



3D Oriented Projective Geometry Through Versors of $\mathbb{R}^{3,3}$

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Abstract. It is possible to set up a correspondence between 3D space and $\mathbb{R}^{3,3}$, interpretable as the space of oriented lines (and screws), such that special projective collineations of the 3D space become represented as rotors in the geometric algebra of $\mathbb{R}^{3,3}$. We show explicitly how various primitive projective transformations (translations, rotations, scalings, perspectivities, Lorentz transformations) are represented, in geometrically meaningful parameterizations of the rotors by their bivectors. Odd versors of this representation represent projective correlations, so (oriented) reflections can only be represented in a non-versor manner. Specifically, we show how a new and useful ‘oriented reflection’ can be defined directly on lines. We compare the resulting framework to the unoriented $\mathbb{R}^{3,3}$ approach of Klawitter (Adv Appl Clifford Algebra, 24:713–736, 2014), and the $\mathbb{R}^{4,4}$ rotor-based approach by Goldman et al. (Adv Appl Clifford Algebra, 25(1):113–149, 2015) in terms of expressiveness and efficiency.

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1. Introduction

1.1. Towards a Geometric Algebra of Projective Geometry

Projective transformations in 3D and 2D are extensively used in computer vision and computer graphics. In those fields, it is common practice to construct a desired projective transformation from certain geometrically meaningful primitive operations (rotations at the origin, translations, scalings, shears, etc.), or to analyze a given transformation in terms of such primitive operations. In applications, the 3D projective transformations are always

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computed by means of the homogeneous coordinates representation. An extra dimension is added to produce a four-dimensional space V^4 , and points and vector from the 3D space are embedded as vectors in V^4 ; the projective transformations become 4×4 matrices, and results need to be interpreted modulo a scaling freedom of the homogeneous embedding. In oriented projective geometry [12], this scaling factor is constrained to be positive, since this represents the practical geometry of computer graphics and machine vision more accurately.

In the homogeneous coordinate framework, Euclidean rigid body motions are treated as a special case of projective transformations. Within geometric algebra, such motions have been represented differently, as a special case of *conformal* transformations. The resulting versor representation of such motions in $\mathbb{R}^{4,1}$, and the accompanying blade representation of sensible Euclidean primitives such as spheres, circles, planes, tangent vectors etc. is beginning to prove very useful. The main advantage over classical approaches of this conformal geometric algebra (CGA) is the automatic structure preservation by the versor representation of the object composition and intersection operations (see e.g. [2]). The resulting simplicity of expression appears to be worth the extra dimensionality of the data representation (for though the 32 dimensions are ultimately all used, all meaningful elements are sparse) [3].

The success of the CGA framework makes it natural to wonder if such a structure-preserving versor representation could not also be found for 3D projective transformations. Since there are 15 degrees of freedom in the homogeneous 4×4 matrices (modulo the scaling freedom), one would expect that the exponentials of the 15 independent bivectors of a suitably chosen six-dimensional space could generate all corresponding versors.

Recently, two possibilities have been explored for a versor representation of projective geometry, both based on existing ideas.

- In [4], Goldman et al. use the algebra of $\mathbb{R}^{4,4}$ to transcribe the 4×4 matrices of the homogeneous representation into versor form, following the general framework on balanced algebras and the suggestion to use it for projective geometry in [1]. They manage to express the most common primitive projective transformations as rotors using explicitly given bivectors. But they also have to perform some non-conventional constructions to incorporate other desired operations like the (non-invertible) stereographic projection. They have reflections as certain versors, but in an unoriented manner (as we will explain in this paper).

Though [4] explores the expression and transformation of quadrics by bivectors (after an unpublished idea by Parkin), in this representation quadrics are not subspaces (as one might have hoped from the analogy of how spheres are represented in CGA). As a consequence, the great advantage of a versor representation (structure preservation of composition of primitives, as in CGA) appears not to materialize—in any case it is not explored in [4]. Also, the dimensionality of the representation in $\mathbb{R}^{4,4}$ seems much too high; there is no explanation of what

the other $\binom{8}{2} - 15 = 13$ bivectors and their corresponding rotors mean, let alone the 256 basis blades of various grades.

- In [6], Klawitter generates the projective automorphisms of Klein's quadric in the projective space $P^5(\mathbb{R})$ by versors of the Clifford algebra $\mathbb{R}^{3,3}$, and relates this to the representation of the 4×4 homogeneous coordinate matrices of projective transformations in 3D. He shows how to convert a versor from $\mathbb{R}^{3,3}$ to a 4×4 matrix, and vice versa. The correspondence is worked out in terms of solving equations of multivector coordinates, and gives little geometrical insight. In particular, the bivector generators of projective transformations, which would provide meaningful parametrizations for practical use, are not included.

Klawitter finds that even versors correspond to projective collineations, and odd versors to projective correlations, and that any regular projective transformation can be constructed from at most six projective correlations.

Geometrically, $\mathbb{R}^{3,3}$ is the space of lines (and some more, as we will see). Klawitter treats $\mathbb{R}^{3,3}$ as a projective space, and therefore ignores the orientation of lines, which makes his results only preliminarily useful to applications, where computations with oriented rays are mandatory (it does allow him some leeway to modify the versor product, permitting the representation of certain non-invertible matrices by rotor-like elements, though not uniquely). For the geometric interpretation of the blades of $\mathbb{R}^{3,3}$, Klawitter refers to the classical work on linear line complexes (see e.g. [10] for an accessible explanation).

In the present paper, we give the $\mathbb{R}^{3,3}$ -bivectors of the primitive projective transformations, thus bringing the $\mathbb{R}^{3,3}$ model of Klawitter closer to the applied flavor exhibited in the Goldman paper on $\mathbb{R}^{4,4}$. The intended practical use, with its emphasis on meaningful orientation of lines, guides our treatment. As with [6], the representation of scenes in terms of lines would be most natural to $\mathbb{R}^{3,3}$, but requires further study to do effectively. Meanwhile, the classical scene representation in terms of points and planes can definitely be supported, since they are naturally included as 3-blades of $\mathbb{R}^{3,3}$.

To make the paper self-contained and readable to practitioners, in the following section we briefly revisit the geometric decomposition of projective transformations in the matrix approach. Then in Sect. 2 we introduce the space of lines $\mathbb{R}^{3,3}$, and endow it with a metric. Studying the effect of projective transformations on lines in Sect. 3, we find that we will unfortunately not be able to represent all projective collineations (reflections are excluded, though we get 'projective correlations' in return). After revisiting the line transformation matrices in Sect. 4, we then explicitly expose the geometric parameterization by bivectors of the rotors of the primitive projective collineations in Sect. 5. Precisely what $\mathbb{R}^{3,3}$ and its orthogonal group $O(3,3)$ allow is explored in Sect. 6. We find that there are useful projective transformations definable in the space of lines that do not correspond to projective transformations of points. Among them are 'oriented reflections', missed in both [4] and [6]. We have no space to treat the line-containing blades fully,

but show how 3-blades can be used to implement 3D points and 3D planes in Sect. 7. The framework for 3D illuminates how to represent projective transformations in some other dimensions, and those are indicated in Sect. 8. In the conclusions of Sect. 9 we discuss how this paper informs the comparison between the $\mathbb{R}^{3,3}$ and $\mathbb{R}^{4,4}$ frameworks. An important new development in the analysis of $\mathbb{R}^{3,3}$ is flagged in the Postscript.

1.2. Decomposition of Projective Transformations

Since a 3D projective transformation can be computed in homogeneous coordinates by means of a 4D linear transformation, its decomposition (or factorization) in terms of primitives can be based on any of the standard matrix decompositions. With the special nature of the extra dimension in the homogeneous coordinate embedding, it is natural to consider the 4×4 matrix as a block matrix, with a 3×3 block A and a 1×1 block δ on the diagonal. Parametrizing the corresponding off-diagonal blocks by vectors \mathbf{b} and \mathbf{c} , the standard *block LU decomposition* is:

$$[H] = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c}^\top & \delta \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{c}^\top A^{-1} & \delta - \mathbf{c}^\top A^{-1} \mathbf{b} \end{bmatrix} \begin{bmatrix} A & \mathbf{b} \\ \mathbf{0}^\top & 1 \end{bmatrix}. \tag{1}$$

In the homogeneous setting, a vector representing a 3D point can be rescaled by a factor α without changing its interpretation; therefore the matrix $[H]$ has a multiplicative degree of freedom. Multiplying $[H]$ by α changes its determinant by α^4 and therefore preserves its sign. Hence any invertible homogeneous coordinate matrix $[H]$ can be rescaled to make its determinant equal to ± 1 without changing the reality it represents. Note that this remains true if one would restrict meaningful rescaling to only positive factors.

The leftmost homogeneous coordinate matrix in the factorization (Eq. 1) is the *perspectivity*¹ P parameterized by a focal plane characterized by the row vector $[\mathbf{f}^\top \ f_0]$:

$$[P] = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{f}^\top & f_0 \end{bmatrix}.$$

Its determinant is $f_0 = \delta - \mathbf{c}^\top A^{-1} \mathbf{b}$; we will soon standardize to $f_0 = 1$ without loss of generality.

The rightmost factor in Eq. (1) is an affine transformation, with determinant equal to $\det(A)$. It is common practice in computer vision (see e.g. [5, p. 42]) to split that affine transformation into a Euclidean similarity (with scaling factor γ , orthogonal transformation R and translation vector \mathbf{t}), and an upper triangular matrix U (giving skewing and non-isotropic scaling):

$$[H] = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{f}^\top & f_0 \end{bmatrix} \begin{bmatrix} \gamma R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}.$$

We will split the affine factor somewhat differently later on, using rotations, translations, ‘squeezes’ (hyperbolic rotations a.k.a. Lorentz transformations, or ‘scissor shears’), and anisotropic scalings. Together with the perspectivity above, those will form our primitive projective transformations.

¹ A perspectivity is a projective transformation fixing the points of a plane (called its *axis*), and leaving invariant all the planes through a point (called its *center*).

Perspective projection will of course not be included in our versor framework, since it is non-invertible and therefore cannot be an orthogonal transformation. However the important *pseudo-perspective* transformation, transforming the scene frustrum in an invertible manner into a standard cube for Z-buffering, *is* included, if we take the viewing direction as positive \mathbf{e}_3 -axis (rather than the conventional negative \mathbf{e}_3). When performing this trivial sign-correction, the pseudo-perspective matrix is of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha - 1 & -\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where α is a positive number related to the frustrum dimensions. This matrix can be decomposed in the manner described above in terms of the primitive operations. The sign correction on the z renders the determinant of the pseudo-perspective matrix positive (and we will learn below why this is important).

2. The Metric Space of Lines $\mathbb{R}^{3,3}$

2.1. Homogeneous Coordinates and Their Grassmann Space V^4

Homogeneous coordinates employ the 4D homogeneous model V^4 of the 3D Euclidean space \mathbb{R}^3 , by augmenting \mathbb{R}^3 with one more dimension. When the extra coordinate in that dimension is zero, the V^4 -vector that results is interpreted as a Euclidean direction; when the extra coordinate is non-zero, the V^4 -vector is interpreted as a (weighted) point. Effectively, the embedding of \mathbb{R}^3 into V^4 in this manner augments our space with *ideal points* (points at infinity, also known as improper points), which act as direction vectors (or displacement vectors) in the 3D space. We can use the join of vectors in the augmented space to represent the lines in 3D; this has the advantage of also modeling the lines at infinity (possible horizons of planes), which we will call *ideal lines* (they are also known as improper lines). Non-ideal elements may be called *finite*.

This V^4 is the well-known space of homogeneous coordinates. It represents the points of our 3D space as 4D vectors. We usually consider that 3D space as having a Euclidean metric so that we can sensibly measure oriented angles and oriented distances in our applications. Yet V^4 itself is not necessarily a metric space. Since it does not have a geometrically meaningful metric to assign, and we will *not* endow it with a geometric algebra (in contrast to [4], who views it as a Euclidean space). But V^4 does have a *Grassmann algebra*, with outer product, so we can make blades, and study the proportionality of parallel blades. We will denote the outer product in V^4 by \wedge , to distinguish it explicitly from the outer product in our space of interest $\mathbb{R}^{3,3}$, for which we employ the usual \wedge -notation. The 2-blades formed in this manner from two representatives of 3D points can be used to represent oriented lines in a 3D space. This is in principle a non-metric construction; yet we will want to perform metric measurements on the properties of the resulting 3D lines

(such as their relative angles). As we will soon see, the proportionality of blades in V^4 permits such quantitative measures.

Using e_0 for the V^4 -vector representing the point at the origin, and using bold font for purely Euclidean vectors (both in the 3D space and its embedded copy within V^4), a weighted point at location \mathbf{p} is represented as the vector $p = \alpha (e_0 + \mathbf{p})$. The line through two weighted points at \mathbf{p} and \mathbf{q} with weights α and β is then represented by the bivector

$$L = p \wedge q = \alpha\beta (e_0 \wedge (\mathbf{q} - \mathbf{p}) + \mathbf{p} \wedge \mathbf{q}) = \alpha\beta (e_0 \wedge \mathbf{u} + \mathbf{p} \wedge \mathbf{u}), \tag{2}$$

where $\mathbf{u} \equiv \mathbf{q} - \mathbf{p}$. If we are consistent about the embedding of homogeneous points by choosing their weights to have the same sign, as is customary in applications [12], the resulting line L is oriented in the direction \mathbf{u} (and clearly distinguished from a line $-L$ through p running in direction $-\mathbf{u}$).

The coefficients of L on the basis:

$$\{e_0 \wedge \mathbf{e}_1, e_0 \wedge \mathbf{e}_2, e_0 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1, \mathbf{e}_1 \wedge \mathbf{e}_2\} \tag{3}$$

are the *Plücker coordinates* of the line; this matches the usual definition, see e.g. [2] and below. A bivector A of $\wedge^2 V^4$ represents a line in 3D (finite or ideal) iff

$$A \wedge A = 0, \tag{4}$$

which is called the Grassmann–Plücker relation. For the line L of Eq. (2) this evaluates to the purely 3D spatial relation $\mathbf{u} \wedge (\mathbf{p} \wedge \mathbf{u}) = 0$, which indeed holds trivially. In terms of the coordinates l_{ij} of L on the basis of Eq. (3), relation (Eq. 4) reads:

$$\ell_{01}\ell_{23} + \ell_{02}\ell_{31} + \ell_{03}\ell_{12} = 0 \tag{5}$$

This demand on the coordinates of an element given on the basis Eq. (3) to be a line is called the (Grassmann–)Plücker condition. As Eq. (4) shows, this condition depends only on the outer product in V^4 ; but we will use the equivalent expression Eq. (5) to endow $\wedge^2 V^4$ with an inner product. Thus $\wedge^2 V^4$ will be a metric space even though V^4 itself remains non-metric.

The Plücker condition is independent of the weight or orientation of the line, so while much of the theory based on that condition remains valid for oriented projective geometry, additional structure about line geometry will be revealed by permitting only positive rescaling in the interpretation, or even no rescaling at all if the weights are deemed relevant.

2.2. Plücker Coordinates of Oriented Lines and Their Space $\mathbb{R}^{3,3}$

For a space of oriented lines (and of more, as we will see), we switch to the six-dimensional space of the Plücker coordinates and call it $\mathbb{R}^{3,3}$ for (metric) reasons which will become clear below. The six basis 2-blades of Eq. (3) in V^4 are now embedded as vectors of $\mathbb{R}^{3,3}$, and we reflect this in the notation. In an obvious one-to-one correspondence with Eq. (3), we write a natural vector basis of $\mathbb{R}^{3,3}$ as:

$$\{ \nu_{01}, \nu_{02}, \nu_{03}, \nu_{23}, \nu_{31}, \nu_{12} \}. \tag{6}$$

The ν notation (Greek n) serves to remind us that these are null vectors of $\mathbb{R}^{3,3}$ (as we will explain below).

An oriented line in \mathbb{R}^3 is represented as a 2-blade L in the Grassmann algebra of V^4 ; its embedding as the vector ℓ of $\mathbb{R}^{3,3}$ will be denoted by the function $\ell = \text{Em}(L)$, with $\text{Em}() : \wedge^2 V^4 \rightarrow \mathbb{R}^{3,3}$, defined as just copying the corresponding coordinates ℓ_{ij} from $\wedge^2 V^4$. For such a vector of $\mathbb{R}^{3,3}$ to represent a line in the original 3D space, these coordinates must satisfy the Plücker condition (Eq. 5). This $\text{Em}()$ is clearly an invertible embedding, we denote its inverse as $\text{Em}^{-1}()$.

For later use, let us spell out $\text{Em}()$ for the finite oriented line (Eq. 2), in the traditionally convenient matrix notation. A point at location \mathbf{p} is represented as a multiple of the unit-weight homogeneous point representation vector $[\mathbf{p}^\top 1]^\top$ on the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, e_0\}$ for V^4 (placing the e_0 -component last, as is customary in the treatment of homogeneous coordinates). Then the weighted oriented line Eq. (2) is represented in $\mathbb{R}^{3,3}$ as:

$$\text{Em} \left(\begin{bmatrix} \alpha \mathbf{p} \\ \alpha \end{bmatrix} \wedge \begin{bmatrix} \beta \mathbf{q} \\ \beta \end{bmatrix} \right) = \alpha \beta \text{Em} (e_0 \wedge \mathbf{u} + \mathbf{p} \wedge \mathbf{u}) = \alpha \beta \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \times \mathbf{u} \end{bmatrix}, \quad (7)$$

where $\mathbf{u} = \mathbf{q} - \mathbf{p}$ and \times denotes the 3D cross product. The coefficients of the 6D vector are given on the ordered basis (Eq. 6).

Mathematically, the set of vectors of $\mathbb{R}^{3,3}$ representing lines (vectors satisfying the Plücker condition) forms a five-dimensional quadric, called the *Klein quadric* (see e.g. [10]). Within oriented projective geometry we will of course not consider this projectively equivalent, by arbitrary rescaling, to a four-dimensional submanifold of a five-dimensional projective space $P^5(\mathbb{R})$, and thus we depart from the traditional treatment also followed by [6].

By Eq. (4), lines are 2-blades of $\wedge^2 V^4$. When considering *general* elements on the 6D basis of bivectors in V^4 given by Eq. (3), we will also consider them as vectors in $\mathbb{R}^{3,3}$, simply by applying our mapping $\text{Em}()$ to copy their coordinates to the null basis Eq. (6) in $\mathbb{R}^{3,3}$. The coordinates of such elements do not necessarily satisfy the Plücker condition, so they are not interpretable as lines. Thus calling $\mathbb{R}^{3,3}$ the space of lines is a misnomer; only a subset, the Klein quadric, deserves to be identified as ‘a manifold of weighted oriented lines’; but ‘space of lines’ is a usefully sloppy mnemonic.

2.3. The Metric Space $\mathbb{R}^{3,3}$

We are going to turn this 6D space $\mathbb{R}^{3,3}$ into a metric space, to provide it with a geometric algebra. This means that we need to define an inner product for it. We let that be inspired by the non-metric outer product nature of V^4 (a Grassmann space), in a well-chosen correspondence (see [7]):

$$\cdot : \mathbb{R}^{3,3} \times \mathbb{R}^{3,3} \mapsto \mathbb{R} : a \cdot b = [\text{Em}^{-1}(a) \wedge \text{Em}^{-1}(b)]. \quad (8)$$

This defines the inner product between elements of the 6D space $\mathbb{R}^{3,3}$ on the left, in terms of a ‘bracket’ taken in the (non-metric) Grassmann algebra $\wedge^2 V^4$ on the right. The bracket of a 4-blade of V^4 is defined as the scalar proportionality factor with the pseudoscalar $I_4 = e_0 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. Thus the bracket is effectively the coordinate of the 4-blade on the basis of 4-vectors specified by I_4 . We therefore use only *ratios of volumes* in V^4 (which may be considered as volume measures in 3D space, as we show below), to

induce a metric of 3D lines, and then employ that as a metric for the vectors representing them in $\mathbb{R}^{3,3}$.

We briefly make the geometry of this inner product explicit for 2-blades of V^4 , and the corresponding weighted oriented lines in the actual 3D space (see also [7]).

- For two finite unit-weight lines $L = (e_0 + \mathbf{p}) \wedge \mathbf{u}$ and $M = (e_0 + \mathbf{q}) \wedge \mathbf{v}$, the value of the inner product $\text{Em}(L) \cdot \text{Em}(M)$ is $[(e_0 + \mathbf{p}) \wedge \mathbf{u} \wedge (e_0 + \mathbf{q}) \wedge \mathbf{v}] = [e_0 \wedge \mathbf{u} \wedge (\mathbf{q} - \mathbf{p}) \wedge \mathbf{v}]$, which is equal to $\det([\mathbf{u} \ (\mathbf{q} - \mathbf{p}) \ \mathbf{v}])$ relative to the Euclidean unit volume element with appropriately chosen orientation $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. Hence this inner product is the (relative) volume spanned by the direction vector \mathbf{u} of L , a relative position vector $(\mathbf{q} - \mathbf{p})$ from a point \mathbf{p} on L to a point \mathbf{q} on M , and the direction vector \mathbf{v} of M . This inner product thus quantifies a combination of metric dissimilarity in position and direction in the Euclidean 3D space: for lines with perpendicular signed distance δ and unit direction vectors making a signed angle ϕ , the inner product is equal to $\delta \sin(\phi)$. We can easily measure distances of perpendicular lines, and angles between lines with unit distance, but *parallel or intersecting lines have a zero inner product*.
- For a finite unit-weight line $L = (e_0 + \mathbf{p}) \wedge \mathbf{u}$ and an ideal line $M = \mathbf{v} \wedge \mathbf{w}$, we obtain $\text{Em}(L) \cdot \text{Em}(M) = \det[\mathbf{u} \ \mathbf{v} \ \mathbf{w}]$, the volume spanned by their directions (this quantity is proportional to the sine of the angle that \mathbf{u} makes with the $\mathbf{v} \wedge \mathbf{w}$ -plane).
- The embeddings of two ideal lines in 3D have an inner product equal to zero. Generally, intersecting lines (finite or ideal) have a zero inner product—and ideal lines always intersect in a common 1-direction (ideal point).
- In this paper, we will give no geometric interpretation to non-lines and their inner product (they can be related to screws and twists).

Under the metric defined by Eq. (8), the *null vectors* (i.e., the vectors a satisfying $a \cdot a = 0$) in the 6D space are precisely identified with lines in \mathbb{R}^3 , due to the Grassmann–Plücker relation (Eq. 4). Reference [7] proves that this metric defined by Eq. (8) is nondegenerate: if $a \cdot b = 0$ for all $b \in \mathbb{R}^{3,3}$, then $a = 0$. Moreover, the 6D space has the metric structure of $\mathbb{R}^{3,3}$. An orthonormal basis to expose explicitly its $\mathbb{R}^{3,3}$ nature is the *unit vector basis* defined as:

$$\{ \epsilon_1, \epsilon_2, \epsilon_3, \bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3 \} \equiv \left\{ \frac{\nu_{01} + \nu_{23}}{\sqrt{2}}, \frac{\nu_{02} + \nu_{31}}{\sqrt{2}}, \frac{\nu_{03} + \nu_{12}}{\sqrt{2}}, \frac{\nu_{01} - \nu_{23}}{\sqrt{2}}, \frac{\nu_{02} - \nu_{31}}{\sqrt{2}}, \frac{\nu_{03} - \nu_{12}}{\sqrt{2}} \right\}. \tag{9}$$

We have $\epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = 1$ and $\bar{\epsilon}_1^2 = \bar{\epsilon}_2^2 = \bar{\epsilon}_3^2 = -1$. (As a practical insight speeding up hand computations: any repeated index in an outer product in V^4 leads to a zero contribution to the inner product in $\mathbb{R}^{3,3}$, and signs are established from even/odd permutations relative to the standard order.)²

² Should you wish to implement this paper, there are some handy coordinate-free constructions inspired by [1]. We can introduce a *complementation bivector* $C = \epsilon_1 \wedge \bar{\epsilon}_1 + \epsilon_2 \wedge \bar{\epsilon}_2 + \epsilon_3 \wedge \bar{\epsilon}_3 = \nu_{23} \wedge \nu_{01} + \nu_{31} \wedge \nu_{02} + \nu_{12} \wedge \nu_{03}$ to find algebraically the ‘complement’ of

The inner product multiplication table on the unit basis shows the (3, 3) signature of the space, thus explaining our notation $\mathbb{R}^{3,3}$:

·	ϵ_1	ϵ_2	ϵ_3	$\bar{\epsilon}_1$	$\bar{\epsilon}_2$	$\bar{\epsilon}_3$
ϵ_1	1	0	0	0	0	0
ϵ_2	0	1	0	0	0	0
ϵ_3	0	0	1	0	0	0
$\bar{\epsilon}_1$	0	0	0	-1	0	0
$\bar{\epsilon}_2$	0	0	0	0	-1	0
$\bar{\epsilon}_3$	0	0	0	0	0	-1

Note that these orthonormal basis vectors for $\mathbb{R}^{3,3}$ do not represent lines—they are clearly not null. The ‘standard’ null basis of Eq. (6)

$$\{ \nu_{01}, \nu_{02}, \nu_{03}, \nu_{23}, \nu_{31}, \nu_{12} \}$$

does consist of representatives of lines: three orthogonal oriented lines through the origin in the coordinate directions, and three orthogonal oriented lines at infinity (three ideal lines, which may be thought of as orthogonal great circles on the celestial sphere of V^4 containing the horizons of the coordinate planes). The inner product multiplication table on the null basis is:

·	ν_{01}	ν_{02}	ν_{03}	ν_{23}	ν_{31}	ν_{12}
ν_{01}	0	0	0	1	0	0
ν_{02}	0	0	0	0	1	0
ν_{03}	0	0	0	0	0	1
ν_{23}	1	0	0	0	0	0
ν_{31}	0	1	0	0	0	0
ν_{12}	0	0	1	0	0	0

This can be encoded as a metric matrix $[M]$. Two lines A, B intersect (possibly in a point at infinity) iff $0 = \text{Em}(A) \cdot \text{Em}(B) = [\text{Em}(A)]^\top [M] [\text{Em}(B)]$, where the latter form implements the inner product of $\mathbb{R}^{3,3}$ in terms of a matrix representation of its vectors on the null basis. A bivector A of $\wedge^2 V^4$ is a line iff $0 = \text{Em}(A) \cdot \text{Em}(A) = [\text{Em}(A)]^\top [M] [\text{Em}(A)]$.

2.4. Projective Duality

There is a projective duality \star (denoted here by a five-pointed star) between points and planes in V^4 . Computationally, duality merely copies the $\{e_0, e_1, e_2, e_3\}$ coordinates of a point to the trivector basis $\{e_1 \wedge e_2 \wedge e_3, -e_0 \wedge e_2 \wedge e_3, -e_0 \wedge e_3 \wedge e_1, -e_0 \wedge e_1 \wedge e_2\}$. Weighted points thus become oriented planes. Mathematically, duality is seen as a mapping from V^4 to its dual space $V^{4\star}$ of 1-forms. These 1-forms can act on elements of V^4 to produce a scalar; their geometric interpretation is as 3D planes.

Footnote 2 continued

an element: $\epsilon_1 = \bar{\epsilon}_1 \cdot C$, $\bar{\epsilon}_1 = \epsilon_1 \cdot C$, etc. The null basis elements are eigenvectors of this complementation operation: $\nu_{01} = \nu_{01} \cdot C$, etc. and $\nu_{23} = -\nu_{23} \cdot C$ etc. Another useful construction is the \underline{M} -mapping introduced in Eq. (15).

The effect of this duality is that a 2-blade of $\wedge^2 V^4$ representing the join of two points P and Q becomes (minus) the 2-blade of $\wedge^2 V^4$ representing the meet of the two planes P^* and Q^* . Transferring to $\mathbb{R}^{3,3}$, this is merely a linear transformation of lines, transforming the basis vector ν_{01} according to: $\nu_{01} = \text{Em}(e_0 \wedge e_1) \leftrightarrow \text{Em}((e_1 e_2 e_3) \cap (-e_0 e_2 e_3)) = \text{Em}(-e_2 \wedge e_3) = \nu_{23}$ etc. Effectively, ‘taking the projective dual’ amounts to the permuting coordinates of the representative vector of the line:

$$\ell_{01} \leftrightarrow \ell_{23}, \quad \ell_{02} \leftrightarrow \ell_{31}, \quad \ell_{03} \leftrightarrow \ell_{12}.$$

From this induced change of basis, we see that the $\mathbb{R}^{3,3}$ -matrix of the projective duality is equal to the metric matrix $[M]$ above, so that (with square brackets denoting matrices):

$$[\text{Em}(A^*)] = [M][\text{Em}(A)].$$

This transformation of lines extends naturally to non-lines on the same basis (Eq. 6), and hence to the entire $\mathbb{R}^{3,3}$ -space. We will denote it by \underline{M} (with matrix $[M]$); note that $\det(\underline{M}) = -1$, and $\underline{M}^2 = 1$.³

This induced transformation \underline{M} is an orthogonal transformation, since it preserves the bracket-based inner product: $\text{Em}(A) \cdot \text{Em}(B) = [A^* \wedge B^*] = [((A^* \cdot B)^*)] = [(B \cdot A^*)^*] = [((B \wedge A)^*)^*] = [B \wedge A] = [A \wedge B] = \text{Em}(A) \cdot \text{Em}(B)$. As a consequence, we need not give \underline{M} special attention; we will meet \underline{M} in the course of constructing versors of arbitrary projective transformations in the $\mathbb{R}^{3,3}$ setting. (Spoiler alert: \underline{M} ’s versor will be $M \equiv \bar{e}_1 \bar{e}_2 \bar{e}_3$, see Eq. (15).)

3. Projective Transformations as Versors

Having used the structure of V^4 to define a metric and a useful basis for $\mathbb{R}^{3,3}$ in the previous chapter, we need no longer consider V^4 , but can focus fully on the space $\mathbb{R}^{3,3}$. Its null vectors (i.e., the vectors on the Klein quadric) are representations of oriented lines in 3D, and if we need to distinguish finite and ideal lines, we can do so using the distinguished 3-blades $O = \nu_{01} \wedge \nu_{02} \wedge \nu_{03}$ (the origin) and $H = \nu_{23} \wedge \nu_{31} \wedge \nu_{12}$ (the horizon). If required, projective duality in V^4 is implemented by the mapping \underline{M} in $\mathbb{R}^{3,3}$.

We now proceed to investigate which projective transformations in 3D space correspond to orthogonal transformations of $\mathbb{R}^{3,3}$. Those that do can then be represented by means of versors, i.e. as a product of vectors, to be used in a sandwiching product to transform general elements of the geometry in a structure-preserving manner. Since there is some variation in how to define the sandwiching product for versors, let us state in advance that we will use:

$$X \mapsto (-1)^{\text{grade}(X)\text{grade}(V)} V X V^{-1}. \tag{10}$$

³ Using \underline{M} , we can relate the classical 1-form view of the inner product of two vectors ℓ and m in $\mathbb{R}^{3,3}$ (see [10]) with our metric formulation: $\ell^*(m) = \ell^* \cdot_E m = [\ell]^\top [M]^\top [m] = [\ell]^\top [M][m] = \ell \cdot m$, where \cdot_E denotes the evaluation of the 1-form ℓ^* on the vector m as effectively a Euclidean inner product of their coordinates.

This convention appears to be the standard usage in geometric algebra and in spinor literature such as [9]. For us, the sign is important, since we want to treat the lines as oriented. Note that in [6], which does not treat lines as oriented, a non-standard sandwiching product is used that differs in sign and in the avoidance of the inverse, so as to allow null elements to be versor-like in their sandwiching action. However, this non-standard usage also blurs the distinction between an even versor and a rotor. It is easily checked that with Eq. (10) as definition, a versor with an even number of vector factors generates a mapping with determinant $+1$, and a versor with an odd number of vector factors generates a mapping with determinant -1 , independent of the signature of the vectors.

For an invertible vector v used as versor, the sandwiching product implements a reflection of the representational space. Any orthogonal transformation in n -D can be represented by at most n reflections (this is Cartan–Dieudonné’s theorem), and hence by a versor of at most grade n . In GA parlance, a normalized even versor R is called a *rotor* if it satisfies $R\tilde{R} = 1$ (where \tilde{R} is the reverse of the versor, the geometric product of its factors in reverse order); rotors can be written as the exponential of a bivector (modulo a sign, and not necessarily uniquely). They represent ‘rotations’ of the representational space which can be done by ‘degrees’ from the identity transformation. To be a rotor, a versor must be even, normalized, and not contain an odd number of vectors of negative signature.

Projective transformations of 3D space are traditionally divided into projective collineations and projective correlations. As [6] showed, this distinction between the two reveals itself in $\mathbb{R}^{3,3}$ as that between even and odd versors. Yet as we analyze this in detail for our definition (Eq. 10), we find that there is a further subclassification we need to make, based on the sign of the determinant of the projective transformation in V^4 .

3.1. Projective Collineations

A (projective) collineation in 3D is representable in the homogeneous coordinate representation as a linear transformation of V^4 . A point remains a point (though possibly improper), and hence an oriented line joining two points transforms to an oriented line joining their images. We would call this a ‘projective transformation in 3D’ in computer vision; in mathematical texts like [10], it would be called ‘a projective automorphism of P^3 ’, the projective space associated with our 3D space, though it is then unfortunately customary to ignore the signs of the correlated embedded points and hence the consistent orientation of the lines they produce.

A linear transformation f on V^4 induces an outer-morphism on its Grassmann algebra. A join $P \wedge Q$ of points P and Q transforms to $f(P \wedge Q) = f(P) \wedge f(Q)$, so f naturally affects the 2-blades and hence the bivectors in $\wedge^2 V^4$. In this manner, it induces a linear transformation F in $\mathbb{R}^{3,3}$ through the embedding: $F(\text{Em}(\)) = \text{Em}(f(\))$. We define $\overline{\text{Em}}[\]$ as the operation mapping functions; then $F(\) = \overline{\text{Em}}[f](\) = \text{Em}(f(\text{Em}^{-1}(\)))$. The determinants of f and $\overline{\text{Em}}[f]$ are directly related:

$$\det(\overline{\text{Em}}[f]) = \det(f)^3. \tag{11}$$

[This is most easily seen for a mapping f that can be diagonalized. Then its determinant in V^4 is the product of the eigenvalues $\det(f) = \lambda_0\lambda_1\lambda_2\lambda_3$. Each eigenvector participates in forming an eigen-2-blade with the three remaining vectors, and after conversion to $\mathbb{R}^{3,3}$ these eigen-2-blades become eigenvectors of $\overline{\text{Em}}[f]$ with eigenvalues $\lambda_0\lambda_1, \lambda_0\lambda_2$, etc.; therefore $\det(\overline{\text{Em}}[f]) = (\lambda_0\lambda_1)(\lambda_0\lambda_2) \cdots (\lambda_1\lambda_2) = (\lambda_0\lambda_1\lambda_2\lambda_3)^3 = \det(f)^3$.]

The inner product of the line representatives may be affected by $F = \overline{\text{Em}}[f]$. We compute, for $a = \text{Em}(A)$ and $b = \text{Em}(B)$:

$$\begin{aligned} F(a) \cdot F(b) &= F(\text{Em}(A)) \cdot F(\text{Em}(B)) \\ &= \text{Em}(f(A)) \cdot \text{Em}(f(B)) \\ &= [f(A) \wedge f(B)] \\ &= [f(A \wedge B)] \\ &= \det(f) [A \wedge B] \\ &= \det(f) \text{Em}(A) \cdot \text{Em}(B) \\ &= \det(f) a \cdot b. \end{aligned} \tag{12}$$

It follows that when we restrict ourselves to transformations f for which $\det(f) = 1$, their representation F as a mapping of $\mathbb{R}^{3,3}$ onto itself is an orthogonal transformation. These are the mappings that especially interest us in this paper; let us call them *special collineations*. As we have seen in Sect. 1.2, only mappings that have a positive determinant can be rescaled to have determinant +1. We can therefore represent only half of the invertible projective collineations as transformations in $\mathbb{R}^{3,3}$. We continue with those mappings with positive determinant; whether it is a practical loss not to have the others is discussed in Sect. 6.2.

As an orthogonal transformation of $\mathbb{R}^{3,3}$, $F = \overline{\text{Em}}[f]$ (with $\det(f) = 1$) could in principle have $\det(F) = +1$ or -1 , but from Eq. (11) only the possibility $\det(F) = 1$ remains. The 3D collineations with determinant 1 (as automorphism of V^4) therefore become special orthogonal transformations in $\mathbb{R}^{3,3}$. Among these special orthogonal transformations are the *rotors* of $\mathbb{R}^{3,3}$, and we will find that these are sufficient to represent all special projective collineations. In slightly different terms, this is the statement “All special projective transformations can be classified by their spinor generators” in [7]. Mathematically, this is known as the ‘accidental Lie group isomorphism’ between $SL(4, \mathbb{R})$ and $\text{Spin}^+(3, 3)$ (see [9, p. 160]).

3.2. Projective Correlations

In projective geometry, there is a duality between points and hyperplanes in V^4 . This has led people to set up dually corresponding structures in V^4 (the space of points, represented by vectors) and its *dual space* V^{4*} (the space of hyperplanes, represented by 1-forms or covectors). We have met the projective duality at the origin of V^4 in Sect. 2.4 and have shown that it is represented as an orthogonal transformation \underline{M} of $\mathbb{R}^{3,3}$ (with matrix $[M]$).

Under a *projective correlation* in 3D, a point is transformed into a plane, and the line that is the join of two points transforms to the line that is (minus) the meet of the two corresponding planes. Therefore a 3D-line still transforms to a 3D-line under a projective correlation. Classically (and in [6]) this is done in an unoriented manner, with arbitrarily weighted points, planes and lines, but a similar definition can be used in oriented projective geometry propagating signs consistently to obtain a transformation between oriented lines.

Any projective correlation may be represented by a linear transformation g of V^4 , followed by the projective duality at the origin to map the result to V^{4*} . As a consequence, the correspondence to orthogonal transformations of $\mathbb{R}^{3,3}$ for general projective correlations is not really different from that of collineations:

$$\begin{aligned} G(a) \cdot G(b) &= \text{Em}(g(A)^*) \cdot \text{Em}(g(B)^*) \\ &= (\underline{M}\text{Em}(g(A))) \cdot (\underline{M}\text{Em}(g(B))) \\ &= \text{Em}(g(A)) \cdot \text{Em}(g(B)) \\ &= \det(g) a \cdot b. \end{aligned} \tag{13}$$

So again, when g has $\det(g) = 1$, the mapping induces an orthogonal transformation on $\mathbb{R}^{3,3}$. Let us call such a mapping a *special correlation*. The total transformation $G = \underline{M}\overline{\text{Em}}[g]$ of g has $\det(G) = \det(\underline{M}) \det(\overline{\text{Em}}[g]) = (-1)(\det(g)^3) = -1$. Since \underline{M} and $\overline{\text{Em}}[g]$ are orthogonal transformations of $\mathbb{R}^{3,3}$, so is G . And since G 's determinant is -1 , it is representable by an odd versor in the geometric algebra of $\mathbb{R}^{3,3}$.

An example of a projective correlation versor is the vector \bar{e}_1 ; using it as a reflection versor v in the standard formulation $x \mapsto -v x v^{-1}$, we find:

$$-\bar{e}_1 \nu_{01} \bar{e}_1^{-1} = \bar{e}_1 \nu_{01} \bar{e}_1 = \nu_{23} \quad \text{and} \quad -\bar{e}_1 \nu_{23} \bar{e}_1^{-1} = \bar{e}_1 \nu_{23} \bar{e}_1 = \nu_{01}. \tag{14}$$

All other basis vectors are unchanged. The general correspondence between the representation of a correlation as a matrix mapping from V^4 to V^{4*} and the corresponding odd versor in its multivector coordinate representation may be found in [6] for Klawitter's version of the versor sandwiching product.

Using the above result for \bar{e}_1 , it is easy to verify that the projective duality mapping \underline{M} can be represented by the odd versor $M \equiv \bar{e}_1 \bar{e}_2 \bar{e}_3$:

$$\underline{M}\ell = -M \ell M^{-1} = -M \ell M. \tag{15}$$

Note that in our notation, M is the versor of the linear transformation \underline{M} with matrix $[M]$.

4. Orthogonal Matrix Representations of Projective Transformations

As a reference for the geometric algebra versor treatment, it is convenient to discuss the orthogonal matrix representation of the projective transformations in the Plücker space $\mathbb{R}^{3,3}$. Since these matrices specify how the basis vectors transform, they help to specify the corresponding versors. This section

is mostly standard background material, though the *perspectivity* of Sect. 4.3 is an uncommon primitive collineation. Since it may be confused with a perspective projection, we decided to expound its geometry in some detail.

4.1. Orthogonality of $\mathbb{R}^{3,3}$ Matrices

As we have seen, special projective transformations on \mathbb{R}^3 can be represented as orthogonal transformations on $\mathbb{R}^{3,3}$. A transformation X acting on vectors of $\mathbb{R}^{3,3}$ is *orthogonal* if it preserves the inner product of $\mathbb{R}^{3,3}$: for any vectors ℓ and m , we should have that $\ell \cdot m = (X\ell) \cdot (Xm)$. Expressing this in terms of matrices involves the metric matrix $[M]$: for all ℓ and m , $[\ell]^\top [M]^\top [m] = [\ell]^\top [X]^\top [M]^\top [X] [m]$. Using $[M]^\top = [M]^{-1} = [M]$, this yields

$$X \text{ orthogonal matrix for } \mathbb{R}^{3,3} \iff [M][X]^\top [M][X] = [I] \tag{16}$$

as orthogonality condition.

4.2. The General Linear Transformation in V^4

Applying the general transformation $\begin{bmatrix} A & \mathbf{b} \\ \mathbf{c}^\top & \delta \end{bmatrix}$ of V^4 to the elements of a line, the line representation itself transforms as well. Let us compute this for a finite line with direction \mathbf{u} passing through a point at location \mathbf{p} :

$$\begin{aligned} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \times \mathbf{u} \end{bmatrix} &= \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} \wedge \begin{bmatrix} \mathbf{u} \\ 0 \end{bmatrix} \\ &\mapsto \begin{bmatrix} A\mathbf{p} + \mathbf{b} \\ \mathbf{c}^\top \mathbf{p} + \delta \end{bmatrix} \wedge \begin{bmatrix} A\mathbf{u} \\ \mathbf{c}^\top \mathbf{u} \end{bmatrix} \\ &= \begin{bmatrix} \delta A\mathbf{u} + (\mathbf{c} \cdot \mathbf{p}) A\mathbf{u} - A\mathbf{p}(\mathbf{c} \cdot \mathbf{u}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{u}) \\ A\mathbf{p} \times A\mathbf{u} + \mathbf{b} \times A\mathbf{u} \end{bmatrix} \\ &= \begin{bmatrix} \delta[A] - \mathbf{bc}^\top & -[A][\mathbf{c}^\times] \\ [\mathbf{b}^\times][A] & \det(A)[A]^{-T} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \times \mathbf{u} \end{bmatrix}. \end{aligned} \tag{17}$$

This uses $-\mathbf{p}(\mathbf{c} \cdot \mathbf{u}) + \mathbf{u}(\mathbf{c} \cdot \mathbf{p}) = -\mathbf{c} \times (\mathbf{p} \times \mathbf{u})$, and denotes the cross product by the cross product matrix $[\mathbf{c}^\times]$, defined by $[\mathbf{c} \times \mathbf{x}] = [\mathbf{c}^\times][\mathbf{x}]$. Also, we used the well-known transformation of a cross product: $A\mathbf{p} \times A\mathbf{u} = \det(A)A^{-T}(\mathbf{p} \times \mathbf{u})$.

4.3. Perspectivities of \mathbb{R}^3 Represented in $\mathbb{R}^{3,3}$

As we have seen, the block LU-decomposition of a general projective transformation leads us to consider certain perspectivities of \mathbb{R}^3 represented by a matrix of the form: $[P] = \begin{bmatrix} I & 0 \\ \mathbf{f}^\top & f_0 \end{bmatrix}$. This $[P]$ has determinant f_0 , and its sign cannot be changed by rescaling, so we must assume $f_0 > 0$ to map to an orthogonal transformation. We can rewrite $[P] = \begin{bmatrix} I & 0 \\ \mathbf{f}^\top & f_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathbf{f}^\top & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbf{0}^\top & f_0 \end{bmatrix}$, and absorb the final factor in the affine part of the decomposition (it is an isotropic rescaling). This brings the perspectivity in the standard primitive form

$$\begin{bmatrix} I & 0 \\ \mathbf{f}^\top & 1 \end{bmatrix}, \tag{18}$$

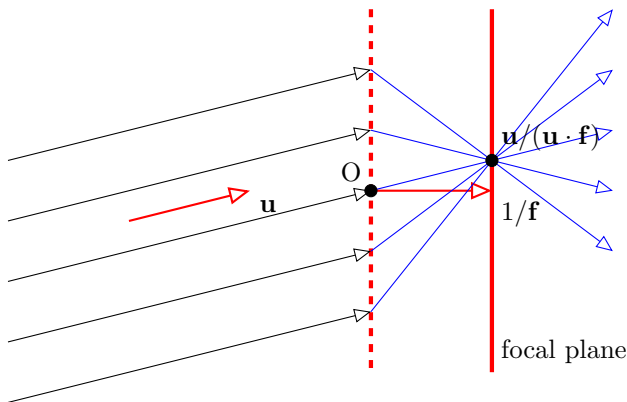


FIGURE 1. The effect of the perspectivity of Eq. (19) on oriented lines. Only parts of the lines drawn, for clarity. All planes through O are invariant (making O the ‘center’ of the perspectivity) and all points on the plane through O with normal vector \mathbf{f} are invariant (making that plane the ‘axis’)

with as corresponding matrix in $\mathbb{R}^{3,3}$:

$$\begin{bmatrix} I & -\mathbf{f}^\times \\ 0 & I \end{bmatrix}. \tag{19}$$

This perspectivity in V^4 should not be confused with a *perspective projection* used in imaging. As its name implies, perspective projection is a non-invertible transformation; in the standard matrix representation, perspective projection looks like Eq. (18) but has a 0 in the (e_0, e_0) entry.

Considering the lines as basic elements, Eq. (19) is the transformation executed on rays by a perfect lens at the origin having a focal plane with orthogonal support vector $1/\mathbf{f}$, see Fig. 1. It refracts all lines in direction \mathbf{u} where they meet the invariant plane through the origin with normal \mathbf{f} (the ‘axis’ of the perspectivity), to run through their common focal point in direction \mathbf{u} in a parallel plane with reciprocal support vector \mathbf{f} (that point is at $\mathbf{u}/(\mathbf{u} \cdot \mathbf{f})$). Considering the points on the original parallel lines of V^4 to be traversed with equal speed, the focal point is reached infinitely slowly by the transformed points (indicating that this is not physical refraction, since it changes the ‘speed of light’ along the rays). A bundle of lines with a common object point transforms to a bundle of lines with a common image point.

5. The Bivector Generators of Collineations

A projective collineation in 3D is represented by a 4×4 matrix acting on V^4 . There is a homogeneous degree of freedom; a mathematically natural way to restrict this is by demanding that the determinant be equal to 1. This

restriction implies that a nonsingular projective transformation is represented as an element of $SL(4)$, which has 15 degrees of freedom (DoF).

We now show that the special projective collineations of \mathbb{R}^3 , which are represented as special orthogonal transformations of $\mathbb{R}^{3,3}$, can be represented as rotors (even versors R satisfying $R\tilde{R} = 1$). We do so by explicitly specifying the bivectors characterizing these rotors as the exponentials of bivectors. The number of independent bivectors in $\mathbb{R}^{3,3}$ is $\binom{6}{2} = 15$, precisely matching the expected 15 degrees of freedom for the projective transformation. Therefore the mapping between special projective collineations and bivectors is one-to-one, giving the usual double cover by plus or minus the exponential of the bivectors.

The common way to partition the set of possible transformations into geometrically meaningful primitive transformations is as: translations (3 DoF), rotations (3 DoF), perspectivities (3 DoF), scalings of the axes (a.k.a. dilations, 3 DoF). The remaining 3 DoF are often assigned to shears; since there are actually 6 DoFs in shears, this is commonly done by an asymmetrical convention (using upper triangular 3×3 matrices encoding ‘skewings’). We will touch upon shears, but then ultimately prefer the 3 DoF ‘squeezes’ (Lorentz transformations) to encode the remaining degrees of freedom.

This section thus establishes the correspondence between the geometric taxonomy of the primitive transformations and their bivector representation. Figure 2 lists the 2-blades formed from these bivectors, the caption specifying the effect they have when their exponentials are used as rotors, encoded by their characteristic parameters. Figure 2 is a guide to the text in the following subsections, notably to keep track of standardized signs of the bivectors of the transformations.

5.1. Translation

Let us find the rotor for the translation over $\mathbf{t} = \tau\mathbf{e}_1$. This translation replaces e_0 by $e_0 + \tau\mathbf{e}_1$ in V^4 , and its action on $\wedge^2 V^4$ is fully defined by specifying what its effect is on the six basis lines of the bivector basis of Eq. (3). You may glean these effects from Eq. (17) (with $[A] = [I]$, $[\mathbf{c}] = [\mathbf{0}]$, $[\mathbf{b}] = \mathbf{e}_1$, $\delta = 1$). But we can also specify these effects directly: $e_0 \wedge \mathbf{e}_1$ should become $(e_0 + \tau\mathbf{e}_1) \wedge \mathbf{e}_1$, and hence remain invariant; $e_0 \wedge \mathbf{e}_2$ should become $(e_0 + \tau\mathbf{e}_1) \wedge \mathbf{e}_2$; and $e_0 \wedge \mathbf{e}_3$ should become $(e_0 + \tau\mathbf{e}_1) \wedge \mathbf{e}_3$. The ideal lines should remain unchanged.

This in turn is mapped by the embedding $\text{Em}(\cdot)$ to $\mathbb{R}^{3,3}$. To represent the translation over $\tau\mathbf{e}_1$, we would therefore like to find a rotor V_{τ_1} such that:

$$V_{\tau_1} \nu_{02} \widetilde{V}_{\tau_1} = \nu_{02} + \tau\nu_{12} \leftrightarrow (e_0 + \tau\mathbf{e}_1) \wedge \mathbf{e}_2 \tag{20}$$

$$V_{\tau_1} \nu_{03} \widetilde{V}_{\tau_1} = \nu_{03} - \tau\nu_{31} \leftrightarrow (e_0 + \tau\mathbf{e}_1) \wedge \mathbf{e}_3, \tag{21}$$

while leaving the other null basis vectors invariant. The lack of trigonometric or hyperbolic functions signifies the use of a null 2-blade, see “Appendix A”. It is easy to verify that the rotor giving the correct transformation is

$$\tau\mathbf{e}_1\text{-translation rotor: } V_{\tau_1} = 1 - \frac{1}{2}\tau\nu_{31} \wedge \nu_{12} = \exp\left(-\frac{1}{2}\tau B_{\tau_1}\right),$$

	ν_{01}	ν_{02}	ν_{03}	ν_{23}	ν_{31}	ν_{12}
ν_{01}	0	$-f_3$	f_2	$-\pi_1$	$-\sigma_{21}$	$-\sigma_{31}$
ν_{02}	f_3	0	$-f_1$	$-\sigma_{12}$	$-\pi_2$	$-\sigma_{32}$
ν_{03}	$-f_2$	f_1	0	$-\sigma_{13}$	$-\sigma_{23}$	$-\pi_3$
ν_{23}	π_1	σ_{12}	σ_{13}	0	τ_3	$-\tau_2$
ν_{31}	σ_{21}	π_2	σ_{23}	$-\tau_3$	0	τ_1
ν_{12}	σ_{31}	σ_{32}	π_3	τ_2	$-\tau_1$	0

FIGURE 2. The correspondence of bivectors to primitive transformations for 3D projective geometry. The f_i denote *perspective* transformations in direction \mathbf{e}_i , the τ_i *translations* in direction \mathbf{e}_i . The π_i denote *pinches* in the \mathbf{e}_i direction [stretching the orthogonal vectors by $\exp(-\pi_i)$]; we ultimately prefer to combine three of those to produce true scaling rotors. The σ_{ij} denotes a *shear* of \mathbf{e}_i in the \mathbf{e}_j -direction; we ultimately prefer to combine the shears over σ_{ij} and σ_{ji} , to produce *rotations* as differences of shears, and *squeezes* (Lorentz transformations) as sums of shears. The *signs* are denoted as follows: for row I with null vector ν_I , column J with null vector ν_J , with table entry ϵ_{IJ} , the 2-blade $\nu_I \wedge \nu_J$ is to be used in the rotor $V = \exp(-\frac{1}{2}\epsilon_{IJ}(\nu_I \wedge \nu_J))$ (note the *minus sign!*), mapping x to $Vx\tilde{V}$

where we introduced the \mathbf{e}_1 -translation null bivector B_{τ_1} :

$$\mathbf{e}_1\text{-translation bivector: } B_{\tau_1} = \nu_{31} \wedge \nu_{12},$$

You can verify this by Eq. (29) in ‘‘Appendix A’’, or use a direct computation, for example:

$$\begin{aligned} V_{\tau_1}\nu_{02}\widetilde{V}_{\tau_1} &= \left(1 - \frac{1}{2}\tau\nu_{31} \wedge \nu_{12}\right) \nu_{02} \left(1 + \frac{1}{2}\tau\nu_{31} \wedge \nu_{12}\right) \\ &= \nu_{02} + \tau\nu_{02} \cdot (\nu_{31} \wedge \nu_{12}) - \frac{1}{4}\tau^2(\nu_{31} \wedge \nu_{12})\nu_{02}(\nu_{31} \wedge \nu_{12}) \\ &= \nu_{02} + \tau\nu_{12} + 0. \end{aligned}$$

A general translation can be composed using similar bivectors for the \mathbf{e}_2 and \mathbf{e}_3 direction, achieved by cyclic permutation of the indices: $B_{\tau_2} = \nu_{12} \wedge \nu_{23}$ and $B_{\tau_3} = \nu_{23} \wedge \nu_{31}$. Since these bivectors all commute, so do the resulting rotors. As a consequence, we can add their bivectors, so a general translation over $\mathbf{t} = \tau_1\mathbf{e}_1 + \tau_2\mathbf{e}_2 + \tau_3\mathbf{e}_3$ is achieved by the rotor:

$$V_{\tau}(\mathbf{t}) = \exp\left(\frac{1}{2}\tau_1\nu_{12} \wedge \nu_{31} + \frac{1}{2}\tau_2\nu_{23} \wedge \nu_{12} + \frac{1}{2}\tau_3\nu_{31} \wedge \nu_{23}\right).$$

Characterizing fully within $\mathbb{R}^{3,3}$, we can define $t \equiv \text{Em}(e_0 \wedge \mathbf{t}) = \tau_1\nu_{01} + \tau_2\nu_{02} + \tau_3\nu_{03}$, and $V_{\tau}(\mathbf{t}) = \exp(-\frac{1}{2}t \cdot H)$ (where $H = \nu_{23} \wedge \nu_{31} \wedge \nu_{12}$, the horizon trivector, see Sect. 7).

5.2. Perspectivities

The perspectivity of Eqs. (18) and (19) works as in Fig. 1: perspectivity transforms an improper point (the direction \mathbf{u}) into a proper point in the same direction, with a scaling factor to make all directions end up in the same focal plane. A perspective transformation for focal length $1/f$ in the \mathbf{e}_1 -direction has the effect in V^4 (or homogeneous coordinates) to change every occurrence of \mathbf{e}_1 to $f\mathbf{e}_0 + \mathbf{e}_1$, and to leave the other basis vectors invariant.

In $\wedge^2 V^4$, this affects the ideal basis lines containing \mathbf{e}_1 , which are $\mathbf{e}_3 \wedge \mathbf{e}_1$ and $\mathbf{e}_1 \wedge \mathbf{e}_2$, changing them to $-(f\mathbf{e}_0 + \mathbf{e}_1) \wedge \mathbf{e}_3$ and $(f\mathbf{e}_0 + \mathbf{e}_1) \wedge \mathbf{e}_2$, respectively, but it affects none of the other basis lines. Therefore we are looking for a rotor in $\mathbb{R}^{3,3}$ that has the effects:

$$V_{f_1} \nu_{31} \widetilde{V}_{f_1} = \nu_{31} - f\nu_{03} \leftrightarrow -(f\mathbf{e}_0 + \mathbf{e}_1) \wedge \mathbf{e}_3 \tag{22}$$

$$V_{f_1} \nu_{12} \widetilde{V}_{f_1} = \nu_{12} + f\nu_{02} \leftrightarrow (f\mathbf{e}_0 + \mathbf{e}_1) \wedge \mathbf{e}_2 \tag{23}$$

Checking the desired scaling factor to correspond to the specific perspective change, we find that we should use as actual rotor

$$f\mathbf{e}_1\text{-perspective rotor: } V_{f_1} = \exp\left(-\frac{1}{2}fB_{f_1}\right) = 1 - \frac{1}{2}fB_{f_1},$$

with

$$\mathbf{e}_1\text{-perspective null bivector: } B_{f_1} = \nu_{03} \wedge \nu_{02}.$$

We can parameterize a general perspective transformation by a vector \mathbf{f} , with each of its components the focal length in the corresponding direction. Since the individual rotors commute, we can add their bivectors, so that the general perspective rotor is:

$$V_{\mathbf{f}} = \exp\left(\frac{1}{2}(f_1 \nu_{02} \wedge \nu_{03} + f_2 \nu_{03} \wedge \nu_{01} + f_3 \nu_{01} \wedge \nu_{02})\right).$$

Characterizing fully within $\mathbb{R}^{3,3}$, we can define $f = \text{Em}((\mathbf{e}_0 \wedge \mathbf{f})^*) = f_1\nu_{23} + f_2\nu_{31} + f_3\nu_{12}$, and $V_{\mathbf{f}} = \exp(\frac{1}{2}f \cdot O)$ (where $O = \nu_{01} \wedge \nu_{02} \wedge \nu_{03}$, the line bundle at the origin, see Sect. 7).

5.3. Shearing

A shear (a.k.a. skew) is like sliding a pack of cards: the higher the card (in a specified direction) the more it needs to move (in another specified direction), see Fig. 3a. Shearing is a fairly involved transformation: there are six elementary shears (on an orthonormal basis: 3 choices for the first direction, and for each of those 2 other directions).

The null 2-blade $B_{\sigma} = \nu_{12} \wedge \nu_{02}$ produces a shearing transformation. Since $B_{\sigma}^2 = 0$, using Eq. (29) the action of the rotor $V_{\sigma} = \exp(-\sigma_{32}B_{\sigma}/2)$ affects the null basis as follows:

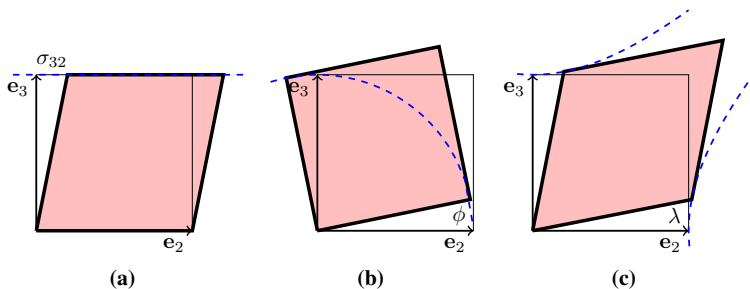


FIGURE 3. Area preserving transformations in the $\mathbf{e}_2 \wedge \mathbf{e}_3$ -plane: **a** a shear, **b** a rotation from two similar shears, **c** a squeeze from two opposite shears (‘scissor shears’ in [4])

$$\begin{aligned}
 V_\sigma \nu_{01} / V_\sigma &= \nu_{01}, \\
 V_\sigma \nu_{02} / V_\sigma &= \nu_{02}, \\
 V_\sigma \nu_{03} / V_\sigma &= \nu_{03} + \sigma_{32} \nu_{02} = \text{Em}(e_0 \wedge (\mathbf{e}_3 + \sigma_{32} \mathbf{e}_2)), \\
 V_\sigma \nu_{23} / V_\sigma &= \nu_{23} = \text{Em}(\mathbf{e}_2 \wedge (\mathbf{e}_3 + \sigma_{32} \mathbf{e}_2)), \\
 V_\sigma \nu_{31} / V_\sigma &= \nu_{31} - \sigma_{32} \nu_{12} = \text{Em}((\mathbf{e}_3 + \sigma_{32} \mathbf{e}_2) \wedge \mathbf{e}_1), \\
 V_\sigma \nu_{12} / V_\sigma &= \nu_{12}.
 \end{aligned}$$

Since all these relationships are reducible to $\mathbf{e}_3 \mapsto \mathbf{e}_3 + \sigma_{32} \mathbf{e}_2$ in V^4 , they indeed specify the shear of the \mathbf{e}_3 direction into the \mathbf{e}_2 -direction, over σ_{32} . In terms of \mathbf{e}_2 and \mathbf{e}_3 , the 2-blade can be characterized as $B_\sigma = \underline{M}[\text{Em}(e_0 \wedge \mathbf{e}_3)] \wedge \text{Em}(e_0 \wedge \mathbf{e}_2)$ (for $\underline{M}[\]$ see Eq. (15)).

There are six basic shears; to specify them we can use σ_{ij} for the amount of shear of the \mathbf{e}_i -direction into the \mathbf{e}_j direction. Note that the order matters, swapping the indices does not merely produce a change of sign! The appropriate bivectors are listed in Fig 2.

5.4. Rotation

The rotation bivector is a bit more involved. We view a rotation as two related shears in the same plane, with the same direction. The orthogonality conditions on the transformation automatically introduce a proper scaling to the transformed vectors to preserve the transformed spanned area, see Fig. 3b.

We now show that the bivector $B_{\phi_{23}}$ for a rotation in the $\mathbf{e}_2 \wedge \mathbf{e}_3$ plane is:

$$B_{\phi_{23}} = \frac{1}{2}(\nu_{02} \wedge \nu_{12} - \nu_{03} \wedge \nu_{31}) = \frac{1}{2}(\epsilon_2 \epsilon_3 - \bar{\epsilon}_2 \bar{\epsilon}_3)$$

$B_{\phi_{23}}$ is the difference of two 2-blades; such structure is a generic pattern of construction for a natural bivector basis of a balanced algebra; it was exposed in [1], and also employed in [4]. This $B_{\phi_{23}}$ is not a blade, as we can tell from its square: $B_{\phi_{23}}^2$ is not a scalar (as we would expect for a 2-blade, see [11]), but of the form scalar plus quadvector:

$$B_{\phi_{23}}^2 = -\frac{1}{2}(1 + \nu_{02} \wedge \nu_{12} \wedge \nu_{03} \wedge \nu_{31}).$$

However, one may verify that $B_{\phi_{23}}^3 = -B_{\phi_{23}}$. The algebraic properties of $B_{\phi_{23}}$ therefore still allow a grouping into trigonometric functions using Eq. (26) in ‘‘Appendix A’’:

$$e^{\phi B_{\phi_{23}}} = 1 + \sin(\phi) B_{\phi_{23}} + (1 - \cos(\phi)) B_{\phi_{23}}^2.$$

Vectors commuting with $B_{\phi_{23}}$ are unchanged by the action of this rotor in the sandwich product. But a vector b not commuting with $B_{\phi_{23}}$ (and so ‘in’ $B_{\phi_{23}}$, such as ν_{02} or ν_{12}) may be verified to satisfy the identities

$$B_{\phi_{23}} b B_{\phi_{23}} = 0 \quad \text{and} \quad B_{\phi_{23}}^2 b + b B_{\phi_{23}}^2 = -b.$$

The non-invertibility of $B_{\phi_{23}}$ means that the latter is not implied by the former. These identities affect the expansion of the application of the rotor to such a vector b , as follows [writing $B = B_{\phi_{23}}$, and $c = \cos(\phi)$ and $s = \sin(\phi)$ for short]:

$$\begin{aligned} e^{-\phi B} b e^{\phi B} &= (1 - sB + (1 - c)B^2) b (1 + sB + (1 - c)B^2) \\ &= b - s(Bb - bB) + (1 - c)(B^2b + bB^2) - s^2BbB \\ &\quad - s(1 - c)(BbB^2 - B^2bB) + (1 - c)^2 B^2 b B^2 \\ &= \cos(\phi) b + \sin(\phi) b^\perp, \end{aligned} \tag{24}$$

where we introduced a B -associated vector to b , namely $b^\perp = bB - Bb$. (Specifically for this bivector $B_{\phi_{23}}$ one may compute: $\nu_{02}^\perp = \nu_{03}$ and $\nu_{03}^\perp = -\nu_{02}$ and $\nu_{31}^\perp = \nu_{12}$ and $\nu_{12}^\perp = -\nu_{31}$, specifying what happens to all vectors linearly dependent on these four.)

Equation (24) shows that $\exp(-\phi B_{\phi_{23}})$ is the rotor for a rotation parameterized by ϕ , leaving the vectors commuting with $B_{\phi_{23}}$ invariant. With the signs and magnitude of $B_{\phi_{23}}$ as chosen, the rotation is positive in the usual sense: $\nu_{02} \mapsto \cos(\phi)\nu_{02} + \sin(\phi)\nu_{03}$ in $\mathbb{R}^{3,3}$ transfers to $\mathbf{e}_2 \mapsto \cos(\phi)\mathbf{e}_2 + \sin(\phi)\mathbf{e}_3$ in V^4 (and similarly for \mathbf{e}_3). Note that this rotor employs the full rotation angle ϕ in the exponent, rather than the required half-angle characterization of rotation rotors (quaternions) in the geometric algebra of \mathbb{R}^3 .

One can derive similar bivectors $B_{\phi_{31}}$ and $B_{\phi_{12}}$ to rotate in the other coordinate planes, by cyclic permutation of the indices. In general, a rotation over an arbitrary plane with normalized 2-blade $\mathbf{I} = b_{23}\mathbf{e}_2 \wedge \mathbf{e}_3 + b_{31}\mathbf{e}_3 \wedge \mathbf{e}_1 + b_{12}\mathbf{e}_1 \wedge \mathbf{e}_2$ (so that $b_{23}^2 + b_{31}^2 + b_{12}^2 = 1$) in the 3D Euclidean space has as its corresponding bivector $B = b_{23}B_{\phi_{23}} + b_{31}B_{\phi_{31}} + b_{12}B_{\phi_{12}}$, and rotor $\exp(-\phi B)$.

5.5. Squeeze (Lorentz Transformation, Scissors Shear)

The constituent 2-blade parts of the rotator bivector are pure shears; a rotation combines an \mathbf{e}_2 axis shear and an \mathbf{e}_3 -axis shear in the same direction. Changing the sign of their combination gives a volume-preserving ‘squeeze’ operation [13]. (In [4] this is called a ‘scissors shear’.)

The pattern of derivation is very similar to that for the rotation, with some sign changes leading to a substitution of hyperbolic functions for trigonometric functions. Since this derivation is not very long, we give it again explicitly.

The bivector $B_{\lambda_{23}}$ for a squeeze in the $\mathbf{e}_2 \wedge \mathbf{e}_3$ plane is:

$$B_{\lambda_{23}} = -\frac{1}{2}(\nu_{02} \wedge \nu_{12} + \nu_{03} \wedge \nu_{31}) = \frac{1}{2}(\epsilon_2 \bar{\epsilon}_3 - \bar{\epsilon}_2 \epsilon_3)$$

$B_{\lambda_{23}}$ is the difference of two 2-blades; such structure is a generic pattern of construction for a natural bivector basis of a balanced algebra; it was exposed in [1], and also employed in [4]. This $B_{\lambda_{23}}$ is not a blade, as we can tell from its square: $B_{\lambda_{23}}^2$ is not a scalar (as we would expect for a 2-blade, see [11]), but of the form scalar plus quadvector:

$$B_{\lambda_{23}}^2 = \frac{1}{2}(1 + \nu_{02} \wedge \nu_{12} \wedge \nu_{03} \wedge \nu_{31}).$$

However, one may verify that $B_{\lambda_{23}}^3 = B_{\lambda_{23}}$. The algebraic properties of $B_{\lambda_{23}}$ therefore still allow a grouping into hyperbolic functions using Eq. (28) in ‘‘Appendix A’’:

$$e^{\lambda B_{\lambda_{23}}} = 1 + \sinh(\lambda) B_{\lambda_{23}} + (\cosh(\lambda) - 1) B_{\lambda_{23}}^2.$$

With suitable sign changes one then computes fully analogously to Eq. (24) that on a vector b not commuting with $B = B_{\lambda_{23}}$, the rotor action is:

$$e^{-\lambda B} b e^{\lambda B} = \cosh(\lambda) b + \sinh(\lambda) b^\perp, \tag{25}$$

where we introduced a ‘ B -associated’ vector to b , namely $b^\perp = bB - Bb$. (Specifically for this bivector $B_{\lambda_{23}}$ one may compute: $\nu_{02}^\perp = \nu_{03}$ and $\nu_{03}^\perp = \nu_{02}$ and $\nu_{31}^\perp = -\nu_{12}$ and $\nu_{12}^\perp = -\nu_{31}$, specifying what happens to all the vectors linearly dependent on these four.)

Equation (25) shows that $\exp(\lambda B_{\lambda_{23}})$ is the rotor for a squeeze parameterized by λ , leaving the vectors commuting with $B_{\lambda_{23}}$ invariant. Its result on a typical element $\nu_{02} \mapsto \cosh(\lambda)\nu_{02} + \sinh(\lambda)\nu_{03}$ relates to the usual transformation $\mathbf{e}_2 \mapsto \cosh(\lambda)\mathbf{e}_2 + \sinh(\lambda)\mathbf{e}_3$ that one would associate with a positive Lorentz transformation at the origin in the $\mathbf{e}_2 \wedge \mathbf{e}_3$ -plane, see Fig. 3c.

One can derive similar bivectors $B_{\lambda_{31}}$ and $B_{\lambda_{12}}$ to squeeze in the other coordinate planes, by cyclic permutation of the indices. In general, a squeeze in an arbitrary plane with normalized 2-blade $\mathbf{I} = b_{23}\mathbf{e}_2 \wedge \mathbf{e}_3 + b_{31}\mathbf{e}_3 \wedge \mathbf{e}_1 + b_{12}\mathbf{e}_1 \wedge \mathbf{e}_2$ (so that $b_{23}^2 + b_{31}^2 + b_{12}^2 = 1$) in the 3D Euclidean space has as its corresponding bivector $B = b_{23}B_{\lambda_{23}} + b_{31}B_{\lambda_{31}} + b_{12}B_{\lambda_{12}}$, and rotor $\exp(-\lambda B)$.

5.6. Pinching: Not Quite Scaling

We saved the directional scaling to last, since this transformation is the most involved to represent. Here the determinant constraint will lead to an interaction of more elementary transformations to produce a desired scaling.

First, 2-blades capable of directional scaling should square to a non-zero positive scalar, normalized to 1. For a scaling in the \mathbf{e}_1 -direction, one might think to use the 2-blade:

$$B_{\pi_1} = \nu_{23} \wedge \nu_{01} = \epsilon_1 \bar{\epsilon}_1.$$

The corresponding rotor for an amount π_1 , which is

$$V_{\pi_1} = e^{-\pi_1 B_{\pi_1/2}} = e^{\pi_1(\nu_{01} \wedge \nu_{23})/2},$$

leaves all null basis elements unchanged except:

$$\nu_{01} \mapsto e^{\pi_1} \nu_{01}, \quad \nu_{23} \mapsto e^{-\pi_1} \nu_{23}.$$

The ν_{01} scaling seems as hoped for, and the ν_{23} scaling is obviously needed to keep the inner product $\nu_{01} \cdot \nu_{23}$ invariant.

However, despite appearances, this rotor V_{π_1} is *not* the scaling in the \mathbf{e}_1 -direction. Rather, by the transfer of matrices in Eq. (17), the $\mathbb{R}^{3,3}$ matrix $\text{diag}(e^{\pi_1}, 1, 1, e^{-\pi_1}, 1, 1)$ of this mapping of the basis vectors is the representation of the V^4 -matrix $\text{diag}(e^{\pi_1/2}, e^{-\pi_1/2}, e^{-\pi_1/2}, e^{\pi_1/2})$, which is homogeneously equivalent to $\text{diag}(1, e^{-\pi_1}, e^{-\pi_1}, 1)$. Therefore in its V^4 interpretation, the rotor V_{π_1} leaves \mathbf{e}_1 invariant and simultaneously shrinks \mathbf{e}_2 and \mathbf{e}_3 both by $e^{-\pi_1}$. This effect inspired the name ‘pinching’. We will argue below pinching is a more natural choice for a primitive projective transformation of lines than the traditional directional scaling used in a homogeneous point representation.

5.7. Directional Scaling (or Dilation)

The \mathbf{e}_1 -scaling in V^4 , in which $\mathbf{e}_1 \mapsto e^\gamma \mathbf{e}_1$ while the other basis vectors remain invariant, has matrix $\text{diag}[e^\gamma, 1, 1, 1]$. Rescaling to unit determinant, this is homogeneously equivalent to the matrix $\text{diag}[e^{3\gamma/4}, e^{-\gamma/4}, e^{-\gamma/4}, e^{-\gamma/4}]$. By Eq. (17), it corresponds to the matrix $\text{diag}[e^{\gamma/2}, e^{-\gamma/2}, e^{-\gamma/2}, e^{-\gamma/2}, e^{\gamma/2}, e^{\gamma/2}]$ in $\mathbb{R}^{3,3}$. This matrix transformation is equivalent to requiring:

$$\begin{aligned} \nu_{01} &\mapsto e^{\gamma/2} \nu_{01}, & \nu_{02} &\mapsto e^{-\gamma/2} \nu_{02}, & \nu_{03} &\mapsto e^{-\gamma/2} \nu_{03}, \\ \nu_{23} &\mapsto e^{-\gamma/2} \nu_{23}, & \nu_{31} &\mapsto e^{\gamma/2} \nu_{31}, & \nu_{12} &\mapsto e^{\gamma/2} \nu_{12}. \end{aligned}$$

These transformations of the basis vectors thus determine the rotor for the \mathbf{e}_1 -scaling by e^γ . Using the pinching results above, the coupled scaling of ν_{01} and ν_{23} by $e^{\gamma/2}$ and $e^{-\gamma/2}$, respectively, would be achieved by the 2-blade B_{π_1} in the rotor $\exp(-\gamma B_{\pi_1}/4) = \exp(\gamma \nu_{01} \wedge \nu_{23}/4)$. We need to combine this rotor with two more pinches in the \mathbf{e}_2 and \mathbf{e}_3 directions (involving analogous 2-blades $B_{\pi_2} = \nu_{31} \wedge \nu_{02}$ and $B_{\pi_3} = \nu_{12} \wedge \nu_{03}$) to perform the pure \mathbf{e}_1 -scaling. These 2-blades all commute, so they can be added to form the bivector of the total scaling rotor. Therefore the pure \mathbf{e}_1 -scaling by e^γ is done by the rotor:

$$\mathbf{e}_1\text{-scaling by } e^\gamma \text{ rotor: } V_{\gamma_1} = \exp(\gamma(\nu_{01} \wedge \nu_{23} - \nu_{02} \wedge \nu_{31} - \nu_{03} \wedge \nu_{12})/4).$$

which can be written as $S_1 = \exp(-B_{\gamma_1} \gamma/2)$ using the \mathbf{e}_1 -scaling bivector:

$$\mathbf{e}_1\text{-scaling bivector: } B_{\gamma_1} = \frac{1}{2}(\nu_{23} \wedge \nu_{01} - \nu_{31} \wedge \nu_{02} - \nu_{12} \wedge \nu_{03})$$

Note that while $B_{\pi_i}^2 = 1$ for the pinching 2-blades, we have $B_{\gamma_1}^3 = (7B_{\gamma_1} - 3I_6)/4$ for the scaling bivector (with I_6 the pseudoscalar of $\mathbb{R}^{3,3}$), so that B_{γ_1} is not a 2-blade, and also not of the next simplest bivector form satisfying $B_{\gamma_1}^3 = B_{\gamma_1}$ studied in the appendix.

A scaling in an arbitrary direction can be characterized by a total logarithmic gain vector $\mathbf{g} = \gamma_1 \mathbf{e}_1 + \gamma_2 \mathbf{e}_2 + \gamma_3 \mathbf{e}_3$. Thus arbitrary scaling may be

achieved by multiplying the rotors for the scalings in the directions of each of the axes. Since the bivectors occurring in the exponential all commute, we can group them:

$$\begin{aligned} \mathbf{g} = & \exp((\gamma_1 - \gamma_2 - \gamma_3) \nu_{01} \wedge \nu_{23} \\ & + (-\gamma_1 + \gamma_2 - \gamma_3) \nu_{02} \wedge \nu_{31} \\ & + (-\gamma_1 - \gamma_2 + \gamma_3) \nu_{03} \wedge \nu_{12})/4). \end{aligned}$$

This is rather involved! For lines, pinches appear more natural than scalings, since the bivectors of pinches are the simpler 2-blades in $\mathbb{R}^{3,3}$. But we should mention that in the $\mathbb{R}^{4,4}$ framework of [4], which operates on points and planes as geometric primitives, scalings *are* the exponentials of 2-blades like $e_1 \bar{e}_1$ in terms of their basis vectors e_1 and \bar{e}_1 .

6. Other Transformations of Oriented Lines

6.1. Non-Projective Orthogonal Transformations of Lines

It is possible to perform transformations of 3D oriented lines that are orthogonal transformations of $\mathbb{R}^{3,3}$, yet do not correspond to projective transformations of 3D space.

A concrete example is the versor $V_1 = \nu_{01} \wedge \nu_{23} = \epsilon_1 \bar{e}_1$ (not to be confused with pinching, which is the exponent of such an element). Its transformation of the basis vectors by $x \mapsto V_1 x V_1^{-1} = V_1 x V_1$ establishes that its matrix is $\text{diag}[-1, 1, 1, -1, 1, 1]$. Its determinant is equal to 1, and it is easy to check that this is an orthogonal matrix, by Eq. (16).

It is not possible to find a projective transformation of V^4 of which this is the embedding. This is easily seen by inspecting the lower right block of this matrix, which has determinant -1 . From Eq. (17) we observe that for a block diagonal projective transformation of V^4 transferred to lines, that lower right block should have determinant $\det(\det(A)[A]^{-T}) = \det(A)^2 > 0$. Therefore V_1 is a counterexample to the idea that all special orthogonal transformations of $\mathbb{R}^{3,3}$ might represent projective transformations of V^4 . A distinguishing characteristic of V_1 is that $V_1^{-1} = V_1$ while $\tilde{V}_1 = -V_1$. This even versor V is therefore not a rotor, since $V_1 \tilde{V}_1 \neq 1$.

In the previous section, we have shown that all rotors in $\mathbb{R}^{3,3}$ (written as exponentials of bivectors) are the representatives of special projective collineations of V^4 , and indeed can represent all of those transformations. Therefore all we need for our representation of (special) projective collineations is the rotor component of the collection of even versors of $\mathbb{R}^{3,3}$; that component of course also contains the identity transformation.

$V_1 = \epsilon_1 \bar{e}_1$ is an even versor that is not a rotor, yet an orthogonal transformation. Looking at the orthogonal group $O(3, 3)$ more closely, it consists of four connected components. An overview of these components and their coset characterization is given in Fig. 5. The component classifications are based on the sign of the determinant of the associated map $x \mapsto \underline{V}[x] = (-1)^{\text{grade}(V)} V x V^{-1}$, and the quantity $V \tilde{V}$, which must be equal to $+1$ for a rotor. Using these criteria, the versors can be classified in

even or odd (based on the sign of the determinant), and on whether they contain an odd or even number of vectors with negative signature (based on the sign of $R\bar{R}$). A typical example of each component is given.

The component containing the identity transformation is isomorphic to $SO(3, 3)$ —this is the component of elements that can be represented by rotors (and hence is isomorphic to $\text{Spin}^+(3, 3)$ in the notation of [9]). Another significant component is the one used to represent the special correlations, the right coset of $SO(3, 3)$ containing the versor $\bar{\epsilon}_1$, our example of Eq. (14). The operation introduced above, generated by the versor $\epsilon_1\bar{\epsilon}_1$, is in neither of these components of $O(3, 3)$; this transformation is in the right coset of $SO(3, 3)$ by the versor $\epsilon_1\bar{\epsilon}_1$. And there is a fourth component, the right coset of $SO(3, 3)$ by ϵ_1 . The practical use of the last two components is not yet clear.

6.2. Upon Reflection

The representation of reflections in $\mathbb{R}^{3,3}$ is more subtle than one would expect.

A first naive implementation would be to take the spatial reflection of points, which is an affine transformation with negative determinant, and embed that as a transformation of lines. For simplicity, consider a reflection in a plane through the origin represented in 3D by the orthogonal 3×3 matrix $[A]$ with determinant -1 . The embedding is, by Eq. (17), $\text{diag}(\delta [A], \det(A) [A]^{-T}) = \text{diag}(\delta [A], -[A])$. This matrix can only represent an orthogonal transformation if $\delta = -1$, by Eq. (12). However, the $\mathbb{R}^{3,3}$ mapping of lines is then $\text{diag}(-[A], -[A])$ with $\det(A) = -1$; this map is indistinguishable from the embedding of the mapping ‘ $-[A]$ ’ in 3D, which is an orthogonal transformation with determinant $+1$, and hence a rotation (over π radians due to its eigenvalues 1 and -1). Thus the option of setting $\delta = -1$ fails to implement a reflection, and therefore we cannot represent the reflection of 3D points as an orthogonal transformation on lines in $\mathbb{R}^{3,3}$. From the geometry of the reflection transformation, this impossibility is fully understandable: the inner product between finite lines is equal to a Euclidean volume; and under a Euclidean reflection, this volume changes sign.

So constraining a reflection of lines to be the transfer to $\mathbb{R}^{3,3}$ of a reflection of points in 3D (or V^4) does not lead to an orthogonal transformation. Rather than revert to a possible non-orthogonal implementation of reflection in $\mathbb{R}^{3,3}$, we should consider what we actually want the effect of a reflection to be, on lines. We would prefer the reflection to keep invariant certain geometrical properties of the lines and their relationships. In particular, when we reflect an object defined by facet planes with a distinguished inside/outside direction, we desire to maintain this consistently in the reflection result, so that we obtain a reflected object bounded by the reflected facet planes in a properly oriented manner.

In $\mathbb{R}^{3,3}$, a plane is represented as a field of lines, defined by a 3-blade $\ell_1 \wedge \ell_2 \wedge \ell_3$ of three pairwise intersecting lines in that plane (see Sect. 7 below). After the ordinary (point-based) reflection $[A]$ in 3D treated above, the lines acquire an orientation that makes their wedge product endow the containing plane with an orientation opposite to what would be consistent

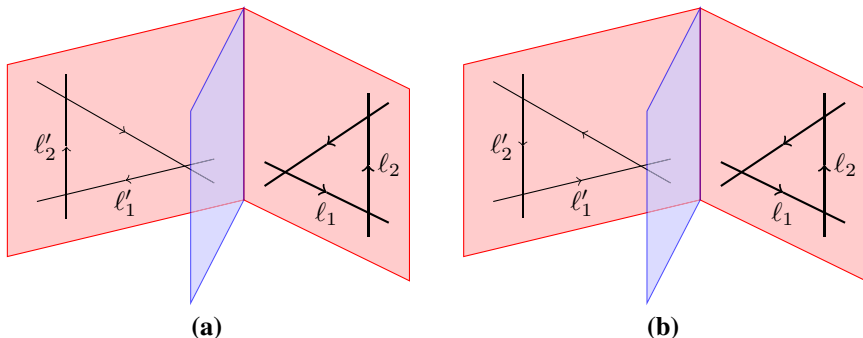


FIGURE 4. Reflection of oriented lines in a plane, $l_i \mapsto l'_i$: **a** point-based, **b** line-based. With the line-based manner of **b**, the orientation of the plane containing the lines reflects sensibly

with the reflection of the normal vector of the original plane (see Fig. 4a). Hence such a reflection would interchange the inside and outside of the plane when used as a facet. To restore the relationship of the desired orientation of the reflected plane to the outer product of the three reflected lines, we should have defined a reflection that gives each the opposite orientation of the point-based reflection (Fig. 4b). The latter transformation is implemented in $\mathbb{R}^{3,3}$ as multiplying by -1 , so the desired line-based reflection transformation in $\mathbb{R}^{3,3}$ (let us denote it by \hat{A}) should have matrix $[\hat{A}] \equiv -\text{diag}([A], \det(A) [A]^{-T}) = \text{diag}(-[A], [A])$. Since the lower right 3×3 matrix now has a negative determinant, this cannot be constructed as the embedding of a transformation on 3D points (for which that determinant is always positive as we reasoned above). This desired line reflection \hat{A} has $\det([\hat{A}]) = -1$, so \hat{A} might be an orthogonal transformation of $\mathbb{R}^{3,3}$. But since $[M][\hat{A}]^T[M][\hat{A}] = \text{diag}(-[I], -[I]) \neq \text{diag}([I], [I])$, it is not. Therefore the desired line reflection is not in the orthogonal group $O(3, 3)$ of Fig. 5, and so the desired line reflection cannot be represented by a versor of $\mathbb{R}^{3,3}$. Nevertheless, this line reflection is a legitimate linear outermorphism of $\mathbb{R}^{3,3}$, and it is good to have it.

In the $\mathbb{R}^{4,4}$ framework used for projective transformations in Goldman [4], the spatial reflection $[A]$ of points (and simultaneously of planes) can be represented as an orthogonal transformation. For example, reflection in the plane with normal e_1 , is done by $R = e_1 \bar{e}_1$ (using Goldman's basis unit vectors for $\mathbb{R}^{4,4}$). Goldman calls R a rotor, though it is not (since $R^{-1} \neq \tilde{R}$), but at least R is a versor. This reflection $[A]$ simultaneously reflects planes (so that the inner product in $\mathbb{R}^{4,4}$ is preserved), and hence $[A]$ cannot be used for an oriented projective geometry (which should covariantly preserve the sides of planes). Therefore its versor nature is actually of limited use: one could not use this versor on composite elements in a consistent orientation-preserving

connected components of $O(3, 3)$	$V \tilde{V} = 1$	$V \tilde{V} = -1$
$\det(\underline{V}) = 1$	<i>special collineations</i> $\pm \text{rotor} \times 1$ $\text{Spin}^+(3, 3) \cong \text{SO}(3, 3)$	$\pm \text{rotor} \times (\epsilon_1 \bar{\epsilon}_1)$
$\det(\underline{V}) = -1$	$\pm \text{rotor} \times \epsilon_1$	<i>special correlations</i> $\pm \text{rotor} \times \bar{\epsilon}_1$ (including duality map \underline{M})

FIGURE 5. The classification of normalized versors into the four components of $O(3, 3)$, with a typical member indicated for each class. Two components are useful for modelling 3D projective geometry: $\text{Spin}^+(3, 3) \cong \text{SO}(3, 3)$ and its right coset by $\bar{\epsilon}_1$

manner. As we have shown above, the more desirable oriented reflection cannot be represented as a point-based operation, so oriented reflection seems naturally absent from the $\mathbb{R}^{4,4}$ framework.

7. Line-Containing 3-Blades of $\mathbb{R}^{3,3}$

Much can be said about the blades of $\mathbb{R}^{3,3}$ and their interpretation as sets of lines. Since these blades are candidates for the natural geometrical primitives with which to describe real world scenes in a manner well suited to the $\mathbb{R}^{3,3}$ versor representation of the special projective transformations, this subject is highly relevant to the theme of this paper. The essence of these blades corresponds to the classical treatment of linear line complexes, linear line congruences and the like, well described and illustrated in [10]. Descriptions in terms of the geometric algebra of $\mathbb{R}^{3,3}$ may be found in [7] and more extensively in [6].

Since the present paper is long already, we will not give a full description of all blades. We provide just some intuition for the richness of the blade representation by considering the 3-blades in $\bigwedge^3 \mathbb{R}^{3,3}$ containing lines. These 3-blades will also tell us how to re-represent the usual classical elements used to describe objects in 3D projective geometry: points and planes.

An easy way to construct a 3-blade R containing lines is as the outer product of three line representatives ℓ_1, ℓ_2, ℓ_3 , so $R = \ell_1 \wedge \ell_2 \wedge \ell_3$. Any linear combination of these lines is then also contained in the 3-blade, and forms its outer product nullspace (i.e., all x such that $x \wedge R = 0$). The null elements among those are lines (they are in the intersection of the 3-blade and the Klein quadric), and the collection of these lines can be used to depict the geometric meaning of the 3-blade R in our 3D-space. We can also consider the inner product nullspace of R ; since it is identical to the outer product nullspace of its dual, it is also called the dual interpretation of R . The lines in the inner product nullspace of R intersect all three lines ℓ_1, ℓ_2 and ℓ_3 .

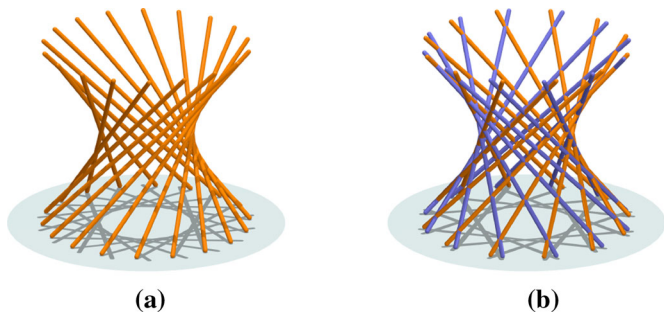


FIGURE 6. From the lecture notes of Hans Havlicek (TUWien, with permission): **a** a regulus, **b** the two dual sets of lines on a regulus

- For three mutually skew lines, the collection R is a *regulus* containing those lines, see Fig. 6a. (A regulus is a ruled surface that is a hyperboloid of revolution—a cooling tower of a nuclear power station—though in general the hyperboloid may be affinely deformed to have an elliptic principal cross section.) The dual in $\mathbb{R}^{3,3}$ of a 3-blade is again a 3-blade; in this case, the dual regulus is formed by all lines intersecting the three lines. This dual regulus is somewhat easier to visualize than the direct regulus. As point sets in 3D space, they are the same, see Fig. 6b.
- For three intersecting lines, their 3-blade is the *bundle of lines* passing through the intersection point. This 3-blade is how a 3D point may be represented in the geometric algebra of $\mathbb{R}^{3,3}$. An example of such a 3-blade is the origin blade $O \equiv \nu_{01} \wedge \nu_{02} \wedge \nu_{03}$. The 3-blade R of a bundle is a null blade (i.e., $R^2 = 0$). A bundle is self-dual: the only lines intersecting all lines in the bundle are those also passing through the intersection point.
- For three coplanar lines, the 3-blade is the *field of lines* consisting of all lines in the common plane. This 3-blade is how we may represent a 3D plane in the geometric algebra of $\mathbb{R}^{3,3}$. An example of such a 3-blade is the horizon blade $H \equiv \nu_{23} \wedge \nu_{31} \wedge \nu_{12}$. The 3-blade of a field is a null blade. A field is self-dual: the only lines intersecting all lines in the plane are those in the plane.
- When one of the three lines intersects the other two, we obtain a special case of line-containing 3-blades, the *double wheel pencil* [7, 10]. Its dual is also a double wheel pencil, and the sets of spatial points contained in their lines are the same in both. The 3-blade of a double wheel pencil is a null blade.

A full specification of how geometric algebra can re-encode practical projective geometry requires a full treatment of its blades. We leave this analysis to another time; but we can already reveal that there are no quadrics among these blades (though the quadric cross section of the regulus may offer opportunities to encode conics in 2D projective geometry).

	ν_{01}	ν_{02}	ν_{03}	ν_{23}	ν_{31}	ν_{12}
ν_{01}	0	0	f_2	$-\pi_1$	$-\sigma_{21}$	0
ν_{02}	0	0	$-f_1$	$-\sigma_{12}$	$-\pi_2$	0
ν_{03}	$-f_2$	f_1	0	0	0	0
ν_{23}	π_1	σ_{12}	0	0	0	$-\tau_2$
ν_{31}	σ_{21}	π_2	0	0	0	τ_1
ν_{12}	0	0	0	τ_2	$-\tau_1$	0

FIGURE 7. The correspondence of bivectors to primitive transformations for 2D projective geometry in the $\mathbf{e}_1 \wedge \mathbf{e}_2$ plane of a 3D space. Notation as in Fig. 2

8. Projective Geometry in Other Dimensions

The projective transformations on n D space can be represented, by means of homogeneous coordinates, as matrices of $SL(n + 1)$. Thus a projective transformation has $(n + 1)^2 - 1 = n(n + 2)$ degrees of freedom. To represent the special collineations precisely as the rotors characterized by bivectors in some m -dimensional space offers $m(m - 1)/2$ degrees of freedom.

Matching the necessary condition of equality of these counts, up to $n = 100$ the only integer solutions for (n, m) are: $(1, 3)$, $(3, 6)$, $(10, 16)$, $(22, 33)$ and $(63, 91)$. Concentrating on the lower dimensions, this solution set includes our 3D case, related to the accidental isomorphism of $SL(4)$ and $Spin^+(3, 3)$. We now see why the isomorphism is called ‘accidental’: it does not generalize to $SL(n)$ for arbitrary n . The matching count also suggests a solution to 1D which we will briefly treat below. But for the practically useful case of 2D projective transformations, there is no solution. To treat 2D projective geometry by means of rotors, we apparently have to see 2D geometry as a special case of 3D geometry.

8.1. 2D Projective Geometry in $\mathbb{R}^{3,3}$

We can consider projective transformations in 2D by concentrating on a single plane in space, and considering the subset of transformations that keep that plane and its normal direction invariant. We again limit ourselves to projective collineations with positive determinant, rescaled to be +1. There are then eight primitive transformations: two translations, two scalings, two perspective transformations, one rotation and one squeeze. These transformations can be represented as orthogonal transformations of $\mathbb{R}^{3,3}$. The bivectors of their rotors are as specified in Fig. 7.

8.2. 1D Projective Geometry in $\mathbb{R}^{3,3}$ and $\mathbb{R}^{2,1}$

Since 1D projective transformations can be represented in V^2 as $SL(2)$, they have 3 DoFs. In our terms, the degrees of freedom would be characterized as one translation parameter, one perspectivity parameter, and one directional scaling (or pinch). The remaining bivectors are as in the table of Fig. 8.

Here the use of $\mathbb{R}^{3,3}$ is overkill, even if the goal is to represent the transformations as rotors. Indeed, there is an accidental isomorphism of $SL(2)$

	ν_{01}	ν_{02}	ν_{03}	ν_{23}	ν_{31}	ν_{12}
ν_{01}	0	0	0	$-\pi_1$	0	0
ν_{02}	0	0	$-f_1$	0	0	0
ν_{03}	0	f_1	0	0	0	0
ν_{23}	π_1	0	0	0	0	0
ν_{31}	0	0	0	0	0	τ_1
ν_{12}	0	0	0	0	$-\tau_1$	0

FIGURE 8. The correspondence of bivectors to primitive transformations for 1D projective geometry, (ab) using the 3D projective geometry representation in $\mathbb{R}^{3,3}$. Notation as in Fig. 2

with the three-dimensional group $\text{Spin}(2, 1)$ which can do the representation more compactly. Since projective geometry in 1D is of very limited practical use, we do not explore this subject any further.

9. Conclusions

This paper is part of a research effort to identify a geometric algebra that has projective transformations as versors. Such an algebra would permit their encoding and application, in a structure-preserving manner, to suitable and geometrically meaningful primitive elements and their compositions. A framework of that kind would emulate the success of conformal geometric algebra for Euclidean and conformal geometry, and could standardize and simplify software for projective geometry, encoding all necessary projective operations, while making non-projectively-covariant (and hence non-geometrical) constructions impossible. That would make such a geometric algebra a dedicated and reliable programming tool.

We and others looking for this algebra have restricted ourselves to 3D. As we have seen in Sect. 8, there are mathematical reasons: the mapping between linear transformations of $(n + 1)$ -D space and orthogonal operators in some well-chosen representation m -D space does not work in general. Fortunately, we live in a 3D space, so this restriction does not limit the applicability of the results unduly.

The present attempt uses the space of oriented lines (and more) $\mathbb{R}^{3,3}$ to provide the primitive elements and operations upon them. This representation allows a geometrically meaningful metric to be introduced (based on the Plücker relation), and the versors in the resulting algebra indeed implement almost all relevant projective transformations. In particular, this representation implements as rotors the *special projective collineations* (‘special’ meaning that their homogeneous coordinate matrix has positive determinant), and as certain odd versors the *special projective correlations* (currently little used in applications). Reflections of 3D space are *not* included—reflections change the sign of the inner product, and are hence not orthogonal transformations

	issue	Goldman et al. [4]	Klawitter [6]	this paper
a	representational space	projective(?) $\mathbb{R}^{4,4}$	projective $\mathbb{R}^{3,3}$	oriented $\mathbb{R}^{3,3}$
b	standard sandwiching	almost	no	yes
c	versors employed	some	all	all
d	blades	not developed	line complexes	line complexes
e	3D modelling by	points, planes	lines	lines
f	quadrics	yes	no	no
g	limitations V^4 -map	none (?)	special only (?)	special only
h	projective correlations	no	yes	yes
i	bivector generators	yes	no	yes
j	oriented reflections	no	no	yes
k	n -D extendible?	yes	no	no
l	unusual constructions	rotor-like perspective projections	non-invertible 'versor' mappings	non- V^4 -matrix operations

FIGURE 9. Comparison of the frameworks for projective transformations as versors

of $\mathbb{R}^{3,3}$, and therefore not versors. However, our analysis revealed a means to implement an *oriented reflection* that preserves inside/outside relationships, by transforming lines directly (and that is not reducible to an operation on points or planes).

We decomposed the special projective collineations into primitive operations, and gave the bivector generators of the corresponding rotors of $\mathbb{R}^{3,3}$. These primitive operations include the usual translation, rotation, scaling along an axis, and shears. Perspectivity (or 'ideal lensing') was a bit unexpected, and we also identified relatively novel primitives like a 'squeeze' (aka Lorentz transformations or scissors shears) and a 'pinch' of lines, both of which assume a simple form in our framework. In this specificity, we expand a recent treatment by Klawitter [6] of an unoriented $\mathbb{R}^{3,3}$ to transcribe 4×4 homogeneous coordinate matrices into (freely rescalable) versors. This expansion towards practically parametrized specification permits a comparison with Goldman et al.'s [4] employment of $\mathbb{R}^{4,4}$ to construct rotor representations of important projective transformations in computer graphics.

The main points of comparison of all three frameworks are listed in the form of a table in Fig. 9. There are five main themes to organize and discuss the various differences in approach or expressiveness. We indicate the labels of the rows of the table they inform.

- *Orientability* (a, g, h) Does one consider arbitrary scaling of the homogeneous representation V^4 in which 3D points become 4D vectors? And does one allow arbitrary scalings for lines as well? Whether we do or not may sacrifice useful geometry in applications. Rays are oriented lines; oriented lines can only be constructed consistently from points with the same sign of their homogeneous scaling freedom (and even the magnitude of these factors has a meaning as the weight or mass of a point).

We have chosen to develop, as much as possible, an *oriented projective geometry* [12], since we know that it could be very convenient in

applications to get the signs right by means of direct (algebraic) computations, rather than fixing the signs afterwards.

By contrast, [6] treats both V^4 and $\mathbb{R}^{3,3}$ as projective spaces, with any non-zero scaling not affecting the 3D geometric interpretation. This scaling preserves a surprising amount of the relevant mathematical structure (such as the difference between collineations and correlations which is essentially an even/odd parity), but we had to revisit some of the results to process the consequences of the relevant orientation signs.

On the other hand, [4] does not treat this issue of signs in much detail. Its authors appear to assume the standard practice in computer graphics to embed points sensibly with positive signs, and interpret signs of constructions as signed distances to the camera plane. More research may be needed on the robustness of that approach; it appears to miss opportunities in the encoding of orientation preserving reflections, which cannot be based on point transformations, and so may not be in $\mathbb{R}^{4,4}$.

- *Orthodoxy of terms, signs and scaling factors* (a, b, c, l) A versor is a product of invertible vectors, a rotor is an even versor such that $R\tilde{R} = 1$. Both [4] and [6] allow themselves leeway in this standardized use of terms, partly because of the sign issue above. This apparent flexibility misses essential structure in the orthogonal transformations of their representative space, and hence of the projective transformations which may or may not be represented. Orientation signs are important in practice!

Non-invertible transformations cannot be represented by versors (or rotors), because they are not orthogonal transformations. Any tweaking of definitions of the versor sandwiching product, or non-linear non-covariant preparatory constructions to seemingly make this possible, leads to an unnecessary confusion. We wish those authors had not reused existing terms in a different meaning.

In the present paper, we have chosen to specify precisely what we can or cannot implement by means of the standard versors in our representational algebra of choice, before we would even think of unusual constructions.

As we mentioned, [6] treats $\mathbb{R}^{3,3}$ as a projective space, and for that reason he permits himself to redefine the sandwiching product first to neglect signs, and then to neglect even invertibility (using null versors in a sandwiching product).

Unfortunately, [4] misuses the term rotors in general (using them also for even versors that cannot be written in exponential form), and moreover makes three different constructions of perspective projection which are called rotors even though they are by design non-invertible.

- *Modelling by blades* (d, e, f) How well suited are the blades of the geometric algebra of choice to model reality? None of the three approaches answers this question fully yet, the focus has so far been on the operations themselves rather than the objects they operate on.

$\mathbb{R}^{4,4}$ can use the vectors of V^4 as 3D points and the vectors of V^{4*} as planes (though with an ungeometrical inner product in each of

those constituent spaces, only the mixed inner product is interpretable). While doing so has the advantage of corresponding directly to the customary usage of homogeneous coordinate vectors, the meaning of the composition into blades of these elements has not been spelled out fully. As [4] indicates, some of the mixed bivectors appear to be usable as quadrics (though composition of quadrics is not treated, and may not be based on standard meet and join operations).

Both [6] and we refer to the literature on linear line complexes (such as [10]) for the interpretation and usage of the blades of $\mathbb{R}^{3,3}$. But while we know that these blades will transform covariantly under our versors, how to use them in practical modelling needs to be spelled out explicitly to see how easy these blades are to use in practice. For now, we can only refer to [10]. At least $\mathbb{R}^{3,3}$ does have points and planes as null trivectors, to correspond to classical representations; yet the modelling possibilities are presumably richer.

- *Expressiveness and efficiency* (c, f, h, i, j, k) Clearly the algebra of the rotors of projective transformations based on $\mathbb{R}^{4,4}$ in [4] is very similar to ours, since these algebras ultimately generate the same transformations. There are some subtle differences (rotations must be based on bivectors satisfying $B^3 = -B$ in our case, in their case in the algebra $\mathbb{R}^{4,4}$ the corresponding primitives can be based on the 2-blades satisfying the simpler $B^2 = -1$). Goldman et al. [4] shows how to incorporate non-oriented reflections in 3D as blade-based versors.

But the main difference between the $\mathbb{R}^{3,3}$ and $\mathbb{R}^{4,4}$ approaches is the dimensionality. The $\binom{6}{2}$ -dimensional bivector basis of $\mathbb{R}^{3,3}$ precisely matches the 15 degrees of freedom of the 4×4 homogeneous coordinate matrices classically representing the 3D projective transformations (with the subtlety that we can only represent matrices with positive determinant). In that sense $\mathbb{R}^{3,3}$ fits like a glove. By contrast, the sensible approach of [4] to view the 4×4 matrices as linear mappings and then use the general framework of [1] for balanced algebras to encode these linear transformations by rotors requires many more dimensions to generate these 15-degrees-of-freedom transformations, and almost half of its $\binom{8}{2}$ -dimensional manifold of rotors remains unused. The orthogonal transformations of $\mathbb{R}^{4,4}$ are simply too rich for 3D projective geometry, and permit non-projectively-covariant expressions. Therefore the $\mathbb{R}^{4,4}$ framework requires the checking on projective relevance of constructions that one had hoped to make intrinsic by a picking a perfectly suited algebra.

With respect to our oriented $\mathbb{R}^{3,3}$ framework relative to the unoriented $\mathbb{R}^{3,3}$ of [6], we already made the point above that oriented rays are essential to applications. While much of the essential structure is the same, the expressiveness and precision of maintaining the orientation signs makes the oriented framework preferable.

- *Novel possibilities* (e, j, l) The embedding of projective geometry into geometric algebra is not done just for art's sake. The richer formal structure might permit novel possibilities to include necessary but extraneous

aspects of practical projective geometry in a structural and integrated manner. We see signs of such possibilities in all three approaches.

Klawitter [6] showed how closely related correlations and collineations really are in the $\mathbb{R}^{3,3}$ framework. The underuse of correlations in literature on machine vision and computer graphics perhaps needs reconsidering. Klawitter [6] also shows how (for unoriented lines) certain non-invertible operations may become tractable by modifying the sandwiching product.

In the present paper, we found a way to implement consistently oriented reflections using the line representation (though not as versors). This implementation could structure and simplify the inside/outside consistency tests in geometric modelling software.

With its much larger space $\mathbb{R}^{4,4}$, [4] may offer many more possibilities. The mixed bivectors from $V^4 \wedge V^{4*}$ to represent quadrics may be a mere glimpse of its capabilities. Still, the extreme overkill of the dimensionality of the representational space and its geometric algebra implies that there are going to be many non-projectively-covariant constructions, as we asserted above.⁴

In summary, our augmentation of the $\mathbb{R}^{3,3}$ framework in [6] to treat oriented projective geometry shows that $\mathbb{R}^{3,3}$ is essentially as expressive as the core of the $\mathbb{R}^{4,4}$ model in giving a structure-preserving rotor representation of primitive projective collineations. $\mathbb{R}^{3,3}$ does so using a much smaller 6-D rather than 8-D representational space (or 64-D rather than 256-D if one counts the dimensions of the geometric algebra), and exhaustively employs all rotors in that space. Even all these rotors of $\mathbb{R}^{3,3}$ do not suffice to generate all transformations that may be of interest, as we found discussing spatial reflection. In $\mathbb{R}^{4,4}$, reflection is a versor; but $\mathbb{R}^{3,3}$ offers the much better possibility to define an *oriented* reflection directly on lines.

To conclude, we believe that oriented $\mathbb{R}^{3,3}$ is a more natural candidate framework for structure-preserving projective geometry than $\mathbb{R}^{4,4}$. How convenient $\mathbb{R}^{3,3}$ really is in practice will depend on the development of sensible object representation methods using its blades.

10. Postscript: New Developments in $\mathbb{R}^{3,3}$

After submission and approval for publication of this paper, at the Barcelona AGACSE conference in July 2015 Lei Huang presented an extended mathematical analysis of $\mathbb{R}^{3,3}$. This work is currently available on arXiv [8]. By augmenting the rotor concept, he apparently manages to incorporate the anti-orthogonal oriented reflections into the framework, and to clarify the components of the augmented $O(3,3)$ and their covers. Thus [8] appears to confirm and complete the theoretical foundations for the $\mathbb{R}^{3,3}$ approach to

⁴ Since we believe that the rather contrived non-covariant $\mathbb{R}^{4,4}$ constructions of [4] for perspective projection should be replaced by more classical covariant geometric algebra projection operators, we have not investigated their $\mathbb{R}^{3,3}$ counterparts in this paper.

oriented projective geometry, resolving most of the weaknesses we mentioned, while not affecting any of its strengths.

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Appendix A. Exponentials of Bivectors as Degree 2 Polynomials

A bivector squaring to a scalar is always a 2-blade, see [11]. The rotor of a 2-blade can be organized into polynomials of degree 1 in the 2-blade, weighted by linear, hyperbolic or trigonometric functions of the scalar strength of the 2-blade.

The bivectors we encounter in this paper may not be blades, and therefore do not necessarily square to a scalar. However, we find that all our bivectors B (except the scaling bivector of Sect. 5.7) fall into one of three classes for their normalized bivector B , namely $B^3 = B$ (sometimes even $B^2 = 1$), $B^3 = -B$ (sometimes even $B^2 = -1$), or $B^2 = 0$. The exponentials of such bivectors can still be written in terms of trigonometric, linear or hyperbolic functions of ϕ weighting polynomials of the bivectors, now of degree 2. We show this property for the three cases.

- $B^3 = -B$: The algebraic properties of B then allow for an obvious grouping into trigonometric functions:

$$\begin{aligned}
 e^{\phi B} &= 1 + \frac{1}{1!}\phi B + \frac{1}{2!}\phi^2 B^2 + \frac{1}{3!}\phi^3 B^3 + \frac{1}{4!}\phi^4 B^4 + \dots \\
 &= 1 + \frac{1}{1!}\phi B + \frac{1}{2!}\phi^2 B^2 - \frac{1}{3!}\phi^3 B - \frac{1}{4!}\phi^4 B^2 + \dots \\
 &= 1 + B^2 + \left(\frac{1}{1!}\phi - \frac{1}{3!}\phi^3 + \dots\right) B - \left(1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 + \dots\right) B^2 \\
 &= 1 + \sin(\phi) B + (1 - \cos(\phi)) B^2.
 \end{aligned}
 \tag{26}$$

The special case for a blade B with $B^2 = -1$ is the familiar $e^{\phi B} = \cos(\phi) + \sin(\phi) B$.

- $B^2 = 0$: The algebraic properties of B then allow for a grouping into a linear function of the parameter:

$$e^{\tau B} = 1 + \tau B.
 \tag{27}$$

- $B^3 = B$: The algebraic properties of B then allow for a grouping into hyperbolic functions.

$$\begin{aligned}
 e^{\lambda B} &= 1 + \frac{1}{1!}\lambda B + \frac{1}{2!}\lambda^2 B^2 + \frac{1}{3!}\lambda^3 B^3 + \frac{1}{4!}\lambda^4 B^4 + \dots \\
 &= 1 + \frac{1}{1!}\lambda B + \frac{1}{2!}\lambda^2 B^2 + \frac{1}{3!}\lambda^3 B + \frac{1}{4!}\lambda^4 B^2 + \dots \\
 &= 1 - B^2 + \left(\frac{1}{1!}\lambda + \frac{1}{3!}\lambda^3 + \dots \right) B + \left(1 + \frac{1}{2!}\lambda^2 + \frac{1}{4!}\lambda^4 + \dots \right) B^2 \\
 &= 1 + \sinh(\lambda) B + (\cosh(\lambda) - 1) B^2.
 \end{aligned}
 \tag{28}$$

The special case for a blade B with $B^2 = 1$ is the familiar $e^{\lambda B} = \cosh(\lambda) + \sinh(\lambda) B$.

We can summarize these cases for $B^3 = \sigma B$ with $\sigma \in \{-1, 0, 1\}$ as:

$$e^{\alpha B} = 1 + s_\sigma(\alpha)B + \sigma(c_\sigma(\alpha) - 1)B^2,$$

with appropriate choice of $s_\sigma(\alpha)$ and $c_\sigma(\alpha)$ as trigonometric, linear or hyperbolic functions of the characteristic parameter α :

$$s_{-1}(\alpha) = \sin(\alpha), \quad s_0(\alpha) = \alpha, \quad s_1(\alpha) = \sinh(\alpha),$$

$$c_{-1}(\alpha) = \cos(\alpha), \quad c_0(\alpha) = 1, \quad c_1(\alpha) = \cosh(\alpha).$$

Note that $c_\sigma(\alpha)^2 - \sigma s_\sigma(\alpha)^2 = 1$ relates the two functions in all cases. When B is a 2-blade, we have $B^2 = \sigma$. In that case, some straightforward rewriting shows that the rotor sandwich product on a vector x can be expressed compactly as:

$$e^{-\alpha B} x e^{\alpha B} = x + 2s_\sigma(\alpha) e^{-\alpha B} (x \cdot B). \tag{29}$$

The second term on the right is therefore the *chord* from x to its transformed version.

References

- [1] Doran, C., Hestenes, D., Sommen, F., Van Acker, N.: Lie groups as spin groups. *J. Math. Phys.* **34**(8), 3642–3669 (1993)
- [2] Dorst, L., Fontijne, D., Mann, S.: *Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry*. Morgan Kaufman, Burlington (2009)
- [3] Fontijne, D.: *Efficient Implementation of Geometric Algebra*. PhD thesis, University of Amsterdam, Amsterdam (2007)
- [4] Goldman, R., Mann, S.: $\mathbb{R}^{4,4}$ as a computational framework for 3-dimensional computer graphics. *Adv. Appl. Clifford Algebra* **25**(1), 113–149 (2015)
- [5] Hartley, R.I., Zisserman, A.: *Multiple View Geometry in Computer Vision*, 2nd edn. Cambridge University Press, Cambridge (2003) (ISBN 0521540518)
- [6] Klawitter, D.: A Clifford algebraic approach to line geometry. *Adv. Appl. Clifford Algebra* **24**, 713–736 (2014)

- [7] Li, H., Zhang, L.: Line geometry in terms of the null geometric algebra over $\mathbb{R}^{3,3}$, and application to the inverse singularity analysis of generalized Stewart platforms. In: Dorst, L., Lasenby, J. (eds.) *Guide to Geometric in Practice*, pp. 253–272. Springer, New York (2011)
- [8] Li, H., Huang, L., Shao, C., Dong, L.: *Three-Dimensional Projective Geometry with Geometric Algebra* (2015). [arXiv:1507.06634v1](https://arxiv.org/abs/1507.06634v1)
- [9] Porteous, I.R.: *Clifford Algebras and the Classical Groups*. Cambridge university Press, Cambridge (1995)
- [10] Pottmann, H., Wallner, J.: *Computational Line Geometry*. Springer, New York (2001)
- [11] Sabadini, I., Sommen, F.: Clifford analysis on the space of vectors, bivectors and ℓ -vectors. In: Qian, T., Hempfling, T., McIntosh, A., Sommen, F. (eds.) *Advances in Analysis and Geometry*. Trends in Mathematics, pp. 161–185. Birkhäuser, Basel (2004)
- [12] Stolfi, J.: *Oriented Projective Geometry*. Academic Press, London (1991)
- [13] Artzy, R.: *Linear Geometry*, 3rd edn., p. 94. Dover Publications (2008). See also Wikipedia entry on ‘Squeeze Mapping’ (2015)

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