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Crossing estimates for symmetric Markov processes

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Abstract. A crossing estimate is established for symmetric Markov processes on general state spaces.

1. Introduction

T. Lyons and W.-A. Zheng [17] used the forward-backward martingale decomposition to prove a crossing estimate for Dirichlet functions along the paths of a stationary symmetric diffusion on $\mathbf{R}^n$ over the time interval $[0, 1]$. Recently, A. Ancona, R. Lyons & Y. Peres [2] established a crossing estimate for Dirichlet functions along the paths of a transient symmetric discrete-time Markov chain or a transient symmetric diffusion on a Riemannian manifold, over the time interval $[0, \infty)$, allowing an arbitrary starting point for the process. In this paper, we show that a crossing estimate can be established for general symmetric right Markov processes – transient or recurrent, over a finite or infinite time interval, for quasi-every starting point. By averaging the starting point with respect to the symmetrizing measure, we obtain a crossing estimate that extends and sharpens the work of Lyons & Zheng [17].

Preliminary material is discussed in Section 2. Our main results are stated and proved in Section 3. To give the reader a taste of these results we now state a special case of Theorem 3.6. Let X be an irreducible symmetric strong Markov process with state space E and symmetry measure m. For $x \in E$, let $\mathbf{P}_x$ denote the law of X under the initial condition $X_0 = x$. Let $f : E \to \mathbf{R}$ be an element of the Dirichlet

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space of X. Such a function f admits a "quasi-continuous" m-version $\widetilde{f}$ with the property that $s \mapsto \widetilde{f}(X_s)$ is right-continuous, a.s. $\mathbf{P}_m$. (Here, $\mathbf{P}_m := \int_E \mathbf{P}_x\, m(dx)$.)

Theorem. *Fix real numbers $a < b$, and let U_t be the number of crossings of the interval $[a, b]$ that are completed by the process $s \to \widetilde{f}(X_s)$ during the time interval $[0, t]$. Let μ be a probability distribution on E admitting a bounded density g with respect to m. Then for any $t > 0$,*

$$\mathbf{E}_\mu U_t := \int_E \mathbf{E}_x(U_t)\, g(x)\, m(dx) \le 2t(b-a)^{-2}\mathscr{E}(f, f) \cdot \sup_{x\in E} g(x), \tag{1.1}$$

where $\mathscr{E}(f, f)$ is the "Dirichlet energy" of f.

A variant of this result, more appropriate when X is transient, appears in Corollary 3.7 (iii). The right side of (1.1) is of the correct order of magnitude (for large t) when X is positive recurrent. When X is null recurrent, it is not uncommon for the left side of (1.1) to grow like a slowly varying function of t. An illuminating discussion, in the case of planar Brownian motion, can be found in [3].

In Section 4 we provide examples for both continuous and discrete-time Markov chains and diffusions on finite- and infinite-dimensional spaces illustrating how our results contain those obtained by Ancona, Lyons & Peres in [2] as special cases.

2. Preliminaries

Let E be homeomorphic to a Borel subset of a compact metric space, and let $\mathscr{B}(E)$ denote the Borel σ-algebra on E. Let m be a σ-finite measure on $\mathscr{B}(E)$ with supp$[m] = E$. Let $X = (\Omega, \mathscr{M}, \mathscr{M}_t, \theta_t, X_t, \mathbf{P}_x)$ be an m-symmetric, irreducible right Markov process with state space E. In more detail, the right-continuous process $[0, +\infty) \ni t \mapsto X_t$ is defined on the sample space $(\Omega, \mathscr{M})$, adapted to the filtration $(\mathscr{M}_t)$, and under the law $\mathbf{P}_x$ is a strong Markov process with initial condition $X_0 = x$. The shift operators θ_t, $t \ge 0$, satisfy $X_s \circ \theta_t = X_{s+t}$ identically for $s, t \ge 0$. Adjoined to the state space E is an isolated point $\partial \notin E$; the process X retires to ∂ at its "lifetime" $\zeta := \inf\{t : X_t = \partial\}$.

The transition operators P_t, $t \ge 0$, are defined by

$$P_t f(x) := \mathbf{E}_x(f(X_t)) = \mathbf{E}_x(f(X_t); t < \zeta).$$

(Here and in the sequel, we use the convention that a function defined on E takes the value 0 at the cemetery point ∂.) The P_t may be viewed as operators on $L^2(E, m)$; as such they form a strongly continuous semigroup of self-adjoint contractions. (This is the "m-symmetry" mentioned earlier.) The associated infinitesimal generator L is defined by

$$Lf := \lim_{t\downarrow 0} t^{-1}[P_t f - f] \tag{2.1}$$

on the domain consisting of those $f \in L^2(E, m)$ for which the limit in (2.1) exists in the strong sense. The (typically unbounded) operator $-L$ is self-adjoint and positive, so it admits a (self-adjoint, positive) square root $\sqrt{-L}$. Let $\mathscr{F}$ be the domain of $\sqrt{-L}$, and define the bilinear form $\mathscr{E}$ on $\mathscr{F}$ by

$$\mathscr{E}(u, v) = (\sqrt{-L}u, \sqrt{-L}v)_{L^2(E,m)}, \quad u, v \in \mathscr{F}.$$

Then $(\mathscr{E}, \mathscr{F})$ is the *symmetric Dirichlet form* on $L^2(E, m)$ associated with the process X.

As was noted in [10], Theorems (16.19) and (16.21) of [12] imply that X is m-special standard. Therefore, by the fundamental work [1] of S. Albeverio and Z.-M. Ma, the Dirichlet form $(\mathscr{E}, \mathscr{F})$ is quasi-regular. (Conversely, it is shown in [1] that given a quasi-regular Dirichlet form on $L^2(E, m)$, there is an associated m-special standard symmetric Markov processes.) It is proved in [5] that a Dirichlet form is quasi-regular if and only if it is quasi-homeomorphic to a regular Dirichlet form on a locally compact separable metric space. Therefore, any result valid for regular Dirichlet forms and invariant under quasi-homeomorphisms is applicable to quasi-regular Dirichlet forms. This "transfer principle" will be used in the sequel without special mention.

The process X will be the object of study in the rest of the paper. In the remainder of this section we recall certain notions that will be particularly important. For further details and discussion the reader is referred to [11] and [18].

For a closed set $F \subset E$, define

$$\mathscr{F}_F := \{f \in \mathscr{F} : \ f = 0 \ m\text{-a.e. on } E \backslash F\},$$

and recall the following definitions (cf. [1], [5] and [18]).

Definition 2.1.

(i) An increasing sequence $\{F_n\}_{n\geq 1}$ of closed subsets of E is an *$\mathscr{E}$-nest* provided $\cup_{n\geq 1}\mathscr{F}_{F_n}$ is $\mathscr{E}_1$-dense in $\mathscr{F}$, where $\mathscr{E}_1 = \mathscr{E} + (\cdot, \cdot)_{L^2(E,m)}$.
(ii) A set $A \subset E$ is said to be *quasi-open* (resp. *quasi-closed*) if there is an $\mathscr{E}$-nest $\{F_k\}_{k\geq 1}$ such that $F_k \cap A$ is relatively open (resp. relatively closed) in F_k for each $k \geq 1$.
(iii) A function $u : E \to [-\infty, +\infty]$ is *quasi-continuous* provided there is an $\mathscr{E}$-nest $\{F_k\}$ such that u restricted to F_k is real-valued and continuous, for each k.
(iv) A set $N \subset E$ is *$\mathscr{E}$-polar* provided there is an $\mathscr{E}$-nest $\{F_k\}_{k\geq 1}$ such that $N \subset \cap_k(E \backslash F_k)$.
(v) A statement involving x is said to hold *quasi-everywhere* (q.e.) on E if the set of x's for which the statement fails to hold is $\mathscr{E}$-polar.

Let $\sigma_N := \inf\{t \geq 0 : X_t \in N\}$ denote the hitting time of N. It is known that an increasing sequence $\{F_n\}_{n\geq 1}$ of closed subsets of E is an $\mathscr{E}$-nest if and only if $\mathbf{P}_m(\lim_n \sigma_{E\backslash F_n} < \zeta) = 0$. In particular, N is $\mathscr{E}$-polar if and only if $\mathbf{P}_m(\sigma_N < \zeta) = 0$. See [1] and [18]. Notice that an $\mathscr{E}$-polar set is necessarily m-negligible.

Each $u \in \mathscr{F}$ has a quasi-continuous m-version $\widetilde{u}$, for which

$$\widetilde{u}(X_t) - \widetilde{u}(X_0) = M_t^u + N_t^u, \qquad t \geq 0; \tag{2.2}$$

see Theorem 5.2.2 of [11]. (Recall the convention $f(\partial) = 0$.) The process M^u is a martingale additive functional of X and N^u is a zero-energy continuous additive functional (CAF) of X. The above decomposition is unique and is called Fukushima's decomposition. M^u can be further decomposed as $M^u = M^{u,c} + M^{u,d}$, where $M^{u,c}$ is the continuous part and $M^{u,d}$ the purely discontinuous part of M^u. The quadratic variation $\langle M^{u,c}\rangle = [M^{u,c}]$ of $M^{u,c}$ is a positive continuous additive functional of X, and its Revuz measure is denoted $\mu^c_{\langle u\rangle}$. A well-known property of continuous martingales implies that if D is quasi-open and $u \in \mathscr{F}$ then

$$\mu^c_{\langle u\rangle}(D) = 0 \text{ if and only if } \widetilde{u} \text{ is constant q.e. on } D. \tag{2.3}$$

The following proposition records the (first) Beurling–Deny formula for quasi-regular Dirichlet forms; see [7] or [15]. If μ is a measure on E charging no $\mathscr{E}$-polar set, then there is a quasi-closed set F carrying μ that is minimal in the sense that if F^* is any other quasi-closed set carrying μ, then $F\backslash F^*$ is $\mathscr{E}$-polar; such an F is unique modulo $\mathscr{E}$-polars, and is called the *quasi-support* of μ. See (4.6.3) and (4.6.4) of [11].

Proposition 2.2. *The symmetric quasi-regular Dirichlet form* $(\mathscr{E},\mathscr{F})$ *can be uniquely decomposed as*

$$\mathscr{E}(u,v) = \mathscr{E}^{(c)}(u,v) + \frac{1}{2}\iint_{E\times E}[\widetilde{u}(x)-\widetilde{u}(y)]\cdot[\widetilde{v}(x)-\widetilde{v}(y)]J(dx,dy) + \int_E \widetilde{u}(x)\widetilde{v}(x)\,\kappa(dx)$$

for $u, v \in \mathscr{F}$, *with* $\mathscr{E}^{(c)}$, J *and* κ *satisfying the following conditions*

(i) $(\mathscr{E}^{(c)},\mathscr{F})$ *is a positive definite symmetric bilinear form with the strong local property:* $\mathscr{E}(u,v) = 0$ *for any* u, v *in* $\mathscr{F}$ *such that* $\widetilde{u}$ *is q.e. constant on a quasi-open superset of the quasi-support of the measure* $|v|\cdot dm$.
(ii) J *is a* σ*-finite symmetric positive measure on* $(E\times E)\backslash d$ *that charges no subset of* $(E\times E)\backslash d$ *whose marginal projection is* $\mathscr{E}$*-polar. (Here* d *denotes the diagonal of* $E\times E$*.)*
(iii) κ *is a* σ*-finite positive measure on* E *charging no* $\mathscr{E}$*-polar set.*

In fact, $\mathscr{E}^{(c)}(u,v) = \frac{1}{2}\mu^{(c)}_{\langle u,v\rangle}(E)$. *Furthermore, every normal contraction operates on the form* $\mathscr{E}^{(c)}$.

The "jumping measure" J is given by $J(dx,dy) = \nu(dx)N(x,dy)$, where (N,H) is a Lévy system for X and ν is the Revuz measure of the PCAF H. The "killing measure" is the Revuz measure of the PCAF obtained by taking the dual predictable projection of the additive functional $t\mapsto 1_{\{\zeta_i\le t\}}$, where ζ_i is the totally inaccessible part of ζ. See Section 2 of [4] and Section 4.5 of [11].

The reflected Dirichlet space $(\mathscr{E}^{\text{ref}},\mathscr{F}^{\text{ref}})$ associated with $(\mathscr{E},\mathscr{F})$ was introduced (for regular Dirichlet forms) by M. L. Silverstein [22, 23] in 1974, as a tool in his study of the boundary theory of symmetric Markov processes. The reflected Dirichlet space can be defined for a quasi-regular Dirichlet form as follows; see [16]. As in [11], define

$$\overset{\bullet}{\mathscr{F}}_{\text{loc}} = \{u : \exists \text{ an increasing sequence } \{D_n\} \text{ of quasi-open sets with } \cup_{n=1}^\infty D_n = E \text{ q.e and a sequence } \{u_n\}\subset\mathscr{F} \text{ such that } u = u_n \text{ } m\text{-a.e. on } D_n,\ \forall n\}.$$

Each $u \in \overset{\bullet}{\mathscr{F}}_{\text{loc}}$ has a quasi-continuous m-version $\widetilde{u}$ on E. By (2.3), $\mu^c_{\langle u\rangle}$ is well-defined for $u \in \overset{\bullet}{\mathscr{F}}_{\text{loc}}$ by declaring $\mu^c_{\langle u\rangle} := \mu^c_{\langle u_n\rangle}$ on D_n. For $u \in \overset{\bullet}{\mathscr{F}}_{\text{loc}}$, define

$$\mathscr{E}^{\text{ref}}(u,u) := \frac{1}{2}\mu^c_{\langle u\rangle}(E) + \frac{1}{2}\int\int_{E\times E}[\widetilde{u}(x)-\widetilde{u}(y)]^2 J(dx,dy) + \int_E \widetilde{u}(x)^2\,\kappa(dx). \tag{2.4}$$

Definition 2.2. The reflected Dirichlet space $(\mathscr{E}^{\text{ref}}, \mathscr{F}^{\text{ref}})$ associated with the quasi-regular Dirichlet space $(\mathscr{E}, \mathscr{F})$ is the set

$$\mathscr{F}^{\text{ref}} := \{u \in \overset{\bullet}{\mathscr{F}}_{\text{loc}} : \mathscr{E}^{\text{ref}}(u,u) < \infty\}$$

equipped with the norm $\mathscr{E}^{\text{ref}}$ determined by (2.4).

For example, let X be the absorbing Brownian motion on $E = (0,1)$. In this context m is Lebesgue measure and each $u \in \mathscr{F}$ is equal m-a.e to an absolutely continuous function $\widetilde{u}$ with $\widetilde{u}(0+) = \widetilde{u}(1-) = 0$ and $\widetilde{u}'$ square integrable over $(0,1)$. The Dirichlet form is given by $\mathscr{E}(u,v) = \frac{1}{2}\int_0^1 \widetilde{u}'(x)\cdot\widetilde{v}'(x)\,m(dx)$. One checks that $u \in \mathscr{F}^{\text{ref}}$ if and only if u admits an absolutely continuous (on $(0,1)$) m-version $\widetilde{u}$ with $\widetilde{u}'$ square integrable over $(0,1)$. Such a function $\widetilde{u}$ necessarily has finite limits at the endpoints of $(0,1)$. Of course, $(\mathscr{E}^{\text{ref}}, \mathscr{F}^{\text{ref}} \cap L^2(E,m))$ can be identified as the Dirichlet form of the Brownian motion on [0,1] with reflection at the endpoints.

Note that if $(\widehat{\mathscr{E}}, \widehat{\mathscr{F}})$ is a regular Dirichlet space on $L^2(\widehat{E}, \widehat{m})$ which is quasi-homeomorphic to $(\mathscr{E}, \mathscr{F})$, then $(\widehat{\mathscr{E}}^{\text{ref}}, \widehat{\mathscr{F}}^{\text{ref}})$ is the image space of $(\mathscr{E}^{\text{ref}}, \mathscr{F}^{\text{ref}})$ under the same quasi-homeomorphic map j. Thus the results established in [4], [22] and [23] for reflected Dirichlet spaces remain valid in the current setting. In particular, it follows from [4] that $\left(\mathscr{E}^{\text{ref}}, \mathscr{F}^{\text{ref}} \cap L^2(E,\, m)\right)$ is a Dirichlet form on $L^2(E,m)$.

An important related object is the *extended Dirichlet space* $(\mathscr{E}, \mathscr{F}_e)$. A function f is in $\mathscr{F}_e$ if and only if there is an $\mathscr{E}$-Cauchy sequence $\{f_n\}_{n\geq 1}$ in $\mathscr{F}$ such that f_n converges to f m-a.e. on E. The sequence $\{f_n\}_{n\geq 1}$ is called an approximating sequence for f. Let $\mathscr{E}(f,f) := \lim_{n\to\infty}\mathscr{E}(f_n,f_n)$, which, by Lemma 1.7 of [23] (or Theorem 1.5.2 of [11]), does not depend on the choice of the approximating sequence. By Theorem 2.1.7 of [11], each function $u \in \mathscr{F}_e$ has a quasi-continuous m-version $\widetilde{u}$, and by Theorem 5.2.2 of [11] the Fukushima decomposition (2.2) holds for $u \in \mathscr{F}_e$. Note that it follows from Theorem 1.9 of [23] (or Theorem 1.6.2 of [11]) that $(\mathscr{E}, \mathscr{F}_e)$ is a Hilbert space if and only if $(\mathscr{E}, \mathscr{F})$ is transient. By Lemma 1.7 of [23] and the assumed irreducibility of $(\mathscr{E}, \mathscr{F})$,

$$\text{if } u \in \mathscr{F}_e \text{ has } \mathscr{E}(u,u) = 0, \text{ then } u \text{ is constant } m\text{-a.e. on } E. \tag{2.5}$$

Silverstein ([23], Theorem 16.2) has shown that $\mathscr{F}^{\text{ref}} = \mathscr{F}_e$ when $(\mathscr{E}, \mathscr{F})$ is recurrent. When $(\mathscr{E}, \mathscr{F})$ is transient, it is known (see [4] and [23]) that the subspace $\mathscr{H} \subset \mathscr{F}^{\text{ref}}$ of harmonic functions of finite $\mathscr{E}^{\text{ref}}$-norm is $\mathscr{E}^{\text{ref}}$-orthogonal to $\mathscr{F}_e$ and that $\mathscr{F}^{\text{ref}} = \mathscr{F}_e \oplus \mathscr{H}$. A function h in $\mathscr{H}$ can be uniquely expressed as $h(x) = \mathbf{E}_x[\phi]$ for $x \in E$, where ϕ is a terminal random variable for the right Markov process associated with $(\mathscr{E}, \mathscr{F})$. A *terminal random variable* is an $\mathscr{M}$-measurable function ϕ on the sample space Ω such that (i) $\mathbf{E}_x|\phi| < \infty$ for q.e.

$x \in E$, (ii) $\phi \circ \theta_t = \phi$ on $\{t < \zeta\}$ for all $t > 0$, and (iii) $\{\phi \neq 0\} \subset \{\zeta_p < \infty\}$. (Here ζ_p is the predictable part of the lifetime ζ.) Let $\{D_n\}_{n\geq 1}$ be an increasing sequence of quasi-open sets with $\cup_{n\geq 1} D_n = E$ q.e.. It follows from Lemma 8.1 of [22] that for q.e. $x \in E$, $\{\widetilde{h}(X_{\tau_{D_n}})\}_{n\geq 1}$ is a $\mathbf{P}_x$-uniformly integrable martingale and

$$\phi = \lim_{n\to\infty} h(X_{\tau_{D_n}}), \qquad \mathbf{P}_x\text{-a.s.} \tag{2.6}$$

for q.e. $x \in E$. Here $\tau_{D_n} := \inf\{t \geq 0 : X_t \notin D_n\}$. In general, $h(X_t)$ is not a martingale. However by (1.16) of [4]

$$M_t^h := h(X_t) - h(X_0) + \phi 1_{\{t\geq\zeta\}} \quad t \geq 0, \tag{2.7}$$

is a martingale additive functional of X having finite energy; in particular, the Revuz measure of $\langle M^h\rangle$ has finite mass. Thus by Theorem 1.4 of [4], M^h is a square-integrable martingale under $\mathbf{P}_x$ for q.e. $x \in E$. Conversely, if ϕ is a terminal random variable of X and $h(x) = \mathbf{E}_x[\phi]$, then $h \in \mathscr{H}$ if and only if the Revuz measure of $\langle M^h\rangle$ has finite mass (see, e.g. [4]). Because $\mathscr{F}^{\rm ref} \subset \dot{\mathscr{F}}_{\rm loc}$, one can use Theorem (5.7)(iii) of [10] and an adaptation of Theorem 1.6 of [4] to see that

$$\lim_{t\to\zeta, t<\zeta} h(X_t) = h(X_{\zeta-})1_{\{\zeta_i<\infty\}} + \phi, \tag{2.8}$$

where, as before, ζ_i is the totally inaccessible part of ζ. (See [10; p. 301] for the fact that the left limit $X_{\zeta-}$ exists in E on $\{\zeta_i < \infty\}$.) Thus

$$M_\zeta^h = \phi - h(X_0) \ \text{ and } \ M_{\zeta-}^h = h(X_{\zeta-})1_{\{\zeta_i<\infty\}} + \phi - h(X_0). \tag{2.9}$$

For later reference we record here the analogous statement for $f \in \mathscr{F}_e$, now a direct consequence of Theorem (5.7)(iii) in [10]:

$$\lim_{t\to\zeta, t<\zeta} \widetilde{f}(X_t) = 1_{\{\zeta_i<\infty\}}\widetilde{f}(X_{\zeta-}) \qquad \mathbf{P}_x\text{-a.s. for q.e. } x \in E. \tag{2.10}$$

In combination with Fukushima's decomposition (2.2), this yields

$$M_\zeta^f - M_{\zeta-}^f = -\widetilde{f}(X_{\zeta-})1_{\{\zeta_i<\infty\}} \tag{2.11}$$

3. Crossing Estimate

Throughout this section we use A and B to denote disjoint quasi-closed subsets of E, neither of which is $\mathscr{E}$-polar. Define

$$\Gamma_{A,B} := \{f \in \mathscr{F}^{\rm ref} : \ 0 \leq \widetilde{f} \leq 1, \ \widetilde{f} = 1 \ \text{ q.e. on } A \text{ and } \widetilde{f} = 0 \ \text{ q.e. on } B\},$$

and notice that $\Gamma_{A,B}$ is convex.

Lemma 3.1. $\Gamma_{A,B}$ *is* $\mathscr{E}^{\rm ref}$*-complete.*

Proof. Suppose first that X is transient. Let $\{f_n\} \subset \Gamma_{A,B}$ be an $\mathcal{E}^{\text{ref}}$-Cauchy sequence. Then $f_n = g_n + h_n$ where $g_n \in \mathcal{F}_e$ and $h_n \in \mathcal{H}$. Because $\mathcal{F}_e$ and $\mathcal{H}$ are $\mathcal{E}^{\text{ref}}$-orthogonal, both $\{g_n\}$ and $\{h_n\}$ are $\mathcal{E}^{\text{ref}}$-Cauchy sequences. As noted just before (2.5), $\mathcal{F}_e$ is $\mathcal{E}^{\text{ref}}$-complete, so there exists $g \in \mathcal{F}_e$ with $\lim_n \mathcal{E}^{\text{ref}}(g_n - g, g_n - g) = 0$. Extracting a subsequence if necessary, we can even assume that

$$\mathbf{P}_x\Big(\lim_{n\to\infty} \sup_{t\geq 0} |\widetilde{g}_n(X_t) - \widetilde{g}(X_t)| = 0\Big) = 1 \quad \text{for q.e. } x \in E;$$

see Theorem 2.1.4 and Lemma 5.1.2 of [11]. In particular, $\widetilde{g}_n(x) \to \widetilde{g}(x)$ for q.e. $x \in E$. At the cost of extracting a further subsequence, we can use Theorem 1.4 of [4] to arrange that

$$\mathbf{P}_x(M_t^{h_n} \text{ converges uniformly in } t \in [0,\infty) \text{ as } n \to \infty) = 1 \quad \text{for q.e. } x \in E.$$

In view of (2.7) we therefore have

$$\mathbf{P}_x(\widetilde{f}_n(X_t) - \widetilde{f}_n(x) \text{ converges uniformly in } t \in [0,\zeta) \text{ as } n \to \infty) = 1 \text{ for q.e. } x \in E.$$

But $(\mathcal{E}, \mathcal{F})$ is irreducible, so by Theorem 4.6.6 of [11],

$$\mathbf{P}_x(\sigma_B < \zeta) > 0 \quad \text{for q.e. } x \in E,$$

and therefore $f(x) := \lim_n \widetilde{f}_n(x)$ exists for q.e. $x \in E$, because $\widetilde{f}_n(X_{\sigma_B}) = 0$ on $\{\sigma_B < \zeta\}$. It follows that $\lim_n \widetilde{h}_n(x) = f(x) - \widetilde{g}(x)$ for q.e. $x \in E$, and then by Lemma 3.1 of [4] that $h := f - \widetilde{g}$ is an element of $\mathcal{H}$ and that $\{h_n\}$ is $\mathcal{E}^{\text{ref}}$-convergent to h. Consequently $\{f_n\}$ is $\mathcal{E}^{\text{ref}}$-convergent to f, from which it follows that f is an element of $\Gamma_{A,B}$. We have shown that every $\mathcal{E}^{\text{ref}}$-Cauchy sequence from $\Gamma_{A,B}$ admits a convergent subsequence; this proves the $\mathcal{E}^{\text{ref}}$-completeness of $\Gamma_{A,B}$ in the transient case.

Now suppose X is recurrent, so that $\mathcal{F}^{\text{ref}} = \mathcal{F}_e$. Fix a strictly positive bounded function γ on E with $\int_E \gamma\, dm < \infty$, and let $\mathcal{F}_e^\gamma$ denote the extended Dirichlet space for the *transient* Dirichlet form $(\mathcal{E}^\gamma, \mathcal{F}^\gamma)$ on $L^2(E,m)$, where $\mathcal{F}^\gamma = \mathcal{F} \cap L^2(E,\gamma)$ and

$$\mathcal{E}^\gamma(u,v) = \mathcal{E}(u,v) + \int_E u(x)v(x)\gamma(x)\, m(dx) \quad \text{for } u, v \in \mathcal{F}^\gamma.$$

(This is the Dirichlet form for X killed at rate γ.) Because $u \in \mathcal{F}$ implies $(u \vee 0) \wedge 1 \in \mathcal{F}$, it is easy to check that $\mathcal{F}_e \cap L^\infty(E,m) \subset \mathcal{F}_e^\gamma$. Of course, $\Gamma_{A,B} \subset \mathcal{F}_e \cap L^\infty(E,m)$. Let $\{f_n\} \subset \Gamma_{A,B}$ be an $\mathcal{E}^{\text{ref}}$-Cauchy sequence. Clearly $\sup_n \mathcal{E}^\gamma(f_n, f_n) < \infty$, so by the Banach-Saks theorem there is a subsequence $\{f_{n(k)}\}$ of $\{f_n\}$ whose Cesàro means $g_k := k^{-1}\sum_{j=1}^k f_{n(j)}$ are $\mathcal{E}^\gamma$-convergent (hence also $\mathcal{E}$-convergent) to some $g \in \mathcal{F}_e^\gamma$. Because $(\mathcal{E}^\gamma, \mathcal{F}^\gamma)$ is transient, the sequence $\{g_k\}$ admits a further subsequence along which the quasi-continuous m-versions $\widetilde{g}_k := k^{-1}\sum_{j=1}^k \widetilde{f}_{n(k)}$ converge $\mathcal{E}^\gamma$-q.e. ($= \mathcal{E}$-q.e.) to $\widetilde{g}$. It follows that $g \in \Gamma_{A,B}$. Now let $\overline{f}$ denote the limit of $\{f_n\}$ in the abstract completion $\overline{\Gamma}_{A,B}$ of $\Gamma_{A,B}$ (endowed with the metric associated with $\mathcal{E}^{\text{ref}}$). Then $g_k \to \overline{f}$ in $\overline{\Gamma}_{A,B}$ as well, so $\overline{f} = g \in \Gamma_{A,B}$. Thus $\Gamma_{A,B}$ is $\mathcal{E}^{\text{ref}}$-complete in the recurrent case. □

For Theorems 3.2 and 3.3 below, we assume that $\Gamma_{A,B}$ is non-empty. Since $\Gamma_{A,B}$ is also convex and $\mathscr{E}^{\mathrm{ref}}$-complete, it admits a unique element $F = F_{A,B}$ of minimal $\mathscr{E}^{\mathrm{ref}}$-norm; this follows by standard arguments because of (2.5). If $(\mathscr{E}, \mathscr{F})$ is transient, then $F = f + h$, where $f \in \mathscr{F}_e$ and $h \in \mathscr{H}$; we set $M^F := M^f + M^h$ where M^f and M^h are defined by (2.2) and (2.7). If $(\mathscr{E}, \mathscr{F})$ is recurrent, then $F \in \mathscr{F}^{\mathrm{ref}} = \mathscr{F}_e$, in which case we let M^F be the martingale additive functional in the Fukushima decomposition (2.2) for $F \in \mathscr{F}_e$. In either case, M^F is a $\mathbf{P}_x$-square integrable martingale for q.e. $x \in E$, and we have the "Fukushima decomposition"

$$\widetilde{F}(X_t) - \widetilde{F}(X_0) = M_t^F + N_t^F - \phi 1_{\{\zeta \le t\}}, \qquad t \ge 0, \tag{3.1}$$

where N^F is a CAF of zero energy and $\phi := 1_{\{\zeta_p < \infty\}} \lim_{t \uparrow \zeta} \widetilde{F}(X_t)$ is the terminal variable associated with the harmonic part of F if X is transient, and simply 0 if X is recurrent; cf. (2.8) and (2.10). Let $[M^F]$ denote the quadratic variation of M^F and let $\langle M^F \rangle$ denote the dual predictable projection of $[M^F]$.

Theorem 3.2. *Let S and T be finite stopping times with $S \le T < \zeta_p$. Then for quasi-every $x \in E$,*

$$\mathbf{E}_x([\widetilde{F}(X_T) - 1]^2 - [\widetilde{F}(X_S) - 1]^2) \le \mathbf{E}_x(\langle M^F \rangle_T - \langle M^F \rangle_S) \tag{3.2}$$

and

$$\mathbf{E}_x(\widetilde{F}(X_T)^2 - \widetilde{F}(X_S)^2) \le \mathbf{E}_x(\langle M^F \rangle_T - \langle M^F \rangle_S). \tag{3.3}$$

Proof. Using arguments found on pp. 322–323 of [20], one sees that $\widetilde{F}$ is equal q.e. to the "condenser potential"

$$G(x) := \mathbf{P}_x(\sigma_A < \sigma_B), \qquad x \in E.$$

The function G defined above is quasi-continuous. Moreover, G is an excessive function of the part process $X^{E \setminus B}$ (X killed at σ_B); in fact, G is the equilibrium potential of A relative to $X^{E \setminus B}$. It follows that N^F in (3.1) is non-increasing on the interval $[0, \sigma_B]$ and decreases only when X is in A, $\mathbf{P}_x$-a.s. for q.e. $x \in E$. The additivity of N^F now implies that N^F is non-increasing throughout the random time set $\{t \ge 0 : X_t \notin B\}$, decreasing only on $\{t \ge 0 : X_t \in A\}$.

Because the quasi-closed sets A and B are disjoint, the quasi-left continuity of X implies that $\mathbf{P}_x(\sigma_A = \sigma_B < \zeta) = 0$ for q.e. $x \in E$. Consequently, $1 - F(x)$ is equal for q.e. x to the complementary condenser potential $H(x) := \mathbf{P}_x(\sigma_B < \sigma_A)$. But clearly

$$H(X_t) - H(X_0) = -M_t^F - N_t^F + \phi 1_{\{\zeta \le t\}}, \qquad t \ge 0,$$

$\mathbf{P}_x$-a.s. for q.e. x, so the preceding argument allows us to deduce that N^F is non-decreasing on $\{t \ge 0 : X_t \notin A\}$, increasing only on $\{t \ge 0 : X_t \in B\}$. It follows that there are positive continuous additive functionals (PCAFs) C^A and C^B such that C^A increases only when X is in A, C^B increases only when X is in B, and

$$\widetilde{F}(X_t) - \widetilde{F}(X_0) = M_t^F - C_t^A + C_t^B - \phi 1_{\{\zeta \le t\}}, \qquad t \ge 0, \tag{3.4}$$

By Itô's formula (see, e.g., Théorème VIII.27 in [6]),

$$\begin{aligned}\widetilde{F}(X_t)^2 - \widetilde{F}(X_0)^2 &= K_t - 2\int_0^t \widetilde{F}(X_s)\,[dC_s^A - dC_s^B] - \phi^2 1_{\{\zeta\le t\}} + \langle M^F\rangle_t, \\ &= K_t - 2C_t^A - \phi^2 1_{\{\zeta\le t\}} + \langle M^F\rangle_t, \qquad (3.5)\end{aligned}$$

where K is a $\mathbf{P}_x$-uniformly integrable martingale for q.e. $x \in E$. The second equality in (3.5) follows because $\widetilde{F} = 1$ q.e. on A and $\widetilde{F} = 0$ q.e. on B. Combining (3.4) and (3.5) we find that

$$[\widetilde{F}(X_t)-1]^2-[\widetilde{F}(X_0)-1]^2 = K'_t+\langle M^F\rangle_t-2C_t^B+(2\phi-\phi^2)1_{\{\zeta\le t\}}, \qquad t \ge 0,$$

$\mathbf{P}_x$-a.s. for q.e. $x \in E$, where K' is another uniformly integrable martingale. Consequently, if $S \le T < \zeta_p$ are stopping times,

$$[\widetilde{F}(X_T) - 1]^2 - [\widetilde{F}(X_S) - 1]^2 = K'_T - K'_S + \langle M^F\rangle_T - \langle M^F\rangle_S - 2(C_T^B - C_S^B),$$

from which (3.2) follows upon taking expectations. Likewise, (3.5) implies

$$\widetilde{F}(X_T)^2 - \widetilde{F}(X_S)^2 = K_T - K_S - 2(C_T^A - C_S^A) + \langle M^F\rangle_T - \langle M^F\rangle_S,$$

from which (3.3) follows upon taking expectations. □

Let U_t be the number of crossings between A and B that the process X completes by time t. More precisely, define stopping times $S_1, S_2, \dots, T_1, T_2, \dots$ by $S_1 := \sigma_A$, $T_1 := S_1+\sigma_B\circ\theta_{S_1}$, and inductively, $S_n := T_{n-1}+\sigma_A\circ\theta_{T_{n-1}}$, $T_n := S_n+\sigma_B\circ\theta_{S_n}$. Let the sequences $\hat{S}_1, \hat{S}_2, \dots, \hat{T}_1, \hat{T}_2, \dots$ be defined analogously with the roles of A and B interchanged. Now put $U_t^{A\to B} := \sup\{n : T_n \le t\}$, $U_t^{B\to A} := \sup\{n : \hat{T}_n \le t\}$, and finally $U_t := U_t^{A\to B} + U_t^{B\to A}$. (Convention: $\sup\emptyset = 0$.)

Theorem 3.3. *If T is a stopping time, then*

$$\mathbf{E}_x(U_T) \le \mathbf{E}_x\langle M^F\rangle_T$$

for q.e. $x \in E$.

Proof. Assume, for the moment, that $T < \zeta_p$. Observe that $\widetilde{F}(X_{S_n}) = 1$ on $\{S_n < \infty\}$ and $\widetilde{F}(X_{T_n}) = 0$ on $\{T_n < \infty\}$, $\mathbf{P}_x$-a.s. for q.e. $x \in E$. Thus, by (3.2),

$$\begin{aligned}\mathbf{P}_x(U_T^{A\to B} \ge n) &= \mathbf{P}_x(T_n \le T) \\ &= \mathbf{E}_x([\widetilde{F}(X_{T_n\wedge T}) - 1]^2 - [\widetilde{F}(X_{S_n\wedge T}) - 1]^2; T_n \le T) \\ &\le \mathbf{E}_x([\widetilde{F}(X_{T_n\wedge T}) - 1]^2 - [\widetilde{F}(X_{S_n\wedge T}) - 1]^2) \\ &\le \mathbf{E}_x(\langle M^F\rangle_{T_n\wedge T} - \langle M^F\rangle_{S_n\wedge T}) \qquad (3.6)\end{aligned}$$

for q.e. $x \in E$. In the same way, (3.3) leads to

$$\mathbf{P}_x(U_T^{B\to A} \ge k) \le \mathbf{E}_x(\langle M^F\rangle_{\hat{T}_k\wedge T} - \langle M^F\rangle_{\hat{S}_k\wedge T}) \qquad (3.7)$$

The desired inequality follows by combining (3.6) with (3.7) after summing as n and k vary over the positive integers – the intervals $[S_n\wedge T, T_n\wedge T]$, $[\hat{S}_k\wedge T, \hat{T}_k\wedge T]$ $(n, k \ge 1)$ have no interior points in common and their union is contained in $[0, T]$.

Turning to the general case, let $\{\sigma_k\}$ be an increasing sequence of bounded stopping times announcing the predictable stopping time ζ_p. By the preceding paragraph,

$$\mathbf{E}_x(U_{T\wedge\sigma_k}) \le \mathbf{E}_x\langle M^F\rangle_{T\wedge\sigma_k}, \qquad k \ge 1.$$

The proof is finished by sending k off to ∞ because $U_{T\wedge\sigma_k}$ increases to U_T as k increases to ∞. □

The following two lemmas record facts needed in the proof of Theorem 3.6, our main result. We write $\mu_{\langle f\rangle}$ for the Revuz measure of the PCAF $\langle M^f\rangle$.

Lemma 3.4. *For* $f \in \mathscr{F}^{\mathrm{ref}}$,

$$\mu_{\langle f\rangle}(E) = 2\mathscr{E}^{\mathrm{ref}}(f,f) - \int \widetilde{f}(x)^2\,\kappa(dx).$$

Proof. Fix $f \in \mathscr{F}^{\mathrm{ref}} \subset \dot{\mathscr{F}}_{\mathrm{loc}}$. Then there exists an increasing sequence $\{D_k\}_{k\ge1}$ of quasi-open sets with $\cup_{k=1}^{\infty} D_k = E$ q.e. and a sequence $\{g_k\} \subset \mathscr{F}$ such that $f = g_k$ a.e. on D_k. By (2.3), $M_t^{f,c} = M^{g_k,c}$ for $t \le \tau_{D_k}$, whence $[M^f]_t = [M^{g_k}]_t$ for $t < \tau_{D_k}$, for all k. Consequently,

$$[M^f]_t = [M^{f,c}]_t + \sum_{0<s\le t}\left(\widetilde{f}(X_s) - \widetilde{f}(X_{s-})\right)^2 \quad \text{for } t < \zeta.$$

At time ζ,

$$\begin{aligned}
[M^f]_\zeta &= [M^{f,c}]_\zeta + \sum_{0<s<\zeta}\left(\widetilde{f}(X_s) - \widetilde{f}(X_{s-})\right)^2 + (M^f_\zeta - M^f_{\zeta-})^2 \\
&= [M^{f,c}]_\zeta + \sum_{0<s<\zeta}\left(\widetilde{f}(X_s) - \widetilde{f}(X_{s-})\right)^2 + 1_{\{\zeta_i<\infty\}}[\widetilde{f}(X_{\zeta-})]^2,
\end{aligned}$$

the second equality following from (2.11). Therefore (see, e.g., [4]),

$$\begin{aligned}
\frac{1}{2}\mu_{\langle f\rangle}(E) &= \frac{1}{2}\mu^c_{\langle f\rangle}(E) + \frac{1}{2}\iint\left(\widetilde{f}(x) - \widetilde{f}(y)\right)^2\, J(dx,dy) + \frac{1}{2}\int \widetilde{f}(x)^2\,\kappa(dx) \\
&= \mathscr{E}^{\mathrm{ref}}(f,f) - \frac{1}{2}\int \widetilde{f}(x)^2\,\kappa(dx).
\end{aligned}$$

□

In what follows we write $G_\alpha := \int_0^\infty e^{-\alpha t} P_t\,dt$ ($\alpha > 0$) for the α-potential operator associated with X. With reference to the following lemma, see Section 5.1 of [11] for a discussion of the Revuz correspondence, which relates smooth measures on E to PCAFs of X.

Lemma 3.5. *Let μ and ν be smooth measures with associated PCAFs C^μ and C^ν. Then $\mathbf{E}_\mu C_t^\nu = \mathbf{E}_\nu C_t^\mu$ for all $t \ge 0$.*

Proof. Any smooth measure μ can be written as a sum of finite smooth measures with bounded 1-potentials. (For example, $\mu = \sum_{n=0}^{\infty} \mu_n$, where $\mu_n := 1_{F_n}\mu$, $F_n = \{(n+1)^{-1} \le \phi < n^{-1}\}$, and ϕ is as in the proof of Lemma 5.1.7 in [11].) Because $(\mu, \nu) \mapsto \mathbf{E}_\mu C_t^\nu$ is additive in either argument, it therefore suffices to consider the case in which $\mu(E) + \nu(E) < \infty$ and $\mathbf{E}_x \int_0^\infty e^{-t}\, dC_t^{\mu+\nu}$ is a q.e. bounded function of x. But then the desired equality follows by Laplace inversion from

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_\mu(C_t^\nu)\, dt &= \alpha^{-1}\mathbf{E}_\mu \int_0^\infty \mathrm{e}^{-\alpha t}\, dC_t^\nu \\ &= \alpha^{-1}\mu(G_\alpha \nu) = \alpha^{-1}\nu(G_\alpha\mu) \\ &= \alpha^{-1}\mathbf{E}_\nu \int_0^\infty \mathrm{e}^{-\alpha t}\, dC_t^\mu = \int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_\nu(C_t^\mu)\, dt,\end{aligned}$$

wherein all terms are finite if $\alpha \ge 1$. The third equality above is an instance of the "Revuz formula" and is a slight extension of formula (9.5) in [12]. □

Theorem 3.6. *Fix $f \in \mathscr{F}^{\mathrm{ref}}$ and $a, b \in \mathbf{R}$ such that $a < b$. Let U_t be the number of crossings of the interval $[a, b]$ that are completed by the process $s \to \widetilde{f}(X_s)$ during the time interval $[0, t]$. Then for any smooth measure μ on E and any $t > 0$,*

$$\mathbf{E}_\mu U_t \le 2(b-a)^{-2}\mathscr{E}^{\mathrm{ref}}(f, f)\, \|\mathbf{E}_\cdot C_t^\mu\|_\infty, \tag{3.8}$$

where C^μ is the PCAF of X with Revuz measure μ, and $\|\mathbf{E}_\cdot C_t^\mu\|_\infty$ is the $L^\infty(E, m)$-norm of $x \mapsto \mathbf{E}_x C_t^\mu$.

Proof. Let $A = \{x \in E : \widetilde{f}(x) \le a\}$ and $B = \{x \in E : \widetilde{f}(x) \ge b\}$. We assume, without loss of generality, that neither A nor B is $\mathscr{E}$-polar. Suppose first that the killing measure κ vanishes. Then $\mathscr{F}^{\mathrm{ref}}$ contains the constant functions, so $u := 0 \vee \frac{b-f}{b-a} \wedge 1$ is an element of $\mathscr{F}^{\mathrm{ref}}$ and is a normal contraction of $(b-f)/(b-a)$. Consequently, $u \in \Gamma_{A,B}$, so the minimal-energy element F of $\Gamma_{A,B}$ exists, and $\mathscr{E}^{\mathrm{ref}}(F, F) \le \mathscr{E}^{\mathrm{ref}}(u, u)$. By Theorem 3.3, $\mathbf{E}_x U_t \le \mathbf{E}_x \langle M^F \rangle_t$. Therefore (using Lemma 3.4 for the third inequality and Lemma 3.5 for the equality),

$$\begin{aligned}\mathbf{E}_\mu U_t &\le \mathbf{E}_\mu \langle M^F \rangle_t = \mathbf{E}_{\mu_{\langle F \rangle}} C_t^\mu \le \|\mathbf{E}_\cdot C_t^\mu\|_\infty\, \mu_{\langle F \rangle}(E) \\ &\le 2\|\mathbf{E}_\cdot C_t^\mu\|_\infty\, \mathscr{E}^{\mathrm{ref}}(F, F) \le 2\|\mathbf{E}_\cdot C_t^\mu\|_\infty\, \mathscr{E}^{\mathrm{ref}}(u, u) \\ &\le 2(b-a)^{-2}\|\mathbf{E}_\cdot C_t^\mu\|_\infty \mathscr{E}^{\mathrm{ref}}(f, f).\end{aligned}$$

When $\kappa \ne 0$ we proceed as follows. Let X^{res} be the Markov process obtained by "resurrecting" (repeatedly) X at its death place $X_{\zeta-}$ whenever $\zeta = \zeta_i < \infty$ (see [14], [19], and Section 18 in [23]). The Dirichlet form $(\mathscr{E}^{\mathrm{res}}, \mathscr{F}^{\mathrm{res}})$ of X^{res} is related to $(\mathscr{E}, \mathscr{F})$ by

$$\mathscr{F} = \{u \in \mathscr{F}^{\mathrm{res}} : \widetilde{u} \in L^2(E, \kappa)\},$$

$$\mathscr{E}(u, v) = \mathscr{E}^{\mathrm{res}}(u, v) + \int_E \widetilde{u}(x)\widetilde{v}(x)\, \kappa(dx), \qquad u, v \in \mathscr{F}.$$

In particular, the killing measure for X^{res} is zero. The process X can be obtained (in law) by killing X^{res} at the time

$$S := \inf\{t : C_t^{\kappa,\text{res}} > -\log U\},$$

where U is uniformly distributed over $(0, 1)$ and independent of X^{res}. Here we use the fact that κ is a smooth measure for X^{res} as well as for X (see [9]); $C^{\kappa,\text{res}}$ is the associated PCAF. Likewise μ is a smooth measure for X^{res}, with associated PCAF $C^{\mu,\text{res}}$. Moreover, $C_{t\wedge S}^{\mu,\text{res}}$ under $\mathbf{P}_x^{\text{res}}$ has the same distribution as C_t^{μ} under $\mathbf{P}_x$ for q.e. $x \in E$. Thus, using the preceding paragraph,

$$\begin{aligned}\mathbf{E}_\mu(U_t) &= \mathbf{E}_\mu^{\text{res}}(U_{t\wedge S}) \\ &\le 2(b-a)^{-2}\|\mathbf{E}_\cdot^{\text{res}} C_{t\wedge S}^{\mu,\text{res}}\|_\infty \mathscr{E}^{\text{res}}(f, f) \\ &\le 2(b-a)^{-2}\|\mathbf{E}_\cdot C_t^{\mu}\|_\infty \mathscr{E}^{\text{ref}}(f, f).\end{aligned}$$

□

Let $G := \int_0^\infty P_t\, dt$ denote the 0-potential operator for X.

Corollary 3.7. *Fix $f \in \mathscr{F}^{\text{ref}}$ and $a, b \in \mathbf{R}$ such that $a < b$. Let U_t be the number of crossings of the interval $[a, b]$ that are completed by the process $s \to \widetilde{f}(X_s)$ during the time interval $[0, t]$. Then:*

(i) $\mathbf{E}_m U_t \le 2t(b-a)^{-2}\mathscr{E}^{\text{ref}}(f, f)$.
(ii) For any positive smooth measure μ, $\mathbf{E}_\mu U_t \le 2(b-a)^{-2}(1+t)\|G_1\mu\|_\infty \mathscr{E}^{\text{ref}}(f, f)$.
(iii) If $(\mathscr{E}, \mathscr{F})$ is transient and μ is a positive smooth measure, then

$$\mathbf{E}_\mu U_\infty \le 2(b-a)^{-2}\|G\mu\|_\infty\, \mathscr{E}^{\text{ref}}(f, f).$$

Proof. Point (i) follows from Theorem 3.6 because the PCAF associated with m is $t \mapsto t \wedge \zeta$. Point (ii) follows from (3.8) by Lemma 5.1.9 of [11]. Finally, (iii) results upon taking $t \to \infty$ in Theorem 3.6 and noting that $\mathbf{E}_x C_\infty^{\mu} = G\mu(x)$. □

Remark 3.8. Corollary 3.7(i) extends and sharpens the crossing estimate of Lyons & Zheng [17], which concerned symmetric diffusions on $\mathbf{R}^n$.

In the remainder of this section, $(\mathscr{E}, \mathscr{F})$ is assumed to be transient. We study the limiting behavior of $\widetilde{f}(X_t)$ for $f \in \mathscr{F}^{\text{ref}}$. Let $\{T_k\}_{k\ge1}$ be an increasing sequence of stopping times of X with limit ζ. Recall that ζ_i is the totally inaccessible part of the lifetime ζ.

Lemma 3.9. *Assume that $(\mathscr{E}, \mathscr{F})$ is transient, and let f be an element of $\mathscr{F}_e$. Then for q.e. $x \in E$*

(i) $\lim_{t\to\zeta, t\lesssim\zeta} \widetilde{f}(X_t) = 1_{\{\zeta_i<\infty\}}\widetilde{f}(X_{\zeta-})$, $\mathbf{P}_x$*-a.s.,*
(ii) $\sup_{t\ge0} |\widetilde{f}(X_t)|$ *is $\mathbf{P}_x$-square integrable, and*
(iii) $\widetilde{f}(X_{T_k})$ converges to 0 in $L^2(\mathbf{P}_x)$ as $k \to \infty$.

Proof. Assertion (i) restates (2.10) – of course, the *existence* of this limit, in $[-\infty, +\infty]$, is also guaranteed by Theorem 3.7(iii). Because X is transient, there is a strictly positive bounded Borel function γ with bounded Green potential $G\gamma$. Defining $\mu(dx) = \gamma(x)m(dx)$, we have

$$\mathbf{P}_\mu\Big(\sup_{t\geq 0} |\widetilde{f}(X_t)| > \lambda\Big) \leq \|G\gamma\|_\infty \operatorname{Cap}(|\widetilde{f}| > \lambda), \qquad \lambda > 0,$$

where Cap is the 0-order capacity associated with X. In addition, by a result of K. Hansson (Theorem 1.6 in [13]; the proof adapts easily to the present situation), we have

$$\int_0^\infty \operatorname{Cap}(|\widetilde{f}| > \lambda) 2\lambda\, d\lambda \leq 4\mathscr{E}(f, f).$$

It follows that

$$\mathbf{P}_\mu\Big(\sup_{t\geq 0}[\widetilde{f}(X_t)]^2\Big) \leq 4\|G\mu\|_\infty \mathscr{E}(f, f). \tag{3.9}$$

This proves assertion (ii), and (iii) follows from (i) and (3.9) because $\{T_k < \zeta, \forall k\} \subset \{\zeta_i = \infty\}$. □

Theorem 3.10. *Let $f \in \mathscr{F}^{\text{ref}}$ have decomposition $f = g + h$, where $g \in \mathscr{F}_e$ and $h = \mathbf{E}_\cdot[\phi] \in \mathscr{H}$. Then for q.e. $x \in E$, $\widetilde{f}(X_t)$ converges $\mathbf{P}_x$-a.s. to $\widetilde{f}(X_{\zeta-}) 1_{\{\zeta_i < \infty\}} + \phi$ as $t \uparrow \zeta$, and $\widetilde{f}(X_{T_k})$ converges to ϕ in $L^2(\mathbf{P}_x)$ as $k \to \infty$.*

Proof. The theorem follows from Lemma 3.9, (2.8), and evident properties of the $\mathbf{P}_x$-square integrable martingale M^h defined by (2.7). □

4. Examples

In this section, we give three examples to illustrate the general results obtained in the last section.

Example 4.1. Let $X = \{X_t, t \geq 0\}$ be an irreducible continuous-time Markov chain on a countable state space E. For $x, y \in E$, define

$$p_t(x, y) := \mathbf{P}_x(X_t = y) \quad \text{and} \quad G(x, y) := \int_0^\infty p_t(x, y)\, dt.$$

For $x, y \in E$ the limit

$$q(x, y) := \lim_{t\downarrow 0} \frac{p_t(x, y) - \delta_{xy}}{t}$$

exists, $0 \leq q(x, y) < \infty$ if $x \neq y$, $-\infty \leq q(x, x) < 0$, and $\sum_{y:y\neq x} q(x, y) \leq -q(x, x)$. The symmetry of X amounts to the "detailed balance" equations:

$$m(x)q(x, y) = m(y)q(y, x), \qquad x \neq y,$$

where $m(x) := m(\{x\})$.

The Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(E, m)$ associated with X is given by

$$\mathscr{E}(f, g) = \frac{1}{2} \sum_{x \neq y} [f(x) - f(y)] \cdot [g(x) - g(y)]\, m(x) q(x, y), \tag{4.1}$$

on the domain $\mathscr{F}$, which contains the completion of the space of finitely supported functions on E endowed with the inner product $\mathscr{E}_1 = \mathscr{E} + (\cdot, \cdot)_{L^2(E,m)}$. It is easy to see that

$$\mathscr{F}^{\text{ref}} = \left\{ f : \sum_{x \neq y} [f(x) - f(y)]^2 m(x) q(x, y) < \infty \right\} \tag{4.2}$$

and that $\mathscr{E}^{\text{ref}}(f, g)$ is given by the expression on the right side of (4.1). Given $x \in E$, the Revuz measure of the PCAF

$$C_t^x := \int_0^t 1_{\{x\}}(X_s)\, ds, \quad t \geq 0,$$

consists of a mass of size $m(x)$ concentrated at x. By the strong Markov property of X applied at the hitting time of $\{x\}$,

$$\mathbf{E}_y C_t^x \leq \mathbf{E}_x C_t^x = \int_0^t \mathbf{P}_x(X_s = x)\, ds = \int_0^t p_s(x, x)\, ds$$

for all $y \in E$ and $t \in [0, \infty)$. Thus for every $f \in \mathscr{F}^{\text{ref}}$, we have by Theorem 3.6,

$$m(x) \mathbf{E}_x U_t \leq 2(b - a)^{-2} \left(\int_0^t p_s(x, x)\, ds \right) \mathscr{E}^{\text{ref}}(f, f), \tag{4.3}$$

where U_t is the number of crossings of $[a, b]$ by $s \mapsto f(X_s)$ during the time interval $[0, t]$. In particular,

$$m(x) \mathbf{E}_x U_\infty \leq 2(b - a)^{-2} G(x, x)\, \mathscr{E}^{\text{ref}}(f, f). \tag{4.4}$$

Example 4.2. Let $Y = \{Y_n, n = 0, 1, 2, \cdots\}$ be an irreducible discrete-time Markov chain on a countable state space E, with transition probability function $p(x, y) = \mathbf{P}(Y_{n+1} = y \mid Y_n = x)$. Define, for $x, y \in E$,

$$p^n(x, y) := \mathbf{P}(Y_{n+k} = y | Y_k = x) \quad \text{and} \quad H(x, y) := \sum_{n=0}^\infty p^n(x, y).$$

Assume that Y is symmetric in the sense that there is a measure m on E with

$$m(x) p(x, y) = m(y) p(y, x), \qquad \text{for all } x \neq y,$$

where $m(x) = m(\{x\})$ as before. Let $(\Pi(t))_{t \geq 0}$ be a unit-rate Poisson process (with $\Pi(0) = 0$) independent of Y. Then

$$X_t := Y_{\Pi(t)}, \qquad t \geq 0,$$

is a continuous-time Markov chain as in Example 4.1, with symmetry measure m and "infinitesimal transition rates" $q(x, y)$ equal to $p(x, y)$ $(x \neq y)$. Notice that

$$G(x, y) = \int_0^\infty \mathbf{P}_x(X_t = y)\, dt = \int_0^\infty \mathbf{P}_x(Y_{\Pi(t)} = y)\, dt = H(x, y)$$

for all $x, y \in E$.

Let $(\mathscr{E}^{\text{ref}}, \mathscr{F}^{\text{ref}})$ and U_t be as as in Example 4.1, with $q(x, y)$ replaced by $p(x, y)$. For $f \in \mathscr{F}^{\text{ref}}$ and real $a < b$, the number U_n^Y of crossings of $[a, b]$ that $k \mapsto f(Y_k)$ completes by time n is equal to U_{T_n}, where $T_n = \inf\{t : \Pi(t) = n\}$. Thus, by Theorem 3.6 and independence,

$$\begin{aligned} m(x)\mathbf{E}_x U_n^Y &= m(x)\mathbf{E}_x(U_{T_n}) \\ &\le 2(b-a)^{-2}\mathbf{E}_x(C_{T_n}^x)\, \mathscr{E}^{\text{ref}}(f, f) \\ &= 2(b-a)^{-2}\sum_{k=0}^{n-1} p^k(x, x)\, \mathscr{E}^{\text{ref}}(f, f), \end{aligned} \tag{4.5}$$

for all $x \in E$. Sending n to infinity in (4.5) we obtain

$$m(x)\mathbf{E}_x U_\infty^Y \le 2(b-a)^{-2} H(x, x)\, \mathscr{E}^{\text{ref}}(f, f), \tag{4.6}$$

which is Theorem 1.3 of [2].

Example 4.3. In this final example suppose that our symmetric right Markov process X is a *diffusion* in the sense that $t \mapsto X_t$ is continuous on $[0, \zeta)$ and ζ is predictable. In this case the Beurling–Deny decomposition simplifies to

$$\mathscr{E}(f, g) = \frac{1}{2}\int_E \Gamma(f, g)(dx), \qquad f, g \in \mathscr{F},$$

where $\Gamma(f, g)$ is the (signed) Revuz measure of the CAF $\langle M^f, M^g\rangle$.

For example, if E is a finite-dimensional Riemannian manifold and X is the associated Brownian motion, then $d\Gamma(f, g)/dm = \nabla f \cdot \nabla g$, where m is the Riemannian volume measure. Similar expressions for $\Gamma(f, g)$ occur when X is the diffusion associated with a divergence-form generator or an infinite-dimensional diffusion of "gradient type"; see, e.g., Section II.3 of [18], and also [8] for examples of diffusions on path and loop spaces over compact Riemannian manifolds.
In the present setting,

$$\mathscr{F}^{\text{ref}} = \left\{ f \in \dot{\mathscr{F}}_{\text{loc}} : \int_E \Gamma(f, f)(dx) < \infty \right\},$$

and if $f \in \mathscr{F}^{\text{ref}}$, then $\Gamma(f, f)$ is the smooth measure associated with $\langle M^f\rangle$. Thus, taking $T = \infty$ in Theorem 3.3, we obtain

$$\mathbf{E}_x U_\infty \le \mathbf{E}_x\langle M^f\rangle_\infty = G(\Gamma(f, f))(x) \;\text{ for q.e. } x \in E, \tag{4.7}$$

where U_∞ is the total number of crossings of $[a, b]$ by $t \mapsto \widetilde{f}(X_t)$.

When E is a finite-dimensional Riemannian manifold and the infinitesimal generator of X is a uniformly elliptic divergence-form operator, the transition operators of X admit continuous densities with respect to the volume measure m, and this additional smoothness allows one to eliminate the exceptional set in (4.7). This yields Corollary 8.4 of [2]. Also, specializing Corollary 3.7(iii), we recover Corollary 8.5 of [2].

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