Probab. Theory Relat. Fields 119, 475–507 (2001)
Digital Object Identifier (DOI) 10.1007/s004400000112

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Phase transition of the principal Dirichlet eigenvalue in a scaled Poissonian potential

Received: 10 December 1999 / Revised version: 14 July 2000/
Published online: 15 February 2001 –

Abstract. We consider d-dimensional Brownian motion in a scaled Poissonian potential and the principal Dirichlet eigenvalue (ground state energy) of the corresponding Schrödinger operator. The scaling is chosen to be of critical order, i.e. it is determined by the typical size of large holes in the Poissonian cloud. We prove existence of a phase transition in dimensions $d \geq 4$: There exists a critical scaling constant for the potential. Below this constant the scaled infinite volume limit of the corresponding principal Dirichlet eigenvalue is linear in the scale. On the other hand, for large values of the scaling constant this limit is strictly smaller than the linear bound. For $d < 4$ we prove that this phase transition does not take place on that scale. Further we show that the analogous picture holds true for the partition sum of the underlying motion process.

0. Introduction and results

In this article, we consider standard Brownian motion in $\mathbb{R}^d$, $d \geq 1$, which evolves in a scaled random potential. The scaled random potential is obtained by translating a fixed shape function W to all the points of a Poissonian cloud with constant intensity $\nu = 1$. Let $\mathbb{P}$ stand for the law of the Poissonian point process $\omega = \sum_i \delta_{x_i} \in \Omega$ (where Ω is the set of all simple pure locally finite point measures on $\mathbb{R}^d$). The random scaled Poissonian potential is then defined as follows, for $x \in \mathbb{R}^d$, $\beta > 0$, $t > 0$ and $\omega \in \Omega$:

$$V_{\beta,t}(x, \omega) \stackrel{\text{def}}{=} \frac{\beta}{(\log t)^{2/d}} V(x, \omega) \stackrel{\text{def}}{=} \frac{\beta}{(\log t)^{2/d}} \sum_i W(x - x_i), \tag{0.1}$$

where we assume that the shape function $W \geq 0$ is measurable, bounded, compactly supported and $\int W(x)dx = 1$. For $z \in \mathbb{R}^d$ let P_z stand for the standard Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ starting from z (its canonical process is denoted by $Z_\cdot$).

Let us for the moment restrict to the unscaled Poissonian potential V. The Feynman-Kac functional $u(t, z) = E_z\left[\exp\left\{-\int_0^t V(Z_s, \omega)ds\right\}\right]$ represents the

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M. V. Wüthrich was supported by the Dutch Association for Mathematical Physics (FOM)

Mathematics Subject Classification (2000): Primary 82B44; Secondary 60K35

bounded weak solution of the random parabolic equation

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u - Vu, \\ u_{t=0} = 1. \end{cases} \tag{0.2}$$

Sznitman [7], Theorem 4.5.1, has proved that there is a set of full $\mathbb{P}$-measure such that for all $z \in \mathbb{R}^d$

$$-\log u(t,z) \sim c(d,1)\frac{t}{(\log t)^{2/d}}, \qquad \text{as } t \to \infty, \tag{0.3}$$

where $c(d,1)$, defined in (4.4.20) of [7], is the constant

$$c(d,1) \stackrel{\text{def}}{=} \lambda_d \left(\frac{v_d}{d}\right)^{2/d}, \tag{0.4}$$

here λ_d denotes the principal Dirichlet eigenvalue on the d-dimensional unit ball to the potential 0, and v_d is the volume of the d-dimensional unit ball. A crucial role in the proof of (0.3) is played by the principal Dirichlet eigenvalues to the potential V on the boxes $(-t,t)^d$. Analysing the asymptotic behavior of these principal Dirichlet eigenvalues, one sees that the main contribution comes from the large holes in the (random) Poissonian potential V. The box $(-t,t)^d$ typically contains a ball having a radius of order $d^{1/d}v_d^{-1/d}(\log t)^{1/d}$ which receives no point of ω (see Sznitman [7], Formula (4.4.38) and Theorem 4.4.6). In this article we examine whether such large holes are still dominant when we rescale the Poissonian potential in an appropriate way (see (0.1)): the costs of confining a Brownian particle to large Poissonian holes now compete with the costs arising in an averaged scaled Poissonian potential; the scaling is chosen such that these two costs are of the same "order".

The main role in this context is played by the principal Dirichlet eigenvalue. It is defined as follows: Choose a measurable potential $\tilde{V}$ which is bounded from below. Then the principal Dirichlet eigenvalue on the non-empty open set $U \subset \mathbb{R}^d$ to $\tilde{V}$ is defined as:

$$\lambda_{\tilde{V}}(U) \stackrel{\text{def}}{=} \inf\left\{\frac{1}{2}\|\nabla\phi\|_2^2 + \int_U \tilde{V}\phi^2 dx : \phi \in C_c^\infty(U), \|\phi\|_2^2 = 1\right\}. \tag{0.5}$$

Rescaling the Poissonian potential properly has the following effect: Consider test functions varying on the scale of large holes of the Poissonian cloud. Then the gradient term (kinetic energy) and the potential term live on the same scale. Therefore we ask, which term "wins" in this setting. Our main results are the following theorems:

Theorem 0.1. *For all $d \geq 1$ and $\beta > 0$,*

$$\mathbb{P}\text{-a.s.} \qquad \limsup_{t\to\infty} (\log t)^{2/d}\, \lambda_{V_{\beta,t}}((-t,t)^d) < c(d,1). \tag{0.6}$$

In fact we prove a slightly stronger quantitative asymptotic bound for $\beta \to \infty$ (see (3.37)). Theorem 0.1 proves that in our context we obtain an eigenvalue which is strictly smaller than in the unscaled case (see [7], Theorem 4.4.6). In the unscaled case one observes that the eigenfunctions essentially live in the large Poissonian holes. In our model, the eigenfunctions prefer large connected regions where the number of Poissonian particles is less than its expectation. These regions are typically larger (by a β-dependent factor) than the holes in Sznitman's context. Henceforth the contribution from the potential term can be compensated by the gradient term in such a way that we obtain a smaller value than in the unscaled picture.

Theorem 0.2. *For $d \geq 4$ there exists $\beta_c > 0$ such that for all $\beta < \beta_c$*

$$\mathbb{P}\text{-a.s.} \qquad \lim_{t\to\infty} (\log t)^{2/d} \lambda_{V_{\beta,t}}((-t,t)^d) = \beta. \tag{0.7}$$

Theorems 0.1 and 0.2 prove that for $d \geq 4$ we observe a phase transition on the scale $(\log t)^{2/d}$: There exists a critical scaling constant. Below this constant the asymptotic behavior of the principal Dirichlet eigenvalue is linear in the scaling: we can choose as test function a C_c^∞-approximation to the normalized constant function on $(-t,t)^d$ to evaluate (0.5); this test function provides already the correct asymptotic behavior in (0.7). This picture changes for large β: we have an upper bound which is strictly smaller than the linear one (see (0.6)); this improved upper bound is obtained using other test functions: these test functions are supported on regions having a volume proportional to $\log t$. The number of particles in these regions has to be less than its expected value.

For $d < 4$ the situation is completely different, namely:

Theorem 0.3. *Let $d < 4$ and $\beta > 0$. Then*

$$\mathbb{P}\text{-a.s.} \qquad \limsup_{t\to\infty} (\log t)^{2/d} \lambda_{V_{\beta,t}}((-t,t)^d) < \beta. \tag{0.8}$$

In Lemma 3.4 we provide a more quantitative bound. Theorem 0.3 was in the beginning quite surprising: Our main tool to prove Theorem 0.2 is the Cwickel–Lieb–Rosenbljum Theorem (see Theorem 9.3 in Simon [6]); it suggests that the critical dimension might be $d = 3$. However a closer look at the below used "grey–scale technique" (proof of Lemma 2.5) shows that for $d = 3$ not the small deep holes cause problems but the large shallow ones. These large shallow holes can not be treated by that Theorem; their effect is in fact so strong that we observe in three dimensions a similar picture as for $d = 1, 2$.

Next we consider the partition sum of Brownian motion in the scaled Poissonian potential (starting at the origin),

$$S^\omega_{t,\beta} \overset{\text{def}}{=} E_0\left[\exp\left\{-\frac{\beta}{(\log t)^{2/d}} \int_0^t V(Z_s,\omega)ds\right\}\right]. \tag{0.9}$$

The time scale t is the natural one, because on this space-time scale the Brownian motion with killing has enough time to experience the whole box $(-t,t)^d$, respectively the large holes in the box $(-t,t)^d$ (whenever such a strategy is favorable for the survival of the Brownian particles). We have the following results:

Theorem 0.4. *For all $d \geq 1$ and $\beta > 0$,*

$$\mathbb{P}\text{-}a.s. \qquad \liminf_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log S_{t,\beta}^{\omega} > -c(d,1). \tag{0.10}$$

For $d \geq 4$ there exists $\beta_c > 0$ such that for all $\beta < \beta_c$

$$\mathbb{P}\text{-}a.s. \qquad \lim_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log S_{t,\beta}^{\omega} = -\beta. \tag{0.11}$$

For $d < 4$ and $\beta > 0$

$$\mathbb{P}\text{-}a.s. \qquad \liminf_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log S_{t,\beta}^{\omega} > -\beta. \tag{0.12}$$

One should compare these results with (0.3). It would also be interesting to examine the path behavior of Brownian motion in a scaled Poissonian potential. However, this question goes beyond the scope of this article. So far, our picture suggests that for $d \geq 4$ and small β the motion process should be diffusive, whereas for large β or $d < 4$ we expect a superdiffusive behavior.

Statements similar to Theorems 0.1–0.4 also hold true for general $W \geq 0$ (measurable, bounded, compactly supported) with $\int W(x)dx > 0$ and general Poissonian intensity $\nu > 0$. We restrict ourselves to the case $\int W(x)dx = 1$ and $\nu = 1$ since it already covers the whole flavor of the problem and since the general case can be recovered by a simple scaling argument.

This article is organised as follows: In Section 1 we give some general results and definitions that we use in the whole article.

In Section 2 we provide the lower bound on the principal Dirichlet eigenvalue in the low-β-regime ($d \geq 4$). This consists of three parts: Part 1: We apply the Cwickel–Lieb–Rosenbljum Theorem (Theorem 9.3 in Simon [6]) to our situation, where we do not have one big hole in the Poissonian cloud but many holes which are separated by large distances (see Lemma 2.2 below). The main tool here is a comparison theorem by Sznitman for principal Dirichlet eigenvalues on different domains (see [7], Theorem 3.1.11). Part 2: We define the notion of big holes. We introduce a "stuffing" function to "repair" the potential in regions, where there are too large holes ((2.28)–(2.30)). In Lemma 2.5 we prove that we can compare the principal Dirichlet eigenvalue of the original potential with the eigenvalue of the repaired potential. The main tools in this part are large deviation estimates for having a big hole in the Poissonian cloud configuration on all "grey-scale" levels. Part 3: Finally we estimate the principal Dirichlet eigenvalue of the repaired potential from below by classical methods.

In Section 3 we give the upper bounds on the principal Dirichlet eigenvalues. The upper bounds are based on a variational principle (Lemma 3.2). This is obtained by the Gärtner-Ellis large deviation theorem (Theorem 2.3.6, [3]) applied to integrals of test functions with respect to the Poissonian cloud configuration. We derive all our upper bounds by optimising this variational principle (for the according β's). This is done in Lemmas 3.3, 3.4 and 3.5. The remarkable thing here is that the relevant optimisation problems on [0, 1] behave qualitatively very differently for $d \le 3$, $d = 4$, and $d > 4$. This emphasizes that $d = 4$ is the critical dimension; it also corresponds to the fact that the "grey–scale" estimates for the lower bound (proof of Lemma 2.5) become easier in dimensions $d \ge 5$ (see the remark after the proof of Lemma 2.5).

In Section 4 we finally give the translation of the results concerning the principal Dirichlet eigenvalue to results about partition sums for Brownian motion in a scaled Poissonian potential.

1. Preliminaries

In this section we do all the preparatory work to prove our results. We start with the following definitions: For $t > 0$, we define

$$\mathscr{T}_t \overset{\text{def}}{=} (-t, t)^d, \tag{1.1}$$

$W_\infty \overset{\text{def}}{=} \sup_{x \in \mathbb{R}^d} W(x)$, and a denotes the minimal radius such that $\operatorname{supp} W \subseteq \bar{B}_a(0)$, where $B_a(0)$ is the open ball with center 0 and radius a. We state the following measurability result:

Lemma 1.1. $\lambda_{V_{\beta,t}}(\mathscr{T}_t)$ *is measurable in* ω *and decreasing in* t.

Proof of Lemma 1.1. The measurability follows from the fact that it suffices to consider a countable collection of test functions $\phi \in C_c^\infty(\mathscr{T}_t)$ with $\|\phi\|_2 = 1$ to evaluate the principal Dirichlet eigenvalue. The decrease in t can easily be seen from (3.1.4) in [7]. □

The following lemma estimates large deviations for Poisson random variables:

Lemma 1.2. *Let* $N \overset{\mathbb{P}}{\sim} \text{Poisson}(\mu)$, $0 < \varepsilon < 1$. *Then*

$$\mathbb{P}[N < (1-\varepsilon)\mu] \le e^{-\varepsilon^2\mu/2}. \tag{1.2}$$

Proof of Lemma 1.2. We use the exponential Chebyshev-inequality for $s \ge 0$:

$$\begin{aligned} \mathbb{P}[N < (1-\varepsilon)\mu] &\le e^{s(1-\varepsilon)\mu}\mathbb{E}[e^{-sN}] \\ &= \exp([s(1-\varepsilon) + e^{-s} - 1]\mu) = \exp(-g(\varepsilon)\mu), \end{aligned} \tag{1.3}$$

where we have set $s = -\log(1-\varepsilon) > 0$, i.e. $e^{-s} - 1 = -\varepsilon$, and $g(\varepsilon) = (1-\varepsilon)\log(1-\varepsilon) + \varepsilon$. We have $g'(\varepsilon) = -\log(1-\varepsilon)$, $g''(\varepsilon) = 1/(1-\varepsilon) \ge 1$, $g(0) = 0$, and $g'(0) = 0$; therefore $g(\varepsilon) \ge \frac{1}{2}\varepsilon^2$. Inserting this into (1.3) proves Lemma 1.2. □

2. Lower bound in the low-β-regime

2.1. Adaptation of the Cwickel–Lieb–Rosenbljum Theorem

At the heart of the proof of the lower bound in Theorem 0.2 lies a nice theorem which is due to Cwickel–Lieb–Rosenbljum (CLR) (see Theorem 9.3 in [6]). Here is also an important step where the calculations in dimensions $d = 1, 2$ break down (see Simon [5]). First we quote the CLR Theorem: We define $V_- \stackrel{\text{def}}{=} \max(0, -V)$.

Theorem 2.1 (Theorem 9.3, [6]). *Let $d \geq 3$. There exists a constant a_d such that for all potentials V (not necessarily positive) with $V \in L^{d/2}(\mathbb{R}^d)$ and*

$$a_d \int V_-(x)^{d/2} dx < 1, \tag{2.1}$$

we have $\lambda_V(\mathbb{R}^d) \geq 0$.

We only need the above theorem for potentials V which are bounded from below (see (0.5)), but indeed it is valid in a more general setting. Our first goal is to adapt this result to a situation where one has many holes in the potential, but the distances between the holes are large.

Let $d \geq 3$, and let $-1 \leq U_j \leq 0$ be supported on $A_j \subset \mathbb{R}^d$. Further we assume

$$a_d \int |U_j|^{d/2} dx < 1, \tag{2.2}$$

and

$$l \stackrel{\text{def}}{=} \inf_{j \neq i} \operatorname{dist}(A_j, A_i) > 1. \tag{2.3}$$

We set $U \stackrel{\text{def}}{=} \sum_j U_j$. Then

Lemma 2.2. *Assume $d \geq 3$, and define $f(l) \stackrel{\text{def}}{=} l^{-2} \log^3 l$. Then there exists $L = L(d) > 1$ such that if $l > L$ then the following holds:*

$$\lambda_U(\mathbb{R}^d) \geq -f(l). \tag{2.4}$$

Proof of Lemma 2.2. Our main goal is to apply Theorem 3.1.11 of [7]. Therefore we have to estimate A, B, C defined in (3.1.36) of [7]. We start with the following definitions: $\mathscr{A} \stackrel{\text{def}}{=} \bigcup_j A_j$, and $\mathscr{O}$ is the open $l/4$-neighborhood of $\mathscr{A}$. Notice that the disjoint holes A_j have also a disjoint $l/4$-neighborhood (which are denoted by $\mathscr{O}_j$). Further we define $V \stackrel{\text{def}}{=} U + 1 \geq 0$ so that we do not have to bother about signs, i.e.

$$\text{for all open sets } \mathscr{U} \subset \mathbb{R}^d, \qquad \lambda_V(\mathscr{U}) \geq 0. \tag{2.5}$$

We claim

$$\lambda_V(\mathscr{O}) \wedge 1 - f(l) \leq \lambda_V(\mathbb{R}^d). \tag{2.6}$$

The inequality (2.6) implies (2.4): due to the CLR Theorem (Theorem 2.1) we know that $\lambda_{U_j}(\mathbb{R}^d) \geq 0$ for all j, hence we have (using (2.6) and (3.1.5) of [7])

$$\begin{aligned}\lambda_U(\mathbb{R}^d) &= \lambda_V(\mathbb{R}^d) - 1\\ &\geq \lambda_V(\mathscr{O}) \wedge 1 - f(l) - 1\\ &= \inf_j \lambda_V(\mathscr{O}_j) \wedge 1 - f(l) - 1 \qquad (2.7)\\ &\geq (\inf_j \lambda_{U_j}(\mathbb{R}^d) + 1) \wedge 1 - f(l) - 1\\ &\geq -f(l).\end{aligned}$$

There remains to prove (2.6). To make our notations consistent with [7] we define $\mathscr{U}_1 \stackrel{\text{def}}{=} \mathscr{O}$ and $\mathscr{U}_2 \stackrel{\text{def}}{=} \mathbb{R}^d$ and $\lambda \stackrel{\text{def}}{=} (\lambda_V(\mathscr{U}_1) \wedge 1 - f(l))_+$.

Either $\lambda = 0$, then (using (2.5))

$$\lambda_V(\mathscr{U}_1) \wedge 1 - \lambda_V(\mathscr{U}_2) \leq \lambda_V(\mathscr{U}_1) \wedge 1 \leq f(l), \qquad (2.8)$$

which finishes the proof in the case $\lambda = 0$.

Or $\lambda > 0$, hence $f(l) < 1$ and $f(l) < \lambda_V(\mathscr{U}_1)$: If $\lambda_V(\mathscr{U}_1) < 1$, then $\lambda_V(\mathscr{U}_1) - f(l) \leq \lambda_V(\mathscr{U}_1)(1 - f(l))$, since $f(l) > 0$, and if $\lambda_V(\mathscr{U}_1) \geq 1$, then $1 - f(l) \leq \lambda_V(\mathscr{U}_1)(1 - f(l))$. Combining these two estimates, we get

$$0 < \lambda = \lambda_V(\mathscr{U}_1) \wedge 1 - f(l) \leq \lambda_V(\mathscr{U}_1)\,(1 - f(l))\,. \qquad (2.9)$$

We define the entrance time $\tau \stackrel{\text{def}}{=} \inf\{s \geq 0, Z_s \in \mathscr{A}\}$ of $Z_.$ into the holes $\mathscr{A}$, the exit time $T_{\mathscr{S}} \stackrel{\text{def}}{=} \inf\{s \geq 0, Z_s \notin \mathscr{S}\}$ of $Z_.$ from an open set $\mathscr{S} \subseteq \mathbb{R}^d$, and

$$S_1 \stackrel{\text{def}}{=} \tau \circ \theta_{T_{\mathscr{U}_1}} + T_{\mathscr{U}_1} \quad \text{and} \quad S_{k+1} \stackrel{\text{def}}{=} S_1 \circ \theta_{S_k} + S_k \text{ for } k \geq 1, \qquad (2.10)$$

where θ_t is the time shift. Because on the time interval $(S_k, S_{k+1}]$ the Brownian motion has to travel at least distance $l/4 > 0$, we have that for all $x \in \mathbb{R}^d$, $\lim_k S_k = \infty$ P_x-a.s. (which is Condition (3.1.38) of [7]). We use Formula (3.1.19) and Corollary 3.1.3 of [7] together with (2.9) (the assumptions of Corollary 3.1.3 of [7] are fulfilled since $f(l) < 1$ (we are in the case $\lambda > 0$)): we obtain for some fixed constant $c_2(d) > 0$ that

$$\begin{aligned}A &\stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}^d} 1 + \int_0^\infty \lambda e^{\lambda u} E_x\left[T_{\mathscr{U}_1} > u, \exp\left\{-\int_0^u V(Z_s)ds\right\}\right] du\\ &\leq \frac{c_2(d)}{f(l)^{\frac{d}{2}+1}} < \infty. \qquad (2.11)\end{aligned}$$

Next we have, again using Corollary 3.1.3 and Formula (3.1.19) of [7] together with (2.9) and the fact that $\lambda \le 1 - f(l) \le \lambda_V(\mathscr{A}^c)(1 - f(l))$,

$$\begin{aligned} B &\stackrel{\text{def}}{=} \sup_{x \notin \mathscr{U}_1} \int_0^\infty \lambda e^{\lambda u} E_x \left[\tau \wedge T_{\mathscr{U}_2} > u, \exp\left\{ -\int_0^u V(Z_s) ds \right\} \right] du \\ &\le 1 + \sup_{x \notin \mathscr{U}_1} \int_0^\infty \lambda e^{\lambda u} E_x \left[T_{\mathscr{A}^c} > u, \exp\left\{ -\int_0^u V(Z_s) ds \right\} \right] du \qquad (2.12) \\ &\le \frac{c_2(d)}{f(l)^{\frac{d}{2}+1}} < \infty. \end{aligned}$$

There remains to estimate

$$C \stackrel{\text{def}}{=} \sup_{x \notin \mathscr{U}_1} E_x \left[\tau < T_{\mathscr{U}_2}, \exp\left\{ \lambda\tau - \int_0^\tau V(Z_s) ds \right\} \right]. \qquad (2.13)$$

On $\mathscr{A}^c$ we have $V = 1$, hence $V - \lambda = 1 - \lambda_V(\mathscr{U}_1) \wedge 1 + f(l) \ge f(l)$. Therefore

$$\begin{aligned} C &\le \sup_{x \notin \mathscr{U}_1} E_x \left[\exp\left\{ -\int_0^\tau f(l) ds \right\} \right] \\ &\le E_0 \left[\exp\left\{ -f(l) T_{B_{l/4}(0)} \right\} \right] \qquad (2.14) \\ &\le 2d \exp\left\{ -(8d)^{-1/2} l f(l)^{1/2} \right\}. \end{aligned}$$

The last estimate in (2.14) can be seen as follows: Choose $r \in \mathbb{R}$ and denote by $Z_s^{(i)}$ the i-th coordinate of the process Z_s ($i = 1, \dots, d$). Then we define $T_r^{(i)} \stackrel{\text{def}}{=} \inf\{s \ge 0 : \ Z_s^{(i)} = r\}$, hence we see that P_0-a.s. for $r > 0$: $T_{B_{d^{1/2}r}(0)} \ge \min_{i=1,\dots,d} \left(T_r^{(i)} \wedge T_{-r}^{(i)} \right)$. Therefore for positive r, μ:

$$\begin{aligned} E_0 \left[\exp\left\{ -\mu T_{B_{d^{1/2}r}(0)} \right\} \right] &\le E_0 \left[\exp\left\{ -\mu \min_{i=1,\dots,d} \left(T_r^{(i)} \wedge T_{-r}^{(i)} \right) \right\} \right] \\ &\le E_0 \left[2 \sum_{i=1}^d \exp\left\{ -\mu T_r^{(i)} \right\} \right] \qquad (2.15) \\ &= 2d E_0 \left[\exp\left\{ -\mu T_r^{(1)} \right\} \right] = 2d \exp\left\{ -\sqrt{2\mu} r \right\}, \end{aligned}$$

where the last equality can be found e.g. in [4], Proposition 8.5, p.96.

Hence there exists $L(d) > 1$ such that if $l > L(d)$:

$$AC \le c_3 \exp\left\{ -\left(\frac{d}{2} + 1 \right) \log f(l) - (8d)^{-1/2} \log^{3/2} l \right\} < 1 \qquad (2.16)$$

with $c_3(d) \stackrel{\text{def}}{=} 2dc_2(d)$. So Theorem 3.1.11 of [7] gives that $\lambda = \lambda_V(\mathscr{U}_1) \wedge 1 - f(l) \le \lambda_V(\mathscr{U}_2)$ for $l > L(d)$, which finishes the proof of (2.6). □

2.2. The grey-scale technique

We use different scales of volumes: $t^d \gg l(t)^d \gg \log t \gg \alpha(t)\log t \gg 1$ (for large values of t); "$a(t) \gg b(t)$" means that $a(t)/b(t) \to \infty$ as $t \to \infty$. The meanings of the scaling functions are roughly:

- t^d is the scale of the "universe box" $\mathscr{T}_t$;
- $l(t)$ is the length scale of the minimal distance between the sets A_i in Lemma 2.2;
- $\log t$ is the scale of the largest hole in the potential on a box $\mathscr{T}_t$;
- $\alpha(t)\log t$ is the scale on which we define the "stuffing" function;
- 1 is the scale of the support of the shape function W.

We set for large values of t (where $[\cdot]$ denotes the integer part):

$$l(t) \stackrel{\text{def}}{=} \frac{t}{[(\log t)^{-2}t]} \quad \text{and} \quad \alpha(t) \stackrel{\text{def}}{=} \frac{l(t)^d}{[l(t)(\log t)^{-1/(2d)}]^d \log t}. \tag{2.17}$$

The asymptotic behavior of these functions as $t \to \infty$ is $l(t) \sim (\log t)^2$, and $\alpha(t) \sim (\log t)^{-1/2}$. Here are the main properties of these scaling functions that we use below: with the notation of Lemma 2.2,

$$(\log t)^{2/d} \cdot f(l(t)) \stackrel{t\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad \alpha(t) \stackrel{t\to\infty}{\longrightarrow} 0, \tag{2.18}$$

but $\alpha(t)$ converges only so slowly that

$$\frac{\log l(t)}{\alpha(t)\log t} \stackrel{t\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad -\frac{\log \alpha(t)}{\alpha(t)\log t} \stackrel{t\to\infty}{\longrightarrow} 0. \tag{2.19}$$

The numbers

$$\frac{1}{2} n_1(t)^{\frac{1}{d}} \stackrel{\text{def}}{=} \frac{t}{l(t)} \qquad \text{and} \qquad n_2(t)^{\frac{1}{d}} \stackrel{\text{def}}{=} \frac{l(t)}{(\alpha(t)\log t)^{\frac{1}{d}}} \tag{2.20}$$

are integers (we want to avoid dealing with fractions of boxes); this is why we have introduced integer parts in the definition of the scaling functions l and α. We split the universe box $\mathscr{T}_t = (-t,t)^d$ into $n_1(t)$ cubes

$$A_j = A_j(t) \stackrel{\text{def}}{=} (-t,\dots,-t) + l(t)j + [0, l(t))^d \tag{2.21}$$

of volume $l(t)^d$,

$$j \in J = J(t) \stackrel{\text{def}}{=} \{0,\dots,n_1(t)^{\frac{1}{d}} - 1\}^d, \tag{2.22}$$

$|J(t)| = n_1(t)$; the union over all A_j coincides with $\mathscr{T}_t$ only up to a null set at the boundary of $\mathscr{T}_t$ (since $\mathscr{T}_t$ is an open box). Next we split each of these boxes A_j into $n_2(t)$ smaller boxes

$$K_{i,j} = K_{i,j}(t) \stackrel{\text{def}}{=} (-t,\dots,-t) + l(t)j + (\alpha(t)\log t)^{\frac{1}{d}} i + [0, (\alpha(t)\log t)^{\frac{1}{d}})^d \tag{2.23}$$

of volume $\alpha(t)\log t$,

$$i \in I = I(t) \stackrel{\text{def}}{=} \{0, \dots, n_2(t)^{\frac{1}{d}} - 1\}^d, \tag{2.24}$$

$|I(t)| = n_2(t)$. We partition $J(t)$ into 2^d classes $J_k(t)$, $k \in \{0,1\}^d$, where $J_k = J_k(t) \stackrel{\text{def}}{=} J(t) \cap (k + (2\mathbb{Z})^d)$; we observe that $\text{dist}(A_i, A_j) \geq l(t)$ for $i, j \in J_k(t)$, $i \neq j$. (See also Figure 1.)

Finally we split these boxes $K_{i,j}$ into even smaller boxes on the scale of the diameter of the potential. We choose the length $\bar{a}(t)$ such that $a \leq \bar{a}(t) \leq 2a$, and such that $(\alpha(t)\log t)^{\frac{1}{d}}/\bar{a}(t)$ is an integer (for t large enough); this is again done to avoid handling with fractions of boxes. We define the boxes

$$C_m \stackrel{\text{def}}{=} m\bar{a}(t) + [0, \bar{a}(t))^d, \tag{2.25}$$

for $m \in \mathbb{Z}^d$, and the index set $\mathscr{C}_{i,j} \stackrel{\text{def}}{=} \{m \in \mathbb{Z}^d : C_m \subseteq K_{i,j}\}$.

We introduce a random "stuffing" function: It has the purpose to "repair" the potential, where the truncated version $V \wedge M$ of V is too small (caused by the

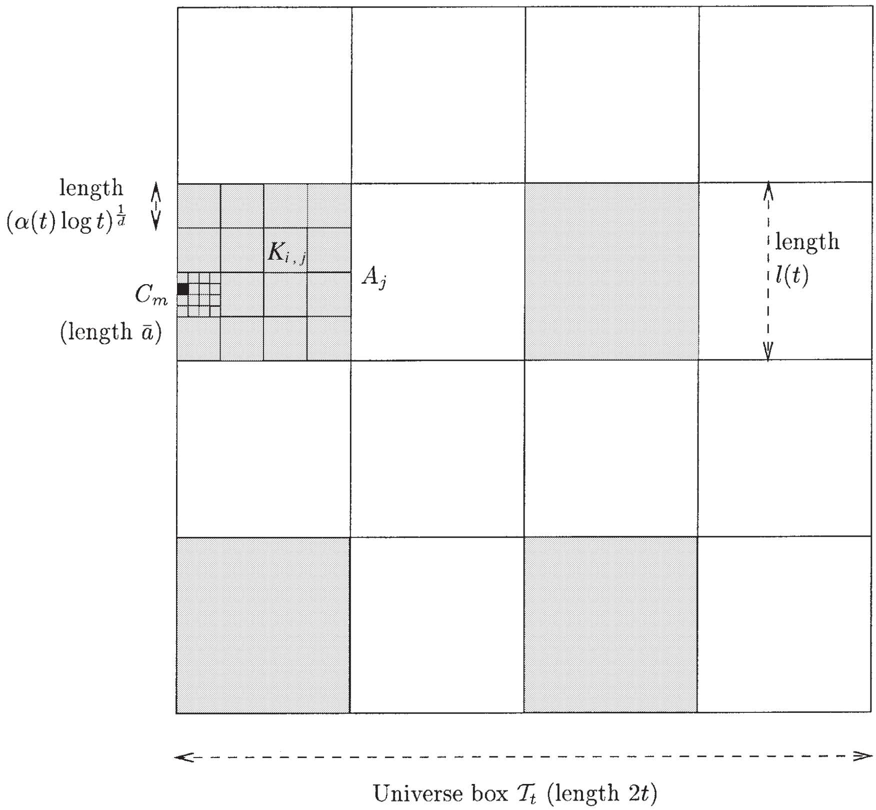

Fig. 1. The universe box $\mathscr{T}_t$, and some boxes A_j, $K_{i,j}$, and C_m. One $\bigcup_{j \in J_k} A_j$ is shaded, and one box C_m is drawn black.

randomness of ω; $M > 1$ denotes a truncation level). The truncated potential may be too small in the box $K_{i,j}$ for two reasons:
1) The total number of points of the Poissonian point process in the box $K_{i,j}$ might be too low; this is measured by the quantity

$$\xi_{i,j} \stackrel{\text{def}}{=} \left(1 - \frac{\omega(K_{i,j})}{|K_{i,j}|}\right) \vee 0. \tag{2.26}$$

2) The points inside the box $K_{i,j}$ might clump too much, leaving holes in other parts of the box. To measure this, we introduce the event

$$F_{i,j}^{(0)} = F_{i,j}^{(0)}(t, M, \eta) \stackrel{\text{def}}{=} \left\{\omega \in \Omega : \sum_{m \in \mathscr{C}_{i,j}} [\omega(C_m) - (\omega(C_m) \wedge \bar{M})] \geq \frac{\eta}{2}|K_{i,j}|\right\}, \tag{2.27}$$

where $\eta > 0$ denotes a (small) allowed tolerance and $\bar{M} \stackrel{\text{def}}{=} M/(3^d W_\infty)$.

We define the random “stuffing” function U (depending on M, η, t, and on the Poissonian cloud configuration ω): for $j \in J$, $k \in \{0,1\}^d$,

$$U_j(x) \stackrel{\text{def}}{=} -\sum_{i \in I} \left[(\xi_{i,j} 1_{\{\xi_{i,j} \geq \eta/2\}}) \vee 1_{F_{i,j}^{(0)}}\right] 1_{K_{i,j}}(x) \qquad \text{(supported on } A_j\text{)}, \tag{2.28}$$

$$\tilde{U}_k \stackrel{\text{def}}{=} \sum_{j \in J_k} U_j \qquad \text{(supported on } \textstyle\bigcup_{j \in J_k} A_j\text{)}, \tag{2.29}$$

$$U(x) \stackrel{\text{def}}{=} \sum_{j \in J} U_j(x) = \sum_{k \in \{0,1\}^d} \tilde{U}_k \qquad \text{(supported on } \mathscr{T}_t\text{)}. \tag{2.30}$$

The intention behind definition (2.28) is: we work with a “grey-scale picture” for repairing the first kind of holes, but a “black-and-white picture” is sufficient to repair the second kind of holes.

The Poissonian cloud configuration $\omega \in \Omega$ is “repaired” by U in the following sense:

$$-\int_{K_{i,j}} U(x)\,dx + \sum_{m \in \mathscr{C}_{i,j}} (\omega(C_m) \wedge \bar{M}) \geq (1-\eta)|K_{i,j}|; \tag{2.31}$$

this is obvious on the event $F_{i,j}^{(0)}$, while on $(F_{i,j}^{(0)})^c$ it follows from

$$\begin{aligned} \sum_{m \in \mathscr{C}_{i,j}} (\omega(C_m) \wedge \bar{M}) &\geq \omega(K_{i,j}) - |K_{i,j}|\frac{\eta}{2} \\ &\geq |K_{i,j}|\left(1 - \xi_{i,j} - \frac{\eta}{2}\right) \\ &\geq |K_{i,j}|(1 - \eta - \xi_{i,j} 1_{\{\xi_{i,j} \geq \eta/2\}}). \end{aligned} \tag{2.32}$$

We observe the following bounds for U:

$$0 \geq U \geq -1. \tag{2.33}$$

We need scaled versions of the functions U_j and $\tilde{U}_k$ too: Using $p = p(d) \stackrel{\text{def}}{=} 2^d + 1$ we define:

$$U_{j,\beta,t} \stackrel{\text{def}}{=} p \frac{\beta}{(\log t)^{2/d}} U_j \qquad \text{and} \qquad \tilde{U}_{k,\beta,t} \stackrel{\text{def}}{=} p \frac{\beta}{(\log t)^{2/d}} \tilde{U}_k. \tag{2.34}$$

Finally we define the "repaired version" of the potential:

$$\tilde{V}^M_{\beta,t} \stackrel{\text{def}}{=} p \frac{\beta}{(\log t)^{2/d}} \left[(V \wedge M - U) 1_{\mathscr{T}_t} + 1_{\mathscr{T}_t^c} \right]. \tag{2.35}$$

We apply the following lemma (in (2.45) below) to the inequality

$$V_{\beta,t} \geq p^{-1} \tilde{V}^M_{\beta,t} + \sum_{k \in \{0,1\}^d} p^{-1} \tilde{U}_{k,\beta,t} \qquad \text{over } \mathscr{T}_t, \tag{2.36}$$

with $p_i = p$ for all i in (2.37):

Lemma 2.3. *Given a lower bound $V \geq \sum_{i=1}^n U_i$ of a potential V over a connected open set $B \subseteq \mathbb{R}^d$ and weights $p_1, \ldots, p_n > 1$ with $\sum_i p_i^{-1} = 1$ we have*

$$\lambda_V(B) \geq \sum_{i=1}^n p_i^{-1} \lambda_{p_i U_i}(B). \tag{2.37}$$

Proof of Lemma 2.3. Let $T_B \stackrel{\text{def}}{=} \inf\{s : Z_s \notin B\}$ denote the exit time of $Z.$ from B. We have for $T > 0$ by monotonicity of the expectation and Hölder's inequality:

$$\begin{aligned} &E_x \left[\exp\left\{ -\int_0^T V(Z_s)\, ds \right\}, T_B > T \right] \\ &\leq \prod_{i=1}^n E_x \left[\exp\left\{ -\int_0^T U_i(Z_s)\, ds \right\}^{p_i}, T_B > T \right]^{\frac{1}{p_i}}. \end{aligned} \tag{2.38}$$

Consequently, the Feynman-Kac representation of the principal Dirichlet eigenvalue implies for every $x \in B$:

$$\begin{aligned} \lambda_V(B) &= -\lim_{T\to\infty} \frac{1}{T} \log E_x \left[\exp\left\{ -\int_0^T V(Z_s)\, ds \right\}, T_B > T \right] \\ &\geq -\lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^n p_i^{-1} \log E_x \left[\exp\left\{ -\int_0^T p_i U_i(Z_s)\, ds \right\}, T_B > T \right] \\ &= \sum_{i=1}^n p_i^{-1} \lambda_{p_i U_i}(B). \end{aligned} \tag{2.39}$$

□

The next lemma bounds the probability that Poissonian points clump too much inside a box $K_{i,j}$:

Lemma 2.4. *For all* $\eta > 0$ *there exists* $M = M(\eta) > 1$ *such that for all (sufficiently large)* t *the following bound holds:*

$$\mathbb{P}[F_{i,j}^{(0)}] \le e^{-(\alpha(t)\log t)/2}. \tag{2.40}$$

Proof of Lemma 2.4. Take a fixed $\lambda > 1/\eta$. The exponential Chebyshev-inequality implies

$$\mathbb{P}[F_{i,j}^{(0)}(t, M, \eta)] \le e^{-\lambda\eta|K_{i,j}|/2}\mathbb{E}\left[\exp\left\{\lambda[\omega(C_0) - (\omega(C_0) \wedge \bar{M})]\right\}\right]^{|K_{i,j}|/|C_0|}. \tag{2.41}$$

By the dominated convergence theorem, the last expectation goes to 1 as $M \to \infty$; recall that $\bar{M}$ is proportional to M. We choose M so large that this expectation is less than $e^{a^d(\lambda\eta-1)/2} \le e^{|C_0|(\lambda\eta-1)/2}$, recall (2.25). We get $\mathbb{P}[F_{i,j}^{(0)}(t, M, \eta)] \le e^{-|K_{i,j}|/2} = e^{-(\alpha(t)\log t)/2}$. This proves the lemma. □

Lemma 2.5. *Assume* $d \ge 4$. *There is a* $\beta_0 = \beta_0(d) > 0$ *such that for all* $\beta \in (0, \beta_0)$ *and* $\eta > 0$ *there exists a* $t_c > 0$ *such that the following holds for all* $t > t_c$*:*

$$\mathbb{P}\left[\lambda_{V_{\beta,t}}(\mathscr{T}_t) \ge p^{-1}\lambda_{\tilde{V}_{\beta,t}^M}(\mathscr{T}_t) - c_4(d)f(l(t))\right] \ge 1 - t^{-2}, \tag{2.42}$$

where $c_4(d) \stackrel{\text{def}}{=} 2^d p^{-1} = 2^d/(2^d + 1)$ *and* $M = M(\eta)$ *is taken from Lemma 2.4.*

Proof of Lemma 2.5. We are going to estimate $\lambda_{\tilde{U}_{k,\beta,t}}(\mathscr{T}_t)$ by inserting the decomposition $\tilde{U}_{k,\beta,t} = \sum_{j\in J_k} U_{j,\beta,t}$ in Lemma 2.2: Choose $\beta_0 \stackrel{\text{def}}{=} \left[(d+3)2^{d/2+3} p^{d/2}a_d\right]^{-2/d}$. We observe for $j \ne j'$, $j, j' \in J_k$,

$$\text{dist}(A_j, A_{j'}) \ge l(t), \tag{2.43}$$

and there exists a $t_0(d, \beta_0) > 0$ such that for all $t > t_0(d, \beta_0)$, $\beta \in (0, \beta_0)$ and all Poissonian cloud configurations ω the pointwise lower bound $\tilde{U}_{k,\beta,t} \ge -1$ is valid; see (2.33) and (2.34). We define the event

$$E_j = E_j(d, \eta, \beta, t) \stackrel{\text{def}}{=} \left\{a_d \int |U_{j,\beta,t}|^{\frac{d}{2}}\, dx < 1\right\}; \tag{2.44}$$

we remark that M is according to Lemma 2.4 a fixed constant depending only on η. Lemma 2.2 implies that for $l(t) > L(d)$ the estimate $\lambda_{\tilde{U}_{k,\beta,t}}(\mathscr{T}_t) \ge -f(l(t))$ holds on the event $\bigcap_{j\in J} E_j$; therefore by Lemma 2.3 and (2.36):

$$\begin{aligned}\lambda_{V_{\beta,t}}(\mathscr{T}_t) &\ge p^{-1}\lambda_{\tilde{V}_{\beta,t}^M}(\mathscr{T}_t) + p^{-1}\sum_{k\in\{0,1\}^d}\lambda_{\tilde{U}_{k,\beta,t}}(\mathscr{T}_t)\\ &\ge p^{-1}\lambda_{\tilde{V}_{\beta,t}^M}(\mathscr{T}_t) - c_4(d)f(l(t)).\end{aligned} \tag{2.45}$$

There remains to derive a lower bound for $\mathbb{P}[\bigcap_{j\in J} E_j]$ which converges to 1 sufficiently fast. Set $h(\xi) \stackrel{\text{def}}{=} \xi^{\frac{d}{2}} 1_{\{\xi\geq\eta/2\}}$ and $c_5(d) \stackrel{\text{def}}{=} 2p^{\frac{d}{2}} a_d$, then

$$E_j^c = \left\{ c_5(d)\frac{\beta^{d/2}}{\log t}\sum_{i\in I}(h(\xi_{i,j}) \vee 1_{F_{i,j}^{(0)}})|K_{i,j}| \geq 2 \right\} \subseteq E_j^{(1)c} \cup E_j^{(2)c}, \tag{2.46}$$

with the events

$$E_j^{(1)c} \stackrel{\text{def}}{=} \left\{ \sum_{i\in I} h(\xi_{i,j}) \geq \frac{c_6}{\alpha(t)} \right\} \quad \text{and} \quad E_j^{(2)c} \stackrel{\text{def}}{=} \left\{ \sum_{i\in I} 1_{F_{i,j}^{(0)}} \geq \frac{c_6}{\alpha(t)} \right\}; \tag{2.47}$$

here we abbreviate $c_6 = c_6(\beta, d) = c_5(d)^{-1}\beta^{-d/2}$; recall $|K_{i,j}| = \alpha(t)\log t$.

We divide h in different "grey-scales":
For $\xi \leq 1$ we set $h_1(\xi) \stackrel{\text{def}}{=} \sum_{n=1}^{N(\eta)} 2^{(-n+1)d/2} 1_{\{2^{-n+1}\geq\xi>2^{-n}\}} \geq h(\xi)$ with $N(\eta) \in \mathbb{N}$ so large that $2^{-N(\eta)} < \eta/2$; one should note that $\xi_{i,j} \leq 1$. Using this we estimate

$$E_j^{(1)c} \subseteq \left\{ \sum_{i\in I} h_1(\xi_{i,j}) \geq \frac{c_6}{\alpha(t)} \right\} \tag{2.48}$$

(We introduce the abbreviations $N_n = N_n(d, \beta, t) \stackrel{\text{def}}{=} \frac{c_6}{\alpha(t)} 2^{(n-1)d/2}$ and $\varepsilon_n \stackrel{\text{def}}{=} 2^{-n}$:)

$$= \left\{ \sum_{n=1}^{N(\eta)} N_n^{-1} \left|\{i \in I : 2\varepsilon_n \geq \xi_{i,j} > \varepsilon_n\}\right| \geq 1 \right\} \tag{2.49}$$

(Define the finite set $R_\eta \stackrel{\text{def}}{=} \{\frac{k}{2N(\eta)} : k \in [0, 2N(\eta)] \cap \mathbb{Z}\}$, and for $r \geq 0$ define $q_\eta(r) \stackrel{\text{def}}{=} \max\{\rho \in R_\eta : \rho \leq r\}$; i.e. $q_\eta(r) \leq r < q_\eta(r) + \frac{1}{2N(\eta)}$ for $0 \leq r \leq 1$, and $q_\eta(r) = 1$ for $r > 1$. Consequently the assumption $\sum_{n=1}^{N(\eta)} r_n \geq 1$ with all $r_n \geq 0$ implies $\sum_{n=1}^{N(\eta)} q_\eta(r_n) \geq 1/2$; this statement is trivial when there exists a $r_n > 1$:)

$$\subseteq \left\{ \sum_{n=1}^{N(\eta)} q_\eta\left(N_n^{-1} \left|\{i \in I : 2\varepsilon_n \geq \xi_{i,j} > \varepsilon_n\}\right|\right) \geq \frac{1}{2} \right\} \tag{2.50}$$

(The next union runs over the finite set
$\mathscr{R}_\eta \stackrel{\text{def}}{=} \{\rho = (\rho_1, \dots, \rho_{N(\eta)}) \in R_\eta^{N(\eta)} : \sum_{n=1}^{N(\eta)} \rho_n \geq \frac{1}{2}\}$:)

$$\subseteq \bigcup_{\rho\in\mathscr{R}_\eta} \bigcap_{n=1}^{N(\eta)} \left\{ \left|\{i \in I : 2\varepsilon_n \geq \xi_{i,j} > \varepsilon_n\}\right| \geq \rho_n N_n \right\} \tag{2.51}$$

(We prepare the application of the van den Berg/Kesten-inequality: Let $\mathscr{I}_{N(\eta)}$ denote the set of all families $(I_1, \dots, I_{N(\eta)})$ of pairwise disjoint subsets of I; some of the I_n may be empty:)

$$= \bigcup_{\rho\in\mathscr{R}_\eta} \bigcup_{(I_n)\in\mathscr{I}_{N(\eta)}} \bigcap_{n=1}^{N(\eta)} \left\{ \left|\{i \in I_n : 2\varepsilon_n \geq \xi_{i,j} > \varepsilon_n\}\right| \geq \rho_n N_n \right\}$$

$$\subseteq \bigcup_{\rho\in\mathscr{R}_\eta} \bigcup_{(I_n)\in\mathscr{I}_{N(\eta)}} \bigcap_{n=1}^{N(\eta)} \left\{ \left|\{i \in I_n : \xi_{i,j} > \varepsilon_n\}\right| \geq \rho_n N_n \right\}. \tag{2.52}$$

Now we apply the van den Berg/Kesten-inequality; see Appendix A for the precise version of the BK-inequality that we use here. (Roughly speaking, the events $\{\omega \in \Omega : \left|\{i \in I : \xi_{i,j}(\omega) > \varepsilon_n\}\right| \geq \rho_n N_n\}$ are decreasing, since the random variables $\xi_{i,j}$ are decreasing functions of the Poissonian cloud configuration ω, and they "need to occur on disjoint domains".) We get:

$$\mathbb{P}\left[\bigcup_{(I_n)\in\mathscr{I}_{N(\eta)}} \bigcap_{n=1}^{N(\eta)} \left\{ \left|\{i \in I_n : \xi_{i,j} > \varepsilon_n\}\right| \geq \rho_n N_n \right\} \right]$$
$$\leq \prod_{n=1}^{N(\eta)} \mathbb{P}\left[\left|\{i \in I : \xi_{i,j} > \varepsilon_n\}\right| \geq \rho_n N_n \right]. \tag{2.53}$$

We introduce the events

$$F_{i,j}^{(n)} \stackrel{\text{def}}{=} \{\xi_{i,j} > \varepsilon_n\} = \{\omega(K_{i,j}) < (1-\varepsilon_n)|K_{i,j}|\}, \tag{2.54}$$

for $n \geq 1$, $i \in I$, $j \in J$; one should not confuse these events $F_{i,j}^{(n)}$, $n \geq 1$ (which take care about filling holes on a "grey-scale level") with the event $F_{i,j}^{(0)}$, which was introduced in (2.27), and which takes care of "clumps" in the Poissonian cloud configuration; however the similar notation was chosen intentionally to treat both kinds of "repairing the potential" at the same time below. We get

$$\mathbb{P}[E_j^{(1)c}] \leq \sum_{\rho\in\mathscr{R}_\eta} \prod_{n=1}^{N(\eta)} \mathbb{P}\left[\left|\{i \in I : \xi_{i,j} > \varepsilon_n\}\right| \geq \rho_n N_n \right]$$
$$= \sum_{\rho\in\mathscr{R}_\eta} \prod_{n=1}^{N(\eta)} \mathbb{P}\left[\sum_{i\in I} 1_{F_{i,j}^{(n)}} \geq \rho_n N_n \right] \tag{2.55}$$

(We may drop the factors in the last product with $\rho_n = 0$, since they are 1: let $\mathscr{M}_\rho \stackrel{\text{def}}{=} \{n \in \mathbb{N} : 1 \leq n \leq N(\eta),\ \rho_n > 0\}$ for $\rho \in \mathscr{R}_\eta$:)

$$= \sum_{\rho\in\mathscr{R}_\eta} \prod_{n\in\mathscr{M}_\rho} \mathbb{P}\left[\sum_{i\in I} 1_{F_{i,j}^{(n)}} \geq \rho_n N_n \right]. \tag{2.56}$$

Next we use Lemma 1.2 to estimate

$$\mathbb{P}[F_{i,j}^{(n)}] \leq \exp\left(-\frac{1}{2}\varepsilon_n^2 |K_{i,j}|\right) = \exp\left(-\frac{1}{2}\varepsilon_n^2 \alpha(t)\log t\right). \tag{2.57}$$

We now treat $E_j^{(1)c}$ and $E_j^{(2)c}$ both at the same time; recall that these two events correspond to the two different reasons to "repair" the potential: To get a uniform notation, we introduce an additional point (call it $*$) to the index set: $\mathscr{R}_\eta^* \stackrel{\text{def}}{=} \{*\} \cup \mathscr{R}_\eta$; the extra point $*$ takes care of $E_j^{(2)c}$. We set $\mathscr{M}_* \stackrel{\text{def}}{=} \{0\}$, $\varepsilon_0 \stackrel{\text{def}}{=} 1$, and

$$M_{\rho,n} = M_{\rho,n}(d,\beta,t) \stackrel{\text{def}}{=} \begin{cases} \rho_n N_n = \frac{2^{(n-1)d/2}\rho_n}{c_5(d)\beta^{d/2}\alpha(t)} & \text{for } \rho \in \mathscr{R}_\eta, n \in \mathscr{M}_\rho, \\ \frac{c_6}{\alpha(t)} = \frac{1}{c_5(d)\beta^{d/2}\alpha(t)} & \text{for } \rho = *, n = 0. \end{cases} \tag{2.58}$$

We join (2.46), (2.47) and (2.56) to obtain

$$\mathbb{P}[E_j^c] \leq \sum_{\rho \in \mathscr{R}_\eta^*} \prod_{n \in \mathscr{M}_\rho} \mathbb{P}\left[\sum_{i \in I} 1_{F_{i,j}^{(n)}} \geq M_{\rho,n}\right]. \tag{2.59}$$

We use the exponential Chebyshev-inequality, the independence of the events $(F_{i,j}^{(n)})_{i\in I}$ ($n \in \mathscr{M}_\rho$, $\rho \in \mathscr{R}_\eta^*$), and the bounds (2.57) and (2.40) to get for $\sigma > 0$:

$$\begin{aligned} &\mathbb{P}\left[\sum_{i \in I} 1_{F_{i,j}^{(n)}} \geq M_{\rho,n}\right] \leq e^{-\sigma M_{\rho,n}} \mathbb{E}\left[\exp\left(\sigma \sum_{i\in I} 1_{F_{i,j}^{(n)}}\right)\right] \\ &= e^{-\sigma M_{\rho,n}} \prod_{i\in I} \mathbb{E}[\exp(\sigma 1_{F_{i,j}^{(n)}})] \leq e^{-\sigma M_{\rho,n}} \prod_{i \in I}(1 + e^\sigma \mathbb{P}[F_{i,j}^{(n)}]) \\ &\leq e^{-\sigma M_{\rho,n}} \prod_{i\in I} \exp(e^\sigma \mathbb{P}[F_{i,j}^{(n)}]) \\ &\leq \exp\left(-\sigma M_{\rho,n} + |I| e^\sigma \exp\left(-\frac{1}{2}\varepsilon_n^2\alpha(t)\log t\right)\right) \end{aligned} \tag{2.60}$$

(We choose the optimal σ, which is determined by $|I|e^\sigma \exp(-\frac{1}{2}\varepsilon_n^2\alpha(t)\log t) = M_{\rho,n}$, i.e. $\sigma = \frac{1}{2}\varepsilon_n^2\alpha(t)\log t - \log|I| + \log M_{\rho,n} > 0$, t large; the fact $\sigma > 0$ can be seen using (2.19) because $\log|I| \leq d \log l(t)$, for large t, and $\log M_{\rho,n} > 0$, for large t:)

$$= \exp\left(-\frac{M_{\rho,n}}{2}\varepsilon_n^2\alpha(t)\log t + M_{\rho,n}\log|I| - M_{\rho,n}\log M_{\rho,n} + M_{\rho,n}\right). \tag{2.61}$$

Therefore we obtain the estimate

$$\mathbb{P}[E_j^c] \le \sum_{\rho\in\mathscr{R}_\eta^*}\prod_{n\in\mathscr{M}_\rho}\exp\Big(-\frac{M_{\rho,n}}{2}\varepsilon_n^2\alpha(t)\log t + M_{\rho,n}\log|I| - M_{\rho,n}\log M_{\rho,n} + M_{\rho,n}\Big), \tag{2.62}$$

and hence

$$\mathbb{P}\Big[\bigcup_{j\in J}E_j^c\Big] \le \sum_{j\in J}\mathbb{P}[E_j^c] \tag{2.63}$$

$$\le \sum_{\rho\in\mathscr{R}_\eta^*}\exp\Bigg(\log|J| + \sum_{n\in\mathscr{M}_\rho}\Big(-\frac{M_{\rho,n}}{2}\varepsilon_n^2\alpha(t)\log t + M_{\rho,n}\log|I| - M_{\rho,n}\log M_{\rho,n} + M_{\rho,n}\Big)\Bigg)$$

(using $\log|J| = d(\log 2 + \log t - \log l(t))$ and $\log|I| = d\log l(t) - \log\alpha(t) - \log\log t$:)

$$= \sum_{\rho\in\mathscr{R}_\eta^*}\exp\Bigg(\Big(d - \sum_{n\in\mathscr{M}_\rho}\frac{M_{\rho,n}(d,\beta,t)}{2}\varepsilon_n^2\alpha(t)\Big)\log t + o_{\eta,\beta,\rho,d,t}\Bigg), \tag{2.64}$$

with the higher order terms

$$o_{n,\eta,\beta,\rho,d,t} \stackrel{\text{def}}{=} dM_{\rho,n}\log l(t) - M_{\rho,n}\log M_{\rho,n} - M_{\rho,n}\log\alpha(t) - M_{\rho,n}\log\log t + M_{\rho,n},$$

$$o_{\eta,\beta,\rho,d,t} \stackrel{\text{def}}{=} d\log 2 - d\log l(t) + \sum_{n\in\mathscr{M}_\rho} o_{n,\eta,\beta,\rho,d,t}.$$

Recall Definition (2.58): $M_{\rho,n}(d,\beta,t)$ is proportional to $\alpha(t)^{-1}$. To estimate the higher order terms, we use the choice of the asymptotic behavior (2.18)–(2.19) of $l(t)$ and $\alpha(t)$; we also use $\log\log t/(\alpha(t)\log t) \stackrel{t\to\infty}{\longrightarrow} 0$, which follows from (2.18) and $l(t)^d \gg \log t$: we get $o_{\eta,\beta,\rho,d,t}/\log t \stackrel{t\to\infty}{\longrightarrow} 0$. Now we examine the leading term in (2.64) for $\rho\in\mathscr{R}_\eta$:

$$\sum_{n\in\mathscr{M}_\rho}\frac{M_{\rho,n}}{2}\varepsilon_n^2\alpha(t) = \sum_{n\in\mathscr{M}_\rho}2c_7(d)\frac{2^{(d/2-2)n}}{\beta^{d/2}}\rho_n \ge \frac{c_7(d)}{\beta^{d/2}}, \tag{2.65}$$

we have set $c_7(d) \stackrel{\text{def}}{=} c_5(d)^{-1}2^{-d/2-2}$, and we have used $d\ge 4$, i.e. $2^{(d/2-2)n}\ge 1$, and $\sum_{n\in\mathscr{M}_\rho}\rho_n \ge \frac{1}{2}$. In the case $\rho = *$ we obtain the right-hand side in (2.65) as a lower bound, too: $M_{*,0}\varepsilon_0^2\alpha(t)/2 \geq c_7(d)/\beta^{d/2}$. It is important to note that this right-hand side in (2.65) does not depend on η; this allows us to choose $\beta_0 = \left[(d+3)2^{d/2+3}p^{d/2}a_d\right]^{-2/d}$ independent of the value of η. Inserting these estimates, we get

$$\mathbb{P}\left[\bigcup_{j\in J} E_j^c\right] \leq \sum_{\rho\in\mathscr{R}_\eta^*} \exp\left(\left(d - c_7(d)\beta^{-d/2}\right)\log t + o_{\eta,\beta,\rho,d,t}\right). \tag{2.66}$$

The sum over ρ is finite; consequently we get the following result: By our choice $\beta_0 = c_7^{2/d}(d+3)^{-2/d}$ we see that for all $\beta \in (0, \beta_0)$ and $\eta > 0$ there is $t_c \geq t_0(d, \beta_0) > 0$ such that the following bound holds for all $t > t_c$:

$$\mathbb{P}\left[\bigcup_{j\in J} E_j(d, \eta, \beta, t)^c\right] \leq t^{-2}. \tag{2.67}$$

This proves Lemma 2.5. □

Remark. The case $d > 4$ could be handled in a simpler way: one may choose one specified ρ only (instead of all $\rho \in \mathscr{R}_\eta$), e.g. $\rho_n = c_8(d, \delta)2^{(d/2-\delta)n}$ for some $0 < \delta < d/2 - 2$ and $c_8(d, \delta)^{-1} \stackrel{\text{def}}{=} \sum_{n=1}^\infty 2^{(d/2-\delta)n}$, drop step (2.50), and replace $\bigcup_{\rho\in\mathscr{R}_\eta}\bigcap_{n=1}^{N(\eta)}$ in (2.51) by $\bigcup_{n=1}^{N(\eta)}$. So for $d > 4$ one does not have to apply the van den Berg/Kesten-inequality. However, this simplification does not suffice for $d = 4$. It is interesting to examine why the method does not apply for $d = 3$: not the small, deep holes change the picture, but the large, shallow ones do (see (2.65)). □

2.3. Lower bound on the principal Dirichlet eigenvalue for the "repaired" potential

Consider the unscaled version of the "repaired" potential

$$\tilde{V}_t^M \stackrel{\text{def}}{=} \frac{(\log t)^{2/d}}{p\beta}\tilde{V}_{\beta,t}^M = (V \wedge M - U)1_{\mathscr{T}_t} + 1_{\mathscr{T}_t^c}; \tag{2.68}$$

$\tilde{V}_{\beta,t}^M$ is defined in (2.35). We abbreviate $\kappa(t) \stackrel{\text{def}}{=} (\alpha(t)\log t)^{1/d} = \operatorname{diam}(K_{i,j})/\sqrt{d}$. This is the scale on which we have defined our "stuffing" function. It scales as follows:

$$1 \ll \kappa(t) \ll (\log t)^{1/d}, \qquad \text{as } t \to \infty. \tag{2.69}$$

The following lemma estimates integrals of the "repaired" potential with respect to (sufficiently regular) test functions:

Lemma 2.6. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ denote a non-negative, uniformly Lipschitz continuous, compactly supported test function with Lipschitz constant ℓ_φ. Assume that t is so large that $\kappa(t) \geq a$. Let the support of φ be contained in a ball with a radius $R \geq \kappa(t)$. Then there is a constant $c_9 = c_9(d) > 0$ such that*

$$\int_{\mathbb{R}^d} \varphi(x) \tilde{V}_t^M(x)\, dx \geq (1-\eta) \left(\int_{\mathbb{R}^d} \varphi(x)\, dx - c_9 R^d \kappa(t) \ell_\varphi \right). \tag{2.70}$$

The test function φ and the radius R may both depend on t; we will choose later the right scale $R = R(t)$ to apply the lemma.

Proof of Lemma 2.6. We define the boxes $K_l \stackrel{\text{def}}{=} \kappa(t)l + [0, \kappa(t))^d$, for all $l \in \mathbb{Z}^d$. We compare this with the definition (2.23) of $K_{i,j}$: each K_l with $K_l \cap \mathcal{T}_t \neq \emptyset$ is some $K_{i,j}$. We remark that $\kappa(t)$ is chosen such that we do not have to deal with fractions of boxes at the boundary of $\mathcal{T}_t$ (see (2.20)). C_m ($m \in \mathbb{Z}^d$) are the boxes defined in (2.25) (they live on the same scale as the support of W). We estimate $\tilde{V}_t^M$ from below by the following sum:

$$\tilde{V}_t^M(x) \geq \sum_{l \in \mathbb{Z}^d} \tilde{V}_t^{l,\bar{M}}(x), \tag{2.71}$$

where (using the abbreviation $b_m^{(M)} \stackrel{\text{def}}{=} (\bar{M}/\omega(C_m)) \wedge 1$ with $b_m^{(M)} = 1$ for $\omega(C_m) = 0$; recall $\bar{M} = M/(3^d W_\infty)$):

$$\tilde{V}_t^{l,\bar{M}}(x) \stackrel{\text{def}}{=} \begin{cases} -1_{K_l}(x)U(x) + \sum\limits_{\substack{m \in \mathbb{Z}^d \\ C_m \cap K_l \neq \emptyset}} b_m^{(M)} \int_{C_m} W(x-y)\, \omega(dy) & \text{for } K_l \cap \mathcal{T}_t \neq \emptyset, \\ 1_{K_l}(x) & \text{for } K_l \cap \mathcal{T}_t = \emptyset. \end{cases} \tag{2.72}$$

To verify (2.71) we remark that on $\mathcal{T}_t^c$ the inequality is clear, whereas on $\mathcal{T}_t$ we observe the following: On $\{x \in \mathcal{T}_t : V(x) \leq M\}$ the inequality follows from $b_m^{(M)} \leq 1$ and the fact that we only increase the potential if we integrate also over the obstacles in the a-neighborhood of $\mathcal{T}_t$ (for "boundary" boxes K_l). On $\{x \in \mathcal{T}_t : V(x) > M\}$:

$$b_m^{(M)} \int_{C_m} W(x-y)\omega(dy) \leq W_\infty b_m^{(M)} \omega(C_m) \leq W_\infty \bar{M}. \tag{2.73}$$

Since the box C_m which contains x has at most $3^d - 1$ neighboring boxes (all the other boxes $C_{m'}$ do note lie within the range of $W(x - \cdot)$) the claim follows from $3^d W_\infty \bar{M} = M = V(x) \wedge M$ (we apply (2.73) to C_m and its $3^d - 1$ neighboring boxes).

The function $\tilde{V}_t^{l,\bar{M}}(x)$ is supported on the a-neighborhood K_l^a of K_l, which has diameter $\operatorname{diam}(K_l^a) = \operatorname{diam}(K_l) + 2a \leq c_{10}\kappa(t)$ with $c_{10} = c_{10}(d) \stackrel{\text{def}}{=} \sqrt{d} + 2$

(by our assumptions on t). We claim that for all $l \in \mathbb{Z}^d$

$$\int_{\mathbb{R}^d} \tilde{V}_t^{l,\bar{M}}(x)\,dx \geq (1-\eta)|K_l|. \tag{2.74}$$

This is clear for l with $K_l \cap \mathscr{T}_t = \emptyset$; for l with $K_l \cap \mathscr{T}_t \neq \emptyset$ we use Fubini's theorem and $\|W\|_1 = 1$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \tilde{V}_t^{l,\bar{M}}(x)\,dx &= -\int_{K_l} U(x)\,dx + \sum_{\substack{m\in\mathbb{Z}^d\\ C_m\cap K_l\neq\emptyset}} b_m^{(M)} \int_{\mathbb{R}^d}\int_{C_m} W(x-y)\,\omega(dy)\,dx \\ &= -\int_{K_l} U(x)\,dx + \sum_{\substack{m\in\mathbb{Z}^d\\ C_m\cap K_l\neq\emptyset}} b_m^{(M)}\omega(C_m) \\ &\geq (1-\eta)|K_l|,\end{aligned} \tag{2.75}$$

where in the last step we have used that $b_m^{(M)}\omega(C_m) = \omega(C_m) \wedge \bar{M}$ and (2.31). Hence

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)\tilde{V}_t^{l,\bar{M}}(x)\,dx &\geq \left(\inf_{y\in K_l^a}\varphi(y)\right)\int_{\mathbb{R}^d} \tilde{V}_t^{l,\bar{M}}(x)\,dx \\ &\geq (1-\eta)|K_l|\left(\inf_{y\in K_l^a}\varphi(y)\right) \\ &\geq (1-\eta)\left(\int_{K_l}\varphi(x)\,dx - c_{10}\kappa(t)\ell_\varphi|K_l|\right).\end{aligned} \tag{2.76}$$

Let $L = L(t) \stackrel{\text{def}}{=} \{l \in \mathbb{Z}^d : K_l^a \cap \operatorname{supp}\varphi \neq \emptyset\}$; the cardinality of this set of indices is bounded by $|L| \leq |B_{R+\operatorname{diam}(K_l^a)}(0)|/|K_l| = |B_1(0)|(R+\operatorname{diam}(K_l)+2a)^d/|K_l| \leq c_{11}R^d/|K_l|$, where $c_{11} = c_{11}(d) \stackrel{\text{def}}{=} |B_1(0)|(3+\sqrt{d})^d$. Summing (2.76) over all $l \in L$ and using (2.71) we get

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)\tilde{V}_t^{M}(x)\,dx &\geq (1-\eta)\sum_{l\in L}\left(\int_{K_l}\varphi(x)\,dx - c_{10}\kappa(t)\ell_\varphi|K_l|\right) \\ &\geq (1-\eta)\left(\int_{\mathbb{R}^d}\varphi(x)\,dx - c_9R^d\kappa(t)\ell_\varphi\right),\end{aligned} \tag{2.77}$$

with $c_9 \stackrel{\text{def}}{=} c_{10}c_{11}$. Lemma 2.6 is proved. □

Proof of the lower bound in Theorem 0.2. As in Lemma 2.5 we choose $\beta < \beta_0$, $\eta > 0$ and t_c such that for all $t > t_c$

$$\mathbb{P}\left[\lambda_{V_{\beta,t}}(\mathscr{T}_t) \geq p^{-1}\lambda_{\tilde{V}_{\beta,t}^M}(\mathscr{T}_t) - c_4(d)f(l(t))\right] \geq 1 - t^{-2}. \tag{2.78}$$

Formula (2.35) and the bound (2.33) on U imply

$$0 \le \tilde V^M_{\beta,t} \le p \frac{\beta}{(\log t)^{2/d}} (M+1). \tag{2.79}$$

We use the following estimate of the principal Dirichlet eigenvalue $\lambda_{\tilde V^M_{\beta,t}}(\mathscr{T}_t)$ from below; see e.g. Proposition 3.1.4 in [7]: For all $T > 0$,

$$\lambda_{\tilde V^M_{\beta,t}}(\mathscr{T}_t) \ge \frac{1}{T}\left(1 - \sup_{x\in\mathscr{T}_t} E_x\left[\exp\left(-\int_0^T \tilde V^M_{\beta,t}(Z_s)\,ds\right),\ T < T_{\mathscr{T}_t}\right]\right). \tag{2.80}$$

We apply this estimate on a time scale $T = T(t)$ with

$$\kappa(t)^2 \ll T(t) \ll (\log t)^{2/d}, \qquad \text{as } t \to \infty. \tag{2.81}$$

We start to estimate the right-hand side of (2.80): Let $x \in \mathscr{T}_t$.

$$\begin{aligned} E_x\left[\exp\left(-\int_0^T \tilde V^M_{\beta,t}(Z_s)\,ds\right),\ T < T_{\mathscr{T}_t}\right] &\le E_x\left[\exp\left(-\int_0^T \tilde V^M_{\beta,t}(Z_s)\,ds\right)\right] \\ &\le 1 - E_x\left[\int_0^T \tilde V^M_{\beta,t}(Z_s)\,ds\right] + \frac{1}{2}\left(\frac{Tp\beta(M+1)}{(\log t)^{2/d}}\right)^2, \end{aligned} \tag{2.82}$$

where we have used $e^{-z} \le 1 - z + \frac{1}{2}z^2$, which is valid for all $z \ge 0$, and (2.79). We observe the following asymptotic behavior of the last summand in (2.82):

$$\frac{1}{2}\left(\frac{T(t)p\beta(M+1)}{(\log t)^{2/d}}\right)^2 \ll \frac{T(t)}{(\log t)^{2/d}}, \qquad \text{as } t \to \infty, \tag{2.83}$$

which is valid for fixed M and β. Let $p(s,x,y) \stackrel{\text{def}}{=} (2\pi s)^{-d/2} e^{-|y-x|^2/(2s)}$ denote the Brownian transition density. Set $\Phi_T(y) \stackrel{\text{def}}{=} \int_0^T p(s,0,y)\,ds$; this function has the scaling property $\Phi_T(y) = T^{1-d/2}\Phi_1(T^{-1/2}y)$, and it fulfills $\int_{\mathbb{R}^d} \Phi_T(y)\,dy = T$. We estimate the middle term in expression (2.82): First we use Fubini's theorem twice:

$$\begin{aligned} E_x\left[\int_0^T \tilde V^M_{\beta,t}(Z_s)\,ds\right] &= \int_0^T \int_{\mathbb{R}^d} \tilde V^M_{\beta,t}(y) p(s,x,y)\,dy\,ds \\ &= \int_{\mathbb{R}^d} \tilde V^M_{\beta,t}(y)\Phi_T(y-x)\,dy \end{aligned} \tag{2.84}$$

(We introduce a compactly supported, uniformly Lipschitz continuous test function $0 \le \varphi \le \Phi_1$ and a scaled version of it: $\varphi_T(y) \stackrel{\text{def}}{=} T^{1-d/2}\varphi(T^{-1/2}y) \le \Phi_T(y)$:)

$$\ge \int_{\mathbb{R}^d} \tilde V^M_{\beta,t}(y)\varphi_T(y-x)\,dy. \tag{2.85}$$

Let φ be supported in a ball of radius $r_\varphi \geq 1$; then φ_T is supported in a ball with radius

$$R = R(t) \stackrel{\text{def}}{=} \sqrt{T(t)}r_\varphi \gg \kappa(t), \qquad \text{as } t \to \infty. \tag{2.86}$$

Lemma 2.6 provides a lower bound for the term in (2.85):

$$\begin{aligned}\int_{\mathbb{R}^d} \tilde{V}^M_{\beta,t}(y)\varphi_T(y-x)\,dy &= \frac{p\beta}{(\log t)^{2/d}} \int_{\mathbb{R}^d} \tilde{V}^M_t(y)\varphi_T(y-x)\,dy \\ &\geq \frac{p\beta}{(\log t)^{2/d}}(1-\eta)\left(\|\varphi_T\|_1 - c_9 T^{d/2} r^d_\varphi \kappa(t)\ell_{\varphi_T}\right) \\ &= \frac{p\beta T}{(\log t)^{2/d}}(1-\eta)\left(\|\varphi\|_1 - c_9 T^{-1/2}\kappa(t) r^d_\varphi \ell_\varphi\right),\end{aligned} \tag{2.87}$$

where we have used the scaling behavior $\ell_{\varphi_T} = T^{(1-d)/2}\ell_\varphi$ of the Lipschitz constants. Using (2.80)–(2.82) we obtain

$$\frac{(\log t)^{2/d}}{p}\lambda_{\tilde{V}^M_{\beta,t}}(\mathscr{T}_t) \geq \beta(1-\eta)\,\|\varphi\|_1 - e_1(t), \tag{2.88}$$

where by our choice of $T(t)$ (see (2.81))

$$e_1(t) \stackrel{\text{def}}{=} \beta(1-\eta)c_9 T^{-1/2}\kappa(t) r^d_\varphi \ell_\varphi + \frac{1}{2}p\beta^2(M+1)^2 \frac{T}{(\log t)^{2/d}} \stackrel{t\to\infty}{\longrightarrow} 0. \tag{2.89}$$

Using (2.78) we see that for all $t > t_c$:

$$\mathbb{P}\left[(\log t)^{2/d}\lambda_{V_{\beta,t}}(\mathscr{T}_t) \geq \beta(1-\eta)\,\|\varphi\|_1 - e(t)\right] \geq 1 - t^{-2}, \tag{2.90}$$

with (see (2.18) and (2.89))

$$e(t) \stackrel{\text{def}}{=} e_1(t) + c_4(d)(\log t)^{2/d} f(l(t)) \stackrel{t\to\infty}{\longrightarrow} 0. \tag{2.91}$$

Applying the Borel-Cantelli Lemma, Lemma 1.1 and the fact that $\lim_{t\to\infty} \log(t-1)/\log t = 1$, we find that for all $\beta < \beta_0$, for all $\eta > 0$, and for all $0 \leq \varphi \leq \Phi_1$ (compactly supported, uniformly Lipschitz continuous)

$$\mathbb{P}\text{-a.s.} \qquad \liminf_{t\to\infty} (\log t)^{2/d}\lambda_{V_{\beta,t}}(\mathscr{T}_t) \geq \beta(1-\eta)\,\|\varphi\|_1 . \tag{2.92}$$

Finally we take the supremum over φ and let $\eta \to 0$ to see that our claim follows for all $\beta < \beta_0$. This finishes the proof of the lower bound in Theorem 0.2: one may take $\beta_c = \beta_0$ in this theorem. □

3. Upper bounds

The derivation of upper bounds is based on a variational principle. This is obtained using the large deviation theorem of Gärtner and Ellis (Theorem 2.3.6 in [3]). First we prepare the application of this large deviation result: We apply it to integrals of test functions with respect to Poissonian cloud configurations. For this reason we examine the following rate functions:
Let ϕ be a bounded measurable test function with compact support. We define the generating function of the Poisson process:

$$\Lambda_\phi(\sigma) \overset{\text{def}}{=} \log \mathbb{E}\left[\exp\left\{\sigma \int_{\mathbb{R}^d} \phi^2 \, d\omega\right\}\right] = \int_{\mathbb{R}^d} (e^{\sigma\phi^2} - 1)\, dx, \qquad \sigma \in \mathbb{R}, \quad (3.1)$$

and its one–dimensional Fenchel–Legendre transform

$$\Lambda^*_\phi(\mu) \overset{\text{def}}{=} \sup_{\sigma \in \mathbb{R}} \left(\sigma\mu - \Lambda_\phi(\sigma)\right). \quad (3.2)$$

We collect some important properties of this function:

Lemma 3.1. *Assume that* $\|\phi\|_2 = 1$.

1. Λ^*_ϕ *is a convex, non-negative, real-analytic function on the interval* $(0, \infty)$ *with the global minimum* $\Lambda^*_\phi(1) = 0$. *Especially,* Λ^*_ϕ *is monotonically decreasing on the interval* $(0, 1)$.
2. *Set* $S \overset{\text{def}}{=} \{x \in \mathbb{R}^d : \phi(x) \neq 0\}$. *Then* $\Lambda^*_\phi(\mu) \overset{\mu\downarrow 0}{\longrightarrow} |S|$ *with a vertical tangent:* $\frac{d}{d\mu}\Lambda^*_\phi(\mu) \overset{\mu\downarrow 0}{\longrightarrow} -\infty$. *More quantitatively: There are constants* $c_{12} = c_{12}(\phi) > 0$ *and* $c_{13} = c_{13}(\phi)$ *such that for all* $0 < \mu < 1$ *the following upper bound holds:*

$$\Lambda^*_\phi(\mu) \le |S| + c_{12}\mu \log \mu - c_{13}\mu. \quad (3.3)$$

Proof of Lemma 3.1.

Proof of 1. The integrand $x \mapsto e^{\sigma\phi(x)^2} - 1$ depends analytically on σ, and the upper bound $x \mapsto \sup_{\sigma\in K} |e^{\sigma\phi(x)^2} - 1|$ is integrable for all compact subsets K of

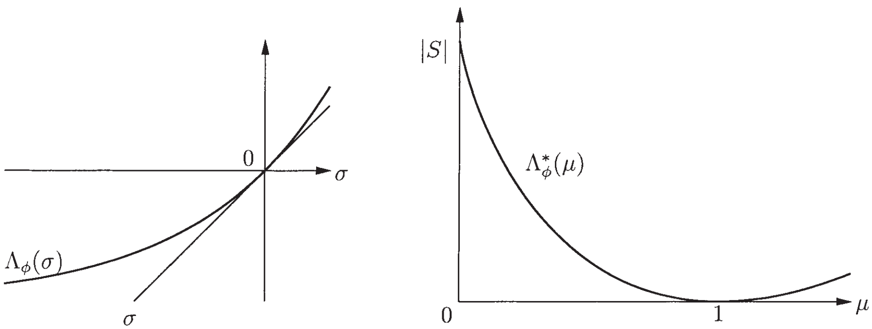

Fig. 2. Typical graphs for Λ_ϕ and Λ^*_ϕ.

the complex plane; hence $\Lambda_\phi(\sigma) < \infty$ for all $\sigma \in \mathbb{R}$, and Λ_ϕ is a real-analytic function. We observe that

$$\Lambda'_\phi(\sigma) = \int_{\mathbb{R}^d} \phi^2 e^{\sigma\phi^2}\, dx \begin{cases} \stackrel{\sigma\to-\infty}{\longrightarrow} 0 \\ \stackrel{\sigma\to+\infty}{\longrightarrow} \infty \end{cases} \tag{3.4}$$

by the dominated convergence theorem and the monotone convergence theorem respectively. Furthermore $\Lambda'_\phi(0) = \|\phi\|_2^2 = 1$. Λ_ϕ is strictly convex since

$$\Lambda''_\phi(\sigma) = \int_{\mathbb{R}^d} \phi^4 e^{\sigma\phi^2}\, dx > 0. \tag{3.5}$$

Consequently the inverse function $\Lambda'^{-1}_\phi : (0,\infty) \to \mathbb{R}$ of Λ'_ϕ is real-analytic as well, and we have the following description of Λ^*_ϕ in terms of this function:

$$\Lambda^*_\phi(\mu) = \Lambda'^{-1}_\phi(\mu)\mu - \Lambda_\phi(\Lambda'^{-1}_\phi(\mu)), \qquad \text{for } 0 < \mu < \infty. \tag{3.6}$$

This shows that Λ^*_ϕ is real-analytic over $(0,\infty)$, too. The convexity of Λ^*_ϕ follows directly from its definition; see [3], Lemma 2.2.5. We evaluate: $\Lambda^*_\phi(1) = -\Lambda_\phi(0) = 0$. This is the global minimum of Λ^*_ϕ, since we have for $0 < \mu = \Lambda'_\phi(\sigma) < \infty$ by (strict) convexity of Λ_ϕ:

$$0 = \Lambda_\phi(0) \ge \Lambda_\phi(\sigma) + (0-\sigma)\Lambda'_\phi(\sigma) = -\Lambda^*_\phi(\mu), \tag{3.7}$$

with equality only for $\mu = 1, \sigma = 0$.

Proof of 2. Differentiation of the equation (3.6) yields $\frac{d}{d\mu}\Lambda^*_\phi(\mu) = \Lambda'^{-1}_\phi(\mu)$ for $\mu \in (0,\infty)$, and using $\Lambda'_\phi(\sigma) \downarrow 0$ as $\sigma \to -\infty$ (see (3.4)) we get $\Lambda'^{-1}_\phi(\mu) \to -\infty$ as $\mu \downarrow 0$.
Let $A \subseteq \mathbb{R}^d$ be measurable, $|A| < \infty$, and $a > 0$. We determine explicitly:

$$\Lambda_{a1_A}(\sigma) = (e^{\sigma a^2} - 1)|A|, \tag{3.8}$$

$$\Lambda^*_{a1_A}(\mu) = |A| + a^{-2}\mu \log(a^{-2}|A|^{-1}\mu) - a^{-2}\mu \stackrel{\mu\downarrow 0}{\longrightarrow} |A|. \tag{3.9}$$

Let $A_\varepsilon \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : |\phi(x)| > \varepsilon\}$ for $\varepsilon > 0$. Set $s \stackrel{\text{def}}{=} \sup|\phi|$, and assume that $\varepsilon > 0$ is so small that $|A_\varepsilon| > 0$. We get:

$$\varepsilon 1_{A_\varepsilon} \le \quad |\phi| \quad \le s1_S, \tag{3.10}$$

$$\Lambda_{\varepsilon 1_{A_\varepsilon}}(\sigma) \ge \Lambda_\phi(\sigma) \ge \Lambda_{s1_S}(\sigma) \qquad \text{for } \sigma \le 0. \tag{3.11}$$

We claim for any two compactly supported test functions ϕ_1, ϕ_2 with $\|\phi_1\|_2 > 0$: If $\Lambda_{\phi_1}(\sigma) \ge \Lambda_{\phi_2}(\sigma)$ holds for all $\sigma \le 0$, then $\Lambda^*_{\phi_1}(\mu) \le \Lambda^*_{\phi_2}(\mu)$ holds for all $\mu \in (0, \|\phi_1\|_2^2)$. To prove this claim for a given μ, we observe that there is a $\sigma \in \mathbb{R}$ with $\Lambda'_{\phi_1}(\sigma) = \mu$ as a consequence of (3.4). (One should note that $\|\phi\|_2 > 0$ suffices to derive (3.4); the precise value $\|\phi\|_2 = 1$ is not used in that derivation.) Our assumption $\mu < \|\phi_1\|_2^2 = \Lambda'_{\phi_1}(0)$ and the strict monotonicity of Λ'_{ϕ_1} imply $\sigma < 0$.

Consequently, using (3.6), $\Lambda^*_{\phi_2}(\mu) \geq \mu\sigma - \Lambda_{\phi_2}(\sigma) \geq \mu\sigma - \Lambda_{\phi_1}(\sigma) = \Lambda^*_{\phi_1}(\mu)$, which proves the above claim. Combining this with (3.11), we get

$$\begin{aligned} \Lambda^*_{\varepsilon 1_{A_\varepsilon}}(\mu) \leq \Lambda^*_\phi(\mu) \qquad & \text{for } 0 < \mu < \varepsilon^2 |A_\varepsilon|, \\ \Lambda^*_\phi(\mu) \leq \Lambda^*_{s1_S}(\mu) \qquad & \text{for } 0 < \mu < \|\phi\|_2^2 = 1. \end{aligned} \tag{3.12}$$

We take the limit $\mu \downarrow 0$ in (3.12):

$$|A_\varepsilon| = \lim_{\mu\downarrow 0} \Lambda^*_{\varepsilon 1_{A_\varepsilon}}(\mu) \leq \liminf_{\mu\downarrow 0} \Lambda^*_\phi(\mu) \leq \limsup_{\mu\downarrow 0} \Lambda^*_\phi(\mu) \leq \lim_{\mu\downarrow 0} \Lambda^*_{s1_S}(\mu) = |S|. \tag{3.13}$$

This implies $\lim_{\mu\downarrow 0} \Lambda^*_\phi(\mu) = |S|$ using $|A_\varepsilon| \overset{\varepsilon\to 0}{\longrightarrow} |S|$. The quantitative bound (3.3) is a consequence of (3.12) and (3.9). Lemma 3.1 is proved. □

The function Λ^*_ϕ plays an essential role in the following variational principle:

Lemma 3.2. *Assume that* $\phi \in H^1(\mathbb{R}^d)$ *is continuous, compactly supported, and normalized:* $\|\phi\|_2 = 1$. *Further assume that* $\mu \in (0, 1)$ *fulfills*

$$\Lambda^*_\phi(\mu) < d. \tag{3.14}$$

Then

$$\mathbb{P}\text{–}a.s. \qquad \limsup_{t\to\infty} (\log t)^{2/d}\, \lambda_{V_{\beta,t}}(\mathscr{T}_t) \leq \frac{1}{2}\,\|\nabla\phi\|_2^2 + \beta\mu. \tag{3.15}$$

Proof of Lemma 3.2. Assume that ϕ is supported in $B_r(0)$, $r > 0$ fixed, with a positive distance between $\operatorname{supp}\phi$ and $B_r(0)^c$. For $t > 0$ we choose a pairwise disjoint family of balls $B_{r(\log t)^{1/d}}(y) \subseteq \mathscr{T}_t$, $y \in Y_{t,r}$, where $\log|Y_{t,r}|/\log t \overset{t\to\infty}{\longrightarrow} d$. (To be specific: one may choose $Y_{t,r} \overset{\text{def}}{=} 2r(\log t)^{1/d}\mathbb{Z}^d \cap \mathscr{T}_{t-r(\log t)^{1/d}}$.) We define $\phi_{y,t}$ to be a scaled and translated version of ϕ supported in $B_{r(\log t)^{1/d}}(y)$:

$$\phi_{y,t}(x) \overset{\text{def}}{=} (\log t)^{-1/2}\phi((\log t)^{-1/d}(x-y)); \tag{3.16}$$

the normalizing factor is chosen such that $\left\|\phi_{y,t}\right\|_2 = 1$. By the variational characterisation (0.5) of the principal Dirichlet eigenvalue we know

$$\lambda_{V_{\beta,t}}(\mathscr{T}_t) \leq \min_{y\in Y_{t,r}} \left(\frac{1}{2}\left\|\nabla\phi_{y,t}\right\|_2^2 + \int_{\mathbb{R}^d} V_{\beta,t}\phi_{y,t}^2\,dx \right); \tag{3.17}$$

and by scaling:

$$(\log t)^{2/d}\lambda_{V_{\beta,t}}(\mathscr{T}_t) \leq \frac{1}{2}\,\|\nabla\phi\|_2^2 + \beta \min_{y\in Y_{t,r}} \int_{\mathbb{R}^d} V\phi_{y,t}^2\,dx. \tag{3.18}$$

We rewrite the last integral, using the notation $(\psi_1^- * \psi_2)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^d} \psi_1(z - x)\psi_2(z)\,dz$:

$$\int_{\mathbb{R}^d} V\phi_{y,t}^2\,dx = \int_{\mathbb{R}^d} W^- * \phi_{y,t}^2\,d\omega. \tag{3.19}$$

Therefore the integral in (3.18) depends only on the points of the Poissonian cloud configuration in an a–neighborhood of $\operatorname{supp}\phi_{y,t}$. This a–neighborhood of $\operatorname{supp}\phi_{y,t}$ is contained in $B_{r(\log t)^{1/d}}(y)$, at least for large t. Since these balls $B_{r(\log t)^{1/d}}(y)$, $y \in Y_{t,r}$, are pairwise disjoint, this implies that $\int_{\mathbb{R}^d} W^- * \phi_{y,t}^2 \, d\omega$, $y \in Y_{t,r}$, are i.i.d. random variables, at least for large t. Using the Laplace transform of a Poisson process we get the generating functions of these random variables; in the calculation we use a scaled version of W, defined by $W_t(x) \stackrel{\text{def}}{=} (\log t) W((\log t)^{1/d} x)$, $\|W_t\|_1 = 1$:

$$\begin{aligned}
&\frac{1}{\log t} \log \mathbb{E}\left[\exp\left\{(\log t)\sigma \int_{\mathbb{R}^d} W^- * \phi_{y,t}^2 \, d\omega\right\}\right] \\
&= \frac{1}{\log t} \int_{\mathbb{R}^d} (\exp\{(\log t)\sigma W^- * \phi_{0,t}^2\} - 1) \, dx \\
&= \int_{\mathbb{R}^d} (\exp\{\sigma W_t^- * \phi^2\} - 1) \, dx \stackrel{t\to\infty}{\longrightarrow} \Lambda_\phi(\sigma);
\end{aligned} \tag{3.20}$$

we have used the dominated convergence theorem: one observes $(W_t^- * \phi^2)(x) \stackrel{t\to\infty}{\longrightarrow} \phi^2(x)$ for all $x \in \mathbb{R}^d$ by continuity of ϕ; further recall that ϕ is compactly supported and bounded. Λ_ϕ is defined and real-analytic on the whole real line (see before (3.4)); therefore the Gärtner–Ellis theorem is applicable (Theorem 2.3.6 in [3]; unfortunately the theorem is stated there for integer parameter sequences only, but this is not essential for our application: for example, one may intermediately introduce factors $[\log t]/\log t \stackrel{t\to\infty}{\longrightarrow} 1$ below):

$$\liminf_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}\left[\int_{\mathbb{R}^d} V\phi_{0,t}^2 \, dx < \mu\right] \geq -\inf_{m<\mu} \Lambda_\phi^*(m) = -\Lambda_\phi^*(\mu); \tag{3.21}$$

we used in the last step that Λ_ϕ^* is monotonically decreasing and continuous on the interval $(0, 1)$. We estimate the probability of the minimum in (3.18) being too large (using some error terms $o_{1,t} \stackrel{t\to\infty}{\longrightarrow} 0$ and $o_{2,t} \stackrel{t\to\infty}{\longrightarrow} 0$) for large t:

$$\begin{aligned}
&\log \mathbb{P}\left[\min_{y\in Y_{t,r}} \int_{\mathbb{R}^d} V\phi_{y,t}^2 \, dx \geq \mu\right] = |Y_{t,r}| \log \mathbb{P}\left[\int_{\mathbb{R}^d} V\phi_{0,t}^2 \, dx \geq \mu\right] \\
&\leq -|Y_{t,r}| \mathbb{P}\left[\int_{\mathbb{R}^d} V\phi_{0,t}^2 \, dx < \mu\right] \leq -|Y_{t,r}| \exp\left\{(\log t)(-\Lambda_\phi^*(\mu) - o_{1,t})\right\} \\
&\leq -\exp\left\{(\log t)(d - \Lambda_\phi^*(\mu) - o_{2,t})\right\} \leq -t^{\vartheta/2} \qquad (t \text{ large}),
\end{aligned} \tag{3.22}$$

where $\vartheta \stackrel{\text{def}}{=} d - \Lambda_\phi^*(\mu) > 0$; see (3.14). We insert this into (3.18) and obtain for large t:

$$\mathbb{P}\left[(\log t)^{2/d} \lambda_{V_{\beta,t}}(\mathscr{T}_t) > \frac{1}{2} \|\nabla\phi\|_2^2 + \beta\mu\right] \leq \exp\left\{-t^{\vartheta/2}\right\}. \tag{3.23}$$

The Borel–Cantelli argument and Lemma 1.1 implies the upper bound (3.15), which is the claim of Lemma 3.2 (see also (2.91)–(2.92)). □

The next lemma proves the upper bound in Theorem 0.2. However, for this upper bound the assumptions $d \geq 4$ and $\beta < \beta_c$ are irrelevant:

Lemma 3.3. *Let $d \geq 1$ be any dimension, and $\beta > 0$. Then*

$$\mathbb{P}\text{–}a.s. \qquad \limsup_{t\to\infty} \, (\log t)^{2/d} \, \lambda_{V_{\beta,t}}(\mathscr{T}_t) \leq \beta. \tag{3.24}$$

Proof of Lemma 3.3. Let ϕ denote an arbitrary test function that fulfills the assumptions in Lemma 3.2. For $r > 0$ we introduce the scaled version

$$\phi_r(x) \stackrel{\text{def}}{=} r^{-d/2}\phi(x/r); \tag{3.25}$$

it scales as follows:

$$\|\phi_r\|_2^2 = 1 \quad \text{and} \quad \|\nabla\phi_r\|_2^2 = r^{-2}\,\|\nabla\phi\|_2^2, \tag{3.26}$$

$$\Lambda_{\phi_r}(\sigma) = r^d \Lambda_\phi(r^{-d}\sigma) \quad \text{and} \quad \Lambda^*_{\phi_r}(\mu) = r^d \Lambda^*_\phi(\mu). \tag{3.27}$$

We choose a function $\mu \mapsto r(\mu)$ for $\mu \in (0,1)$ such that

$$r(\mu) \stackrel{\mu\uparrow 1}{\longrightarrow} \infty \qquad \text{and} \qquad \Lambda^*_{\phi_{r(\mu)}}(\mu) = r(\mu)^d \Lambda^*_\phi(\mu) \stackrel{\mu\uparrow 1}{\longrightarrow} 0; \tag{3.28}$$

this is possible since $\lim_{\mu\uparrow 1} \Lambda^*_\phi(\mu) = 0$. The scaling rules (3.26) imply

$$\frac{1}{2}\left\|\nabla\phi_{r(\mu)}\right\|_2^2 + \beta\mu = \frac{\|\nabla\phi\|_2^2}{2r(\mu)^2} + \beta\mu \stackrel{\mu\uparrow 1}{\longrightarrow} \beta. \tag{3.29}$$

The upper bound (3.24) is now a consequence of Lemma 3.2. □

The next lemma improves the upper bound (3.24) for low dimensions: We strengthen Theorem 0.3 slightly by including a quantitative bound:

Lemma 3.4. *Let $d < 4$. For every $b_1 > 0$ there is a $c_{14} > 0$ such that for every $\beta \in (0, b_1)$:*

$$\mathbb{P}\text{–}a.s. \qquad \limsup_{t\to\infty} (\log t)^{2/d} \lambda_{V_{\beta,t}}(\mathscr{T}_t) \leq \beta - c_{14}\beta^{4/(4-d)}. \tag{3.30}$$

Proof of Lemma 3.4. We use the same setup as in the proof of Lemma 3.3. This time, a more detailed analysis of $\Lambda^*_\phi(\mu)$ near $\mu = 1$ is required:

The Taylor expansion of Λ^*_ϕ around its global minimum at $\mu_0 = 1$ provides an upper bound for Λ^*_ϕ in some ε_ϕ–neighborhood of 1: There are constants $c_{15} = c_{15}(\phi) > 0$ and $\varepsilon_\phi \in (0,1]$ such that for all μ with $|\mu - 1| < \varepsilon_\phi$ we have

$$\Lambda^*_\phi(\mu) \leq c_{15}(\mu - 1)^2. \tag{3.31}$$

When we plug (3.26), (3.27), and (3.31) into Lemma 3.2, we see that

$$\mathbb{P}\text{–a.s.} \qquad \limsup_{t\to\infty} (\log t)^{2/d} \lambda_{V_{\beta,t}}(\mathscr{T}_t) \leq \frac{1}{2r^2}\,\|\nabla\phi\|_2^2 + \beta\mu, \tag{3.32}$$

whenever

$$c_{15} r^d (1-\mu)^2 < d \qquad \text{and} \qquad 0 < 1-\mu < \varepsilon_\phi. \tag{3.33}$$

Given $b_1 > 0$ we choose first $c_{16} = c_{16}(b_1, \phi) > 0$ so small that $c_{16} b_1^{d/(4-d)} < \varepsilon_\phi$ and

$$c_{15}^{2/d} d^{-2/d} < 2 \, \|\nabla\phi\|_2^{-2} \, c_{16}^{1-4/d}; \tag{3.34}$$

recall $d < 4$. Then we choose $c_{17} = c_{17}(b_1, \phi) > 0$ so that

$$c_{15}^{2/d} d^{-2/d} c_{16}^{4/d} < c_{17}^{-2} < 2 \, \|\nabla\phi\|_2^{-2} \, c_{16}; \tag{3.35}$$

the choice (3.34) of c_{16} guarantees that such a c_{17} exists. Finally we choose $\beta \in (0, b_1)$ and set $1 - \mu = c_{16} \beta^{d/(4-d)} < \varepsilon_\phi$ and $r = c_{17} \beta^{-2/(4-d)}$. With these choices the conditions (3.33) are fulfilled, and

$$\frac{1}{2r^2} \, \|\nabla\phi\|_2^2 + \beta\mu = \beta - c_{14} \beta^{4/(4-d)}, \tag{3.36}$$

where $c_{14} \stackrel{\text{def}}{=} c_{16} - \frac{1}{2} \, \|\nabla\phi\|_2^2 \, c_{17}^{-2} > 0$. In view of bound (3.32) this finishes the proof of Lemma 3.4. □

Finally we prove the upper bound in the large–β–regime: A consequence of Sznitman's Theorem 4.4.6, [7], is: $\mathbb{P}$-a.s. $\limsup_{t\to\infty} \, (\log t)^{2/d} \, \lambda_{V_{\beta,t}}(\mathscr{T}_t) \le c(d, 1)$. We prove an upper bound which is a little bit smaller than $c(d, 1)$ for all finite β. We state a slightly sharpened version of Theorem 0.1: we include a quantitative upper bound for $\beta \to \infty$:

Lemma 3.5. *For all $\beta > 0$ the following asymptotic upper bound holds: There are positive constants $c_{18} = c_{18}(d)$, $c_{19} = c_{19}(d)$ and $b_2 = b_2(d)$, such that for all $\beta > b_2$:*

$$\mathbb{P}\text{-a.s.} \qquad \limsup_{t\to\infty} \, (\log t)^{2/d} \, \lambda_{V_{\beta,t}}(\mathscr{T}_t) \le c(d, 1) - c_{18} e^{-c_{19}\beta}. \tag{3.37}$$

Proof of Theorem 0.1 and Lemma 3.5. This time we analyze the variational principle (3.14)–(3.15) for μ close to 0 and a special choice of ϕ: Let $\phi \in H^1(\mathbb{R}^d)$ denote the (normalized) principal Dirichlet eigenfunction of $-\frac{1}{2}\Delta$ on the unit ball; we extend this eigenfunction by 0 outside of this ball. Lemma 3.2, the quantitative upper bound (3.3) for $\Lambda^*(\phi)$, and the scaling properties (3.26) and (3.27) yield

$$\mathbb{P}\text{–a.s.} \qquad \limsup_{t\to\infty} (\log t)^{2/d} \lambda_{V_{\beta,t}}(\mathscr{T}_t) \le \frac{\lambda_d}{r^2} + \beta\mu, \tag{3.38}$$

whenever

$$m \stackrel{\text{def}}{=} \frac{r^d}{d}(v_d + c_{12}\mu \log\mu - c_{13}\mu) < 1, \tag{3.39}$$

and μ is small enough ($0 < \mu < 1$); recall the notation $\lambda_d = \lambda_{V=0}(B_1(0)) = \frac{1}{2}\|\nabla\phi\|_2^2$ and $v_d = |B_1(0)|$. We use the definition of m in (3.39) to rewrite the right-hand side (3.38):

$$\frac{\lambda_d}{r^2} + \beta\mu = m^{-2/d}\lambda_d d^{-2/d}(v_d + c_{12}\mu\log\mu - c_{13}\mu)^{2/d} + \beta\mu. \tag{3.40}$$

Next we use Lipschitz continuity of $x \mapsto x^{2/d}$ at $x = v_d$: there exist constants $c_{20} = c_{20}(d) > 0$, $c_{21} = c_{21}(d)$, $\mu_0 = \mu_0(d) > 0$, and $m_0 = m_0(d) \in (0, 1)$ such that for all $\mu \in (0, \mu_0)$ and all $m \in (m_0, 1)$ (recall $c(d, 1) = \lambda_d d^{-2/d} v_d^{2/d}$):

$$\frac{\lambda_d}{r^2} + \beta\mu \le m^{-2/d}c(d, 1) + c_{20}\mu\log\mu - c_{21}\mu + \beta\mu \tag{3.41}$$

(We substitute the optimal value $\mu = e^{(-\beta - c_{20} + c_{21})/c_{20}}$, for $\beta > (c_{21} - c_{20}(1 + \log\mu_0)) \vee 0$:)

$$= m^{-2/d}c(d, 1) - c_{20}e^{(-\beta - c_{20} + c_{21})/c_{20}}. \tag{3.42}$$

Finally we let $m \uparrow 1$. This proves the asymptotic bound (3.37) for $b_2 = (c_{21} - c_{20}(1 + \log\mu_0)) \vee 0$, and an appropriate choice of c_{18} and c_{19}. Theorem 0.1 is a consequence of (3.37) and the monotonicity of $\beta \mapsto \lambda_{V_{\beta,t}}(\mathcal{T}_t)$. □

4. Asymptotic behavior of the partition sum

In this section we give the relations between the principal Dirichlet eigenvalue on $\mathcal{T}_t = (-t, t)^d$ and the partition sum $S^{\omega}_{t,\beta}$ for Brownian motion in a scaled Poissonian potential.

Proof of Theorem 0.4. First we prove the upper bound in (0.11). Using Theorem 3.1.2 of [7] we see that (where $T_{\mathcal{T}_t} \stackrel{\text{def}}{=} \inf\{s \ge 0, Z_s \notin \mathcal{T}_t\}$ is the exit time from $\mathcal{T}_t$)

$$\begin{aligned} S^{\omega}_{t,\beta} &\le P_0\left[T_{\mathcal{T}_t} \le t\right] + E_0\left[\exp\left\{-\int_0^t V_{\beta,t}(Z_s, \omega)ds\right\}, T_{\mathcal{T}_t} > t\right] \\ &\le 4d\exp\{-t/2\} + c_{22}(d)\left((\lambda_{V_{\beta,t}}(\mathcal{T}_t)\,t)^{d/2} + 1\right)\exp\left\{-\lambda_{V_{\beta,t}}(\mathcal{T}_t)\,t\right\}, \end{aligned} \tag{4.1}$$

where the first term on the right-hand side of (4.1) has been estimated by the standard one-dimensional estimate using the reflection principle. We remark that the leading term is the second one, the exponent $\lambda_{V_{\beta,t}}(\mathcal{T}_t)\,t$ grows slower than of order t as $t \to \infty$. Choosing β small enough, we see that our upper bound in (0.11) follows from the asymptotic behavior of $\lambda_{V_{\beta,t}}(\mathcal{T}_t)$ (see (0.7)).

So let us come to the lower bounds of Theorem 0.4. We imitate the proof of Theorem 6.1.1 of [7] for our scaled potential: Define $s = s(t) \stackrel{\text{def}}{=} t(\log t)^{-3/d}$ and choose $\mathcal{T}_s = (-s, s)^d$, then we denote by $\tilde{\mathcal{T}}_s$ the open $\sqrt{d}$-neighborhood of $\mathcal{T}_s$.

Hence for all large t: $\mathscr{T}_s \subset \tilde{\mathscr{T}}_s \subset \mathscr{T}_t$, and $|\tilde{\mathscr{T}}_s| \le (2t)^d$. Let ϕ_t be a non-negative normalized principal Dirichlet eigenfunction to the associated problem on $\mathscr{T}_s$, $\lambda_{V_{\beta,t}}(\mathscr{T}_s) = \inf\{\|\nabla\phi\|_2^2/2 + \int_{\mathscr{T}_s} V_{\beta,t}\phi^2 dx \; : \; \phi \in C_c^\infty(\mathscr{T}_s), \int_{\mathscr{T}_s} \phi^2\, dx = 1\}$. Then we choose $y_t \in \mathscr{T}_s$ such that (6.1.17) of [7] holds, i.e. such that ϕ_t^2 puts enough mass close to y_t in the sense that $\int_{y_t+[-1,1]^d} \phi_t^2(x)dx \ge 1/(2|\tilde{\mathscr{T}}_s|)$. Such a y_t always exists, this can be seen by covering $\mathscr{T}_s$ with boxes of unit size and using $\int_{\mathscr{T}_s} \phi_t^2\, dx = 1$. Using Proposition 3.1.12 of [7], we obtain for $t \ge 2$ and $A_t \overset{\text{def}}{=} y_t + [-1,1]^d$:

$$
\begin{aligned}
S^{\omega}_{t,\beta} \ge \; & \inf_{A_t \times A_t} r_{\mathbb{R}^d, V_{\beta,t}}(2,\cdot,\cdot) \\
& \cdot \frac{1}{2|\tilde{\mathscr{T}}_s|} E_0\left[\exp\left\{-\int_0^{H(A_t)} V_{\beta,t}(Z_s,\omega)ds\right\}, H(A_t) < \infty\right] \\
& \exp\left\{-\lambda_{V_{\beta,t}}(\mathscr{T}_s)\, t\right\},
\end{aligned}
\tag{4.2}
$$

where $H(A_t)$ is the entrance time of $Z_\cdot$ into A_t, and for an open set $U \subset \mathbb{R}^d$ we define

$$
r_{U,V_{\beta,t}}(u,x,y) \overset{\text{def}}{=} p(u,x,y) E^u_{x,y}\left[\exp\left\{-\int_0^u V_{\beta,t}(Z_s,\omega)ds\right\}, T_U > u\right],
\tag{4.3}
$$

with $p(u,x,y)$ the Brownian transition density and $P^u_{x,y}$ the Brownian bridge measure (from x to y in time u), for a reference see [7], pp. 137–140. Estimating the first term in (4.2) we obtain

$$
\begin{aligned}
\inf_{A_t \times A_t} r_{\mathbb{R}^d, V_{\beta,t}}(2,\cdot,\cdot) &\ge \inf_{A_t \times A_t} r_{y_t+(-2,2)^d, V_{\beta,t}}(2,\cdot,\cdot) \\
&\ge \inf_{[-1,1]^d \times [-1,1]^d} r_{(-2,2)^d, V=0}(2,\cdot,\cdot) \\
&\qquad \exp\left\{-2 \sup_{y_t+(-2,2)^d} V_{\beta,t}\right\}.
\end{aligned}
\tag{4.4}
$$

For all large t, $y_t + (-2,2)^d \subset \mathscr{T}_t$; hence we estimate the last term in (4.4), using (4.5.12) of [7], by

$$
\mathbb{P}\text{-a.s.} \qquad \sup_{\mathscr{T}_t} V_{\beta,t} = o\left((\log t)^{1-2/d}\right), \qquad \text{as } t \to \infty.
\tag{4.5}
$$

We come back to the remaining terms in (4.2): Using a shape theorem (Theorem 5.2.5, [7]) we see that there exists a constant $\alpha = \alpha(d,W)$ such that for almost every ω and for all large t ($H(A_t) \le H(\bar{B}_1(y_t))$); see also (6.5.5) of [7])

$$
\begin{aligned}
& E_0\left[\exp\left\{-\int_0^{H(A_t)} V_{\beta,t}(Z_s,\omega)ds\right\}, H(A_t) < \infty\right] \\
& \quad \ge E_0\left[\exp\left\{-\int_0^{H(\bar{B}_1(y_t))} V(Z_s,\omega)ds\right\}, H(\bar{B}_1(y_t)) < \infty\right] \\
& \quad \ge \exp\{-\alpha s\}.
\end{aligned}
\tag{4.6}
$$

Collecting (4.2)–(4.6) we obtain $\mathbb{P}$-a.s. for all large t (using $|\tilde{\mathscr{T}}_s| \le (2t)^d$)

$$S^{\omega}_{t,\beta} \ge c_{23}(d) \exp\left\{-\lambda_{V_{\beta,t}}(\mathscr{T}_s)\, t - \alpha s - (\log t)^{1-2/d} - d \log t\right\}. \tag{4.7}$$

Then the claims follow from the remark that the leading orders of $\lambda_{V_{\beta,t}}(\mathscr{T}_s)$ and $\lambda_{V_{\beta,t}}(\mathscr{T}_t)$ are the same as $t \to \infty$: For large t we have $s \le t$, hence using the monotonicity of the principal Dirichlet eigenvalue $\lambda_{V_{\beta,t}}(\mathscr{T}_s) \le \lambda_{V_{\beta,s}}(\mathscr{T}_s)$. Using $(\log t/\log s)^{2/d} \to 1$ as $t \to \infty$, we obtain $\mathbb{P}$-a.s.

$$\limsup_{t\to\infty} (\log t)^{2/d}\, \lambda_{V_{\beta,t}}(\mathscr{T}_s) \le \limsup_{s\to\infty} (\log s)^{2/d}\, \lambda_{V_{\beta,s}}(\mathscr{T}_s). \tag{4.8}$$

This together with the asymptotic upper bounds on $\lambda_{V_{\beta,s}}(\mathscr{T}_s)$ (see (0.6), (0.7) and (0.8)) finishes the proof of Theorem 0.4. □

A. Van den Berg/Kesten-inequality

In this appendix we prove that the van den Berg/Kesten-inequality applies to our situation (the goal is to apply Theorem 4.2 (a) of [1]). We work with the notations of Section 2. Fix $j \in J$. We consider the random variables $\xi_{i,j} = \left(1 - \frac{\omega(K_{i,j})}{|K_{i,j}|}\right) \vee 0$ for $i \in I$. The $(\xi_{i,j})_{i\in I}$ are i.i.d. with values in the finite set $S_i \stackrel{\text{def}}{=} \left\{\left(1 - \frac{l}{|K_{i,j}|}\right) \vee 0 : l \in \mathbb{N}_0\right\}$ for all $i \in I$. We define the finite cartesian product $\Omega_I \stackrel{\text{def}}{=} \prod_{i\in I} S_i$. Hence the common distribution $\mathbb{P}_I$ of the random vector $(\xi_{i,j})_{i\in I}$ is a product measure over Ω_I. For $\xi \in \Omega_I$ and $K \subseteq I$ we define the cylinder event

$$[\xi]_K \stackrel{\text{def}}{=} \left\{\xi' \in \Omega_I : \xi'_i = \xi_i \text{ for all } i \in K\right\}. \tag{A.1}$$

Finally we define disjoint occurence: Choose $N \in \mathbb{N}$ and $A_n \subseteq \Omega_I$ for $n = 1, \dots, N$. We let $\mathscr{I}_N$ denote the set of all families $(I_1, \dots, I_N)$ of pairwise disjoint subsets of I; then we define

$$\square_{n=1}^{N} A_n \stackrel{\text{def}}{=} \bigcup_{(I_n)\in\mathscr{I}_N} \bigcap_{n=1}^{N} \left\{\xi \in \Omega_I : [\xi]_{I_n} \subseteq A_n\right\}. \tag{A.2}$$

In the special case $N = 2$ this coincides with the operation $A_1 \square A_2$ defined in (2.4) in [1].

Lemma A.1. *Choose $N \in \mathbb{N}$ fixed. We assume that $A_n \subseteq \Omega_I$ ($n = 1, \dots, N$) are increasing events. Then*

$$\mathbb{P}_I\left[\square_{n=1}^{N} A_n\right] \le \prod_{n=1}^{N} \mathbb{P}_I[A_n]. \tag{A.3}$$

Proof of Lemma A.1. The proof goes by induction. The case $N = 1$ is obvious from $\square_{n=1}^{1} A_n = A_1$. Induction step, $N \to N+1$: Theorem 4.2 (a) of [1] tells us $\mathbb{P}_I[A \square B] \le \mathbb{P}_I[A]\mathbb{P}_I[B]$ if A and B are both an intersection of an increasing and a decreasing event (especially if they both are increasing events). To apply this theorem for the induction step, we need to show

$$\square_{n=1}^{N} A_n \text{ is an increasing event,} \tag{A.4}$$

$$\square_{n=1}^{N+1} A_n = \left(\square_{n=1}^{N} A_n\right) \square A_{N+1}. \tag{A.5}$$

For $\xi \in \Omega_I$ and $K \subseteq I$, there is an (unique) minimum $\underline{\xi}_K$ in $[\xi]_K$; it is given by $\left(\underline{\xi}_K\right)_i \stackrel{\text{def}}{=} \xi_i 1_K(i)$. We start with the proof of (A.4): Since the A_n are increasing, we can write

$$\square_{n=1}^{N} A_n = \bigcup_{(I_n)\in\mathscr{I}_N} \bigcap_{n=1}^{N} \left\{\xi \in \Omega_I : \underline{\xi}_{I_n} \in A_n\right\}. \tag{A.6}$$

But then (A.4) easily follows from (A.6), using again the fact that the A_n's are increasing. So there remains to prove (A.5):

$$\begin{aligned}
\left(\square_{n=1}^{N} A_n\right) \square A_{N+1} &= \bigcup_{(K, I_{N+1})\in\mathscr{I}_2} \left\{\xi \in \Omega_I : \underline{\xi}_K \in \square_{n=1}^{N} A_n \text{and } \underline{\xi}_{I_{N+1}} \in A_{N+1}\right\} \\
&= \bigcup_{(K, I_{N+1})\in\mathscr{I}_2} \bigcup_{(I_1,\dots,I_N)\in\mathscr{I}_N} \bigcap_{n=1}^{N} \left\{\xi \in \Omega_I : \underline{\underline{\xi}_K}_{I_n} \in A_n\right\} \\
&\quad \cap \left\{\xi \in \Omega_I : \underline{\xi}_{I_{N+1}} \in A_{N+1}\right\},
\end{aligned} \tag{A.7}$$

but since $\underline{\underline{\xi}_K}_{I_n} = \underline{\xi}_{K\cap I_n}$ it suffices to consider $I_1, \dots, I_N$ being pairwise disjoint subsets of K, hence they are also disjoint from I_{N+1}, and our claim (A.5) follows. This finishes the proof of Lemma A.1. □

In our application (2.53) of Lemma A.1 we set $A_n = \{\xi \in \Omega_I : |\{i \in I : \xi_i > \varepsilon_n\}| \ge \rho_n N_n\}$, which are *increasing* events. (One should not confuse this with $\{\omega \in \Omega : (\xi_{i,j}(\omega))_{i\in I} \in A_n\}$ being *decreasing* events: the map $\omega \mapsto (\xi_{i,j}(\omega))_{i\in I}$ reverses the partial order.) The link from Lemma A.1 to formula (2.53) is given by the identity

$$\square_{n=1}^{N(\eta)} A_n = \bigcup_{(I_n)\in\mathscr{I}_{N(\eta)}} \bigcap_{n=1}^{N(\eta)} \{\xi \in \Omega_I : |\{i \in I_n : \xi_i > \varepsilon_n\}| \ge \rho_n N_n\}. \tag{A.8}$$

Acknowledgements. We thank A.-S. Sznitman for introducing us into this nice problem. We also want to thank him for giving us these beautiful lectures on Brownian motion and random media at ETH Zürich. Further we thank F. den Hollander for interesting discussions, especially for drawing our attention to [2].

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