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The Natural Operators Similar to the Twisted Courant Bracket One

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Abstract. Given natural numbers $m \geq 3$ and $p \geq 3$, all $\mathcal{M}f_m$-natural operators A_H sending p-forms $H \in \Omega^p(M)$ on m-manifolds M into bilinear operators $A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M)$ transforming pairs of couples of vector fields and 1-forms on M into couples of vector fields and 1-forms on M are founded. If $m \geq 3$ and $p \geq 3$, then that any (similar as above) $\mathcal{M}f_m$-natural operator A which is defined only for closed p-forms H can be extended uniquely to the one A which is defined for all p-forms H is observed. If $p = 3$ and $m \geq 3$, all $\mathcal{M}f_m$-natural operators A (as above) such that A_H satisfies the Leibniz rule for all closed 3-forms H on m-manifolds M are extracted. The twisted Courant bracket $[-,-]_H$ for all closed 3-forms H on m-manifolds M gives the most important example of such $\mathcal{M}f_m$-natural operator A.

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1. Introduction

The "doubled" tangent bundle $T \oplus T^*$ over m-dimensional manifolds (m-manifolds) is full of interest because it has the natural inner product, and the Courant bracket, see [1]. Besides, generalized complex structures are defined on $T \oplus T^*$, generalizing both (usual) complex and symplectic structures, see e.g. [3,4].

In Sect. 2, the description from [2] of all $\mathcal{M}f_m$-natural bilinear operators

$$A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M),$$

transforming pairs of couples of vector fields and 1-forms on m-manifolds M into couples of vector fields and 1-forms on M will be shortly cited. The most important example of such $\mathcal{M}f_m$-natural bilinear operator A is given by the Courant bracket $[-,-]^C$, see Example 2.2. This Courant bracket was used in [1] to define the concept of Dirac structures being hybrid of both symplectic and Poisson structures.

In Sect. 2 we also deduce that the "trivial" Lie algebroid $(TM \oplus T^*M, 0, 0)$ is the only $\mathcal{M}f_m$-natural Lie algebroid $(EM, [[-,-]], a)$ with $EM := TM \oplus T^*M$.

In Sect. 3, using essentially the results from [2], if $m \geq 3$ and $p \geq 3$, we find all $\mathcal{M}f_m$-natural operators A sending p-forms $H \in \Omega^p(M)$ on m-manifolds M into bilinear maps

$$A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M).$$

The most important example of such A is given by the H-twisted Courant bracket $[-,-]_H$ for all 3-forms H on m-manifolds M, see Example 3.2. Properties of $[-,-]_H$ (as the Leibniz rule for closed 3-forms H) were used in [7,8] to define the concept of exact Courant algebroid.

In Sect. 4, we observe that if $m \geq 3$ and $p \geq 3$, then any (similar as above) $\mathcal{M}f_m$-natural operator A which is defined only for closed p-forms H can be extended uniquely to the one A which is defined for all p-forms H.

In Sect. 5, if $p = 3$ we extract all $\mathcal{M}f_m$-natural operators A as above satisfying the Leibniz rule

$$A_H(\rho_1, A_H(\rho_2, \rho_3)) = A_H(A_H(\rho_1, \rho_2), \rho_3) + A_H(\rho_2, A_H(\rho_1, \rho_3)),$$

for any closed $H \in \Omega^3(M)$, $\rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M)$ and $M \in obj(\mathcal{M}f_m)$.

From now on, (x^i) $(i = 1, ..., m)$ denote the usual coordinates on $\mathbf{R}^m$ and $\partial_i = \frac{\partial}{\partial x^i}$ are the canonical vector fields on $\mathbf{R}^m$.

All manifolds considered in this paper are assumed to be finite dimensional second countable Hausdorff without boundary and smooth (of class $\mathcal{C}^\infty$). Maps between manifolds are assumed to be smooth (of class $\mathcal{C}^\infty$)

2. The Natural Bilinear Operators Similar to the Courant Bracket

The general concept of natural operators can be found in the fundamental monograph [5]. In the paper, we need two particular cases of natural operators presented in Definitions 2.1 (below) and 3.1 (in the next section).

Let $\mathcal{M}f_m$ be the category of m-dimensional $\mathcal{C}^\infty$ manifolds as objects and their immersions of class $\mathcal{C}^\infty$ as morphisms ($\mathcal{M}f_m$-maps).

Definition 2.1. A natural (called also $\mathcal{M}f_m$-natural) operator A sending pairs of couples of vector fields and 1-forms on m-manifolds M into couples of vector fields and 1-forms on M is a $\mathcal{M}f_m$-invariant family of operators (functions)

$$A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M),$$

for all m-manifolds M, where $\mathcal{X}(M) \oplus \Omega^1(M)$ is the vector space of couples (X, ω) of vector fields X on M and 1-forms ω on M. Such $\mathcal{M}f_m$-natural operator A is called bilinear if A is bilinear (i.e., $A(\rho^1, -)$ and $A(-, \rho^2)$ are linear (over the field $\mathbf{R}$ of real numbers) functions $\mathcal{X}(M) \oplus \Omega^1(M) \to \mathcal{X}(M) \oplus \Omega^1(M)$ for any fixed $\rho^1, \rho^2 \in \mathcal{X}(M) \oplus \Omega^1(M)$) for any m-manifold M. Such $\mathcal{M}f_m$-natural operator A is called skew-symmetric if A is skew-symmetric for any m-manifold M.

The $\mathcal{M}f_m$-invariance of A means that if $(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$ are φ-related by an $\mathcal{M}f_m$-map $\varphi : M \to \overline{M}$ (i.e., $\overline{X}^i \circ \varphi = T\varphi \circ X^i$ and $\overline{\omega}^i \circ \varphi = T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$.

The most important example of such $\mathcal{M}f_m$-natural bilinear operator A is given by the (skew-symmetric) Courant bracket $[-,-]^C$ for any m-manifold M.

Example 2.2. On the vector bundle $TM \oplus T^*M$ there exist canonical symmetric and skew-symmetric pairings

$$\left\langle X^1 \oplus \omega^1, X^2 \oplus \omega^2 \right\rangle_{\pm} = \frac{1}{2}(i_{X^2}\omega^1 \pm i_{X^1}\omega^2)$$

for any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \mathcal{X}(M) \oplus \Omega^1(M)$, where i is the interior derivative. Further, the (skew-symmetric) Courant bracket is given by

$$\begin{aligned}&[X^1 \oplus \omega^1, X^2 \oplus \omega^2]^C \\ &\quad = [X^1, X^2] \oplus \left(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1 + \mathrm{d}\left\langle X^1 \oplus \omega^1, X^2 \oplus \omega^2 \right\rangle_{-}\right)\end{aligned}$$

for any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \mathcal{X}(M) \oplus \Omega^1(M)$, where $[-,-]$ is the usual bracket on vector fields, $\mathcal{L}$ is the Lie derivative and d is the exterior derivative.

Theorem 2.3 [2]. *If $m \geq 2$, any $\mathcal{M}f_m$-natural bilinear operator A in the sense of Definition 2.1 is of the form*

$$\begin{aligned}&A(\rho^1, \rho^2) \\ &= a[X^1, X^2] \oplus \left(b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + b_3\mathrm{d}\left\langle \rho^1, \rho^2 \right\rangle_{+} + b_4\mathrm{d}\left\langle \rho^1, \rho^2 \right\rangle_{-}\right)\end{aligned}$$

for (uniquely determined by A) real numbers a, b_1, b_2, b_3, b_4, where $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ are arbitrary, and where $\langle -,- \rangle_+$ and $\langle -,- \rangle_-$ are as in Example 2.2.

Corollary 2.4 [2]. *If $m \geq 2$, any $\mathcal{M}f_m$-natural skew-symmetric bilinear operator A in the sense of Definition 2.1 is of the form*

$$\begin{aligned}&A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \\ &= a[X^1, X^2] \oplus (b(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1) + c\mathrm{d}\left\langle X^2 \oplus \omega^1, X^1 \oplus \omega^2 \right\rangle_{-})\end{aligned}$$

for (uniquely determined by A) real numbers a, b, c.

Roughly speaking, Corollary 2.4 says that if $m \geq 2$, then any $\mathcal{M}f_m$-natural skew-symmetric bilinear operator A in the sense of Definition 2.1 coincides with the one given by Courant bracket $[-,-]^C$ up to three real constants.

Definition 2.5. A $\mathcal{M}f_m$-natural bilinear operator A in the sense of Definition 2.1 satisfies the Leibniz rule if

$$A(\rho_1, A(\rho_2, \rho_3)) = A(A(\rho_1, \rho_2), \rho_3) + A(\rho_2, A(\rho_1, \rho_3))$$

for all $\rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M)$ and all m-manifolds M.

Of course, in the case of skew-symmetric bilinear A the Leibniz rule is equivalent to the Jacobi identity $\sum_{\mathrm{cycl}(\rho_1,\rho_2,\rho_3)} A(\rho_1, A(A(\rho_2, \rho_3)) = 0$.

Example 2.6. The (not skew-symmetric) Courant bracket given by

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_0 := [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - i_{X^2}\mathrm{d}\omega^1),$$

where $X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$, satisfies the Leibniz rule, see [7,8].

The Courant bracket $[-,-]^C$ from Example 2.2 does not satisfy the Leibniz rule.

Theorem 2.7 [2]. *If $m \geq 2$, any $\mathcal{M}f_m$-natural bilinear operator A in the sense of Definition 2.1 satisfying the Leibniz rule is one of the following ones:*

$$A^{\langle 1,a\rangle}(\rho^1,\rho^2) = a[X^1,X^2] \oplus 0,$$
$$A^{\langle 2,a\rangle}(\rho^1,\rho^2) = a[X^1,X^2] \oplus (a(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1)),$$
$$A^{\langle 3,a\rangle}(\rho^1,\rho^2) = a[X^1,X^2] \oplus a\mathcal{L}_{X^1}\omega^2,$$
$$A^{\langle 4,a,0\rangle}(\rho^1,\rho^2) = a[X^1,X^2] \oplus (a(\mathcal{L}_{X^1}\omega^2 - i_{X^2}\mathrm{d}\omega^1)),$$

where a is an arbitrary real number, and where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$.

Corollary 2.8. *If $m \geq 2$, the Courant bracket $[-,-]_0$ from Example 2.6 for m-manifolds M is the unique $\mathcal{M}f_m$-natural bilinear operator A in the sense of Definition 2.1 satisfying the conditions:*

(*A*1) $A(\rho_1, A(\rho_2,\rho_3)) = A(A(\rho_1,\rho_2),\rho_3) + A(\rho_2, A(\rho_1,\rho_3))$,
(*A*2) $\pi A(\rho_1,\rho_2) = [\pi\rho_1, \pi\rho_2]$,
(*A*3) $A(\rho_1,\rho_1) = i_0\mathrm{d}\left<\rho_1,\rho_1\right>_+$,

for all $\rho_1,\rho_2,\rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M)$ and all m-manifolds M, where $\left<-,-\right>_+$ is the pairing of Example 2.2, $\pi : TM \oplus T^*M \to TM$ is the fibred projection given by $\pi(v,\omega) = v$ and $i_0 : T^*M \to TM \oplus T^*M$ is the fibred embedding $i_0(\omega) = (0,\omega)$.

Consequently, if $m \geq 2$, then a $\mathcal{M}f_m$-natural bilinear operator A in the sense of Definition 2.1 satisfying the conditions (A1)–(A3) satisfies the conditions:

(*A*4) $\pi\rho_1\left<\rho_2,\rho_3\right>_+ = \left<A(\rho_1,\rho_2),\rho_3\right>_+ + \left<\rho_2, A(\rho_1,\rho_3)\right>_+$,
(*A*5) $A(\rho_1, f\rho_2) = \pi\rho_1(f)\rho_2 + fA(\rho_1,\rho_2)$

for all $\rho_1,\rho_2 \in \mathcal{X}(M) \oplus \Omega^1(M)$, all $f \in \mathcal{C}^\infty(M)$ and all m-manifolds M (i.e., putting $[[-,-]] := A$ we get an exact Courant algebroid $E = (TM \oplus T^*M, [[-,-]], \left<-,-\right>_+, \pi, i_0)$ in the sense of [8] for any m-manifold M).

Proof. By Theorem 2.7, the conditions $(A1)$ and $(A2)$ imply that $A = A^{\langle 1,1\rangle}$ or $A = A^{\langle 2,1\rangle}$ or $A = A^{\langle 3,1\rangle}$ or $A = A^{\langle 4,1,0\rangle}$. On the other hand if $\rho_1 = X \oplus \omega$, then $i_0\mathrm{d}\left<\rho_1,\rho_1\right>_+ = 0 \oplus \mathrm{d}i_X\omega$ and $A^{\langle 1,1\rangle}(\rho_1,\rho_1) = 0 \oplus 0$ and $A^{\langle 2,1\rangle}(\rho_1,\rho_1) = 0 \oplus 0$ and $A^{\langle 3,1\rangle}(\rho_1,\rho_1) = 0 \oplus \mathcal{L}_X\omega$ and $A^{\langle 4,1,0\rangle}(\rho_1,\rho_1) = 0 \oplus \mathrm{d}i_X\omega$. Then $A = A^{\langle 4,1,0\rangle}$. □

Corollary 2.9. *If $m \geq 2$, any $\mathcal{M}f_m$-natural Lie algebra brackets on $\mathcal{X}(M) \oplus \Omega^1(M)$ [i.e., $\mathcal{M}f_m$-natural skew-symmetric bilinear operator satisfying the*

Jacobi identity (Leibniz rule) is the constant multiple of the one of the following two Lie algebra brackets:

$$[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_1 = [X^1, X^2] \oplus 0,$$
$$[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_2 = [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1).$$

At the end of this section we are going to describe completely all Lie algebroids $(TM \otimes T^*M, [[-,-]], a)$ which are invariant with respect to immersions ($\mathcal{M}f_m$-maps). The concept of Lie algebroids can be found in the fundamental book [6].

Of course, the anchor $a : TM \oplus T^*M \to TM$ for all m-manifolds M must be $\mathcal{M}f_m$-natural transformation [i.e., $Tf \circ a = a \circ (Tf \oplus T^*f)$ for any $\mathcal{M}f_m$-map $f : M \to M^1$] and fibre linear. By Corollary 2.9, $[[-,-]] = \mu[[-,-]]_1$ or $[[-,-]] = \mu[[-,-]]_2$ for some $\mu \in \mathbf{R}$.

Lemma 2.10. *Any $\mathcal{M}f_m$-natural transformation $a : TM \oplus T^*M \to TM$ which is fibre linear is the constant multiple of the fibre projection $\pi : TM \oplus T^*M \to TM$.*

Proof. Clearly, a is determined by the values $< \eta, a_x(v,\omega) > \in \mathbf{R}$ for all $\omega, \eta \in T^*_xM$, $v \in T_xM$, $x \in M$, $M \in \mathrm{Obj}(\mathcal{M}f_m)$. By the standard chart arguments, we may assume $M = \mathbf{R}^m$, $x = 0$, $\eta = \mathrm{d}_0x^1$. We can write $< \mathrm{d}_0x^1, a_0(v,\omega) > = \sum_i \alpha_i v^i + \sum_j \beta^j \omega_j$, where v^i are the coordinates of v and ω_j are the coordinates of ω, and where α_i and β^j are the real numbers determined by a_0. Then using the invariance of a_0 with respect to the maps $(\tau^1x^1, ..., \tau^m x^m)$ for $\tau^1 > 0, ..., \tau^m > 0$ we deduce that $\alpha_2 = \cdots = \alpha_m = 0$ and $\beta_1 = \cdots = \beta_m = 0$. Then the vector space of all a in question is at most 1-dimensional. Thus the dimension argument completes the proof. □

So, $a = k\pi$ for some real number k. It must be $a([[X^1 \oplus 0, X^2 \oplus 0]]) = [a(X^1 \oplus 0), a(X^2 \oplus 0)]$ for any vector fields X^1 and X^2 on M. This gives the condition $k\mu[X^1, X^2] = k^2[X^1, X^2]$. Then $k\mu = k^2$, and then ($k = 0$ and μ arbitrary) or ($k \neq 0$ and $\mu = k$). Consider two cases:

1. $[[-,-]] = \mu[[-,-]]_1$. Let $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$. It must be $[[\rho^1, f\rho^2]] = a(\rho^1)(f)\rho^2 + f[[\rho^1, \rho^2]]$. Considering the $\Omega^1(M)$-parts of both sides of this equality we get $0 = kX^1(f)\omega^2 + 0$ for any vector fields X^1, X^2 on M any map $f : M \to \mathbf{R}$ and any $\omega^1, \omega^2 \in \Omega^1(M)$. Then $k = 0$. Then considering the $\mathcal{X}(M)$-parts we get $\mu[X^1, fX^2] = f\mu[X^1, X^2]$. Then $\mu X^1(f)X^2 = 0$ for all vector fields X^1 and X^2 on M and all maps $f : M \to \mathbf{R}$, i.e., $\mu = 0$.

2. $[[-,-]] = \mu[[-,-]]_2$. Let $\rho^1 = 0 \oplus \omega^1$ and $\rho^2 = X^2 \oplus 0$. It must be $[[\rho^1, f\rho^2]] = a(\rho^1)(f)\rho^2 + f[[\rho^1, \rho^2]]$. Considering the $\Omega^1(M)$-parts of both sides of this equality we get $-\mu\mathcal{L}_{fX^2}\omega^1 = -\mu f\mathcal{L}_{X^2}\omega^1$. Then $\mu = 0$ or $\mathrm{d}i_{fX^2}\omega^1 + i_{fX^2}\mathrm{d}\omega^1 = f\mathrm{d}i_{X^2}\omega^1 + fi_{X^2}\mathrm{d}\omega^1$. Putting $\omega^1 = \mathrm{d}g$ we get $\mu = 0$ or $\mathrm{d}(i_{fX^2}\mathrm{d}g) = f\mathrm{d}i_{X^2}\mathrm{d}g$. Then $\mu = 0$ or $\mathrm{d}(fX^2g) = f\mathrm{d}(X^2g)$. Then $\mu = 0$ or $X^2(g)\mathrm{d}f = 0$ for any X^2, g, f in question. Putting $X^2 = \frac{\partial}{\partial x^1}$ and $f = g = x^1$ we get $\mu = 0$ or $\mathrm{d}x^1 = 0$. Then $\mu = 0$, and then $k = \mu = 0$.

On the other hand one can directly show that $(TM \oplus T^*M, 0[[-,-]]_1, 0\pi)$ is a Lie algebroid. Thus we have

Proposition 2.11. *If $m \geq 2$, $(TM \otimes T^*M, 0, 0)$ is the only invariant with respect to $\mathcal{M}f_m$-maps Lie algebroid $(EM, [[-,-]], a)$ with $EM = TM \oplus T^*M$.*

3. The Natural Operators Similar to the Twisted Courant Bracket

Definition 3.1. A $\mathcal{M}f_m$-natural operator A sending p-forms $H \in \Omega^p(M)$ on m-manifolds M into bilinear operators

$$A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M),$$

is a $\mathcal{M}f_m$-invariant family of regular operators (functions)

$$A : \Omega^p(M) \to Lin_2((\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)), \mathcal{X}(M) \oplus \Omega^1(M))$$

for all m-manifolds M, where $Lin_2(U \times V, W)$ denotes the vector space of all bilinear (over $\mathbf{R}$) functions $U \times V \to W$ for any real vector spaces U, V, W.

The $\mathcal{M}f_m$-invariance of A means that if $H^1 \in \Omega^p(M)$ and $H^2 \in \Omega^p(\overline{M})$ are φ-related and $(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$ are φ-related by an $\mathcal{M}f_m$-map $\varphi : M \to \overline{M}$, then so are $A_{H^1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A_{H^2}(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$.

The regularity of A means that it transforms smoothly parametrized families $(H_t, X^1_t \oplus \omega^1_t, X^2_t \oplus \omega^2_t)$ into smoothly parametrized families $A_{H_t}(X^1_t \oplus \omega^1_t, X^2_t \oplus \omega^2_t)$.

Example 3.2. The most important example of $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1 for $p = 3$ is given by the H-twisted Courant bracket

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_H := [X^1, X^2] \oplus \left(\mathcal{L}_{X^1}\omega^2 - i_{X^2}\mathrm{d}\omega^1 + i_{X^1}i_{X^2}H\right)$$

for all 3-forms $H \in \Omega^3(M)$ and all m-manifolds M. We call this $\mathcal{M}f_m$-natural operator the twisted Courant bracket $\mathcal{M}f_m$-natural operator.

Example 3.3. The operator given by $[-,-]_{\mathrm{d}H}$ for all $H \in \Omega^2(M)$ and all m-manifolds M is a $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1 for $p = 2$.

The main result of this section is the following

Theorem 3.4. *Assume $m \geq 3$. Then we have:*

1. *Any $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.1 for $p = 2$ such that $A_H = A_{H+\mathrm{d}H^1}$ for any $H \in \Omega^2(M)$ and any $H^1 \in \Omega^1(M)$ is of the form*

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$
$$\oplus \left(b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + b_3\mathrm{d}\left\langle\rho^1, \rho^2\right\rangle_+ + b_4\mathrm{d}\left\langle\rho^1, \rho^2\right\rangle_- + c i_{X^1}i_{X^2}\mathrm{d}H\right),$$

for (uniquely determined by A) reals $a, b_1, ..., c$, where 2-forms $H \in \Omega^2(M)$, pairs $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ and m-manifolds M are arbitrary.

2. *Any $\mathcal{M}f_m$-natural operator (not necessarily satisfying $A_H = A_{H+\mathrm{d}H^1}$) in the sense of Definition 3.1 for $p = 3$ is of the form*

$$A_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus \left(b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + b_3\mathrm{d}\left\langle\rho^1, \rho^2\right\rangle_+ + b_4\mathrm{d}\left\langle\rho^1, \rho^2\right\rangle_- + c i_{X^1} i_{X^2} H\right),$$

   for (uniquely determined by A) reals $a, b_1, ..., c$, where 3-forms $H \in \Omega^3(M)$, pairs $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ and m-manifolds M are arbitrary.

3. *If $p \geq 4$, any $\mathcal{M}f_m$-natural operator (not necessarily satisfying $A_H = A_{H+\mathrm{d}H^1}$) in the sense of Definition 3.1 is of the form*

$$A_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus \left(b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + b_3\mathrm{d}\left\langle\rho^1, \rho^2\right\rangle_+ + b_4\mathrm{d}\left\langle\rho^1, \rho^2\right\rangle_-\right)$$

   for (uniquely determined by A) reals $a, b_1, ..., b_4$, where p-forms $H \in \Omega^p(M)$, pairs $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ and m-manifolds M are arbitrary.

Proof. Clearly, A_0, where 0 is the zero p-form, can be treated as the bilinear operator in the sense of Definition 2.1. Then A_0 is described in Theorem 2.3. So we can replace A by $A - A_0$. In other words, we have assumption $A_0 = 0$.

By the invariance, A is determined by the values $A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}$ for all $H \in \Omega^p(\mathbf{R}^m), X^i \oplus \omega^i \in \mathcal{X}(\mathbf{R}^m) \oplus \Omega^1(\mathbf{R}^m)$. Put

$$A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0} = \left(A^1_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, A^2_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}\right),$$

where $A^1_H(...)_{|0} \in T_0\mathbf{R}^m$ and $A^2_H(...)_{|0} \in T^*\mathbf{R}^m$. Then A is determined by

$$\left\langle A^1_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \eta\right\rangle \in \mathbf{R} \text{ and } \left\langle A^2_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \mu\right\rangle \in \mathbf{R}$$

for all $H \in \Omega^p(\mathbf{R}^m), X^i \oplus \omega^i \in \mathcal{X}(\mathbf{R}^m) \oplus \Omega^1(\mathbf{R}^m)$, $\eta \in T^*_0\mathbf{R}^m, \mu \in T_0\mathbf{R}^m$, $i = 1, 2$.

By the non-linear Peetre theorem, see [5], A is of finite order. It means that there is a finite number r such that from $(j^r_x H = j^r_x \overline{H}, j^r_x(\rho^i) = j^r_x(\overline{\rho}^i)$, $i = 1, 2)$ it follows $A_H(\rho^1, \rho^2)_{|x} = A_{\overline{H}}(\overline{\rho}^1, \overline{\rho}^2)_{|x}$. So, we may assume that $H, X^1, X^2, \omega^1, \omega^2$ are polynomials of degree not more than r.

Using the invariance of A with respect to the homotheties and the bilinearity of A_H (for given H) we obtain homogeneity condition

$$\left\langle A^1_{\left(\frac{1}{t}\mathrm{id}\right)_* H}\left(t\left(\tfrac{1}{t}\mathrm{id}\right)_* X^1 \oplus t\left(\tfrac{1}{t}\mathrm{id}\right)_*\omega^1, t\left(\tfrac{1}{t}\mathrm{id}\right)_* X^2 \oplus t\left(\tfrac{1}{t}\mathrm{id}\right)_*\omega^2\right)_{|0}, \eta\right\rangle = t\left\langle A^1_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \eta\right\rangle.$$

Then, by the homogeneous function theorem, since A is of finite order and regular and $A_0 = 0$ and $p \geq 2$, we have $\left\langle A^1_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \eta\right\rangle = 0$.

Using the same arguments we get homogeneity condition

$$\Big\langle A^2_{\left(\frac{1}{t}\mathrm{id}\right)_* H}\left(t\left(\tfrac{1}{t}\mathrm{id}\right)_* X^1 \oplus t\left(\tfrac{1}{t}\mathrm{id}\right)_* \omega^1, t\left(\tfrac{1}{t}\mathrm{id}\right)_* X^2 \oplus t\left(\tfrac{1}{t}\mathrm{id}\right)_* \omega^2\right)_{|0}, \mu\Big\rangle = t^3\left\langle A^2_H(X^1\oplus\omega^1, X^2\oplus\omega^2)_{|0}, \mu\right\rangle .$$

Then, if $p=2$, by the homogeneous function theorem and the bilinearity of A_H and the assumptions $A_0=0$ and $A_H=A_{H+\mathrm{d}H^1}$, the value $\left\langle A^2_H(X^1\oplus\omega^1, X^2\oplus\omega^2)_{|0}, \mu\right\rangle$ depends quadrilinearly on ${X^1}_{|0}$, ${X^2}_{|0}$, $j^1_0(H-H_{|0})$ and μ, only. By $m\geq 3$ and the regularity of A, we may assume that $X^1_{|0}$, $X^2_{|0}$ and μ are linearly independent. Then by the invariance we may assume $X^1_{|0}=\partial_{1|0}$, $X^2_{|0}=\partial_{2|0}$ and $\mu=\partial_{3|0}$. Then A is determined by the values $\left\langle A^2_{x^{i_1}\mathrm{d}x^{i_2}\wedge\mathrm{d}x^{i_3}}(\partial_1\oplus 0,\partial_2\oplus 0),\partial_{3|0}\right\rangle$ for all $i_1=1,...,m$ and i_2,i_3 with $1\leq i_2<i_3\leq m$. Then using the invariance of A with respect to $\tau\mathrm{id}$ for $\tau^i>0$ we deduce that only $v:=\left\langle A^2_{x^1\mathrm{d}x^2\wedge\mathrm{d}x^3}(\partial_1\oplus 0,\partial_2\oplus 0),\partial_{3|0}\right\rangle$, $w:=\left\langle A^2_{x^2\mathrm{d}x^1\wedge\mathrm{d}x^3}(\partial_1\oplus 0,\partial_2\oplus 0),\partial_{3|0}\right\rangle$, $z:=\left\langle A^2_{x^3\mathrm{d}x^1\wedge\mathrm{d}x^2}(\partial_1\oplus 0,\partial_2\oplus 0),\partial_{3|0}\right\rangle$ may be not-zero. But $x^1\mathrm{d}x^2\wedge\mathrm{d}x^3 = -x^2\mathrm{d}x^1\wedge\mathrm{d}x^3+\mathrm{d}(...)$. So, $v=-w$. Similarly, $v=-z$. Therefore the vector space of all A in question with $A_0=0$ and $A_H=A_{H+\mathrm{d}H^1}$ is at most one-dimensional. The part (1) of the theorem is complete. If $p=3$, then (by almost the same arguments as for $p=2$) A is determined by the values $\left\langle A^2_{\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\mathrm{d}x^{i_3}}(\partial_1\oplus 0,\partial_2\oplus 0),\partial_{3|0}\right\rangle\in\mathbf{R}$ for all i_1,i_2,i_3 with $1\leq i_1<i_2<i_3\leq m$. Then using the invariance with respect to $(\tau^1x^1,...\tau^mx^m)$ for $\tau^i>0$ we deduce that only the value $\left\langle A^2_{\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3}(\partial_1\oplus 0,\partial_2\oplus 0),\partial_{3|0}\right\rangle\in\mathbb{R}$ may be not-zero. Therefore the vector space of all A in question with $A_0=0$ is one-dimensional (generated by the natural operator $0\oplus i_{X^1}i_{X^2}H$).

If $p\geq 4$, then (similarly as for $p=2$) $< A^2_H(X^1\oplus\omega^1, X^2\oplus\omega^2)_{|0}, \mu >= 0$. Theorem 3.4 is complete. □

Corollary 3.5. *If $m\geq 3$, any $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.1 for $p=3$ such that A_H is skew-symmetric for any $H\in\Omega^3(M)$ and any m-manifold M is of the form*

$$A_H(X^1\oplus\omega^1, X^2\oplus\omega^2) = a[X^1,X^2] \oplus\Big(b(\mathcal{L}_{X^1}\omega^2-\mathcal{L}_{X^2}\omega^1)+c\mathrm{d}\left\langle X^2\oplus\omega^1, X^1\oplus\omega^2\right\rangle_- + e i_{X^1}i_{X^2}H\Big)$$

for (uniquely determined by A) real numbers a,b,c,e.

Roughly speaking, Corollary 3.5 says that any $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.1 such that A_H is skew-symmetric for any $H\in\Omega^3(M)$ and any m-manifold M coincides with the "skew-symmetrization" of the twisted Courant bracket $\mathcal{M}f_m$-natural operator up to four real constants a,b,c,e.

Corollary 3.6. *If $m\geq 3$, then the twisted Courant bracket $\mathcal{M}f_m$-natural operator from Example 3.2 is the unique $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.1 for $p=3$ satisfying the following properties:*

$(B1)$ $A_0(\rho_1, \rho_2) = [\rho_1, \rho_2]_0$,
$(B2)$ $A_H(X \oplus 0, Y \oplus 0) = [X, Y] \oplus i_X i_Y H$
for all closed $H \in \Omega^3_{cl}(M)$, *all* $\rho_1, \rho_2, X \oplus 0, Y \oplus 0 \in \mathcal{X}(M) \oplus \Omega^1(M)$ *and all* m*-manifolds* M*, where* $[-,-]_0$ *is the* $\mathcal{M}f_m$*-natural bilinear operator given by the (not skew-symmetric) Courant bracket as in Example* 2.6*.*

Proof. Clearly, the twisted Courant bracket $\mathcal{M}f_m$-natural operator satisfies $(B1)$ and $(B2)$. Consider A in question satisfying $(B1)$ and $(B2)$. Then by Theorem 3.4, there exist uniquely determined reals $a, b_1, ..., c$ such that for all $H \in \Omega^3(M)$ and m-manifolds M

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$
$$\oplus \Big(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 \mathrm{d}\left\langle \rho^1, \rho^2\right\rangle_+ + b_4 \mathrm{d}\left\langle \rho^1, \rho^2\right\rangle_- + c i_{X^1} i_{X^2} H\Big),$$

where $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ are arbitrary. Putting $\omega^1 = \omega^2 = 0$ we get $A_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus c i_{X^1} i_{X^2} H$. Then condition $(B2)$ implies $c = 1$. Putting $H = 0$ we get

$$A_0(\rho^1, \rho^2) = a[X^1, X^2] \oplus \Big(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 \mathrm{d}\left\langle \rho^1, \rho^2\right\rangle_+ + b_4 \mathrm{d}\left\langle \rho^1, \rho^2\right\rangle_-\Big)$$

for all $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ and all m-manifolds M. But A_0 is a $\mathcal{M}f_m$-natural bilinear operator in the sense of Definition 2.1. Then a, b_1, b_2, b_3, b_4 are uniquely determined because of Theorem 2.3. Then a, $b_1, ..., c$ are uniquely determined. So, A is uniquely determined by conditions $(B1)$ and $(B2)$. □

4. The Natural Operators Similar to the Twisted Courant Bracket and Defined for Closed p-Forms Only

In the previous section, we considered $\mathcal{M}f_m$-natural operators A which are defined for all p-forms H. In this section, we observe what happens if A are defined for closed p-forms H, only. We start with the following

Definition 4.1. A $\mathcal{M}f_m$-natural operator A sending closed p-forms $H \in \Omega^p_{cl}(M)$ on m-manifolds M into bilinear operators

$$A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M),$$

is a $\mathcal{M}f_m$-invariant family of regular operators (functions)

$$A : \Omega^p_{cl}(M) \to Lin_2((\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)), \mathcal{X}(M) \oplus \Omega^1(M)),$$

for all m-manifolds M.

We have the following corollary of Theorem 3.4.

Corollary 4.2. *Assume* $m \geq 3$*. Then we have:*

1. *If* $p = 3$*, any* $\mathcal{M}f_m$*-natural operator in the sense of Definition* 4.1 *is of the form*

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$
$$\oplus (b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 \mathrm{d}\left\langle \rho^1, \rho^2\right\rangle_+ + b_4 \mathrm{d}\left\langle \rho^1, \rho^2\right\rangle_- + c i_{X^1} i_{X^2} H),$$

for uniquely determined by A reals $a, b_1, ..., c$, where closed 3*-forms $H \in \Omega^3_{cl}(M)$, pairs $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ and m-manifolds M are arbitrary.*

2. *If $p \geq 4$, any $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 is of the form*

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$
$$\oplus \left(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 \mathrm{d}\left\langle \rho^1, \rho^2 \right\rangle_+ + b_4 \mathrm{d}\left\langle \rho^1, \rho^2 \right\rangle_-\right)$$

for uniquely determined by A reals $a, b_1, ..., b_4$, where closed p-forms $H \in \Omega^p_{cl}(M)$, pairs $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$ and m-manifolds M are arbitrary.

Proof. Let A be a $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 for p. Define a $\mathcal{M}f_m$-natural operator A^1 in the sense of Definition 3.1 for $p-1$ by $A^1_{\tilde{H}} = A_{\mathrm{d}\tilde{H}}$. Then $A^1_{\tilde{H}+\mathrm{d}H_1} = A^1_{\tilde{H}}$ for any $\tilde{H} \in \Omega^{p-1}(M)$ and $H_1 \in \Omega^{p-2}(M)$.

If $p = 3$, then by Theorem 3.4, A^1 is of the form

$$A^1_{\tilde{H}}(\rho^1, \rho^2) = a[X^1, X^2]$$
$$\oplus \left(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 \mathrm{d}\left\langle \rho^1, \rho^2 \right\rangle_+ + b_4 \mathrm{d}\left\langle \rho^1, \rho^2 \right\rangle_- + c i_{X^1} i_{X^2} \mathrm{d}\tilde{H}\right)$$

for uniquely determined reals $a, b_1, ..., c$ and all $\tilde{H} \in \Omega^2(M)$, where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. Then

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$
$$\oplus \left(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 \mathrm{d}\left\langle \rho^1, \rho^2 \right\rangle_+ + b_4 \mathrm{d}\left\langle \rho^1, \rho^2 \right\rangle_- + c i_{X^1} i_{X^2} H\right)$$

for all exact 3-forms H, where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. But by the locality of A and the Poincare lemma we may replace the phrase "all exact 3-forms" by "all closed 3-forms".

If $p \geq 4$, then by Theorem 3.4, A^1 is of the form

$$A^1_{\tilde{H}}(\rho^1, \rho^2) = a[X^1, X^2]$$
$$\oplus \left(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 \mathrm{d}\left\langle \rho^1, \rho^2 \right\rangle_+ + b_4 \mathrm{d}\left\langle \rho^1, \rho^2 \right\rangle_- + c i_{X^1} i_{X^2} \mathrm{d}\tilde{H}\right)$$

for uniquely determined reals $a, b_1, ..., c$ (with arbitrary c if $p = 4$ and with $c = 0$ if $p \geq 5$) and all $\tilde{H} \in \Omega^{p-1}(M)$, where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. The condition $A^1_{\tilde{H}} = A^1_{\tilde{H}+\mathrm{d}H_1}$ implies $c i_{X^1} i_{X^2} \mathrm{d}H_1 = 0$ for any $H_1 \in \Omega^{p-2}(M)$. If $p = 4$, putting $X^1 = \partial_1$, $X^2 = \partial_2$ and $H_1 = x^1 \mathrm{d}x^2 \wedge \mathrm{d}x^3$, we get $c(-\mathrm{d}x^3) = 0$, i.e., $c = 0$. If $p \geq 5$, then $c = 0$, see above. Next, we proceed similarly as in the case $p = 3$. □

The above corollary and Theorem 3.4 imply

Theorem 4.3. *If $m \geq 3$ and $p \geq 3$ then any $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 can be extended uniquely to a $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1.*

Roughly speaking, if $m \geq 3$ and $p \geq 3$, then any $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 can be treated as the $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1, and vice-versa.

5. The Natural Operators Similar to the Twisted Courant Bracket and Satisfying the Leibniz Rule for Closed 3-Forms

Definition 5.1. A $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.1 (or equivalently in the sense of Definition 4.1) satisfies the Leibniz rule for closed p-forms if

$$A_H(\rho_1, A_H(\rho_2,\rho_3)) = A_H(A_H(\rho_1,\rho_2),\rho_3) + A_H(\rho_2, A_H(\rho_1,\rho_3))$$

for all $\rho_1,\rho_2,\rho_3 \in \mathcal{X}(M)\oplus\Omega^1(M)$, all closed p-forms $H\in\Omega^p_{cl}(M)$ and all m-manifolds M.

Example 5.2. The twisted Courant bracket $\mathcal{M}f_m$-natural operator presented in Example 3.2 satisfies the Leibniz rule for closed 3-forms, see [3,8].

Theorem 5.3. *If $m\geq 3$, any $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.1 (or equivalently of Definition 4.1) for $p=3$ satisfying the Leibniz rule for closed 3-forms is one of the $\mathcal{M}f_m$-natural operators:*

$$\begin{aligned}
A^{\langle 1,a\rangle}_H(\rho_1,\rho_2) &= a[X^1,X^2]\oplus 0,\\
A^{\langle 2,a\rangle}_H(\rho^1,\rho^2) &= a[X^1,X^2]\oplus (a(\mathcal{L}_{X^1}\omega^2-\mathcal{L}_{X^2}\omega^1)),\\
A^{\langle 3,a\rangle}_H(\rho^1,\rho^2) &= a[X^1,X^2]\oplus (a\mathcal{L}_{X^1}\omega^2),\\
A^{\langle 4,a,e\rangle}_H(\rho^1,\rho^2) &= a[X^1,X^2]\oplus (a(\mathcal{L}_{X^1}\omega^2 - i_{X^2}\mathrm{d}\omega^1)+e i_{X^1}i_{X^2}H),
\end{aligned}$$

where $\rho^1=X^1\oplus\omega^1$ and $\rho^2=X^2\oplus\omega^2$, and a and e are arbitrary real numbers.

Proof. Let A be a $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1 for $p=3$ such that A_H satisfies the Leibniz rule for any closed $H\in\Omega^3_{cl}(M)$. By Theorem 3.4, A is of the form

$$\begin{aligned}
A_H(X^1\oplus\omega^1, X^2\oplus\omega^2) = a[X^1,X^2]\\
\oplus(b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + c_1\mathrm{d}i_{X^2}\omega^1 + c_2\mathrm{d}i_{X^1}\omega^2 + e i_{X^1}i_{X^2}H),
\end{aligned}$$

for (uniquely determined by A) real numbers a,b_1,b_2,c_1,c_2,e. Then for any $X^1,X^2,X^3\in\mathcal{X}(M)$ and $\omega^1,\omega^2,\omega^3\in\Omega^1(M)$ we have

$$\begin{aligned}
A_H(X^1\oplus\omega^1, A_H(X^2\oplus\omega^2, X^3\oplus\omega^3)) &= a^2[X^1,[X^2,X^3]]\oplus\Omega,\\
A_H(A_H(X^1\oplus\omega^1, X^2\oplus\omega^2), X^3\oplus\omega^3) &= a^2[[X^1,X^2],X^3]\oplus\Theta,\\
A_H(X^2\oplus\omega^2, A_H(X^1\oplus\omega^1, X^3\oplus\omega^3)) &= a^2[X^2,[X^1,X^3]]\oplus\mathcal{T},
\end{aligned}$$

where

$$\begin{aligned}\Omega &= b_1\mathcal{L}_{a[X^2,X^3]}\omega^1 + c_1\mathrm{d}i_{a[X^2,X^3]}\omega^1 + ei_{X^1}i_{a[X^2,X^3]}H \\ &\quad + b_2\mathcal{L}_{X^1}(b_1\mathcal{L}_{X^3}\omega^2 + b_2\mathcal{L}_{X^2}\omega^3 + c_1\mathrm{d}i_{X^3}\omega^2 + c_2\mathrm{d}i_{X^2}\omega^3 + ei_{X^2}i_{X^3}H) \\ &\quad + c_2\mathrm{d}i_{X^1}(b_1\mathcal{L}_{X^3}\omega^2 + b_2\mathcal{L}_{X^2}\omega^3 + c_1\mathrm{d}i_{X^3}\omega^2 + c_2\mathrm{d}i_{X^2}\omega^3 + ei_{X^2}i_{X^3}H), \\ \Theta &= b_2\mathcal{L}_{a[X^1,X^2]}\omega^3 + c_2\mathrm{d}i_{a[X^1,X^2]}\omega^3 + ei_{a[X^1,X^2]}i_{X^3}H \\ &\quad + b_1\mathcal{L}_{X^3}(b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + c_1\mathrm{d}i_{X^2}\omega^1 + c_2\mathrm{d}i_{X^1}\omega^2 + e_{X^1}i_{X^2}H) \\ &\quad + c_1\mathrm{d}i_{X^3}(b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + c_1\mathrm{d}i_{X^2}\omega^1 + c_2\mathrm{d}i_{X^1}\omega^2 + ei_{X^1}i_{X^2}H), \\ \mathcal{T} &= b_1\mathcal{L}_{a[X^1,X^3]}\omega^2 + c_1\mathrm{d}i_{a[X^1,X^3]}\omega^2 + ei_{X^2}i_{a[X^1,X^3]}H \\ &\quad + b_2\mathcal{L}_{X^2}(b_1\mathcal{L}_{X^3}\omega^1 + b_2\mathcal{L}_{X^1}\omega^3 + c_1\mathrm{d}i_{X^3}\omega^1 + c_2\mathrm{d}i_{X^1}\omega^3 + ei_{X^1}i_{X^3}H) \\ &\quad + c_2\mathrm{d}i_{X^2}(b_1\mathcal{L}_{X^3}\omega^1 + b_2\mathcal{L}_{X^1}\omega^3 + c_1\mathrm{d}i_{X^3}\omega^1 + c_2\mathrm{d}i_{X^1}\omega^3 + ei_{X^1}i_{X^3}H).\end{aligned}$$

The Leibniz rule of A_H is equivalent to $\Omega = \Theta + \mathcal{T}$.

Putting $H = 0$, we are in the situation of Theorem 2.7. Then by Theorem 2.7 (i.e., by Theorem 3.2 in [2]) we get $(b_1, b_2, c_1, c_2) = (0, 0, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (0, a, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (-a, a, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (-a, a, a, 0)$. More, A_0 for such (b_1, b_2, c_1, c_2) satisfies the Leibniz rule.

Therefore (as $c_2 = 0$) the Leibniz rule of A_H is equivalent to the equality

$$\begin{aligned}&eai_{X^1}i_{[X^2,X^3]}H + b_2e\mathcal{L}_{X^1}i_{X^2}i_{X^3}H \\ &\quad = eai_{[X^1,X^2]}i_{X^3}H + b_1e\mathcal{L}_{X^3}i_{X^1}i_{X^2}H + c_1e\mathrm{d}i_{X^3}i_{X^1}i_{X^2}H \\ &\qquad + eai_{X^2}i_{[X^1,X^3]}H + b_2e\mathcal{L}_{X^2}i_{X^1}i_{X^3}H.\end{aligned}$$

If $(b_1, b_2, c_1, c_2) = (0, 0, 0, 0)$, the above equality is equivalent to

$$eai_{X^1}i_{[X^2,X^3]}H = eai_{[X^1,X^2]}i_{X^3}H + eai_{X^2}i_{[X^1,X^3]}H.$$

Putting $X^1 = \partial_1$, $X^2 = \partial_1 + x^1\partial_3$ and $X^3 = \partial_2$ we have $[X^2, X^3] = 0$, $[X^1, X^3] = 0$ and $[X^1, X^2] = \partial_3$, and then $0 = eai_{\partial_3}i_{\partial_2}H$ for any closed H (for example for $H = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$). Consequently $e = 0$ or $a = 0$.

If $(b_1, b_2, c_1, c_2) = (0, a, 0, 0)$, the above equality is equivalent to

$$\begin{aligned}&eai_{X^1}i_{[X^2,X^3]}H + ea\mathcal{L}_{X^1}i_{X^2}i_{X^3}H \\ &\quad = eai_{[X^1,X^2]}i_{X^3}H + eai_{X^2}i_{[X^1,X^3]}H + ea\mathcal{L}_{X^2}i_{X^1}i_{X^3}H.\end{aligned}$$

Putting $X^1 = \partial_1$, $X^2 = \partial_2$ and $X^3 = \partial_3$ and $H = x^2\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ (it is closed) we have $[X^2, X^3] = 0$, $[X^1, X^2] = 0$, $[X^1, X^3] = 0$, $\mathcal{L}_{X^2}i_{X^1}i_{X^3}H = \mathcal{L}_{\partial_2}x^2\mathrm{d}x^2 = \mathrm{d}x^2$ and $\mathcal{L}_{X^1}i_{X^2}i_{X^3}H = \mathcal{L}_{\partial_1}(-x^2\mathrm{d}x^1) = 0$. Then $ea\mathrm{d}x^2 = 0$. So, $a = 0$ or $e = 0$.

If $(b_1, b_2, c_1, c_2) = (-a, a, 0, 0)$, the above equality is equivalent to

$$\begin{aligned}&eai_{X^1}i_{[X^2,X^3]}H + ea\mathcal{L}_{X^1}i_{X^2}i_{X^3}H \\ &\quad = eai_{[X^1,X^2]}i_{X^3}H - ea\mathcal{L}_{X^3}i_{X^1}i_{X^2}H + eai_{X^2}i_{[X^1,X^3]}H + ea\mathcal{L}_{X^2}i_{X^1}i_{X^3}H.\end{aligned}$$

Putting $X^1 = \partial_1$, $X^2 = \partial_2$ and $X^3 = \partial_3$ and $H = x^2\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ we have (see above) $[X^2, X^3] = 0$, $[X^1, X^2] = 0$, $[X^1, X^3] = 0$, $\mathcal{L}_{X^2}i_{X^1}i_{X^3}H = \mathrm{d}x^2$, $\mathcal{L}_{X^1}i_{X^2}i_{X^3}H = 0$ and $\mathcal{L}_{X^3}i_{X^1}i_{X^2}H = \mathcal{L}_{\partial_3}(-x^2\mathrm{d}x^3) = 0$. Then $ea\mathrm{d}x^2 = 0$. So, $a = 0$ or $e = 0$.

If $(b_1, b_2, c_1, c_2) = (-a, a, a, 0)$, the above equality is equivalent to

$$ea \sum \left\{ i_{X^1} i_{[X^2,X^3]} H + \mathcal{L}_{X^1} i_{X^2} i_{X^3} H \right\} = ead i_{X^1} i_{X^2} i_{X^3} H,$$

where $\sum$ is the cyclic sum $\sum_{cycl(X^1,X^2,X^3)}$. Then e is arbitrary real number because from $\mathrm{d}H = 0$ it follows

$$\sum \left\{ i_{X^1} i_{[X^2,X^3]} H + \mathcal{L}_{X^1} i_{X^2} i_{X^3} H \right\} = \mathrm{d} i_{X^1} i_{X^2} i_{X^3} H.$$

Indeed, using $\mathrm{d}H = 0$ and $i_{[X^1,X^4]} = \mathcal{L}_{X^1} i_{X^4} - i_{X^4} \mathcal{L}_{X^1}$ and the well-known formula expressing $\mathrm{d}H(X^1, X^2, X^3, X^4)$, we have

$$\begin{aligned} &\sum \left\{ i_{X^4} i_{X^1} i_{[X^2,X^3]} H + i_{X^4} \mathcal{L}_{X^1} i_{X^2} i_{X^3} H \right\} \\ &= \sum \left\{ i_{X^4} i_{X^1} i_{[X^2,X^3]} H + \mathcal{L}_{X^1} i_{X^4} i_{X^2} i_{X^3} H - i_{[X^1,X^4]} i_{X^2} i_{X^3} H \right\} \\ &= 6 \sum \{ H([X^2, X^3], X^1, X^4) + X^1 H(X^3, X^2, X^4) \\ &\qquad - H(X^3, X^2, [X^1, X^4]) \} \\ &= -24 \mathrm{d}H(X^1, X^2, X^3, X^4) + 6X^4 H(X^3, X^2, X^1) = i_{X^4} \mathrm{d} i_{X^1} i_{X^2} i_{X^3} H. \end{aligned}$$

Summing up, given a real number $a \neq 0$ we have $(b_1, b_2, c_1, c_2, e) = (0, 0, 0, 0, 0)$ or $(b_1, b_2, c_1, c_2, e) = (0, a, 0, 0, 0)$ or $(b_1, b_2, c_1, c_2, e) = (-a, a, 0, 0, 0,)$ or $(b_1, b_2, c_1, c_2, e) = (-a, a, a, 0, e)$, where e may be arbitrary real number. If $a = 0$ we have $(b_1, b_2, c_1, c_2, e) = (0, 0, 0, 0, e)$, where e may be arbitrary. Theorem 5.3 is complete. □

Corollary 5.4. *If $m \geq 3$, then the twisted Courant bracket $\mathcal{M}f_m$-natural operator from Example 3.2 is the unique $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.1 for $p = 3$ satisfying the following conditions:*

$(C1)$ $A_H(\rho_1, A_H(\rho_2, \rho_3)) = A_H(A_H(\rho_1, \rho_2), \rho_3) + A_H(\rho_2, A_H(\rho_1, \rho_3))$,
$(C2)$ $A_H(X \oplus 0, Y \oplus 0) = [X, Y] \oplus i_X i_Y H$

for all $\rho_1, \rho_2, \rho_3, X \oplus 0, Y \oplus 0 \in \mathcal{X}(M) \oplus \Omega^1(M)$, all closed $H \in \Omega^3_{cl}(M)$ and all m-manifolds M.

Proof. Indeed, the condition $(C1)$ and Theorem 5.3 imply that $A = A^{\langle 1,a \rangle}$ or $A = A^{\langle 2,a \rangle}$ or $A = A^{\langle 3,a \rangle}$ or $A = A^{\langle 4,a,e \rangle}$ for some real numbers a and e. Then $(C2)$ implies that $A = A^{\langle 4,a,e \rangle}$ and $a = 1$ and $e = 1$ because $A^{\langle 1,a \rangle}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus 0$ and $A^{\langle 2,a \rangle}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus 0$ and $A^{\langle 3,a \rangle}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus 0$ and $A^{\langle 4,a,e \rangle}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus e i_X i_Y H$. □

Corollary 5.5. *If $m \geq 3$, any $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.1 for $p = 3$ such that A_H is a Lie algebra bracket (i.e., it is skew-symmetric, bilinear and satisfying the Leibniz rule) for all closed 3-forms $H \in \Omega^3_{cl}(M)$ and all m-manifolds M is one of the $\mathcal{M}f_m$-natural operators:*

$$\begin{aligned} A^{\langle 1,a \rangle}_H(\rho_1, \rho_2) &= a[X^1, X^2] \oplus 0, \\ A^{\langle 2,a \rangle}_H(\rho^1, \rho^2) &= a[X^1, X^2] \oplus (a(\mathcal{L}_{X^1} \omega^2 - \mathcal{L}_{X^2} \omega^1)), \\ A^{\langle 4,0,e \rangle}_H(\rho^1, \rho^2) &= 0 \oplus e i_{X^1} i_{X^2} H, \end{aligned}$$

where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$, and a and e are arbitrary real numbers.

Proof. It follows from Theorem 5.3. □

Corollary 5.6. *If $m \geq 3$, any $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.1 for $p = 3$ satisfying the Leibniz rule for all 3-forms H (or for all closed 3-forms and at least one non-closed 3-form) is one of the $\mathcal{M}f_m$-natural operators:*

$$\begin{aligned} A_H^{\langle 1,a\rangle}(\rho_1,\rho_2) &= a[X^1,X^2]\oplus 0,\\ A_H^{\langle 2,a\rangle}(\rho^1,\rho^2) &= a[X^1,X^2]\oplus (a(\mathcal{L}_{X^1}\omega^2-\mathcal{L}_{X^2}\omega^1)),\\ A_H^{\langle 3,a\rangle}(\rho^1,\rho^2) &= a[X^1,X^2]\oplus (a\mathcal{L}_{X^1}\omega^2),\\ A_H^{\langle 4,a,0\rangle}(\rho^1,\rho^2) &= a[X^1,X^2]\oplus (a(\mathcal{L}_{X^1}\omega^2-i_{X^2}\mathrm{d}\omega^1)),\\ A_H^{\langle 4,0,e\rangle}(\rho^1,\rho^2) &= 0\oplus e i_{X^1}i_{X^2}H, \end{aligned}$$

where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$, and a and e are arbitrary real numbers.

Proof. It follows from Theorem 5.3 and its proof. □

Remark 5.7. It is well-known that given closed 3-form $H \in \Omega^3_{cl}(M)$ on a m-manifold M, the twisted Courant bracket $[-,-]_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M)$ is bilinear and satisfies the properties (A1)–(A5) from Corollary 2.8 for all $\rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M)$ and all $f \in \mathcal{C}^\infty(M)$, see [3,8], but $[-,-]_H \neq [-,-]_0$ if $H \neq 0$. Is it a contradiction with the uniqueness from Corollary 2.8? No, it is not. Indeed, $[-,-]_H$ is not extendable to a $\mathcal{M}f_m$-natural bilinear operator in the sense of Definition 2.1 because it is invariant only with respect to $\mathcal{M}f_m$-maps $\varphi : M \to M$ preserving H, in fact.

Remark 5.8. By Corollary 5.5, given a closed 3-form H on M, the skew-symmetric bracket $[[X^1\oplus\omega^1, X^2\oplus\omega^2]]^{(H)} := 0\oplus i_{X^1}i_{X^2}H$ satisfies the Leibniz rule. One can easily directly verify that $(TM \oplus T^*M, e[[-,-]]^{(H)}, 0\pi)$ for arbitrary fixed $e \in \mathbf{R}$ and closed 3-form H is a Lie algebroid canonically depending on H. So, if we have a closed 3-form H on a m-manifold M, we can construct canonical (in H) Lie algebroids $(EM, [[-,-]]^{[H]}, a^{[H]})$ with $EM = TM \oplus T^*M$ different than the one from Proposition 2.11.

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References

[1] Courant, T.: Dirac manifolds. Trans. Am. Math. Soc. **319**(631), 631–661 (1990)

[2] Doupovec, M., Kurek, J., Mikulski, W.M.: The natural brackets on couples of vector fields and 1-forms. Turk. J. Math. **42**(2), 1853–1862 (2018)

[3] Gualtieri, M.: Generalized complex geometry. Ann. Math. **174**(1), 75–123 (2011)

[4] Hitchin, N.: Generalized Calabi–Yau manifolds. Q. J. Math. **54**(3), 281–308 (2003)

[5] Kolář, I., Michor, P.W., Slovák, J.: Natural Operations in Differential Geometry. Springer, Berlin (1993)

[6] Mackenzie, K.C.H.: General Theory of Lie Groupoids and Lie Algebroids. London Math. Soc., Lecture Note 213. Cambridge University Press, Cambridge (2005)

[7] Liu, Z.J., Weinstein, A., Xu, P.: Main triples for Lie bialgebroids. J. Differ. Geom. **45**, 547–574 (1997)

[8] Ševera, P., Weinstein, A.: Poisson geometry with a 3-form bacground. Progr. Teoret. Phys. Suppl., 145–154 (2001)

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