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Correction to: Solutions of Complex Fermat-Type Partial Difference and Differential-Difference Equations

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Abstract. We give a correction to Theorem 1.2 in a previous paper [Mediterr. J. Math. (2018) 15:227]. Two examples are given to explain the corrected conclusion.
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1. Introduction and main result

Recently, the present authors originally considered solutions of complex partial differential-difference equations of the Fermat type by making use of Nevanlinna theory. Unfortunately, there was an error in the proof of [1, Theorem 1.2] (that is lines -1 to -3 on the Page 11), and thus its conclusion was stated wrong. Here we correct it as follows.s

Theorem 1.1. *Let $c = (c_1, c_2)$ be a constant in $\mathbb{C}^2$. Then any transcendental entire solution with finite order of the partial difference-differential equation of the Fermat type*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f^2(z_1 + c_1, z_2 + c_2) = 1 \tag{1}$$

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has the form of $f(z_1, z_2) = \sin(Az_1 + Bz_2 + H(z_2))$, *where* A, B *are constants on* $\mathbb{C}$ *satisfying* $A^2 = 1$ *and* $Ae^{i(Ac_1+Bc_2)} = 1$, *and* $H(z_2)$ *is a polynomial in one variable* z_2 *such that* $H(z_2) \equiv H(z_2 + c_2)$. *In the special case whenever* $c_2 \neq 0$, *we have* $f(z_1, z_2) = \sin(Az_1 + Bz_2 + Constant)$.

We show the details of the proof as follows.

Proof. Assume the f is a transcendental entire solution with finite order of equation (1), then

$$\left[\frac{\partial f(z_1, z_2)}{\partial z_1} + if(z_1 + c_1, z_2 + c_2)\right]\left[\frac{\partial f(z_1, z_2)}{\partial z_1} - if(z_1 + c_1, z_2 + c_2)\right] = 1.$$

From the equation we see that both $\frac{\partial f(z_1,z_2)}{\partial z_1} + if(z_1 + c_1, z_2 + c_2)$ and $\frac{\partial f(z_1,z_2)}{\partial z_1} - if(z_1 + c_1, z_2 + c_2)$ have no zeros in $\mathbb{C}^2$. Hence similarly as the proof of [1, Theorem 1.4], we may also assume that

$$\frac{\partial f(z_1, z_2)}{\partial z_1} + if(z_1 + c_1, z_2 + c_2) = e^{ip(z)}$$

and

$$\frac{\partial f(z_1, z_2)}{\partial z_1} - if(z_1 + c_1, z_2 + c_2) = e^{-ip(z)}$$

where p is a nonconstant entire function on $\mathbb{C}^2$, which gives

$$f(z_1 + c_1, z_2 + c_2) = \frac{e^{ip(z)} - e^{-ip(z)}}{2i} = \sin p(z).$$

Furthermore, it follows immediately from [1, Lemma 3.3] for any variable z_j $(j \in 1, 2)$ that p should be a polynomial function on $\mathbb{C}^2$. Hence, p is a nonconstant polynomial on $\mathbb{C}^2$. From these equations above, we get from [1, Lemma 3.2] that

$$\begin{aligned}\frac{\partial f(z_1 + c_1, z_2 + c_2)}{\partial z_1} &= \frac{e^{ip(z_1+c_1,z_2+c_2)} + e^{-ip(z_1+c_1,z_2+c_2)}}{2}\\ &= \frac{\partial p(z_1, z_2)}{\partial z_1} \cdot \frac{e^{ip(z_1,z_2)} + e^{-ip(z_1,z_2)}}{2},\end{aligned}$$

that is

$$\begin{aligned}&\frac{\partial p(z_1, z_2)}{\partial z_1} e^{ip(z_1,z_2)+ip(z_1+c_1,z_2+c_2)} + \frac{\partial p(z_1, z_2)}{\partial z_1} e^{ip(z_1+c_1,z_2+c_2)-ip(z_1,z_2)}\\ &-e^{i2p(z_1+c_1,z_2+c_2)} = 1. \qquad (2)\end{aligned}$$

From the assertion that p is a nonconstant polynomial, we see that $ip(z_1, z_2) + ip(z_1 + c_1, z_2 + c_2)$ can not be a constant. This implies that both $e^{i2p(z_1+c_1,z_2+c_2)}$ and $e^{ip(z_1,z_2)+ip(z_1+c_1,z_2+c_2)}$ must be nonconstant and transcendental on $\mathbb{C}^2$, and that

$$\frac{\partial p(z_1, z_2)}{\partial z_1} e^{ip(z_1,z_2)+ip(z_1+c_1,z_2+c_2)}$$

can not be a constant. Furthermore, note that

$$N(r, e^{i2p(z_1+c_1,z_2+c_2)}) = N(r, \frac{1}{e^{i2p(z_1+c_1,z_2+c_2)}}) = 0$$

$$N(r, \frac{\partial p(z_1,z_2)}{\partial z_1} e^{ip(z_1,z_2)+ip(z_1+c_1,z_2+c_2)}) = 0$$
$$N(r, \frac{1}{\frac{\partial p(z_1,z_2)}{\partial z_1} e^{ip(z_1,z_2)+ip(z_1+c_1,z_2+c_2)}}) = o(T(r,f)).$$

Hence, we can get from [1, Lemma 3.1] that

$$\frac{\partial p(z_1,z_2)}{\partial z_1} e^{ip(z_1+c_1,z_2+c_2)-ip(z_1,z_2)} \equiv 1. \tag{3}$$

Rewrite it to be

$$\frac{\partial p(z_1,z_2)}{\partial z_1} \equiv e^{ip(z_1,z_2)-ip(z_1+c_1,z_2+c_2)},$$

which implies $ip(z_1,z_2)-ip(z_1+c_1,z_2+c_2)$, and thus $e^{ip(z_1,z_2)-ip(z_1+c_1,z_2+c_2)}$, must be a constant. Otherwise, we obtain a contradiction from the fact that the left of the above equation is nontranscendental but the right is transcendental. Assume that

$$\frac{\partial p(z_1,z_2)}{\partial z_1} \equiv e^{ip(z_1,z_2)-ip(z_1+c_1,z_2+c_2)} \equiv A,$$

where A is a nonzero constant in $\mathbb{C}$. Submitting (3) into (2) gives

$$\frac{\partial p(z_1,z_2)}{\partial z_1} \equiv e^{ip(z_1+c_1,z_2+c_2)-ip(z_1,z_2)} \equiv \frac{1}{A}.$$

Hence $A^2 = 1$. Further, by $\frac{\partial p(z_1,z_2)}{\partial z_1} \equiv A$, we know that $p(z_1,z_2)$ is only a nonconstant polynomial of the form

$$p(z_1,z_2) = Az_1 + g(z_2),$$

where $g(z_2)$ should be a polynomial function in one variable z_2 (Note that the present authors made a mistake of $p(z_1,z_2) = Az_1 + B$ with a constant B in the original proof in [1], and thus the following is different from the original proof). Since $p(z_1,z_2) - p(z_1+c_1,z_2+c_2) = -iLnA$, we get that

$$g(z_2) - g(z_2+c_2) = Ac_1 - iLnA.$$

We may write $g(z_2) = Bz_2 + h(z_2)$ such that $A^2 = 1$ and

$$Ae^{i(Ac_1+Bc_2)} = 1,$$

where $h(z_2)$ is a polynomial in one variable z_2. This implies $Ac_1 + Bc_2 = -k\pi (k \in \mathbb{N})$ and $h(z_2) \equiv h(z_2+c_2)$. Hence,

$$\begin{aligned} f(z_1,z_2) &= \sin(Az_1 - Ac_1 + Bz_2 - Bc_2 + h(z_2-c_2)) \\ &= \sin(Az_1 + Bz_2 - (Ac_1+Bc_2) + h(z_2)) \\ &= \sin(Az_1 + Bz_2 + k\pi + h(z_2)) \\ &:= \sin(Az_1 + Bz_2 + H(z_2)), \end{aligned}$$

where $H(z_2)$ is a polynomial in one variable z_2 satisfying $H(z_2) \equiv H(z_2+c_2)$. It is clear that $H(z_2)$ should be a constant whenever $c_2 \neq 0$. □

We give two examples to explain the conclusion of the theorem.

Example 1.2. Let $A = 1$, $B = 2$, and let two constants c_1 and c_2 satisfy $e^{ic_1} = 1$ and $c_2 = 0$. Then $Ae^{i(Ac_1+Bc_2)} = 1$. The entire function $f(z) = \sin(z_1 + 2z_2 + z_2^3 + 1)$ satisfies the Fermat type partial differential difference equation

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f^2(z_1 + c_1, z_2 + c_2) = 1$$

in $\mathbb{C}^2$, where $c = (c_1, c_2)$. This shows that the function $H(z_2)$ in the conclusion of Theorem 1.1 may be a nonconstant polynomial whenever $c_2 = 0$.

Example 1.3. Let $A = 1$, $B = 2i$, and let two constants c_1 and c_2 satisfy $c_1+2ic_2 = 0$. Then $Ae^{i(Ac_1+Bc_2)} = 1$. The entire funcion $f(z) = \sin(z_1+2iz_2)$ satisfies the Fermat type partial differential difference equation

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f^2(z_1 + c_1, z_2 + c_2) = 1$$

in $\mathbb{C}^2$, where $c = (c_1, c_2)$. This shows that the function $H(z_2)$ in the conclusion of Theorem 1.1 may be a constant whenever $c_2 \neq 0$ or not.

If there are no differences, that is $c = (0, 0)$, then Theorem 1.1 implies the following corollary.

Corollary 1.4. *Any transcendental entire solution with finite order of the partial differential equation of the Fermat type*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f^2(z_1, z_2) = 1 \tag{4}$$

has the form of $f(z_1, z_2) = \sin(z_1 + g(z_2))$, *where* $g(z_2)$ *is a polynomial in one variable* z_2.

Reference

[1] Xu, L., Cao, T.B.: Solutions of Complex Fermat-Type Partial Difference and Differential-Difference Equations. Mediterr. J. Math. **15**, 227 (2018). https://doi.org/10.1007/s00009-018-1274-x

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