text{for } x \leqslant a, \\ x & \text{for } x \in (a,b), \\ r_0(x-b)+b & \text{for } x \geqslant b, \end{cases} \tag{21}$$
where $a, b \in \operatorname{cl} I$ *with* $a \leqslant b$ *and* r_0 *stands for the difference from* 1 *positive root of Eq.* (14)*, in the case where* k *is an odd number and* n *is an even number with* $n \neq 2k$*;*

(iii) $f(x) = r_0 x + c$*, where* r_0 *stands for the negative root of Eq.* (14)*, or* f *is of form* (21)*, where* $a, b \in \operatorname{cl} I$ *with* $a \leqslant b$ *and* r_0 *stands for the difference from* 1 *positive root of Eq.* (14)*, in the case where* k *is an even number and* n *is an odd number or both* k *and* n *are odd numbers.*

Proof. The idea of the proof in each of the considered cases is the same and run in the following way. First, we determine all real roots of Eq. (14) with their multiplicities and localize all its non-real roots. This step is done in Lemmas 4.2 and 4.3. Next, according to [2, Theorem 3.6, Corollary 3.7 and Theorem 4.3], we reduce Eq. (3) to a simpler one, i.e. to an equation which has the same surjective and continuous solutions. Finally, we solve the simpler equation applying Lemmas 3.2 and 4.4. □

Remark 4.6. In the case where $I = \mathbb{R}$ the surjectivity assumption in Theorem 4.5 is satisfied automatically. However, this assumption is not essential when considering the dual equation.

5. Examples, remarks and problems

In this section we give some examples showing how our main results work.

Corollary 5.1. *Assume that* $p \neq 0$ *and* $I \subset (0, \infty)$*. Then* $f \in C(I)$ *satisfies*
$$f^n(x) = \left(\frac{x^p + [f(x)]^p + \cdots + [f^{n-1}(x)]^p}{n} \right)^{1/p}$$
if and only if

(i) $f(x) = x$ *in the case where* n *is an odd number;*

(ii) *$f(x) = x$ or $f(x) = (r_0 x^p + c)^{1/p}$, where c is a constant such that $f(I) \subset I$ and r_0 stands for the negative root of Eq. (11), in the case where n is an even number.*

Proof. It is enough to use Theorem 3.3 and Remark 1.1 with φ of the form $\varphi(x) = x^{1/p}$. ☐

Corollary 5.2. *Assume that $I \subset (0, \infty)$. Then $f \in C(I)$ satisfies*

$$f^n(x) = \sqrt[n]{x f(x) \dots f^{n-1}(x)}$$

if and only if

(i) *$f(x) = x$ in the case where n is an odd number;*

(ii) *$f(x) = x$ or $f(x) = cx^{r_0}$, where c is a constant such that $f(I) \subset I$ and r_0 stands for the negative root of Eq. (11), in the case where n is an even number.*

Proof. It is enough to use Theorem 3.3 and Remark 1.1 with $\varphi = \exp$. ☐

Remark 5.3. In the case where $I = \mathbb{R}$ Eq. (10) is the dual equation to (8). However, as we have shown in Corollary 5.1, there exist non-surjective solutions to (10).

Corollary 5.4. *Assume that $p \neq 0$ and $I \subset (0, \infty)$. Then $f \in C(I)$ satisfies*

$$\frac{[f(x)]^p + [f^2(x)]^p + \cdots + [f^n(x)]^p}{n} = x^p \tag{22}$$

if and only if $f(x) = x$.

Proof. It is enough to use Theorem 3.1 and Remark 1.1 with φ of the form $\varphi(x) = x^{1/p}$. The root r_0 has no influence on solutions to (22), because the interval $\{x^{1/p} \colon x \in I\}$ cannot be equal to the whole real line. ☐

Corollary 5.5. *Assume that $I \subset (0, \infty)$. Then $f \in C(I)$ satisfies*

$$f(x) f^2(x) \dots f^n(x) = x^n$$

if and only if

(i) *$f(x) = x$ in the case where n is an odd number or $I \neq (0, \infty)$;*

(ii) *$f(x) = x$ or $f(x) = cx^{r_0}$, where c is a constant such that $f(I) \subset I$ and r_0 stands for the negative root of Eq. (9), in the case where n is an even number and $I = (0, \infty)$.*

Proof. It is enough to use Theorem 3.1 and Remark 1.1 with $\varphi = \exp$. ☐

Corollary 5.6. *Assume that $p \neq 0$, $m \in \mathbb{N}$ and $I \subset (0, \infty)$. Then $f \in C(I)$ satisfies*

$$\frac{[f^m(x)]^p + [f^{2m}(x)]^p + \cdots + [f^{nm}(x)]^p}{n} = x^p \tag{23}$$

if and only if

(i) $f(x) = x$ *in the case where* m *is an odd number or* I *is neither open nor closed;*
(ii) $f(x) = x$ *or*

$$f(x) = \begin{cases} f_0(x) & \textit{for every } x \leqslant a, \\ f_0^{-1}(x) & \textit{for every } x < a, \end{cases} \tag{24}$$

where a *is an arbitrary interior point of* I *and* f_0 *is an arbitrary continuous and strictly decreasing function defined on* $I \cap (0, a]$ *such that* $\lim_{x \to \inf I} f_0(x) = \sup I$ *and* $f_0(a) = a$*, in the case where* m *is an even number and* I *is open or closed.*

Proof. From Corollary 5.4 we conclude that

$$f^m(x) = x. \tag{25}$$

Now it is enough to apply [5, Theorem 15.3, Theorem 15.2 and Lemma 15.2] jointly with the fact that there is no continuous strictly decreasing function satisfying (25) defined on an interval which is neither open nor closed. □

Corollary 5.7. *Assume* $m \in \mathbb{N}$ *and* I *is bounded. Then a bijection* $f \in C(I)$ *satisfies*

$$\frac{\exp\left(f^m(x)\right) + \exp\left(f^{m+1}(x)\right) + \cdots + \exp\left(f^{m+4n-2}(x)\right)}{4n-1} = \exp\left(f^{m+2n-1}(x)\right)$$

if and only if

(i) $f(x) = x$ *in the case where* m *is an odd number or* I *is neither open nor closed;*
(ii) $f(x) = x$ *or* f *is of the form* (24)*, where* a *is an arbitrary interior point of* I *and* f_0 *is an arbitrary continuous and strictly decreasing function defined on* $I \cap (-\infty, a]$ *such that* $\lim_{x \to \inf I} f_0(x) = \sup I$ *and* $f_0(a) = a$*, in the case where* m *is an even number and* I *is open or closed.*

Proof. Applying assertion (i) of Theorem 4.5 and Remark 1.1 with $\varphi = \log$ we conclude that (25) holds. Now we argue as in Corollary 5.6. □

We finish this paper with two problems motivated by the main results.

Problem 5.8. Can we omit the surjectivity assumption on f in Theorem 4.5?

Problem 5.9. Determine all (bijections) $f \in C(I)$ satisfying (3) in the case where both the numbers k and n are even.

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