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Aharonov–Bohm Effect in Scattering by Point-like Magnetic Fields at Large Separation

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Abstract. The aim is to study the Aharonov–Bohm effect in the scattering by two point–like magnetic fields at large separation in two dimensions. We analyze the asymptotic behavior of scattering amplitude when the distance between the centers of two fields goes to infinity. The obtained result heavily depends on the fluxes of fields and on incident and final directions.

1 Introduction

Magnetic potentials have a direct significance to the motion of particles in quantum mechanics. This property is known as the Aharonov–Bohm effect ([3]) and a lot of physical literatures can be found in the recent book [2]. In this work we consider the scattering by two δ-like magnetic fields at large separation in two dimensions and we analyze the asymptotic behavior of scattering amplitude when the distance between the centers of two fields goes to infinity. Even if a field is compactly supported, the corresponding magnetic potential is not expected to fall off rapidly. In general, it has the long–range property at infinity. We study how the Aharonov–Bohm effect is reflected in the scattering by magnetic fields at large separation.

We work in the two dimensional space $\boldsymbol{R}^2$ throughout the entire discussion. We denote by $x = (x_1, x_2)$ a generic point, and we write

$$H(A) = (-i\nabla - A)^2 = \sum_{j=1}^{2}(-i\partial_j - a_j)^2, \quad \partial_j = \partial/\partial x_j,$$

for the Schrödinger operator with magnetic potential $A(x) = (a_1(x), a_2(x)) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. The magnetic field $b(x)$ is defined as $b = \nabla\times A = \partial_1 a_2 - \partial_2 a_1$, and the quantity $\alpha = (2\pi)^{-1}\int b(x)\,dx$ is called the total flux of field b, where the integration with no domain attached is taken over the whole space. We often use this abbreviation.

We begin by making a brief review on the scattering theory for the Hamiltonian with magnetic field supported on a single point. Such a Hamiltonian is regarded as one of solvable models in quantum mechanics and the explicit form of scattering amplitude has been already calculated ([3,17]). In section 2 we are going to discuss the subject in some detail. Let $2\pi\alpha\delta(x)$ be the magnetic field with flux α and center at the origin. The magnetic potential $A_\alpha(x)$ associated with the field is given by

$$A_\alpha(x) = \alpha\left(-x_2/|x|^2, x_1/|x|^2\right) = \alpha\left(-\partial_2 \log|x|, \partial_1 \log|x|\right).$$

In fact, a simple calculation yields $\nabla \times A_\alpha = \alpha\,\Delta \log|x| = 2\pi\alpha\delta(x)$. If we denote by $\gamma(x)$ the azimuth angle from the positive x_1 axis, then A_α is written in the different form

$$A_\alpha(x) = \alpha\,\nabla\gamma(x) = \alpha\left(-x_2/|x|^2, x_1/|x|^2\right). \tag{1.1}$$

This representation is important. The same relation remains true for the azimuth angle $\gamma(x;\omega)$ from direction $\omega \in S^1$, where S^1 is the unit circle.

Let $H_0 = -\Delta$ be the free Hamiltonian and define H_α by $H_\alpha = H(A_\alpha)$. The potential $A_\alpha(x)$ has a strong singularity at the origin and it is known ([1,7]) that the operator formally defined is not essentially self–adjoint in $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$. We have to impose some boundary conditions at the origin. The operator H_α becomes self–adjoint in $L^2 = L^2(\boldsymbol{R}^2)$ under the condition $\lim_{|x|\to 0}|u(x)| < \infty$, and it is called the Aharonov–Bohm Hamiltonian. If, in particular, $\alpha \notin \boldsymbol{Z}$ is not an integer, the limit is convergent to zero $\lim_{|x|\to 0}|u(x)| = 0$. We now denote by $f(\omega \to \tilde\omega; E, H_\alpha, H_0)$ the scattering amplitude for the scattering from initial direction ω to final one $\tilde\omega$ at energy $E > 0$. If we identify the coordinates over S^1 with the azimuth angles from the positive x_1 axis, then the amplitude is given by

$$f(\omega \to \tilde\omega) = c(E)\Big((\cos\alpha\pi - 1)\delta(\tilde\omega - \omega) - (i/\pi)\sin\alpha\pi\, e^{i[\alpha](\tilde\omega-\omega)} F_0(\tilde\omega - \omega)\Big) \tag{1.2}$$

with $c(E) = (2\pi/i\sqrt{E})^{1/2}$, where the Gauss notation $[\alpha]$ denotes the maximal integer not exceeding α and $F_0(\theta) = \text{v.p.}\, e^{i\theta}/(e^{i\theta} - 1)$.

We move to the scattering by two δ–like magnetic fields. Let $2\pi\alpha_1\delta(x)$ and $2\pi\alpha_2\delta(x-d)$ be given magnetic fields with centers at the origin and $d \in \boldsymbol{R}^2$ respectively. We consider the Hamiltonian

$$H_d = H(A_{\alpha_1} + A_{\alpha_2,d}), \quad A_{\alpha_2,d}(x) = A_{\alpha_2}(x-d),$$

where

$$A_{\alpha_j}(x) = \alpha_j \nabla\gamma(x) = \alpha_j\left(-x_2/|x|^2, x_1/|x|^2\right) \tag{1.3}$$

is the magnetic potential associated with the field $2\pi\alpha_j\delta(x)$. In section 7, we will study the basic spectral problems such as the self–adjointness, the absence of bound states, the principle of limiting absorption and the asymptotic completeness of wave operators for H_d. According to the result there, H_d becomes self–adjoint with domain

$$\mathcal{D}(H_d) = \{u \in L^2 : H_d u \in L^2,\ \lim_{|x|\to 0}|u(x)| < \infty,\ \lim_{|x-d|\to 0}|u(x)| < \infty\}, \tag{1.4}$$

where $H_d u$ is understood in the distributional sense. We set

$$H_j = H(A_{\alpha_j}), \quad j = 1, 2,$$

and we denote by $f_d(\omega \to \tilde{\omega}; E)$ and $f_j(\omega \to \tilde{\omega}; E)$ the scattering amplitude for the pair (H_d, H_0) and (H_j, H_0) respectively. By (1.2), the scattering amplitude for (H_j, H_0) is explicitly calculated as

$$f_j(\omega \to \tilde{\omega}; E) = -c(E)(i/\pi) \sin \alpha_j \pi e^{i[\alpha_j](\tilde{\omega}-\omega)} F_0(\tilde{\omega} - \omega)$$

for $\omega \neq \tilde{\omega}$.

The aim here is to study the asymptotic behavior as $|d| \to \infty$ of $f_d(\omega \to \tilde{\omega}; E)$. If we make a change of variables $x \to |d|y$, then this becomes the problem on the asymptotic behavior at high energy $|d|^2 E$ of scattering amplitude for the Hamiltonian $H(A_{\alpha_1} + \tilde{A}_{\alpha_2})$, where $\tilde{A}_{\alpha_2}(x) = \alpha_2 \nabla \gamma(x - \hat{d})$ and $\hat{d} = d/|d| \in S^1$. We fix the notation. We define $\tau(x; \omega, \tilde{\omega})$ by

$$\tau(x; \omega, \tilde{\omega}) = \gamma(x; \omega) - \gamma(x; -\tilde{\omega})$$

and we interpret $\exp(i\alpha\gamma(x; \omega))$ with $\omega = x/|x|$ as

$$\exp(i\alpha\gamma(x; \omega)) := (1 + \exp(i2\alpha\pi))/2 = \cos\, \alpha\pi \times \exp(i\alpha\pi).$$

The obtained result is formulated as the following theorem.

Theorem 1.1 *Let the notation be as above and let*

$$f_{2,d}(\omega \to \tilde{\omega}; E) = \exp(-i\sqrt{E} d \cdot (\tilde{\omega} - \omega)) f_2(\omega \to \tilde{\omega}; E)$$

be the scattering amplitude for the pair $(H_{2,d}, H_0)$, $H_{2,d} = H(A_{\alpha_2,d})$. *Fix the direction* $\hat{d} = d/|d|$. *If* $\omega \neq \tilde{\omega}$, *then* $f_d(\omega \to \tilde{\omega}; E)$ *behaves like*

$$\begin{aligned} f_d(\omega \to \tilde{\omega}; E) &= \exp(i\alpha_2 \tau(-d; \omega, \tilde{\omega})) f_1(\omega \to \tilde{\omega}; E) \\ &+ \exp(i\alpha_1 \tau(d; \omega, \tilde{\omega})) f_{2,d}(\omega \to \tilde{\omega}; E) + o(1) \end{aligned}$$

as $|d| \to \infty$. *In particular, the backward scattering amplitudes obey*

$$f_d(\omega \to -\omega; E) = f_1(\omega \to -\omega; E) + f_{2,d}(\omega \to -\omega; E) + o(1)$$

for $\omega \neq \pm\hat{d}$, *and*

$$\begin{aligned} f_d(\hat{d} \to -\hat{d}; E) &= f_1(\hat{d} \to -\hat{d}; E) + (\cos \alpha_1 \pi)^2 f_{2,d}(\hat{d} \to -\hat{d}; E) + o(1), \\ f_d(-\hat{d} \to \hat{d}; E) &= (\cos \alpha_2 \pi)^2 f_1(-\hat{d} \to \hat{d}; E) + f_{2,d}(-\hat{d} \to \hat{d}; E) + o(1). \end{aligned}$$

As stated at the beginning, the motion of quantum particles is subject to the influence of magnetic potentials as well as of magnetic fields. This quantum property can be found in the asymptotic formula above. In fact, the first field $2\pi\alpha_1\delta(x)$ has an influence upon the scattering by the second one through the phase factor $\exp(i\alpha_1\tau(d;\omega,\tilde{\omega}))$ in front of $f_{2,d}(\omega \to \tilde{\omega}; E)$, although the centers of two fields are far away from each other. This can be seen more clearly in the backward scattering amplitude $f_d(\hat{d} \to -\hat{d}; E)$ or $f_d(-\hat{d} \to \hat{d}; E)$. If, in particular, the flux α_1 is a half–integer, then the scattering by the second field does not make any contribution to the leading term of the asymptotic formula for $f_d(\hat{d} \to -\hat{d}; E)$.

Many literatures can be found in the book [4] for the spectral and scattering theory of Schrödinger operators with potentials supported on a discrete set of points, and the work [11] has recently dealt with the problem on the asymptotic behavior of scattering amplitude for the Schrödinger operator $-\Delta + V_1(x) + V_2(x - d)$ with potentials falling off rapidly at infinity. In the case of potential scattering, we do not have to modify phase factors and the asymptotic formula is completely split into the sum of two scattering amplitudes corresponding to potentials V_1 and $V_2(\cdot - d)$. However the case is quite different in the scattering by magnetic fields. Roughly speaking, the difficulty comes from the long–range property of magnetic potentials. Several new devices are required at many stages of the argument. The micro-local resolvent estimates for H_d and the asymptotic behavior at infinity of the eigenfunction of $H_1 = H(A_{\alpha_1})$ or H_2 play an important role in proving the theorem. We end the section by making a brief comment on the extension to the case of scattering by point–like magnetic fields supported on several points. This is a natural problem. The analysis heavily depends on the location of centers and on initial and final directions. Some new difficulties may arise. However the idea developed here is thought to be useful to such a generalization. We are going to discuss the detailed matter elsewhere.

2 Scattering by δ–like magnetic field

The present section is devoted to the scattering theory for the Schrödinger operator with point–like magnetic field supported on a single point. Such an operator is called the Aharonov–Bohm Hamiltonian.

2.1. We first make a review on the results from [3,17]. We consider the Hamiltonian

$$H_\alpha = H(A_\alpha), \quad A_\alpha(x) = \alpha\nabla\gamma(x) = \alpha\left(-x_2/|x|^2, x_1/|x|^2\right),$$

which has the δ–like field $2\pi\alpha\delta(x)$ at the origin. We know ([1,7]) that H_α is self–adjoint with domain

$$\mathcal{D}(H_\alpha) = \{u \in L^2 : H_\alpha u \in L^2, \ \lim_{|x|\to 0} |u(x)| < \infty\},$$

$H_\alpha u$ being understood in $\mathcal{D}'$, and that the wave operator

$$W_\pm(H_\alpha, H_0) = s - \lim_{t\to\pm\infty} \exp(itH_\alpha)\exp(-itH_0) : L^2 \to L^2$$

exists and is asymptotically complete : $\operatorname{Ran} W_\pm(H_\alpha, H_0) = L^2$. Hence the scattering operator

$$S(H_\alpha, H_0) = W_+^*(H_\alpha, H_0)W_-(H_\alpha, H_0) : L^2 \to L^2$$

can be defined as a unitary operator. We use the notation $\cdot$ to denote the scalar product in $\boldsymbol{R}^2$. Let $\varphi_0(x;\lambda,\omega) = \exp(i\sqrt{\lambda}\, x\cdot\omega)$ be the generalized eigenfunction of the free Hamiltonian $H_0 = -\Delta$, where $\lambda > 0$ and $\omega \in S^1$. The unitary mapping $F : L^2 \to L^2((0,\infty); d\lambda) \otimes L^2(S^1)$ defined by

$$(Fu)(\lambda,\omega) = 2^{-1/2}(2\pi)^{-1} \int \bar{\varphi}_0(x;\lambda,\omega)u(x)\,dx \tag{2.1}$$

decomposes $S(H_\alpha, H_0)$ into the direct integral

$$S(H_\alpha, H_0) \simeq FS(H_\alpha, H_0)F^* = \int_0^\infty \oplus\, S(\lambda; H_\alpha, H_0)\,d\lambda,$$

where the fiber $S(\lambda; H_\alpha, H_0) : L^2(S^1) \to L^2(S^1)$ is called the scattering matrix at energy $\lambda > 0$ and it acts as

$$\left(S(\lambda; H_\alpha, H_0)(Fu)(\lambda, \cdot)\right)(\omega) = \left(FS(H_\alpha, H_0)u\right)(\lambda,\omega)$$

for $u \in L^2$.

We calculate the generalized eigenfunction $\varphi_\mp(x;\lambda,\omega)$ of H_α to derive the integral kernel of $S(\lambda; H_\alpha, H_0)$. The operator H_α is rotationally invariant. We work in the polar coordinate system (r,θ). Let Λ_l, $l \in \boldsymbol{Z}$, be the eigenspace associated with eigenvalue l of operator $-i\partial/\partial\theta$ acting on $L^2(S^1)$. Then

$$L^2((0,\infty); dr) \otimes L^2(S^1) = \sum_{l\in Z} \oplus \left(L^2((0,\infty); dr) \otimes \Lambda_l\right).$$

We define the unitary mapping

$$(Uu)(r,\theta) = r^{1/2}u(r\theta) : L^2 \to L^2((0,\infty); dr) \otimes L^2(S^1).$$

The mapping U yields the partial wave expansion

$$H_\alpha \simeq UH_\alpha U^* = \sum_{l\in Z} \oplus\, (H_{l\alpha} \otimes Id),$$

where Id is the identity operator and

$$H_{l\alpha} = -\partial_r^2 + (\nu^2 - 1/4)r^{-2}, \quad \nu = |l - \alpha|,$$

is self–adjoint with domain

$$\mathcal{D}(H_{l\alpha}) = \{u \in L^2((0,\infty); dr) : H_{l\alpha} u \in L^2((0,\infty); dr), \ \lim_{r\to 0} r^{-1/2}|u(r)| < \infty\}.$$

The eigenfunction $\varphi_\mp$ is formally defined as $\varphi_\mp = W_\pm(H_\alpha, H_0)\varphi_0$ by using the intertwining property of wave operators. However this does not have the precise meaning, because $\varphi_0(x;\lambda,\omega)$ is not in L^2. The precise definition requires the expansion formula

$$\varphi_0(x;\lambda,\omega) = \sum_{l\in Z} \exp(i|l|\pi/2)\exp(il\gamma(x;\omega))J_{|l|}(\sqrt{\lambda}|x|) \tag{2.2}$$

in terms of the Bessel functions $J_p(r)$. The function $J_p(r)$ satisfies the asymptotic formula

$$J_p(r) = (2/\pi)^{1/2} r^{-1/2}\cos(r-(2p+1)\pi/4)\Big(1+g_N(r)\Big) + O(r^{-N}), \quad r\to\infty,$$

for any $N \gg 1$ large enough, where $g_N(r)$ obeys $(d/dr)^k g_N(r) = O(r^{-1-k})$. If we set

$$e_{\mp l}(r) = \exp(\pm i|l|\pi/2)J_{|l|}(r) - \exp(\pm i\nu\pi/2)J_\nu(r),$$

then

$$e_{\mp l}(r) = \exp(\mp ir)\left(C_{\mp l} r^{-1/2} + O(r^{-3/2})\right) + \exp(\pm ir)O(r^{-3/2})$$

for some constant $C_{\mp l} \neq 0$. Hence $e_{-l}(r)$ satisfies the incoming radiation condition $e'_{-l} + ie_{-l} = O(r^{-3/2})$ at infinity, while $e_{+l}(r)$ satisfies the outgoing radiation condition $e'_{+l} - ie_{+l} = O(r^{-3/2})$. The simple relation

$$\exp(il\gamma(x;-\omega)) = \exp(i|l|\pi + il\gamma(x;\omega))$$

holds between the azimuth angles $\gamma(x;\omega)$ and $\gamma(x;-\omega)$. If we take account of (2.2), then the eigenfunction $\varphi_\mp$ is given by

$$\varphi_\mp(x;\lambda,\omega) = \sum_{l\in Z} \exp(\pm i\nu\pi/2)\exp(il\gamma(x;\pm\omega))J_\nu(\sqrt{\lambda}|x|) \tag{2.3}$$

with $\nu = |l-\alpha|$ again. We can easily see that the series converges locally uniformly and that $\varphi_\mp$ satisfies $H_\alpha\varphi_\mp = \lambda\varphi_\mp$.

We often identify the coordinates over the unit circle S^1 with the azimuth angles from the positive x_1 axis. The scattering matrix $S(\lambda; H_\alpha, H_0)$ has the property

$$S(\lambda; H_\alpha, H_0) : \overline{\varphi}_+(x;\lambda,\cdot) \to \overline{\varphi}_-(x;\lambda,\cdot).$$

A simple computation yields

$$\exp(i\nu\pi/2)\exp(-il\gamma(x;-\omega)) = \exp(i(\nu-l)\pi)\exp(-i\nu\pi/2)\exp(-il\gamma(x;\omega))$$

and hence the kernel of $S(\lambda; H_\alpha, H_0)$ is calculated as

$$S(\theta', \theta; \lambda, H_\alpha, H_0) = (2\pi)^{-1} \sum_{l \in Z} \exp(i(l-\nu)\pi) \exp(il(\theta' - \theta)).$$

According to [17], the sum on the right side equals

$$\sum_{l \in Z} \exp(i(l-\nu)\pi) \exp(il\theta) = 2\pi \Big(\cos\alpha\pi\, \delta(\theta) - (i/\pi) \sin\alpha\pi e^{i[\alpha]\theta} F_0(\theta) \Big),$$

where $F_0(\theta) = \text{v.p.}\, e^{i\theta}/(e^{i\theta} - 1)$. Thus we can obtain the representation (1.2) of amplitude

$$f(\omega \to \tilde{\omega}; E, H_\alpha, H_0) = c(E) \Big(S(\tilde{\omega}, \omega; E, H_\alpha, H_0) - \delta(\tilde{\omega} - \omega) \Big)$$

for the scattering from initial direction ω into final one $\tilde{\omega}$ at energy $E > 0$, where $c(E) = (2\pi/i\sqrt{E})^{1/2}$.

2.2. The asymptotic behavior as $|x| \to \infty$ of eigenfunction $\varphi_\mp(x; \lambda, \omega)$ plays an important role in proving the main theorem. It has been already known in the physical literatures [3,5,14]. However we shall prove the following proposition in section 6 because of its importance.

Proposition 2.1 *The eigenfunction $\varphi_\mp(x; \lambda, \omega)$ has the following asymptotic properties at infinity.*

(1) *Assume that $|x/|x| - \omega| > c > 0$. Then $\varphi_+(x; \lambda, \omega)$ behaves like*

$$\begin{aligned} \varphi_+(x; \lambda, \omega) \quad &= \quad \exp\left(i\alpha\left(\gamma(x; \omega) - \pi\right)\right) \exp(i\sqrt{\lambda} x \cdot \omega) \\ &+ \;\; e^{i\sqrt{\lambda}|x|} |x|^{-1/2} \Big(\sum_{j=0}^{N-1} c_{+j}(x) |x|^{-j} \Big) + O(|x|^{-(N+1/2)}), \end{aligned}$$

where the coefficient $c_{+j}(x)$ obeys the bound $|\partial_x^\beta c_{+j}| = O(|x|^{-|\beta|})$.

(2) *If $|x/|x| + \omega| > c > 0$, then a similar formula*

$$\begin{aligned} \varphi_-(x; \lambda, \omega) \quad &= \quad \exp\left(i\alpha\left(\gamma(x; -\omega) - \pi\right)\right) \exp(i\sqrt{\lambda} x \cdot \omega) \\ &+ \;\; e^{-i\sqrt{\lambda}|x|} |x|^{-1/2} \Big(\sum_{j=0}^{N-1} c_{-j}(x) |x|^{-j} \Big) + O(|x|^{-(N+1/2)}) \end{aligned}$$

holds true for the incoming eigenfunction $\varphi_-(x; \lambda, \omega)$.

(3) *Assume that $1/2 < q \le 1$. If $0 < |x/|x| - \omega| < c|x|^{-q}$ for some $c > 0$, then*

$$\varphi_+(x; \lambda, \omega) = \cos\alpha\pi \times \exp(i\sqrt{\lambda} x \cdot \omega) + O(|x|^{-\nu})$$

with $\nu = 2(q-1/2)/3 > 0$*, and if* $0 < |x/|x| + \omega| < c|x|^{-q}$*, then*

$$\varphi_-(x;\lambda,\omega) = \cos\alpha\pi \times \exp(i\sqrt{\lambda}x\cdot\omega) + O(|x|^{-\nu})$$

for the same ν *as above.*

(4) $\varphi_\mp(x;\lambda,\omega)$ *is bounded uniformly in* x*.*

2.3. We represent the amplitude $f(\omega \to \tilde{\omega}; E, H_\alpha, H_0)$ in terms of resolvent $R(E + i0; H_\alpha)$. We know that the boundary values

$$R(\lambda \pm i0; H_\alpha) = \lim_{\varepsilon\downarrow 0} R(\lambda \pm i\varepsilon; H_\alpha), \quad R(\zeta; H_\alpha) = (H_\alpha - \zeta)^{-1},$$

to the positive real axis exist (principle of limiting absorption) and

$$R(\lambda \pm i0; H_\alpha) : L^2_s(\boldsymbol{R}^2) = L^2(\boldsymbol{R}^2; \langle x\rangle^{2s}dx) \to L^2_{-s}(\boldsymbol{R}^2) \tag{2.4}$$

is bounded for $s > 1/2$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. This is verified by use of the commutator method due to Mourre [13] (see Proposition 7.3 in section 7).

We now introduce a basic cut–off function. Let $\chi \in C_0^\infty[0,\infty)$ be a smooth function such that $\chi(s) \ge 0$ and

$$\chi(s) = 1 \quad \text{for} \ \ 0 \le s \le 1, \qquad \chi(s) = 0 \quad \text{for} \ \ s > 2. \tag{2.5}$$

We fix $E > 0$ and we choose δ, $0 < \delta \ll 1$, sufficiently small. We define

$$\beta_0(\xi) = \chi(2|\xi - \sqrt{E}\omega|/\delta^2)$$

for initial direction ω. We further take a nonnegative function $j_0 \in C^\infty(\boldsymbol{R}^2)$ such that

$$\operatorname{supp} j_0 \subset \Sigma(R, -\omega, \delta), \qquad j_0 = 1 \ \text{ on } \ \Sigma(2R, -\omega, \delta/2), \tag{2.6}$$

and $\partial_x^\beta j_0(x) = O(|x|^{-|\beta|})$ at infinity, where

$$\Sigma(R,\omega,\delta) = \{x : |x| > R, \ \ |x/|x| - \omega| < \delta\}, \quad R > 0.$$

Recall that the azimuth angle $\gamma(x;\omega)$ satisfies (1.1). Hence we have

$$\exp(-i\alpha\gamma(x;\omega))H_\alpha \exp(i\alpha\gamma(x;\omega)) = H(A_\alpha - \alpha\nabla\gamma) = H_0 \tag{2.7}$$

on $\Sigma(R, -\omega, \delta)$.

The next lemma is well known ([15]). We skip the proof.

Lemma 2.1 *Let* $f \in L^2$*. Then the free solution* $\exp(-itH_0)f$ *behaves like*

$$(\exp(-itH_0)f)(x) = (2it)^{-1}\exp(i|x|^2/4t)\widehat{f}(x/2t) + o(1), \quad |t| \to \infty,$$

in L^2*, where* $\widehat{f}(\xi) = (2\pi)^{-1}\int e^{-ix\cdot\xi}f(x)\,dx$ *is the Fourier transform.*

Let K_1 and K_2 be two self–adjoint operators in L^2. We introduce the new notation

$$W_\pm(K_2, K_1; J) = s - \lim_{t\to\pm\infty} \exp(itK_2) J \exp(-itK_1)$$

for a bounded operator J on L^2. Let $\beta_0(\xi)$ and $j_0(x)$ be as above. We set $J = j_0^2 \beta_0(D_x)^2$. Then

$$W_-(H_\alpha, H_0)\beta_0(D_x)^2 = W_-(H_\alpha, H_0; J)$$

by Lemma 2.1, so that we have the decomposition

$$W_-(H_\alpha, H_0)\beta_0(D_x)^2 = W_-(H_\alpha, H_0; J_0) W_-(H_0, H_0; J_1), \tag{2.8}$$

where

$$J_0 = j_0 \exp(i\alpha\gamma(x;\omega))\beta_0(D_x), \quad J_1 = j_0 \exp(-i\alpha\gamma(x;\omega))\beta_0(D_x).$$

The existence of $W_-(H_0, H_0; J_1)$ follows from Lemma 2.1, while the existence of $W_-(H_\alpha, H_0; J_0)$ is verified by use of (2.7). The same argument applies to final direction $\tilde\omega$. We define

$$\tilde\beta_0(\xi) = \chi(2|\xi - \sqrt{E}\tilde\omega|/\delta^2)$$

and we take a function $\tilde j_0 \in C^\infty(\boldsymbol{R}^2)$ such that

$$\operatorname{supp} \tilde j_0 \subset \Sigma(R, \tilde\omega, \delta), \qquad \tilde j_0 = 1 \ \text{ on } \ \Sigma(2R, \tilde\omega, \delta/2). \tag{2.9}$$

If we set

$$\tilde J_0 = \tilde j_0 \exp(i\alpha\gamma(x;-\tilde\omega))\tilde\beta_0(D_x), \quad \tilde J_1 = \tilde j_0 \exp(-i\alpha\gamma(x;-\tilde\omega))\tilde\beta_0(D_x),$$

then we obtain

$$W_+(H_\alpha, H_0)\tilde\beta_0(D_x)^2 = W_+(H_\alpha, H_0; \tilde J_0) W_+(H_0, H_0; \tilde J_1). \tag{2.10}$$

We combine (2.8) and (2.10) to obtain that

$$\tilde\beta_0(D_x)^2 S(H_\alpha, H_0)\beta_0(D_x)^2 = W_+^*(H_0, H_0; \tilde J_1) S_0(H_\alpha, H_0) W_-(H_0, H_0; J_1), \tag{2.11}$$

where

$$S_0(H_\alpha, H_0) = W_+^*(H_\alpha, H_0; \tilde J_0) W_-(H_\alpha, H_0; J_0).$$

The operator $S_0(H_\alpha, H_0)$ also has the direct integral decomposition, because it commutes with H_0. We denote by $S_0(\lambda; H_\alpha, H_0) : L^2(S^1) \to L^2(S^1)$ the fiber and by $S_0(\theta', \theta; \lambda, H_\alpha, H_0)$ the kernel of $S_0(\lambda; H_\alpha, H_0)$. By Lemma 2.1, $W_-(H_0, H_0; J_1)$ acts as the multiplication

$$FW_-(H_0, H_0; J_1)F^* = \exp(-i\alpha\gamma(-\theta;\omega))\beta_0(\sqrt{\lambda}\theta)\times$$

on $L^2((0,\infty);d\lambda)\otimes L^2(S^1)$, where $F : L^2 \to L^2((0,\infty);d\lambda)\otimes L^2(S^1)$ is the unitary mapping defined by (2.1). Similarly

$$FW_+(H_0,H_0;\tilde{J}_1)F^* = \exp(-i\alpha\gamma(\theta;-\tilde{\omega}))\tilde{\beta}_0(\sqrt{\lambda}\theta)\times .$$

Since $e^{-i\alpha\gamma(-\omega;\omega)}\beta_0(\sqrt{E}\omega) = e^{-i\alpha\pi}$ and $e^{-i\alpha\gamma(\tilde{\omega};-\tilde{\omega})}\tilde{\beta}_0(\sqrt{E}\tilde{\omega}) = e^{-i\alpha\pi}$, we have

$$S(\tilde{\omega},\omega;E,H_\alpha,H_0) = S_0(\tilde{\omega},\omega;E,H_\alpha,H_0) \tag{2.12}$$

by (2.11). We derive the representation for $S_0(\theta',\theta;E,H_\alpha,H_0)$ on the right side. The derivation is based on the idea due to [10]. We calculate $T = H_\alpha J_0 - J_0 H_0$ as

$$T = \exp(i\alpha\gamma(x;\omega))\,(H_0 j_0 - j_0 H_0)\,\beta_0(D_x) = \exp(i\alpha\gamma(x;\omega))[H_0, j_0]\beta_0(D_x)$$

by use of (2.7). Similarly we have

$$\tilde{T} = H_\alpha \tilde{J}_0 - \tilde{J}_0 H_0 = \exp(i\alpha\gamma(x;-\tilde{\omega}))[H_0,\tilde{j}_0]\tilde{\beta}_0(D_x).$$

Since $W_+(H_\alpha,H_0;J_0) = 0$ by Lemma 2.1, it follows that

$$W_-(H_\alpha,H_0;J_0) = -i\int \exp(itH_\alpha)T\exp(-itH_0)\,dt.$$

If we make use of this relation, then we obtain the representation

$$S_0(\lambda;H_\alpha,H_0) = 2\pi i F(\lambda)\left(-\tilde{J}_0^* T + \tilde{T}^* R(\lambda+i0;H_\alpha)T\right)F^*(\lambda) \tag{2.13}$$

in exactly the same way as [10, Theorem 3.3], where $F(\lambda) : L^2_s(\boldsymbol{R}^2) \to L^2(S^1),\ s > 1/2$, is the trace operator defined by

$$(F(\lambda)u)(\theta) = (Fu)(\mu,\theta)|_{\mu=\lambda}.$$

We write $\varphi_0(\omega,\lambda)$ for $\varphi_0(x;\omega,E) = \exp(i\sqrt{\lambda}x\cdot\omega)$ and denote by $(\ ,\ )$ the L^2 scalar product. The next lemma immediately follows from (2.12).

Lemma 2.2 *Assume that $\omega \neq \tilde{\omega}$. Then*

$$\begin{aligned} f(\omega\to\tilde{\omega};E,H_\alpha,H_0) &= -(ic(E)/4\pi)(T\varphi_0(\omega,E),\tilde{J}_0\varphi_0(\tilde{\omega},E)) \\ &+ (ic(E)/4\pi)(R(E+i0;H_\alpha)T\varphi_0(\omega,E),\tilde{T}\varphi_0(\tilde{\omega},E)). \end{aligned}$$

We fix σ, $0<\sigma\ll 1$, small enough and take $R = |d|^\sigma$, $|d|\gg 1$, in (2.6) and (2.9). We may assume that j_0 obeys $\partial_x^\beta j_0(x) = O(|x|^{-|\beta|})$ uniformly in d ; similarly for $\tilde{j}_0$. The operators $\tilde{J}_0$, T and $\tilde{T}$ are all pseudo-differential operators. If $\omega\neq\tilde{\omega}$, then we can choose δ so small that the support of symbols $T(x,\xi)$ and $\tilde{J}_0(x,\xi)$ does not intersect with each other. Hence it follows that

$$(T\varphi_0(\omega,E),\tilde{J}_0\varphi_0(\tilde{\omega},E)) = O(|d|^{-N}),\quad |d|\to\infty,$$

for any $N \gg 1$. Thus we have

$$f(\omega \to \tilde{\omega}; E, H_\alpha, H_0) = (ic(E)/4\pi)(R(E+i0; H_\alpha)T\varphi_0(\omega, E), \tilde{T}\varphi_0(\tilde{\omega}, E)) + o(1)$$

as $|d| \to \infty$. We continue to analyze the behavior as $|d| \to \infty$ of the term on the right side. We decompose $T = T(x, D_x)$ into

$$T = \chi_0 T + (1-\chi_0)T = T_0 + T_1,$$

where

$$\chi_0(x) = \chi(|x|/2|d|^\sigma) \tag{2.14}$$

for cut–off function $\chi \in C_0^\infty(0,\infty)$ with property (2.5). By (2.6), ∇j_0 vanishes on $\Sigma(2R, -\omega, \delta/2)$ with $R = |d|^\sigma$. Hence the symbol $T_1(x,\xi)$ has the support in the outgoing region

$$\operatorname{supp} T_1 \subset \{(x,\xi) : |x| > 2|d|^\sigma, \ \ |\xi - \sqrt{E}\omega| < \delta^2, \ \ x\cdot\xi > (-1+\delta/3)|x||\xi|\}.$$

The particle with initial state $(x,\xi) \in \operatorname{supp} T_1$ at $t = 0$ moves like the free particle and it does not pass in a neighborhood of the origin for $t \geq 0$. In fact, we have

$$|x+t\xi|^2 \geq |x|^2 - 2t(1-\delta/3)|x|\,|\xi| + t^2|\xi|^2 \geq c\,(|x| + t|\xi|)^2\,, \quad c > 0.$$

Thus the outgoing particle does not take momentum around $\sqrt{E}\tilde{\omega}$, so that

$$(R(E+i0; H_\alpha)T_1\varphi_0(\omega, E), \tilde{T}\varphi_0(\tilde{\omega}, E)) = O(|d|^{-N})$$

by the micro-local resolvent estimate ([9, Theorems 1 and 2]). Similarly we decompose $\tilde{T}$ into $\tilde{T} = \tilde{T}_0 + \tilde{T}_1$. Then we obtain

$$(R(E+i0; H_\alpha)T_0\varphi_0(\omega, E), \tilde{T}_1\varphi_0(\tilde{\omega}, E)) = O(|d|^{-N}).$$

A similar argument has been used in the semi–classical analysis on scattering amplitudes ([16]). The magnetic potential $A_\alpha(x)$ has a singularity at the origin, but the classical particle starting from $(x,\xi) \in \operatorname{supp} T_1$ or $(x,\xi) \in \operatorname{supp} \tilde{T}_1$ does not pass over the origin. Thus the argument there applies to H_α without any essential changes. The next lemma is obtained as a consequence of Lemma 2.2.

Lemma 2.3 *Let j_0, $\tilde{j}_0$ be as in (2.6) and (2.9) respectively and let χ_0 be defined by (2.14). Assume that $\omega \neq \tilde{\omega}$. Then*

$$f(\omega \to \tilde{\omega}; E, H_\alpha, H_0) = (ic(E)/4\pi)(R(E+i0; H_\alpha)T_0\varphi_0(\omega, E), \tilde{T}_0\varphi_0(\tilde{\omega}, E)) + o(1)$$

as $|d| \to \infty$, where T_0 acts as

$$T_0\varphi_0(\omega, E) = e^{i\alpha\gamma(x;\omega)}\chi_0[H_0, j_0]\varphi_0(\omega, E)$$

on $\varphi_0(\omega, E) = \varphi_0(x;\omega, E) = \exp(i\sqrt{E}x\cdot\omega)$, and $\tilde{T}_0$ acts as

$$\tilde{T}_0\varphi_0(\tilde{\omega}, E) = e^{i\alpha\gamma(x;-\tilde{\omega})}\chi_0[H_0, \tilde{j}_0]\varphi_0(\tilde{\omega}, E).$$

2.4. The main idea to prove the theorem is to represent the scattering amplitude $f_d(\omega \to \tilde{\omega}; E)$ in terms of the eigenfunction of $H_1 = H(A_{\alpha_1})$ or H_2. This subsection is devoted to a preliminary step towards the representation.

The eigenfunction $\varphi_\mp(x; \lambda, \omega)$ of H_α is defined by (2.3). We denote by $F_\pm : L^2 \to L^2((0,\infty); d\lambda) \otimes L^2(S^1)$ the unitary mapping

$$(F_\pm u)(\lambda, \theta) = 2^{-1/2}(2\pi)^{-1} \int \bar{\varphi}_\pm(x; \lambda, \theta) u(x)\, dx$$

and by $F_\pm(\lambda) : L^2_s(\boldsymbol{R}^2) \to L^2(S^1),\ s > 1/2$, the trace operator

$$(F_\pm(\lambda)u)(\theta) = (F_\pm u)(\mu, \theta)|_{\mu=\lambda}.$$

According to the stationary scattering theory, we know that

$$W_\mp(H_\alpha, H_0) = F_\pm^* F \tag{2.15}$$

and hence it follows that

$$F_\pm(\lambda) W_\mp(H_\alpha, H_0) u = F(\lambda) u, \quad \text{a. e.}\, \lambda > 0, \tag{2.16}$$

for $u \in L^2$. We now consider a function of the form

$$v_l(x) = f_l(r) e^{il\theta}, \qquad (F v_l)(\lambda, \theta) = g_l(\lambda) e^{il\theta}, \tag{2.17}$$

for $l \in \boldsymbol{Z}$, where $f_l \in \mathcal{S}[0,\infty)$ (Schwartz space) and

$$g_l(\lambda) = 2^{-1/2} e^{-i|l|\pi/2} \int_0^\infty J_{|l|}(\sqrt{\lambda} r) f_l(r) r\, dr.$$

We assume that $g_l \in C_0^\infty(0,\infty)$ is supported away from the origin.

Lemma 2.4 *Let v_l be as above. Then*

$$\langle x \rangle^N W_\pm(H_\alpha, H_0) v_l \in L^2$$

for any $N \gg 1$.

Proof. By (2.15), we have

$$(W_+(H_\alpha, H_0) v_l)(x) = \left(F_-^* F v_l\right)(x) = f_{-l}(r) e^{il\theta},$$

where

$$f_{-l}(r) = 2^{-1/2} e^{i\nu\pi/2} \int_0^\infty J_\nu(\sqrt{\lambda} r) g_l(\lambda)\, d\lambda$$

with $\nu = |l - \alpha|$. The Bessel function $J_p(r)$ obeys the asymptotic formula

$$J_p(r) = e^{ir} h_{+p}(r) + e^{-ir} h_{-p}(r) \tag{2.18}$$

at infinity, where $\partial_r^m h_{\pm p}(r) = O(r^{-1/2-m})$. By assumption, $g_l \in C_0^\infty(0,\infty)$ has compact support away from the origin. Hence the lemma follows by repeated use of partial integration. □

Lemma 2.5 *One has*

$$\|\langle x\rangle^{-m}\exp(-itH_\alpha)W_\pm(H_\alpha,H_0)v_l\|_{L^2}=O(|t|^{-m}),\quad |t|\to\infty,$$

for $m\ge 0$.

Proof. We divide $\boldsymbol{R}^2$ into the two regions $\{x:|x|>c\,|t|\}$ and $\{x:|x|<c\,|t|\}$ for some $c>0$. It is easy to see that the term in the lemma satisfies the desired bound $O(|t|^{-m})$ over the region $\{x:|x|>c\,|t|\}$. It follows from (2.15) that

$$\Big(\exp(-itH_\alpha)W_+(H_\alpha,H_0)v_l\Big)(x)=2^{-1/2}e^{i\nu\pi/2}\int_0^\infty J_\nu(\sqrt{\lambda}r)e^{-it\lambda}g_l(\lambda)\,d\lambda e^{il\theta}.$$

Assume that $|x|<c\,|t|$. Then we can take $c>0$ so small that the integral above obeys the bound $O(|t|^{-N})$ for any $N\gg 1$. This is again obtained by repeated use of partial integration. Thus the proof is complete. □

Lemma 2.6 *Let* $\beta_0(\xi)=\chi(2|\xi-\sqrt{E}\omega|/\delta^2)$ *be as before and let* $j_\pm(x)$ *be a bounded function vanishing in a conical neighborhood of* $\pm\omega$. *Then one can choose* $\delta>0$ *so small that*

$$\|j_+\beta_0(D_x)\exp(-itH_\alpha)W_\pm(H_\alpha,H_0)v_l\|_{L^2}=O(|t|^{-N}),\quad t\to\infty,$$
$$\|j_-\beta_0(D_x)\exp(-itH_\alpha)W_\pm(H_\alpha,H_0)v_l\|_{L^2}=O(|t|^{-N}),\quad t\to-\infty,$$

for any $N\gg 1$.

Proof. We give only a sketch for a proof. The proof is again done by repeated use of partial integration. We show that the term

$$I=j_+\beta_0(D_x)\exp(-itH_\alpha)W_-(H_\alpha,H_0)v_l$$

obeys the bound $O(|t|^{-N})$ as $t\to\infty$. A similar argument applies to the other terms. If we take account of (2.18), then I is expressed as the sum of two oscillatory integrals of the form

$$I_\pm=\int\int\int_0^\infty \exp(i\psi_\pm(x,\xi,y,\lambda;t))f_\pm(x,\xi,y,\lambda)\,d\lambda\,dy\,d\xi\,e^{il\theta},$$

where

$$\psi_\pm(x,\xi,y,\lambda;t)=(x-y)\cdot\xi\pm\sqrt{\lambda}|y|-t\lambda,\quad t\gg 1.$$

We consider the integral I_+ only. The amplitude function f_+ is supported in a small neighborhood of $\sqrt{E}\omega$ in variables ξ and has compact support away from the origin in variable λ, while the stationary point (ξ,y,λ) of the phase function ψ_+ has to fulfill the relations

$$y=x,\quad \xi=\sqrt{\lambda}y/|y|,\quad |y|=2\sqrt{\lambda}t$$

for $x \in \operatorname{supp} j_+$. If we take $\delta > 0$ small enough, then we see that such a stationary point does not exist. This yields the desired bound. □

Remark 2.1 If $v_l \in L^2$ takes the form $v_l = \left(F_-^* g e^{il\theta}\right)(x)$ or $v_l = \left(F_+^* g e^{il\theta}\right)(x)$ for $g(\lambda) \in C_0^\infty(0,\infty)$ supported away from the origin, then we can show in exactly the same way as above that $\|\langle x\rangle^{-m} \exp(-itH_\alpha)v_l\|_{L^2} = O(|t|^{-m})$ and

$$\|j_+\beta_0(D_x)\exp(-itH_\alpha)v_l\|_{L^2} = O(|t|^{-N}), \quad t \to \infty,$$
$$\|j_-\beta_0(D_x)\exp(-itH_\alpha)v_l\|_{L^2} = O(|t|^{-N}), \quad t \to -\infty.$$

The totality of such v_l is dense in L^2. As an immediate consequence, we have $W_+(H_d, H_\alpha; J_+) = 0$ for $J_+ = j_+\beta_0(D_x)$.

3 Proof of main theorem : reduction to basic lemmas

In this section we prove the main theorem (Theorem 1.1) by reduction to three lemmas (Lemmas 3.2 $\sim$ 3.4). The proof of these lemmas is given in section 4, and section 5 is devoted to proving the estimates for resolvent $R(E + i0; H_d)$ which play a central role in the proof of the lemmas. As previously stated, we prove the self–adjointness, the absence of bound states, the principle of limiting absorption and the asymptotic completeness of wave operators for H_d in section 7. We use these facts without further references.

3.1. The perturbation $H_d - H_0$ between H_d and $H_0 = -\Delta$ is of long–range class. However we can show that the ordinary wave operator

$$W_\pm(H_d, H_0) = s - \lim_{t\to\pm\infty} \exp(itH_d)\exp(-itH_0) : L^2 \to L^2$$

exists and it is asymptotically complete

$$\operatorname{Ran} W_-(H_d, H_0) = \operatorname{Ran} W_+(H_d, H_0) = L^2.$$

Hence the scattering operator

$$S(H_d, H_0) = W_+^*(H_d, H_0)W_-(H_d, H_0) : L^2 \to L^2$$

can be defined as a unitary operator and it has the direct integral decomposition

$$S(H_d, H_0) \simeq FS(H_d, H_0)F^* = \int_0^\infty \oplus S(\lambda; H_d, H_0)\, d\lambda.$$

If we denote by $S(\theta', \theta; \lambda, H_d, H_0)$ the kernel of fiber $S(\lambda; H_d, H_0) : L^2(S^1) \to L^2(S^1)$, then the scattering amplitude $f_d(\omega \to \tilde\omega; E)$ in question is defined by

$$f_d(\omega \to \tilde\omega; E) = c(E)\left(S(\tilde\omega, \omega; E, H_d, H_0) - \delta(\tilde\omega - \omega)\right)$$

with $c(E) = (2\pi/i\sqrt{E})^{1/2}$ again. If, in particular, $\omega \neq \tilde{\omega}$, then

$$f_d(\omega \to \tilde{\omega}; E) = c(E)\, S(\tilde{\omega}, \omega; E, H_d, H_0).$$

The first step toward the proof of Theorem 1.1 is to represent $f_d(\omega \to \tilde{\omega}; E)$ in a convenient form. We always assume that $\omega \neq \tilde{\omega}$. We keep the same notation as in section 2. Let j_0 and $\tilde{j}_0$ be as in (2.6) and (2.9), where R is taken as $R = |d|^\sigma$ for $0 < \sigma \ll 1$ fixed small enough. We set

$$\chi_\infty(x) = 1 - \chi(2|x|/|d|^\sigma),$$

so that $\chi_\infty(x) = 1$ for $|x| > |d|^\sigma$. We further define the following operators :

$$\begin{aligned} J_{0d} &= \exp(i\alpha_2\gamma(x-d;\omega)) j_{0d}\chi_\infty\beta_0(D_x)\chi_\infty, \\ J_{1d} &= \exp(-i\alpha_2\gamma(x-d;\omega)) j_{0d}\beta_0(D_x), \end{aligned}$$

where $j_{0d}(x) = j_0(x-d)$. Then $W_-(H_d, H_0)\beta_0(D_x)^2$ is decomposed into

$$W_-(H_d, H_0)\beta_0(D_x)^2 = W_-(H_d, H_1; J_{0d}) W_-(H_1, H_0) W_-(H_0, H_0; J_{1d}).$$

By Lemma 2.1, $W_-(H_0, H_0; J_{1d})$ is realized as the multiplication

$$F W_-(H_0, H_0; J_{1d}) F^* = e^{-i\alpha_2\gamma(-\theta;\omega)}\beta_0(\sqrt{\lambda}\theta)\times$$

on $L^2((0,\infty); d\lambda) \otimes L^2(S^1)$. A similar relation

$$W_+(H_d, H_0)\tilde{\beta}_0(D_x)^2 = W_+(H_d, H_1; \tilde{J}_{0d}) W_+(H_1, H_0) W_+(H_0, H_0; \tilde{J}_{1d})$$

holds for the wave operator $W_+(H_d, H_0)$, where

$$\begin{aligned} \tilde{J}_{0d} &= \exp(i\alpha_2\gamma(x-d;-\tilde{\omega})) \tilde{j}_{0d}\chi_\infty\tilde{\beta}_0(D_x)\chi_\infty, \\ \tilde{J}_{1d} &= \exp(-i\alpha_2\gamma(x-d;-\tilde{\omega})) \tilde{j}_{0d}\tilde{\beta}_0(D_x). \end{aligned}$$

The eigenfunction $\varphi_{\mp 1}(x;\theta,\lambda)$ of $H_1 = H(A_{\alpha_1})$ is defined by (2.3) with α replaced by α_1. We write $F_{\pm 1} : L^2 \to L^2((0,\infty); d\lambda) \otimes L^2(S^1)$ for the unitary mapping associated with $\varphi_{\pm 1}$ and $F_{\pm 1}(\lambda) : L^2_s(\boldsymbol{R}^2) \to L^2(S^1)$, $s > 1/2$, for the trace operator. Then it follows from (2.15) and (2.16) that $W_\mp(H_1, H_0) = F^*_{\pm 1} F$ and

$$F_{\pm 1}(\lambda) W_\mp(H_1, H_0) u = F(\lambda) u, \quad \text{a. e. } \lambda > 0, \tag{3.1}$$

for $u \in L^2$. We now define $S_0 : L^2 \to L^2$ as

$$S_0 = W_+^*(H_1, H_0) W_+^*(H_d, H_1; \tilde{J}_{0d}) W_-(H_d, H_1; J_{0d}) W_-(H_1, H_0).$$

Since S_0 commutes with H_0, it has the direct integral decomposition. We denote by $S_0(\lambda) : L^2(S^1) \to L^2(S^1)$ the fiber of S_0.

Lemma 3.1 *Let the notation be as above. Then the fiber $S_0(\lambda)$ is represented as*

$$S_0(\lambda) = 2\pi i F_{-1}(\lambda)\left(-\tilde{J}_{0d}^* T_d + \tilde{T}_d^* R(\lambda + i0; H_d) T_d\right) F_{+1}^*(\lambda),$$

where

$$T_d = H_d J_{0d} - J_{0d} H_1, \qquad \tilde{T}_d = H_d \tilde{J}_{0d} - \tilde{J}_{0d} H_1.$$

Before going into the proof, we calculate T_d and $\tilde{T}_d$ in the lemma. Both the operators are realized as a pseudo-differential operator. We write $\gamma_d = \gamma(x-d;\omega)$ and $\beta_0 = \beta_0(D_x)$ for brevity. Since

$$e^{-i\alpha_2\gamma_d} H_d e^{i\alpha_2\gamma_d} = e^{-i\alpha_2\gamma_d} H(A_{\alpha_1} + A_{\alpha_2,d}) e^{i\alpha_2\gamma_d} = H(A_{\alpha_1}) = H_1$$

on the support of j_{0d}, we have

$$T_d = e^{i\alpha_2\gamma_d}\left([H_1, j_{0d}]\chi_\infty\beta_0\chi_\infty + j_{0d}[H_1, \chi_\infty\beta_0\chi_\infty]\right).$$

We set $Q = H_1 - H_0$. The coefficients of Q have a singularity at the origin only. Since $\chi_\infty = \chi_\infty(|x|)$ is rotationally invariant, it is easy to see that $[Q, \chi_\infty] = 0$. Hence we can calculate the second commutator as

$$\begin{aligned}
[H_1, \chi_\infty\beta_0\chi_\infty] &= [H_0, \chi_\infty\beta_0\chi_\infty] + [Q, \chi_\infty\beta_0\chi_\infty] \\
&= [H_0, \chi_\infty]\beta_0\chi_\infty + \chi_\infty\beta_0[H_0, \chi_\infty] + \chi_\infty[Q, \beta_0]\chi_\infty \\
&= [H_0, \chi_\infty]\beta_0\chi_\infty + \chi_\infty\beta_0[H_0, \chi_\infty] + [\chi_\infty Q, \beta_0]\chi_\infty + [\beta_0, \chi_\infty]Q\chi_\infty.
\end{aligned}$$

Thus T_d admits the decomposition

$$T_d = \Gamma_{1d} + \Gamma_{2d} + \Gamma_{3d}, \tag{3.2}$$

where

$$\begin{aligned}
\Gamma_{1d} &= e^{i\alpha_2\gamma(x-d;\omega)} j_{0d}\left([H_0, \chi_\infty]\beta_0\chi_\infty + \chi_\infty\beta_0[H_0, \chi_\infty]\right), \\
\Gamma_{2d} &= e^{i\alpha_2\gamma(x-d;\omega)} [H_1, j_{0d}]\chi_\infty\beta_0\chi_\infty, \\
\Gamma_{3d} &= e^{i\alpha_2\gamma(x-d;\omega)} j_{0d}\left([\chi_\infty Q, \beta_0]\chi_\infty + [\beta_0, \chi_\infty]Q\chi_\infty\right)
\end{aligned}$$

with $Q = H_1 - H_0$. Similarly

$$\tilde{T}_d = \tilde{\Gamma}_{1d} + \tilde{\Gamma}_{2d} + \tilde{\Gamma}_{3d}, \tag{3.3}$$

where

$$\begin{aligned}
\tilde{\Gamma}_{1d} &= e^{i\alpha_2\gamma(x-d;-\tilde{\omega})} \tilde{j}_{0d}\left([H_0, \chi_\infty]\tilde{\beta}_0\chi_\infty + \chi_\infty\tilde{\beta}_0[H_0, \chi_\infty]\right), \\
\tilde{\Gamma}_{2d} &= e^{i\alpha_2\gamma(x-d;-\tilde{\omega})} [H_1, \tilde{j}_{0d}]\chi_\infty\tilde{\beta}_0\chi_\infty, \\
\tilde{\Gamma}_{3d} &= e^{i\alpha_2\gamma(x-d;-\tilde{\omega})} \tilde{j}_{0d}\left([\chi_\infty Q, \tilde{\beta}_0]\chi_\infty + [\tilde{\beta}_0, \chi_\infty]Q\chi_\infty\right).
\end{aligned}$$

We see in the course of the proof of Theorem 1.1 in this section that

$$F_{-1}(\lambda)\tilde{\Gamma}_{kd}^{*}R(\lambda+i0;H_d)\Gamma_{jd}F_{+1}^{*}(\lambda):L^2(S^1)\to L^2(S^1),\quad 1\le j,k\le 3,$$

are all bounded, and hence the relation in Lemma 3.1 makes sense. In fact, each operator is implicitly shown to have a bounded kernel as an integral operator.

Proof of Lemma 3.1. The dependence on d does not matter throughout the proof. We use the following simplified notation :

$$W_\pm = W_\pm(H_1,H_0),\quad V_\pm = W_\pm(H_d,H_1;J_{0d}),\quad \tilde{V}_\pm = W_\pm(H_d,H_1;\tilde{J}_{0d})$$

and

$$U_1(t)=\exp(-itH_1),\quad U(t)=\exp(-itH_d).$$

The proof is based on the same idea as used to derive (2.13) (see [10,15]). We consider the integral

$$(S_0u,v)=\int_0^\infty < S_0(\lambda)F(\lambda)u,F(\lambda)v > \,d\lambda$$

for $u,\ v\in L^2$, where $<\ ,\ >$ denotes the L^2 scalar product in $L^2(S^1)$. According to the notation above, we have

$$(S_0u,v)=(V_-W_-u,\tilde{V}_+W_+v).$$

We assume for the moment that u and v take the form

$$u(x)=f_l(r)e^{il\theta},\qquad v(x)=f_m(r)e^{im\theta} \tag{3.4}$$

as in (2.17). Then Lemma 2.4 implies that $\langle x\rangle^N W_\pm u\in L^2$, and it follows from Lemmas 2.5 and 2.6 that $\|T_dU_1(t)W_\pm u\|_{L^2}=O(|t|^{-2})$ as $|t|\to\infty$. These facts enable us to justify the rather formal computation below.

Since $V_+=0$ (see Remark 2.1), we can write V_- in the integral form

$$V_-=-i\int U(-t)T_dU_1(t)\,dt$$

and hence we obtain

$$(S_0u,v)=-i\int (T_dU_1(t)W_-u,\tilde{V}_+U_1(t)W_+v)\,dt$$

by the intertwining property $U(t)\tilde{V}_+=\tilde{V}_+U_1(t)$. If we further make use of the relation

$$\tilde{V}_+=\tilde{J}_{0d}+i\int_0^\infty U(-s)\tilde{T}_dU_1(s)\,ds,$$

then we have

$$\begin{aligned}(S_0u, v) &= -i\int(\tilde{J}_{0d}^*T_dU_1(t)W_-u, U_1(t)W_+v)\,dt\\ &- \int\int_0^\infty(\tilde{T}_d^*U(s)T_dU_1(t)W_-u, U_1(t+s)W_+v)\,dt\,ds.\end{aligned}$$

We denote by I_1 the first integral on the right side and by I_2 the second one. We calculate I_1 as

$$\begin{aligned}I_1 &= -i\int\int_0^\infty < F_{-1}(\lambda)\tilde{J}_{0d}^*T_dU_1(t)W_-u, F_{-1}(\lambda)U_1(t)W_+v > \,d\lambda\,dt\\ &= -i\int\int_0^\infty < F_{-1}(\lambda)\tilde{J}_{0d}^*T_d\left(e^{it\lambda}U_1(t)\right)W_-u, F_{-1}(\lambda)W_+v > \,d\lambda\,dt\\ &= -i\lim_{\varepsilon\downarrow 0}\int\int_0^\infty < F_{-1}(\lambda)\tilde{J}_{0d}^*T_d\left(e^{-\varepsilon|t|}e^{it\lambda}U_1(t)\right)W_-u, F_{-1}(\lambda)W_+v > \,d\lambda\,dt.\end{aligned}$$

The formula

$$\lim_{\varepsilon\to 0}\int e^{-\varepsilon|t|}e^{it\lambda}U_1(t)\,dt = i\left(R(\lambda - i0; H_1) - R(\lambda + i0; H_1)\right) = 2\pi F_{\pm 1}(\lambda)^*F_{\pm 1}(\lambda)$$

is well known in the stationary scattering theory. Hence it follows from (3.1) that

$$I_1 = 2\pi i\int_0^\infty < -F_{-1}(\lambda)\tilde{J}_{0d}^*T_dF_{+1}(\lambda)^*F(\lambda)u, F(\lambda)v > \,d\lambda.$$

A similar computation gives

$$I_2 = 2\pi i\int_0^\infty < F_{-1}(\lambda)\tilde{T}_d^*R(\lambda + i0; H_d)T_dF_{+1}(\lambda)^*F(\lambda)u, F(\lambda)v > \,d\lambda,$$

where the resolvent $R(\lambda + i0; H_d)$ comes from the integration in variable s. We combine the two relations above to obtain that

$$\int_0^\infty < S_0(\lambda)F(\lambda)u, F(\lambda)v > \,d\lambda =$$
$$2\pi i\int_0^\infty < F_{-1}(\lambda)\left(-\tilde{J}_{0d}^*T_d + \tilde{T}_d^*R(\lambda + i0; H_d)T_d\right)F_{+1}(\lambda)^*F(\lambda)u, F(\lambda)v > \,d\lambda$$

for $u,\ v$ as in (3.4). The Fourier expansion and the limit procedure show that this relation remains true for $u,\ v \in L^2$ such that $(Fu)(\lambda,\theta) = g(\lambda)\eta(\theta)$ and $(Fv)(\lambda,\theta) = \tilde{g}(\lambda)\tilde{\eta}(\theta)$, where $\eta,\ \tilde{\eta} \in C^\infty(S^1)$, and $g,\ \tilde{g} \in C_0^\infty(0,\infty)$ have compact support away from the origin. This completes the proof. □

We write $S_0(\theta',\theta;\lambda)$ for the kernel of fiber $S_0(\lambda)$. As is easily seen,

$$S(\tilde{\omega},\omega; E, H_d, H_0) = S_0(\tilde{\omega},\omega; E)$$

and hence it follows from Lemma 3.1 that

$$\begin{aligned} f_d(\omega \to \tilde{\omega}; E) &= -(ic(E)/4\pi)(T_d\varphi_{+1}(\omega, E), \tilde{J}_{0d}\varphi_{-1}(\tilde{\omega}, E)) \\ &+ (ic(E)/4\pi)(R(E + i0; H_d)T_d\varphi_{+1}(\omega, E), \tilde{T}_d\varphi_{-1}(\tilde{\omega}, E)), \end{aligned}$$

where $\varphi_{\pm 1}(\omega, E) = \varphi_{\pm 1}(x; \omega, E)$. By Proposition 2.1, $\varphi_{\pm 1}(x; \omega, E)$ is bounded uniformly in $x \in \boldsymbol{R}^2$. Roughly speaking, the support of symbols $T_d(x, \xi)$ and $\tilde{J}_{0d}(x, \xi)$ does not intersect with each other, provided that $\omega \neq \tilde{\omega}$. A simple calculus of pseudo-differential operators yields that

$$(T_d\varphi_{+1}(\omega, E), \tilde{J}_{0d}\varphi_{-1}(\tilde{\omega}, E)) = O(|d|^{-N})$$

and hence we have

$$f_d(\omega \to \tilde{\omega}; E) = (ic(E)/4\pi)(R(E + i0; H_d)T_d\varphi_{+1}(\omega, E), \tilde{T}_d\varphi_{-1}(\tilde{\omega}, E)) + o(1). \tag{3.5}$$

3.2. The second step is to study the behavior as $|d| \to \infty$ of the term on the right side of (3.5) by making use of estimates on resolvent $R(E + i0; H_d)$. We introduce the new notation to formulate the resolvent estimates. Let $0 < \sigma \ll 1$ be still fixed small enough and write $\hat{x}$ for direction $x/|x|$. We set

$$B_{1d} = \{x : |x| < C|d|^\sigma\}, \qquad B_{2d} = \{x : |x - d| < C|d|^\sigma\}$$

and

$$\Lambda_d = \{x : |x| > \delta|d|^\sigma,\ \ |\hat{x} - \hat{d}| < \delta,\ \ |x - d| > \delta|d|^\sigma,\ \ |\widehat{(x - d)} + \hat{d}| < \delta\}$$

for some $C \gg 1$, and we denote by $b_{1d},\ b_{2d}$ and λ_d the characteristic function of $B_{1d},\ B_{2d}$ and Λ_d respectively. We further denote by $\|\ \|$ the norm of bounded operators acting on L^2, and we use the notation $\|Q_d\| \simeq O(|d|^\nu)$ when $Q_d : L^2 \to L^2$ obeys the bound $\|Q_d\| \le c_\varepsilon |d|^{\nu+\varepsilon}$, $|d| \gg 1$, for any $\varepsilon > 0$. The proof of the main theorem is based on the following three lemmas.

Lemma 3.2 *Let r_L be the pseudo-differential operator defined by*

$$r_L = r_L(x, D_x) = (|x|^2 + |d|^2)^{-L/2}\langle D_x\rangle^{-L} \tag{3.6}$$

for $L \gg 1$. Then one has :

(1) $\|r_L R(E + i0; H_d)b_{1d}\| = O(|d|^{-L/2})$ *; similarly for b_{2d} and λ_d.*

(2) $\|r_L R(E + i0; H_d)r_L\| = O(|d|^{-L})$.

The estimates in the lemma are very rough. This lemma is used to control error terms which arise in constructing outgoing and incoming approximations to the resolvent $R(E+i0;H_d)$. According to the principle of limiting absorption (Proposition 7.3), we know that $R(E+i0;H_d)$ is bounded from $L^2_s(\boldsymbol{R}^2)$ to $L^2_{-s}(\boldsymbol{R}^2)$ for $s>1/2$, but we do not here intend to pursue how sharp the resolvent estimate can be made. The proof of the theorem does not require such a sharp estimate.

Lemma 3.3 *One has*

$$\|b_{1d}R(E+i0;H_d)b_{2d}\| \simeq O(|d|^{-1/2+4\sigma})$$

and

$$\|b_{1d}\Big(R(E+i0;H_d)-R(E+i0;H_1)\Big)b_{1d}\| \simeq O(|d|^{-1+7\sigma}),$$

$$\|b_{2d}\Big(R(E+i0;H_d)-R(E+i0;H_{2,d})\Big)b_{2d}\| \simeq O(|d|^{-1+7\sigma}).$$

Lemma 3.4 *Write $\gamma_d(x)$ for $\gamma(x-d;\hat{d})$. Then one has*

$$\|b_{2d}R(E+i0;H_d)\lambda_d\langle x\rangle^{-1}\| \simeq O(|d|^{-1/2+3\sigma})$$

and

$$\|b_{1d}\left(R(E+i0;H_d)-e^{i\alpha_2\gamma_d}R(E+i0;H_1)e^{-i\alpha_2\gamma_d}\right)\lambda_d\langle x\rangle^{-1}\| \simeq O(|d|^{-1+6\sigma}),$$

$$\|\langle x\rangle^{-1}\lambda_d\left(R(E+i0;H_d)-e^{i\alpha_2\gamma_d}R(E+i0;H_1)e^{-i\alpha_2\gamma_d}\right)\lambda_d\langle x\rangle^{-1}\| \simeq O(|d|^{-1+5\sigma}).$$

Remark 3.1 All the lemmas remain true for $R(E-i0;H_d)$. Thus Lemma 3.2 shows

$$\|b_{1d}R(E+i0;H_d)r_L\| = O(|d|^{-L/2})$$

by adjoint. In the argument below, we use such an immediate consequence without further references.

We shall complete the proof of Theorem 1.1, accepting these lemmas as proved. To fix the idea, we prove the theorem for $f_d(\hat{d}\to-\hat{d};E)$ only. If $\omega=-\hat{d}$, we represent $f_d(-\hat{d}\to\tilde{\omega};E)$ in terms of the eigenfunction $\varphi_{\mp 2}(x;\theta,\lambda)$ of $H_2=H(A_{\alpha_2})$ and the other cases are more easier to deal with. If, in fact, $\omega\neq\pm\hat{d}$ and $\tilde{\omega}\neq\pm\hat{d}$, then the situation becomes much simpler and the proof does not require Lemma 3.4.

Let Γ_{jd} and $\tilde{\Gamma}_{jd}$ be as in (3.2) and (3.3) respectively. We set

$$\gamma_{jk} = (ic(E)/4\pi)(R(E+i0;H_d)\Gamma_{jd}\varphi_{+1},\tilde{\Gamma}_{kd}\varphi_{-1})$$

for $1 \le j,\ k \le 3$, where $\varphi_{+1} = \varphi_{+1}(x; \hat{d}, E)$ and $\varphi_{-1} = \varphi_{-1}(x; -\hat{d}, E)$. To prove the theorem, we have only to show that :

$$\gamma_{jk} = o(1), \quad j \ne k, \tag{3.7}$$

$$\gamma_{33} = o(1) \tag{3.8}$$

and

$$\gamma_{11} = f_1(\hat{d} \to -\hat{d}; E) + o(1) \tag{3.9}$$

$$\gamma_{22} = (\cos \alpha_1 \pi)^2 f_{2,d}(\hat{d} \to -\hat{d}; E) + o(1). \tag{3.10}$$

When $\omega = \hat{d}$ and $\tilde{\omega} = -\hat{d}$, we may take the two functions j_0 and $\tilde{j}_0$ in such a way that these functions coincide with each other. Thus we assume that $j_0 = \tilde{j}_0$. The three lemmas above can be seen to remain true for the smooth functions

$$b_{1d}(x) = \chi(|x|/C|d|^\sigma), \qquad b_{2d}(x) = \chi(|x-d|/C|d|^\sigma)$$

and

$$\lambda_d(x) = \Big(1 - \chi(2|x|/\delta|d|^\sigma)\Big)\chi(|\hat{x} - \hat{d}|/\delta)\Big(1 - \chi(2|x-d|/\delta|d|^\sigma)\Big)\chi(|\widehat{(x-d)} + \hat{d}|/\delta)$$

associated with the three sets $B_{1d},\ B_{2d}$ and Λ_d respectively. We use the notation $b_{1d},\ b_{2d}$ and λ_d with the meaning ascribed above throughout the proof of (3.7) $\sim$ (3.10). We begin by (3.8). The proof is based on the following lemma.

Lemma 3.5 *Let $r_L = r_L(x, D_x)$, $L \gg 1$, be defined by (3.6) and let $\lambda_d(x)$ be as above. Then $\Gamma_{3d}\varphi_{+1}$ and $\tilde{\Gamma}_{3d}\varphi_{-1}$ take the form*

$$\Gamma_{3d}\varphi_{+1} = \lambda_d \Gamma_{3d}\varphi_{+1} + r_L e_d, \qquad \tilde{\Gamma}_{3d}\varphi_{-1} = \lambda_d \tilde{\Gamma}_{3d}\varphi_{-1} + r_L \tilde{e}_d,$$

where the L^2 norm of remainder terms e_d and $\tilde{e}_d$ is bounded uniformly in d.

Proof. The proof uses Proposition 2.1. Roughly speaking, the symbol $\Gamma_{3d}(x,\xi)$ has support on $\operatorname{supp} j_{0d}$ in variables x and on $\operatorname{supp} \nabla\beta_0$ in variables ξ. By (2.6), $j_{0d}(x) = j_0(x-d)$ has support in $\{x : x - d \in \Sigma(|d|^\sigma, -\hat{d}, \delta)\}$, and $\nabla\beta_0$ has support in $\{\xi : \delta^2/2 < |\xi - \sqrt{E}\hat{d}| < \delta^2\}$ for the incident direction $\hat{d}$. If $\beta(\xi)$ vanishes around $\xi = \sqrt{E}\hat{d}$, then $\beta(D_x)\exp(i\sqrt{E}x\cdot\hat{d}) = 0$, and if $x \in \operatorname{supp} j_{0d} \cap \Lambda_d^c$ and $\xi \in \operatorname{supp} \nabla\beta_0$, then

$$\left|\nabla\left(\sqrt{E}|x| - \xi\cdot x\right)\right| = \left|\sqrt{E}\hat{x} - \xi\right| > c > 0.$$

Thus the first relation follows from Proposition 2.1 (1) and (4). A similar argument applies to the second one and the proof is complete. □

Lemma 3.5 implies (3.8). The symbols $\Gamma_{3d}(x,\xi)$ and $\tilde{\Gamma}_{3d}(x,\xi)$ fall off with order $O(|x|^{-2})$ at infinity uniformly in d. By Proposition 2.1 (4), $\langle x\rangle\lambda_d\Gamma_{3d}\varphi_{+1}$

and $\langle x\rangle\lambda_d\tilde{\Gamma}_{3d}\varphi_{-1}$ are of order $O(\log|d|)$ in the L^2 norm, and by the principle of limiting absorption,

$$\langle x\rangle^{-\rho}R(E+i0;H_1)\langle x\rangle^{-\rho}:L^2\to L^2$$

is bounded for any $\rho>1/2$. Hence (3.8) follows from Lemmas 3.2 and 3.4.

To prove (3.7), we further prove one lemma. We write β_0, $\tilde{\beta}_0$, β_1 and $\tilde{\beta}_1$ for the pseudo-differential operators with symbols

$$\beta_0(\xi)=\chi(2|\xi-\sqrt{E}\hat{d}|/\delta^2),\qquad \tilde{\beta}_0(\xi)=\chi(2|\xi+\sqrt{E}\hat{d}|/\delta^2),$$
$$\beta_1(\xi)=\chi(|\xi-\sqrt{E}\hat{d}|/\delta^2),\qquad \tilde{\beta}_1(\xi)=\chi(|\xi+\sqrt{E}\hat{d}|/\delta^2),$$

respectively. By definition, $\beta_1\beta_0=\beta_0$ and $\tilde{\beta}_1\tilde{\beta}_0=\tilde{\beta}_0$. Let $\lambda(x)$ be a smooth function such that $\partial_x^\beta\lambda=O(|x|^{-|\beta|})$ and

$$\operatorname{supp}\lambda\subset\{x:|x-d|>C|d|^\sigma,\ |\widehat{(x-d)}+\hat{d}|>\delta\}$$

for $C\gg 1$. We construct an outgoing approximation for $R(E+i0;H_d)\lambda\beta_0$ and an incoming one for $R(E-i0;H_d)\lambda\tilde{\beta}_0$. To do this, we take a function $j\in C^\infty(\boldsymbol{R}^2)$ such that $\partial_x^\beta j=O(|x|^{-|\beta|})$ and

$$\operatorname{supp}j\subset\{x:|x-d|>|d|^\sigma,\ |\widehat{(x-d)}+\hat{d}|>\delta/4\}$$

and $j(x)=1$ on $\{x:|x-d|>2|d|^\sigma,\ |\widehat{(x-d)}+\hat{d}|>\delta/2\}$. Hence $j=1$ on the support of λ.

Lemma 3.6 *Let the notation be as above and let $\theta_d(x)$ be defined by*

$$\theta_d(x)=\alpha_1\gamma(x;-\hat{d})+\alpha_2\gamma(x-d;-\hat{d}).$$

Then one has

$$R(E+i0;H_d)\lambda\beta_0=j\exp(i\theta_d)R(E+i0;H_0)\beta_1\exp(-i\theta_d)\lambda\beta_0+R(E+i0;H_d)\tilde{r}_L,$$
$$R(E-i0;H_d)\lambda\tilde{\beta}_0=j\exp(i\theta_d)R(E-i0;H_0)\tilde{\beta}_1\exp(-i\theta_d)\lambda\tilde{\beta}_0+R(E-i0;H_d)\tilde{r}_L$$

for $L\gg 1$, where $\tilde{r}_L$ denotes an operator such that

$$\tilde{r}_L\langle D_x\rangle^L(|x|^2+|d|^2)^{L/2},\quad \langle D_x\rangle^L(|x|^2+|d|^2)^{L/2}\tilde{r}_L:L^2\to L^2 \tag{3.11}$$

are bounded uniformly in d.

Proof. We prove only the first relation. We calculate

$$(H_d-E)j\exp(i\theta_d)=\exp(i\theta_d)(H_0-E)j$$

by use of a relation similar to (2.7). Hence

$$\begin{aligned}(H_d - E)j\exp(i\theta_d)R(E+i0;H_0)\beta_1\exp(-i\theta_d)\lambda\beta_0 \\ = \lambda\beta_0 + \tilde{r}_L + \exp(i\theta_d)[H_0,j]R(E+i0;H_0)\beta_1\exp(-i\theta_d)\lambda\beta_0.\end{aligned}$$

The resolvent $R(E+i0;H_0)$ is represented in the integral form

$$R(E+i0;H_0) = i\int_0^\infty e^{itE}\exp(-itH_0)\,dt.$$

If we choose δ small enough, then the free particle with initial state $(x,\xi) \in \operatorname{supp}\lambda \times \operatorname{supp}\beta_1$ does not pass over $\operatorname{supp}\nabla j$ for $t > 0$, so that we can put

$$\tilde{r}_L = \exp(i\theta_d)[H_0,j]R(E+i0;H_0)\beta_1\exp(-i\theta_d)\lambda\beta_0$$

for the remainder term on the right side of the above relation. In fact, this can be shown in the standard way using partial integral repeatedly. Thus the proof is complete. □

We proceed to the proof of (3.7). We first consider the term γ_{13}. Recall that $\chi_\infty = 1 - \chi(2|x|/|d|^\sigma)$, so that $\nabla\chi_\infty$ has support on $\{x : |d|^\sigma/2 < |x| < |d|^\sigma\} \subset B_{1d}$. Since $\Gamma_{1d}\varphi_{+1}$ is uniformly bounded in L^2, we have

$$\gamma_{13} = (ic(E)/4\pi)(e^{i\alpha_2\gamma_d}R(E+i0;H_1)e^{-i\alpha_2\gamma_d}\Gamma_{1d}\varphi_{+1}, \lambda_d\tilde{\Gamma}_{3d}\varphi_{-1}) + o(1)$$

by Lemmas 3.2, 3.4 and 3.5, where $\gamma_d(x) = \gamma(x-d;\hat{d})$. We construct approximations for resolvent $R(E \pm i0;H_1)$. Let

$$\lambda_{1d}(x) = \Big(1 - \chi(4|x|/|d|^\sigma)\Big)\chi(|x|/|d|^\sigma)\chi(|\hat{x}+\hat{d}|/\delta)$$

be the smooth function associated with the set

$$\Lambda_{1d} = \{x : |d|^\sigma/2 < |x| < |d|^\sigma,\ \ |\hat{x}+\hat{d}| < \delta\}.$$

Assume that $x \in \operatorname{supp}\nabla\chi_\infty$ satisfies $|\hat{x}+\hat{d}| > \delta$ and $\xi \in \operatorname{supp}\beta_0$. Then it follows that $|x+t\xi| > c\,(t+|x|)$, $c > 0$, for $t > 0$. Hence the particle starting from initial state (x,ξ) at $t = 0$ moves like the free particle and it does not take momentum around $-\sqrt{E}\hat{d} \in \operatorname{supp}\tilde{\beta}_0$. This enables us to construct an outgoing approximation in the form

$$\tilde{\Gamma}_{3d}^*\lambda_d e^{i\alpha_2\gamma_d}R(E+i0;H_1)e^{-i\alpha_2\gamma_d}(1-\lambda_{1d})\Gamma_{1d} = \tilde{r}_L + \tilde{\Gamma}_{3d}^*\lambda_d e^{i\alpha_2\gamma_d}R(E+i0;H_1)\tilde{r}_L$$

for any $L \gg 1$. The construction is based on the same idea as in the proof of Lemma 3.6. Thus we obtain

$$\gamma_{13} = (ic(E)/4\pi)(\lambda_{1d}\Gamma_{1d}\varphi_{+1}, e^{i\alpha_2\gamma_d}R(E-i0;H_1)e^{-i\alpha_2\gamma_d}\lambda_d\tilde{\Gamma}_{3d}\varphi_{-1}) + o(1).$$

We further construct an incoming approximation for $R(E-i0;H_1)$. If $x \in \Lambda_d$ and $\xi \in \operatorname{supp} \tilde{\beta}_0$, then the particle with initial state (x,ξ) does not pass over Λ_{1d} for $t<0$. Hence we get $\gamma_{13} = o(1)$ by constructing an approximation

$$\lambda_{1d} e^{i\alpha_2\gamma_d} R(E-i0;H_1) e^{-i\alpha_2\gamma_d} \lambda_d \tilde{\Gamma}_{3d} = \tilde{r}_L + \lambda_{1d} e^{i\alpha_2\gamma_d} R(E-i0;H_1)\tilde{r}_L.$$

Similarly we can show $\gamma_{31} = o(1)$.

Next we consider the term γ_{23}. Recall that ∇j_{0d}, $j_{0d} = j_0(x-d)$, has support on

$$\{x : x-d \in \Sigma(|d|^\sigma, -\hat{d}, \delta) \setminus \Sigma(2|d|^\sigma, -\hat{d}, \delta/2)\}.$$

We construct an outgoing approximation for $R(E+i0;H_d)(1-b_{2d})\Gamma_{2d}$. By Lemma 3.6, the approximation takes the form

$$\tilde{\Gamma}^*_{3d} R(E+i0;H_d)(1-b_{2d})\Gamma_{2d} = \tilde{r}_L + \tilde{\Gamma}^*_{3d} R(E+i0;H_d)\tilde{r}_L,$$

and hence we have

$$\gamma_{23} = (ic(E)/4\pi)(R(E+i0;H_d) b_{2d}\Gamma_{2d}\varphi_{+1}, \lambda_d\tilde{\Gamma}_{3d}\varphi_{-1}) + o(1)$$

by Lemmas 3.2 and 3.5. Since $b_{2d}\Gamma_{2d}\varphi_{+1}$ is uniformly bounded in L^2, the desired bound $\gamma_{23} = o(1)$ follows from Lemma 3.4. A similar argument applies to the other terms γ_{21}, γ_{12} and γ_{32}. Thus (3.7) is verified.

We prove (3.9). We first apply Lemma 3.3 to obtain

$$\gamma_{11} = (ic(E)/4\pi)(R(E+i0;H_1)\Gamma_{1d}\varphi_{+1}, \tilde{\Gamma}_{1d}\varphi_{-1}) + o(1).$$

Next we construct an outgoing approximation for $R(E+i0;H_1)(1-\lambda_{1d})\Gamma_{1d}$ and an incoming one for $R(E-i0;H_1)(1-\lambda_{1d})\tilde{\Gamma}_{1d}$ as in Lemma 3.6. Then we get

$$\gamma_{11} = (ic(E)/4\pi)(R(E+i0;H_1)\lambda_{1d}\Gamma_{1d}\varphi_{+1}, \lambda_{1d}\tilde{\Gamma}_{1d}\varphi_{-1}) + o(1). \tag{3.12}$$

The set Λ_{1d} does not contain a conical neighborhood of direction $\hat{d}$. Hence it follows from Proposition 2.1 (1) that

$$\varphi_{+1} = \varphi_{+1}(x;\hat{d},E) = e^{i\alpha_1(\gamma(x;\hat{d})-\pi)}\varphi_0(\hat{d},E) + e^{i\sqrt{E}|x|}O(|x|^{-1/2})$$

on Λ_{1d}, where $\varphi_0(\hat{d},E) = \exp(i\sqrt{E}x\cdot\hat{d})$. If $\xi \in \operatorname{supp}\beta_0$, then

$$\left|\nabla\left(\sqrt{E}|x| - \xi\cdot x\right)\right| = \left|\sqrt{E}\hat{x} - \xi\right| > c > 0$$

for $x \in \Lambda_{1d}$. This implies that the remainder term is negligible. We note that $j_{0d} = 1$ and

$$e^{i\alpha_2\gamma(x-d;\hat{d})} = e^{i\alpha_2\pi} + O(|d|^{-1+\sigma})$$

on Λ_{1d}. Since $\beta_0(D_x)\varphi_0 = \varphi_0$ for $\varphi_0 = \varphi_0(\hat{d}, E)$, we have

$$\lambda_{1d}\Gamma_{1d}\varphi_{+1} = \lambda_{1d}\left(e^{i(\alpha_2-\alpha_1)\pi}e^{i\alpha_1\gamma(x;\hat{d})}[H_0, \chi_\infty^2]\varphi_0(\hat{d}, E) + O(|d|^{-1+\sigma})\right).$$

Similarly

$$\lambda_{1d}\tilde{\Gamma}_{1d}\varphi_{-1} = \lambda_{1d}\left(e^{i(\alpha_2-\alpha_1)\pi}e^{i\alpha_1\gamma(x;\hat{d})}[H_0, \chi_\infty^2]\varphi_0(-\hat{d}, E) + O(|d|^{-1+\sigma})\right).$$

Hence we have

$$\gamma_{11} = (ic(E)/4\pi)(R(E+i0; H_1)\lambda_{1d}\Phi_{1d}(\hat{d}, E), \lambda_{1d}\Phi_{1d}(-\hat{d}, E)) + o(1),$$

where

$$\Phi_{1d}(\omega, E) = \Phi_{1d}(x; \omega, E) = e^{i\alpha_1\gamma(x;\hat{d})}[H_0, \chi_\infty^2]\varphi_0(\omega, E).$$

We further obtain

$$\gamma_{11} = (ic(E)/4\pi)(R(E+i0; H_1)\Phi_{1d}(\hat{d}, E), \Phi_{1d}(-\hat{d}, E)) + o(1)$$

by repeating the same argument as used to derive (3.12). We split $[H_0, \chi_\infty^2]$ into

$$[H_0, \chi_\infty^2] = \chi(|x|/2|d|^\sigma)\Big([H_0, j_1\chi_\infty^2] + [H_0, (1-j_1)\chi_\infty^2]\Big),$$

where $j_1 \in C^\infty(\boldsymbol{R}^2)$ is a real function such that $\partial_x^\beta j_1 = O(|x|^{-|\beta|})$ and

$$\operatorname{supp} j_1 \subset \Sigma(|d|^\sigma/4, -\hat{d}, \delta), \quad j_1 = 1 \text{ on } \Sigma(|d|^\sigma/2, -\hat{d}, \delta/2).$$

We see that only the first commutator makes a contribution. This can be shown by constructing outgoing and incoming approximations for the second commutator. Thus (3.9) is obtained by Lemma 2.3 with $j_0 = \tilde{j}_0 = j_1\chi_\infty^2$.

The proof of (3.10) is similar but is slightly different. By Lemma 3.6, we construct an outgoing approximation

$$\tilde{\Gamma}_{2d}^* R(E+i0; H_d)(1-b_{2d})\Gamma_{2d} = \tilde{r}_L + \tilde{\Gamma}_{2d}^* R(E+i0; H_d)\tilde{r}_L$$

and an incoming approximation

$$R(E-i0; H_d)(1-b_{2d})\tilde{\Gamma}_{2d} = je^{i\theta_d}R(E-i0; H_0)\tilde{\beta}_1 e^{-i\theta_d}(1-b_{2d})\tilde{\Gamma}_{2d} + R(E-i0; H_d)\tilde{r}_L.$$

We know by the resolvent estimate of [9] that

$$\langle x\rangle^{-s-\tau}R(E-i0; H_0)\tilde{\beta}_1(1-b_{2d})\langle x\rangle^s : L^2 \to L^2, \quad s > 0,$$

is bounded for $\tau > 1$. Hence we have

$$\gamma_{22} = (ic(E)/4\pi)(R(E+i0; H_d)b_{2d}\Gamma_{2d}\varphi_{+1}, b_{2d}\tilde{\Gamma}_{2d}\varphi_{-1}) + o(1)$$

by Lemma 3.2, and it follows from Lemma 3.3 that

$$\gamma_{22} = (ic(E)/4\pi)(R(E+i0;H_{2,d})b_{2d}\Gamma_{2d}\varphi_{+1}, b_{2d}\tilde{\Gamma}_{2d}\varphi_{-1}) + o(1).$$

Let $\Lambda_{2d} = \{x : |d|^\sigma < |x-d| < C|d|^\sigma, \ \ |\widehat{(x-d)} + \hat{d}| < \delta\}$ for $C \gg 1$, and denote by

$$\lambda_{2d}(x) = \Big(1 - \chi(2|x-d|/|d|^\sigma)\Big)\chi(|x-d|/C|d|^\sigma)\chi(|\widehat{(x-d)} + \hat{d}|/\delta)$$

the smooth function associated with Λ_{2d}. Then we obtain

$$\gamma_{22} = (ic(E)/4\pi)(R(E+i0;H_{2,d})\lambda_{2d}\Gamma_{2d}\varphi_{+1}, \lambda_{2d}\tilde{\Gamma}_{2d}\varphi_{-1}) + o(1)$$

in the same way as (3.12). By the principle of limiting absorption,

$$\langle x-d\rangle^{-\rho} R(E+i0;H_{2,d})\langle x-d\rangle^{-\rho} : L^2 \to L^2$$

is bounded uniformly in d for any $\rho > 1/2$, and by Proposition 2.1 (3) with $q = 1-\sigma$, the eigenfunction $\varphi_{\pm 1}$ behaves like

$$\begin{aligned} \varphi_{+1} &= \varphi_{+1}(x;\hat{d},E) = \cos\alpha_1\pi \times \varphi_0(x;\hat{d},E) + O(|d|^{-\nu}), \\ \varphi_{-1} &= \varphi_{-1}(x;-\hat{d},E) = \cos\alpha_1\pi \times \varphi_0(x;-\hat{d},E) + O(|d|^{-\nu}) \end{aligned}$$

on Λ_{2d}, where $\nu = 2(1/2-\sigma)/3$. Since $\langle x-d\rangle^\rho \le c\,|d|^{\rho\sigma}$ on Λ_{2d} and $2\rho\sigma < \nu$ for σ small enough, we have

$$\begin{aligned} \gamma_{22} = (\cos\alpha_1\pi)^2 (ic(E)/4\pi)(R(E+i0;H_{2,d}) \\ \lambda_{2d}\Gamma_{2d}\varphi_0(\hat{d},E), \lambda_{2d}\tilde{\Gamma}_{2d}\varphi_0(-\hat{d},E)) + o(1). \end{aligned}$$

The commutator $[H_1, j_{0d}]$ is calculated as

$$\begin{aligned} [H_1, j_{0d}] &= [H(A_{\alpha_1}), j_{0d}] = e^{i\alpha_1\gamma(x;-\hat{d})}[H_0, j_{0d}]e^{-i\alpha_1\gamma(x;-\hat{d})} \\ &= \Big(e^{i\alpha_1\pi} + O(|d|^{-1+\sigma})\Big)[H_0, j_{0d}]\Big(e^{-i\alpha_1\pi} + O(|d|^{-1+\sigma})\Big) \end{aligned}$$

on Λ_{2d}. We have assumed that $j_0(x-d) = \tilde{j}_0(x-d)$. Note that $\chi_\infty = 1$ on $\operatorname{supp}\nabla j_{0d}$. Hence we have

$$\gamma_{22} = (\cos\alpha_1\pi)^2 (ic(E)/4\pi)(R(E+i0;H_{2,d})\lambda_{2d}\Phi_{2d}(\hat{d},E), \lambda_{2d}\Phi_{2d}(-\hat{d},E)) + o(1),$$

where

$$\Phi_{2d}(\omega,E) = \Phi_{2d}(x;\omega,E) = e^{i\alpha_2\gamma(x-d;\hat{d})}[H_0, j_{0d}]\varphi_0(\omega,E).$$

Thus (3.10) is obtained from Lemma 2.3 after the change of variables $x-d \to x$.

4 Completion : proof of Lemmas 3.2, 3.3 and 3.4

In this section we prove the three lemmas and complete the proof of Theorem 1.1.

4.1. The proof of the lemmas requires several auxiliary operators. We first define these operators. We fix $0 < \sigma_1,\ \sigma_2 \ll 1$ small enough, and we define the following two sets

$$\Pi_{1d} = \{x : |x| < C|d|^{\sigma_1}\} \cup \{x : |x| \ge C|d|^{\sigma_1},\ \ |\hat{x} + \hat{d}| < |d|^{-\sigma_1/2}\}, \tag{4.1}$$

$$\Pi_{2d} = \{x : |x-d| < C|d|^{\sigma_2}\} \cup \{x : |x-d| \ge C|d|^{\sigma_2},\ \ |\widehat{(x-d)} - \hat{d}| < |d|^{-\sigma_2/2}\}$$

for $C \gg 1$. These two sets are disjoint with each other for $|d| \gg 1$.

Let $\zeta_{jd} \in C^\infty(\boldsymbol{R}),\ 1 \le j \le 2$, be a real periodic function with period 2π such that $\zeta_{jd}(s) = \alpha_j s$ for $s \in (|d|^{-\sigma_j/2}, 2\pi - |d|^{-\sigma_j/2})$ and $|(d/ds)^m \zeta_{jd}(s)| \le C_m |d|^{m\sigma_j/2}$ for $C_m > 0$ independent of d. We define a smooth real function η_{1d} by $\eta_{1d}(x) = 0$ for $|x| < |d|^{\sigma_1}/2$ and by

$$\eta_{1d}(x) = \zeta_{1d}(\gamma(x; -\hat{d}))$$

for $|x| > |d|^{\sigma_1}$. We may assume that η_{1d} satisfies

$$|\partial_x^\beta \eta_{1d}(x)| \le C_\beta |d|^{|\beta|\sigma_1/2} |x|^{-|\beta|} \le \tilde{C}_\beta \langle x \rangle^{-|\beta|/2} \tag{4.2}$$

uniformly in d. By definition, we have

$$\nabla \eta_{1d}(x) = \zeta_{1d}'(\gamma(x; -\hat{d})) \nabla \gamma(x; -\hat{d}) = \zeta_{1d}'(\gamma(x; -\hat{d}))\,(-x_2/|x|^2, x_1/|x|^2) \tag{4.3}$$

and hence

$$\nabla \eta_{1d}(x) = \alpha_1\,(-x_2/|x|^2, x_1/|x|^2) \tag{4.4}$$

for $x \in \Pi_{1d}^{\mathrm{c}}$, where Π_{1d}^{c} is the complement of Π_{1d}. Similarly we define η_{2d} by

$$\eta_{2d}(x) = \zeta_{2d}(\gamma(x-d; \hat{d}))$$

for $|x-d| > |d|^{\sigma_2}$ and by $\eta_{2d}(x) = 0$ for $|x-d| < |d|^{\sigma_2}/2$. We set $p_{1d}(x) = \exp(i\eta_{1d}(x))$ and $q_{1d}(x) = 1/p_{1d}(x)$. By (4.2), we have

$$|\partial_x^\beta p_{1d}(x)| + |\partial_x^\beta q_{1d}(x)| \le C_\beta \langle x \rangle^{-|\beta|/2} \tag{4.5}$$

uniformly in d. If $x \in \Pi_{1d}^{\mathrm{c}}$, then

$$p_{1d}(x) = \exp(i\alpha_1 \gamma(x; -\hat{d})), \quad q_{1d}(x) = \exp(-i\alpha_1 \gamma(x; -\hat{d})).$$

Similarly we define $p_{2d}(x) = \exp(i\eta_{2d}(x))$ and $q_{2d}(x) = 1/p_{2d}(x)$. Then

$$|\partial_x^\beta p_{2d}(x)| + |\partial_x^\beta q_{2d}(x)| \le C_\beta \langle x-d \rangle^{-|\beta|/2}$$

and

$$p_{2d}(x) = \exp(i\alpha_2\gamma(x-d;\hat{d})), \quad q_{2d}(x) = \exp(-i\alpha_2\gamma(x-d;\hat{d}))$$

for $x \in \Pi_{2d}^{\mathrm{c}}$.

We now introduce the following three operators

$$\begin{aligned} K_{1d} &= p_{2d}H_1q_{2d} = p_{2d}H(A_{\alpha_1})q_{2d} = H(A_{\alpha_1} + \nabla\eta_{2d}), \\ K_{2d} &= p_{1d}H_{2,d}q_{1d} = p_{1d}H(A_{\alpha_2,d})q_{1d} = H(\nabla\eta_{1d} + A_{\alpha_2,d}) \end{aligned}$$

and $K_{0d} = p_dH_0q_d = H(\nabla\eta_{1d} + \nabla\eta_{2d})$ as basic auxiliary operators, where $p_d = p_{1d}p_{2d}$ and $q_d = q_{1d}q_{2d}$. The operator K_{0d} has the domain $\mathcal{D}(K_{0d}) = H^2(\boldsymbol{R}^2)$, $H^s(\boldsymbol{R}^2)$ being the Sobolev space of order s, while K_{1d} and K_{2d} have the domain

$$\begin{aligned} \mathcal{D}(K_{1d}) &= \{u \in L^2 : K_{1d}u \in L^2, \ \lim_{|x|\to 0} |u(x)| < \infty\}, \\ \mathcal{D}(K_{2d}) &= \{u \in L^2 : K_{2d}u \in L^2, \ \lim_{|x-d|\to 0} |u(x)| < \infty\}. \end{aligned}$$

We consider the difference $W_{1d} = K_{1d} - K_{0d}$. By (4.4), $A_{\alpha_1} = \nabla\eta_{1d}$ on Π_{1d}^{c}, and hence $W_{1d} = 0$ there. Similarly we have

$$H_d - K_{2d} = H(A_{\alpha_1} + A_{\alpha_2,d}) - K_{2d} = 0$$

on Π_{1d}^{c}. Since $A_{\alpha_2,d}(x) = A_{\alpha_2}(x-d) = \nabla\eta_2(x-d)$ on Π_{1d}, we also have

$$H_d - K_{2d} = K_{1d} - K_{0d} = W_{1d}$$

on Π_{1d}. A similar argument applies to $W_{2d} = K_{2d} - K_{0d}$. Thus we can obtain the following relations

$$H_d = K_{1d} + W_{2d}, \qquad H_d = K_{2d} + W_{1d}. \tag{4.6}$$

The difference W_{jd} is a differential operator of first order. For example, W_{1d} takes the form

$$W_{1d} = 2ie_{1d}(x)\cdot\nabla + e_{0d}(x) \tag{4.7}$$

and the coefficients have support in Π_{1d} and singularity at $x = 0$ only. By (4.2) and (4.3), e_{1d} and e_{0d} satisfy

$$e_{1d}(x) = \left(\alpha_1 - \zeta_{1d}'(\gamma(x;-\hat{d}))\right)\nabla\gamma = O(|d|^{\sigma_1/2})\nabla\gamma \tag{4.8}$$

with $\gamma = \gamma(x;-\hat{d})$ and

$$e_{0d}(x) = O(|d|^{\sigma_1})|x|^{-2} \tag{4.9}$$

for $|x| > |d|^{\sigma_1}$, and by (4.5), we have

$$|\partial_x^\beta e_{0d}(x)| + |\partial_x^\beta e_{1d}(x)| \le C_\beta\langle x\rangle^{-|\beta|/2} \tag{4.10}$$

for $|x| > 1$ uniformly in d. The coefficients of W_{2d} have similar properties. They have support in Π_{2d} and singularity at $x = d$ only.

The domain of K_{1d} or K_{2d} is different from that of K_{0d}, and the ordinary resolvent identity is not expected to hold for (K_{jd}, K_{0d}). However we can derive the following relation

$$\psi_j R(E+i0; K_{jd}) = R(E+i0; K_{0d})\psi_j - R(E+i0; K_{0d})U_{jd}R(E+i0; K_{jd}) \quad (4.11)$$

for $j = 1, 2$, where ψ_1 and ψ_2 are smooth bounded functions vanishing around $x = 0$ and $x = d$ respectively, and

$$U_{jd} = -[K_{jd}, \psi_j] + W_{jd}\psi_j. \quad (4.12)$$

We often use the relation with

$$\psi_1(x) = 1 - \chi(|x|/\delta|d|^{\sigma_1}), \qquad \psi_2(x) = 1 - \chi(|x-d|/\delta|d|^{\sigma_2}) \quad (4.13)$$

in later application. We shall show (4.11) in a rather formal way. We write the solution u to equation $(K_{0d} - E)\, u = \psi_1 f$ as

$$u = \psi_1 R(E+i0; K_{1d})f + v.$$

Since $K_{0d} = K_{1d} - W_{1d}$, the remainder v obeys

$$(K_{0d} - E)\, v = \left(-[K_{1d}, \psi_1] + W_{1d}\psi_1\right) R(E+i0; K_{1d})f.$$

This yields the desired relation. Similarly we can show that

$$R(E+i0; H_d)\psi_2 = \psi_2 R(E+i0; K_{1d}) - R(E+i0; H_d)V_{2d}R(E+i0; K_{1d}), \quad (4.14)$$

$$R(E+i0; H_d)\psi_1 = \psi_1 R(E+i0; K_{2d}) - R(E+i0; H_d)V_{1d}R(E+i0; K_{2d}), \quad (4.15)$$

where

$$V_{2d} = [K_{1d}, \psi_2] + W_{2d}\psi_2, \quad V_{1d} = [K_{2d}, \psi_1] + W_{1d}\psi_1. \quad (4.16)$$

If ψ_j is taken as in (4.13), then V_{jd} has properties similar to W_{jd}. The only difference is that the coefficients of V_{jd} are all smooth and bounded uniformly in d. The operator U_{jd} defined by (4.12) has also similar properties.

The argument below requires the Green kernel $G_d(x, y; E)$ of $R(E+i0; K_{0d})$. The resolvent $R(E+i0; H_0)$ has the kernel

$$G_0(x, y : E) = (i/4)H_0^{(1)}(\sqrt{E}\,|x-y|),$$

where $H_0^{(1)}(z)$ is the Hankel function of first kind and order zero. As is well known, $H_0^{(1)}(z)$ behaves like

$$H_0^{(1)}(z) = (2/\pi)^{1/2} \exp(i(z - \pi/4))z^{-1/2}\left(1 + O(|z|^{-1})\right)$$

at infinity. Hence $G_d(x, y; E)$ behaves like

$$G_d = c_0(E)p_d(x)\exp(i\sqrt{E}|x-y|)|x-y|^{-1/2}q_d(y)\left(1+O(|x-y|^{-1})\right) \tag{4.17}$$

as $|x-y| \to \infty$, where $c_0(E) = (1/8\pi)^{1/2}\exp(i\pi/4)E^{-1/4}$.

4.2. Let σ, $0 < \sigma \ll 1$, be fixed small enough as in Lemmas 3.2, 3.3 and 3.4. Throughout the argument in this subsection, K_{1d}, K_{2d} and K_{0d} are defined with $\sigma_1 = \sigma_2 = \sigma$. We prove several lemmas on the resolvent estimates for these operators before going into the proof of the three lemmas. The functions b_{1d}, b_{2d} and λ_d again denote the characteristic functions of sets B_{1d}, B_{2d} and Λ_d respectively.

Lemma 4.1

$$\|b_{2d}R(E+i0;K_{0d})b_{1d}\| = O(|d|^{-1/2+2\sigma}),$$
$$\|b_{2d}R(E+i0;K_{0d})\lambda_d\langle x\rangle^{-1}\| \simeq O(|d|^{-1/2+\sigma}).$$

Proof. To prove the first bound, we evaluate the Hilbert–Schmidt norm of the operator. Since the kernel $G_d(x, y; E)$ of $R(E + i0; K_{0d})$ obeys (4.17), this bound follows at once. To prove the second bound, we decompose λ_d into the sum

$$\lambda_d(x) = \lambda_d(x)\Big(\chi(|x-d|/\delta|d|) + (1-\chi(|x-d|/\delta|d|))\Big) = \mu_{2d}(x) + \mu_{1d}(x).$$

By the principle of limiting absorption, we have

$$\langle x-d\rangle^{-\rho}R(E+i0;K_{0d})\langle x-d\rangle^{-\rho} : L^2 \to L^2$$

is bounded for any $\rho > 1/2$. Since $|x| > c\,|d|$ on the support of μ_{2d} for some $c > 0$, we can choose ρ so close to $1/2$ that

$$\|b_{2d}R(E+i0;K_{0d})\mu_{2d}\langle x\rangle^{-1}\| = O(|d|^{-1+\rho+\rho\sigma}) \simeq O(|d|^{-1/2+\sigma/2}).$$

On the other hand, we obtain

$$\|b_{2d}R(E+i0;K_{0d})\mu_{1d}\langle x\rangle^{-1}\| \simeq O(|d|^{-1/2+\sigma})$$

by evaluating the Hilbert–Schmidt norm. This yields the desired bound. □

Lemma 4.2 *Let*

$$V_{1d} = [K_{2d}, \psi_1] + W_{1d}\psi_1, \quad \psi_1(x) = 1-\chi(|x|/\delta|d|^{\sigma}),$$

be defined by (4.16) with $\sigma_1 = \sigma$. *Take* $\rho > 1/2$ *close enough to 1/2. Then*

$$\|\langle x\rangle^{\rho}V_{1d}R(E+i0;K_{0d})r_L\| = O(|d|^{-L/2}),$$

where r_L *is the pseudo-differential operator defined by (3.6).*

Proof. The proof is based on the fact that the free Hamiltonian H_0 and $\partial/\partial\theta$ commute each other. By definition, we have $R(E+i0;K_{0d}) = p_d R(E+i0;H_0)q_d$, where $p_d = p_{1d}p_{2d}$ and $q_d = 1/p_d$. By (4.7), (4.8) and (4.9), V_{1d} takes the form

$$V_{1d} = O(|d|^{\sigma/2})\nabla\gamma\cdot\nabla + O(|d|^{\sigma})|x|^{-2}, \quad \gamma = \gamma(x;-\hat{d}),$$

in $\{x : |x| > |d|^{\sigma}\}$. The differential operator $\nabla\gamma\cdot\nabla$ can be written as

$$\nabla\gamma\cdot\nabla = |x|^{-2}\Big(-x_2\partial_1 + x_1\partial_2\Big) = |x|^{-2}\,\partial/\partial\theta$$

and p_d satisfies the estimate

$$\nabla p_d = |d|^{\sigma/2}\Big(O(|x|^{-1}) + O(|x-d|^{-1})\Big).$$

If we take account of these facts, the lemma is easily verified. □

We work in the phase space $\boldsymbol{R}^2_x \times \boldsymbol{R}^2_\xi$. We introduce a smooth nonnegative partition of unity over $\boldsymbol{R}^2_\xi$. The partition $\{\beta_\pm, \beta_\infty\}$ is normalized by

$$\beta_+(\xi) + \beta_-(\xi) + \beta_\infty(\xi) = 1 \tag{4.18}$$

and has the following properties : $\operatorname{supp}\beta_\infty \subset \{\xi : |\xi|^2 < E/2 \ \text{ or } \ |\xi|^2 > 2E\}$ and

$$\operatorname{supp}\beta_+ \subset \{\xi : E/3 < |\xi|^2 < 3E, \ \ \hat{\xi}\cdot\hat{d} > -1/4\}$$
$$\operatorname{supp}\beta_- \subset \{\xi : E/3 < |\xi|^2 < 3E, \ \ \hat{\xi}\cdot\hat{d} < 1/4\}.$$

The proof of the two lemmas below is based on the micro-local estimates for the resolvent of auxiliary operators. We make repeated use of a similar idea in the future discussion.

Lemma 4.3

$$\begin{aligned} \|b_{2d}R(E+i0;K_{1d})b_{1d}\| &\simeq O(|d|^{-1/2+3\sigma}), \\ \|b_{1d}R(E+i0;K_{2d})b_{2d}\| &\simeq O(|d|^{-1/2+3\sigma}) \end{aligned}$$

and

$$\|b_{2d}R(E+i0;K_{1d})\lambda_d\langle x\rangle^{-1}\| \simeq O(|d|^{-1/2+2\sigma}).$$

Proof. We prove the first bound only. The second and third bounds are obtained in a similar way. Let ψ_1 be as in Lemma 4.2. We use (4.11) for the function ψ_1. Since $\psi_1 b_{2d} = b_{2d}$, we have

$$\begin{aligned} b_{2d}R(E+i0;K_{1d})b_{1d} &= b_{2d}R(E+i0;K_{0d})\psi_1 b_{1d} \\ &- b_{2d}R(E+i0;K_{0d})U_{1d}R(E+i0;K_{1d})b_{1d}. \end{aligned}$$

By Lemma 4.1, the first operator on the right side obeys the bound $O(|d|^{-1/2+2\sigma})$. To evaluate the second operator, we decompose U_{1d} into the sum of four operators

$$U_{1d} = g_{1d}^2 U_{1d} + U_\infty(x, D_x) + U_+(x, D_x) + U_-(x, D_x), \tag{4.19}$$

where $g_{1d}(x) = \chi(|x|/M|d|^\sigma)$ for $M \gg 1$, and

$$U_\pm(x, D_x) = (1 - g_{1d}^2)U_{1d}\beta_\pm(D_x), \quad U_\infty(x, D_x) = (1 - g_{1d}^2)U_{1d}\beta_\infty(D_x).$$

We have

$$\|b_{2d}R(E + i0; K_{0d})g_{1d}\| = O(|d|^{-1/2+2\sigma})$$

in the same way as in the proof of Lemma 4.1. By the principle of limiting absorption,

$$\langle x\rangle^{-\rho} R(E + i0; K_{1d})\langle x\rangle^{-\rho} : L^2 \to L^2$$

is bounded for any $\rho > 1/2$. Since the coefficients of U_{1d} vanish around $x = 0$ and are bounded uniformly in d, we have

$$\|g_{1d}U_{1d}R(E + i0; K_{1d})b_{1d}\| \simeq O(|d|^\sigma)$$

by elliptic estimate. Thus

$$\|b_{2d}R(E + i0; K_{0d})g_{1d}^2 U_{1d}R(E + i0; K_{1d})b_{1d}\| \simeq O(|d|^{-1/2+3\sigma}).$$

We now assume that $x \in \Pi_{1d}$ and $|x| > M|d|^\sigma$, where Π_{1d} is defined by (4.1) with $\sigma_1 = \sigma$. Then the symbol of $K_{0d} - E$ takes the form $|\xi|^2 - E$ approximately. If $\xi \in \operatorname{supp}\beta_\infty$, then it has a bounded inverse. Since Π_{1d} and B_{2d} do not intersect with each other, we have by the standard calculus of pseudo-differential operators that

$$b_{2d}R(E + i0; K_{0d})U_\infty = \tilde r_N + b_{2d}R(E + i0; K_{0d})\tilde r_N$$

for any $N \gg 1$, where $\tilde r_N$ again denotes a bounded operator having the property (3.11). Hence

$$\|b_{2d}R(E + i0; K_{0d})U_\infty R(E + i0; K_{1d})b_{1d}\| = O(|d|^{-N}).$$

We still assume that $x \in \Pi_{1d}$ and $|x| > M|d|^\sigma$. If $\xi \in \operatorname{supp}\beta_-$, then the free particle with initial state (x, ξ) at $t = 0$ never passes over B_{2d} for $t > 0$. Hence we have

$$\|b_{2d}R(E + i0; K_{0d})U_- R(E + i0; K_{1d})b_{1d}\| = O(|d|^{-N})$$

by use of the micro-local estimate on the resolvent $R(E+i0; K_{0d})$. If, on the other hand, $\xi \in \operatorname{supp}\beta_+$, then we can take $M \gg 1$ so large that the incoming particle with state (x, ξ) at $t = 0$ never passes over B_{1d} for $t < 0$. This enables us to construct an incoming approximation for

$$U_+ R(E + i0; K_{1d})b_{1d} = \Big(b_{1d}R(E - i0; K_{1d})U_+^*\Big)^*.$$

We use an argument similar to that in the proof of Lemma 3.6. Then the approximation is constructed in the form

$$U_+R(E+i0;K_{1d})b_{1d} = \tilde{r}_N + \tilde{r}_N R(E+i0;K_{1d})b_{1d}$$

and hence we get

$$\|b_{2d}R(E+i0;K_{0d})U_+R(E+i0;K_{1d})b_{1d}\| = O(|d|^{-N}).$$

Thus the desired bound is obtained. □

Lemma 4.4 *Let $\rho > 1/2$ and V_{1d} be as in Lemma 4.2. Then*

$$\|\langle x\rangle^\rho V_{1d}R(E+i0;K_{2d})r_L\| = O(|d|^{-L/2}).$$

Proof. We use (4.11) with $\psi_2 = 1-\chi(|x-d|/\delta|d|^\sigma)$. Since $V_{1d}\psi_2 = V_{1d}$, we have

$$\begin{aligned}\langle x\rangle^\rho V_{1d}R(E+i0;K_{2d})r_L &= \langle x\rangle^\rho V_{1d}R(E+i0;K_{0d})\psi_2 r_L \\ &- \langle x\rangle^\rho V_{1d}R(E+i0;K_{0d})U_{2d}R(E+i0;K_{2d})r_L.\end{aligned}$$

By Lemma 4.2, the first operator obeys $O(|d|^{-L/2})$. We decompose U_{2d} into the sum of three operators

$$U_{2d} = U_{2d}\Big(\beta_\infty(D_x) + \beta_+(D_x) + \beta_-(D_x)\Big).$$

The coefficients of U_{2d} have support in $\{x : |x-d| > \delta|d|^\sigma\}$. If we repeat the same argument as in the proof of Lemma 4.3, then we obtain

$$\|\langle x\rangle^\rho V_{1d}R(E+i0;K_{0d})U_{2d}\beta_\infty R(E+i0;K_{2d})r_L\| = O(|d|^{-L}),$$
$$\|\langle x\rangle^\rho V_{1d}R(E+i0;K_{0d})U_{2d}\beta_+ R(E+i0;K_{2d})r_L\| = O(|d|^{-L})$$

by Lemma 4.2. We know by the micro-local resolvent estimate ([9, Theorem 1]) that

$$\langle x-d\rangle^s U_{2d}\beta_-(D_x)R(E+i0;K_{2d})\langle x-d\rangle^{-s-\tau} : L^2 \to L^2, \quad s \ge 0,$$

is bounded for $\tau > 1$. Hence this, together with Lemma 4.2, yields

$$\|\langle x\rangle^\rho V_{1d}R(E+i0;K_{0d})U_{2d}\beta_- R(E+i0;K_{2d})r_L\| = O(|d|^{-L/2}).$$

Thus the proof is complete. □

The following two propositions play a basic role in proving the three lemmas.

Proposition 4.1 *Define* Π_{1d} *and* Π_{2d} *with* $\sigma_1 = \sigma_2 = \sigma$ *and denote by* $\pi_{jd}(x)$, $j = 1, 2$, *the characteristic function of* Π_{jd}. *Let* $\rho > 1/2$. *Then one has :*

(1) $\|r_L R(E + i0; H_d)\pi_{1d}\langle x\rangle^{-\rho}\| = O(|d|^{-L/2})$.

(2) $\|r_L R(E + i0; H_d)\pi_{2d}\langle x - d\rangle^{-\rho}\| = O(|d|^{-L/2})$.

Proposition 4.2

$$\|b_{2d}R(E + i0; H_d)b_{1d}\| = O(|d|^{3\sigma}).$$

4.3. We proceed to proving the three lemmas in question, accepting the two propositions above as proved. The proof of the propositions is done in section 5. Throughout the proof of the lemmas, $\psi_1(x)$ and $\psi_2(x)$ are defined by (4.13) with $\sigma_1 = \sigma_2 = \sigma$.

Proof of Lemma 3.2. First it is clear from Proposition 4.1 that $r_L R(E+i0; H_d)b_{jd}$ obeys the desired bound. We consider the operator $Q = r_L R(E + i0; H_d)r_L$. We decompose Q into the sum

$$Q = r_L R(E + i0; H_d)\psi_1 r_L + r_L R(E + i0; H_d)(1 - \psi_1)r_L = Q_1 + Q_2.$$

The function $1 - \psi_1(x) = \chi(|x|/\delta|d|^\sigma)$ has support around $x = 0$, and it satisfies $W_{2d}(1 - \psi_1) = 0$. We use (4.15) for Q_1 and (4.14) for Q_2. Then

$$\begin{aligned} Q_1 &= r_L\psi_1 R(E + i0; K_{2d})r_L - r_L R(E + i0; H_d)V_{1d}R(E + i0; K_{2d})r_L, \\ Q_2 &= r_L(1 - \psi_1)R(E + i0; K_{1d})r_L - r_L R(E + i0; H_d)\tilde{V}_{2d}R(E + i0; K_{1d})r_L, \end{aligned}$$

where $\tilde{V}_{2d} = -[K_{1d}, \psi_1]$. We decompose V_{1d} into $V_{1d} = (\pi_{1d}\langle x\rangle^{-\rho})\,(\langle x\rangle^\rho V_{1d})$, and we use Lemma 4.4 and Proposition 4.1. Then we obtain $\|Q_1\| = O(|d|^{-L})$. Since the coefficients of $\tilde{V}_{2d}$ have support around $x = 0$, we have also $\|Q_2\| = O(|d|^{-L})$ by Proposition 4.1 again. Thus

$$\|r_L R(E + i0; H_d)r_L\| = O(|d|^{-L}) \tag{4.20}$$

and (2) is proved. Next we consider the operator $R = r_L R(E + i0; H_d)\lambda_d$. By (4.14), R is represented as

$$R = r_L\psi_2 R(E + i0; K_{1d})\lambda_d - r_L R(E + i0; H_d)V_{2d}R(E + i0; K_{1d})\lambda_d.$$

The first operator is easy to evaluate. This obeys the bound $O(|d|^{-L/2})$. To evaluate the second operator, we decompose V_{2d} into the sum of four operators

$$V_{2d} = g_{2d}^2 V_{2d} + V_\infty(x, D_x) + V_+(x, D_x) + V_-(x, D_x), \tag{4.21}$$

where $g_{2d}(x) = \chi(|x - d|/M|d|^\sigma)$ for $M \gg 1$, and

$$V_\pm(x, D_x) = (1 - g_{2d}^2)V_{2d}\beta_\pm(D_x), \quad V_\infty(x, D_x) = (1 - g_{2d}^2)V_{2d}\beta_\infty(D_x).$$

According to the decomposition above, we set

$$\begin{aligned} R_0 &= r_L R(E+i0;H_d) g_{2d}^2 V_{2d} R(E+i0;K_{1d})\lambda_d, \\ R_\infty &= r_L R(E+i0;H_d) V_\infty R(E+i0;K_{1d})\lambda_d, \\ R_\pm &= r_L R(E+i0;H_d) V_\pm R(E+i0;K_{1d})\lambda_d. \end{aligned}$$

Since $g_{2d} = O(|d|)\langle x\rangle^{-1}$, it follows that $\|g_{2d}R(E+i0;K_{1d})\lambda_d\| = O(|d|^\nu)$ for some $\nu > 0$, and hence $\|R_0\| = O(|d|^{-L/2})$ by Proposition 4.1. We use the micro-local analysis for the operators R_∞ and $R_\pm$. A simple calculus of pseudo-differential operators yields

$$V_\infty R(E+i0;K_{1d})\lambda_d = \tilde{r}_N + \tilde{r}_N R(E+i0;K_{1d})\lambda_d.$$

Hence it follows from (4.20) that $\|R_\infty\| = O(|d|^{-L})$. Assume that $x \in \Pi_{2d}$ and $|x| > M|d|^\sigma$. If $\xi \in \operatorname{supp}\beta_-$, then we can take $M \gg 1$ so large that the incoming free particle with state (x,ξ) at $t = 0$ does not pass over Λ_d for $t < 0$. Hence we can construct an incoming approximation

$$V_- R(E+i0;K_{1d})\lambda_d = \tilde{r}_N + \tilde{r}_N R(E+i0;K_{1d})\lambda_d.$$

If we again use (4.20), then we get $\|R_-\| = O(|d|^{-L})$. To deal with R_+, we construct an outgoing approximation in the form

$$R(E+i0;H_d)V_+ = j\exp(i\theta_d)R(E+i0;H_0)\tilde{\beta}_+\exp(-i\theta_d)V_+ + R(E+i0;H_d)\tilde{r}_N$$

by an argument similar to that in the proof of Lemma 3.6, where $\tilde{\beta}_+ \in C_0^\infty(\boldsymbol{R}^2_\xi)$ satisfies $\tilde{\beta}_+\beta_+ = \beta_+$, and $j(x)$ and $\theta_d(x)$ are used with the meaning ascribed in Lemma 3.6. The first operator obeys

$$\|r_L R(E+i0;H_0)\tilde{\beta}_+\exp(-i\theta_d)V_+\| = O(|d|^{-L/2})$$

by the micro-local resolvent estimate ([9, Theorem 1]), and the remainder operator is evaluated as $O(|d|^{-L})$ by (4.20). Hence we have $\|R_+\| = O(|d|^{-L/2})$. This completes the proof. □

For later reference, we here note that the proof of Lemma 3.2 does not use Proposition 4.2. Hence we can use Lemma 3.2 to prove Proposition 4.2.

Proof of Lemma 3.3. By (4.14) and (4.15), we have the following three relations :

$$\begin{aligned} b_{2d}R(E+i0;H_d)b_{1d} = b_{2d}\psi_2 R(E+i0;K_{1d})b_{1d} \\ -\ b_{2d}R(E+i0;H_d)V_{2d}R(E+i0;K_{1d})b_{1d}, \end{aligned}$$

$$\begin{aligned} b_{1d}\left(R(E+i0;H_d) - R(E+i0;K_{1d})\right)b_{1d} \\ = -b_{1d}R(E+i0;H_d)V_{2d}R(E+i0;K_{1d})b_{1d}, \end{aligned}$$

$$\begin{aligned} & b_{2d}\left(R(E+i0;H_d)-R(E+i0;K_{2d})\right)b_{2d} \\ & \quad = \; -b_{2d}R(E+i0;H_d)V_{1d}R(E+i0;K_{2d})b_{2d}. \end{aligned}$$

We decompose V_{1d} as in (4.19) with $g_{1d} = \chi(|x|/M|d|^\sigma)$ and V_{2d} as in (4.21) with $g_{2d} = \chi(|x-d|/M|d|^\sigma)$, and we construct outgoing and incoming approximations. The construction is based on the same idea as in the proof of Lemma 3.6. For example, the approximation for $b_{2d}R(E+i0;H_d)V_+$ is constructed in the form

$$b_{2d}R(E+i0;H_d)V_+ = \tilde{r}_L + b_{2d}R(E+i0;H_d)\tilde{r}_L$$

and hence it follows from Lemma 3.2 that

$$\|b_{2d}R(E+i0;H_d)V_+R(E+i0;K_{1d})b_{1d}\| = O(|d|^{-L}).$$

Thus we repeat the same argument as used in the proof of Lemmas 4.3, 4.4 and 3.2 to obtain the following three inequalities :

$$\begin{aligned} & \|b_{2d}R(E+i0;H_d)b_{1d}\| \\ & \quad \le \; C_\varepsilon|d|^{-1/2+3\sigma+\varepsilon}\Big(1+\|b_{2d}R(E+i0;H_d)g_{2d}\|\Big) + C_L|d|^{-L}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} & \|b_{1d}\left(R(E+i0;H_d)-R(E+i0;K_{1d})\right)b_{1d}\| \\ & \quad \le \; C_\varepsilon|d|^{-1/2+3\sigma+\varepsilon}\|b_{1d}R(E+i0;H_d)g_{2d}\| + C_L|d|^{-L}, \end{aligned} \tag{4.23}$$

$$\begin{aligned} & \|b_{2d}\left(R(E+i0;H_d)-R(E+i0;K_{2d})\right)b_{2d}\| \\ & \quad \le \; C_\varepsilon|d|^{-1/2+3\sigma+\varepsilon}\|b_{2d}R(E+i0;H_d)g_{1d}\| + C_L|d|^{-L} \end{aligned} \tag{4.24}$$

for $L \gg 1$ and any $\varepsilon,\ 0<\varepsilon\ll 1$. By Proposition 4.2, we have

$$\|b_{2d}R(E+i0;H_d)g_{1d}\| + \|b_{1d}R(E+i0;H_d)g_{2d}\| = O(|d|^{3\sigma}).$$

The desired bound is derived by combining this estimate with the three inequalities above. In fact, (4.23) and (4.24) imply that

$$\|b_{jd}R(E+i0;H_d)b_{jd}\| \simeq O(|d|^\sigma)$$

for $j=1,2$. We may assume that this is still valid for g_{jd}, so that we have

$$\|b_{2d}R(E+i0;H_d)b_{1d}\| \simeq O(|d|^{-1/2+4\sigma})$$

by (4.22). This is also valid for g_{1d} and g_{2d}. Thus it again follows from (4.23) and (4.24) that

$$\|b_{jd}\Big(R(E+i0;H_d)-R(E+i0;K_{jd})\Big)b_{jd}\| \simeq O(|d|^{-1+7\sigma})$$

for $j = 1, 2$. The operator $R(E + i0; K_{1d})$ is represented as

$$R(E + i0; K_{1d}) = p_{2d} R(E + i0; H_1) q_{2d}, \quad q_{2d} = 1/p_{2d}.$$

The function p_{2d} behaves like

$$p_{2d}(x) = e^{i\alpha_2 \gamma(x-d;\hat{d})} = e^{i\alpha_2 \gamma(-d;\hat{d})} + O(|d|^{-1+\sigma}) = e^{i\alpha_2 \pi} + O(|d|^{-1+\sigma})$$

on B_{1d} ($= \operatorname{supp} b_{1d}$). Similarly $q_{2d}(x) = e^{-i\alpha_2 \pi} + O(|d|^{-1+\sigma})$. Thus

$$\|b_{1d}\Big(R(E + i0; H_d) - R(E + i0; H_1)\Big)b_{1d}\| \simeq O(|d|^{-1+7\sigma}).$$

A similar bound is true for $b_{2d} R(E + i0; H_d) b_{2d}$, and the proof of the lemma is complete. □

Proof of Lemma 3.4. The lemma is verified in almost the same way as in the proof of Lemma 3.3. We give only a sketch for a proof. We keep the same notation as above. The following three identities are obtained from (4.14) and (4.15) :

$$\begin{aligned} b_{2d} R(E + i0; H_d) \lambda_d \langle x \rangle^{-1} &= b_{2d} \psi_2 R(E + i0; K_{1d}) \lambda_d \langle x \rangle^{-1} \\ &- \ b_{2d} R(E + i0; H_d) V_{2d} R(E + i0; K_{1d}) \lambda_d \langle x \rangle^{-1}, \end{aligned}$$

$$\begin{aligned} & b_{1d} \left(R(E + i0; H_d) - R(E + i0; K_{1d}) \right) \lambda_d \langle x \rangle^{-1} \\ & \quad = \ -b_{1d} R(E + i0; H_d) V_{2d} R(E + i0; K_{1d}) \lambda_d \langle x \rangle^{-1}, \end{aligned}$$

$$\begin{aligned} & \langle x \rangle^{-1} \lambda_d \left(R(E + i0; H_d) - R(E + i0; K_{1d}) \right) \lambda_d \langle x \rangle^{-1} \\ & \quad = \ -\langle x \rangle^{-1} \lambda_d R(E + i0; H_d) V_{2d} R(E + i0; K_{1d}) \lambda_d \langle x \rangle^{-1}. \end{aligned}$$

From these relations, we get the following three inequalities :

$$\begin{aligned} & \|b_{2d} R(E + i0; H_d) \lambda_d \langle x \rangle^{-1}\| \\ & \quad \le \ C_\varepsilon |d|^{-1/2+2\sigma+\varepsilon} \Big(1 + \|b_{2d} R(E + i0; H_d) g_{2d}\|\Big) + C_L |d|^{-L}, \end{aligned}$$

$$\begin{aligned} & \|b_{1d} \left(R(E + i0; H_d) - R(E + i0; K_{1d}) \right) \lambda_d \langle x \rangle^{-1}\| \\ & \quad \le \ C_\varepsilon |d|^{-1/2+2\sigma+\varepsilon} \|b_{1d} R(E + i0; H_d) g_{2d}\| + C_L |d|^{-L}, \end{aligned}$$

$$\begin{aligned} & \|\langle x \rangle^{-1} \lambda_d \left(R(E + i0; H_d) - R(E + i0; K_{2d}) \right) \lambda_d \langle x \rangle^{-1}\| \\ & \quad \le \ C_\varepsilon |d|^{-1/2+2\sigma+\varepsilon} \|\langle x \rangle^{-1} \lambda_d R(E + i0; H_d) g_{2d}\| + C_L |d|^{-L}. \end{aligned}$$

It follows from Lemma 3.3 that

$$\|b_{2d} R(E + i0; H_d) g_{2d}\| \simeq O(|d|^{\sigma}), \quad \|b_{1d} R(E + i0; H_d) g_{2d}\| \simeq O(|d|^{-1/2+4\sigma})$$

and hence we have

$$\|b_{2d}R(E+i0;H_d)\lambda_d\langle x\rangle^{-1}\| \simeq O(|d|^{-1/2+3\sigma}) \tag{4.25}$$

and

$$\|b_{1d}\Big(R(E+i0;H_d)-R(E+i0;K_{1d})\Big)\lambda_d\langle x\rangle^{-1}\| \simeq O(|d|^{-1+6\sigma}).$$

If we further make use of (4.25), then we obtain

$$\|\langle x\rangle^{-1}\lambda_d\Big(R(E+i0;H_d)-R(E+i0;K_{1d})\Big)\lambda_d\langle x\rangle^{-1}\| \simeq O(|d|^{-1+5\sigma}).$$

Thus the lemma is proved. □

5 Resolvent estimates

The present section is devoted to proving Propositions 4.1 and 4.2. Throughout the section, we fix σ_1 as $\sigma \le \sigma_1 \ll 1$ and take ρ as

$$1/2 < \rho < \sigma_1/4 + 1/2. \tag{5.1}$$

On the other hand, σ_2 is assumed to satisfy

$$0 < \sigma_2 < (\sigma_1/4 - (\rho - 1/2))/3 \tag{5.2}$$

for $\rho > 1/2$ as above. We further use the notation $h_{2d}(x)$ to denote the characteristic function of the set $\{x : |x-d| < C|d|^\kappa\}$ for some $C \gg 1$ large enough and $0 < \kappa \ll 1$ small enough.

5.1. The argument here is based on the following proposition.

Proposition 5.1 *Assume that ρ fulfills (5.1). Define*

$$\tilde W_{1d} = \psi_1 W_{1d}, \qquad \psi_1(x) = 1 - \chi(|x|/|d|^{\sigma_1}).$$

Then

$$\|\langle x\rangle^\rho \tilde W_{1d} R(E+i0;K_{0d})h_{2d}\| = O(|d|^{-\nu})$$

with $\nu = \sigma_1/4 - (\rho - 1/2) - \kappa$.

The proof of this proposition heavily depends on the special form of the differential operator W_{1d}. By (4.7), it takes the form $W_{1d} = 2ie_{1d}\cdot\nabla + e_{0d}$, where

$$e_{1d}(x) = \Big(\alpha_1 - \zeta_{1d}'(\gamma(x;-\hat d))\Big)\nabla\gamma = O(|d|^{\sigma_1/2})\nabla\gamma, \quad \gamma = \gamma(x;-\hat d),$$

and $e_{0d}(x) = O(|d|^{\sigma_1})|x|^{-2}$ in $\{x : |x| > |d|^{\sigma_1}\}$.

Lemma 5.1 *Recall that* π_{1d} *denotes the characteristic function of* Π_{1d}. *Then*

$$\|\langle x\rangle^{\rho-2}\pi_{1d}R(E+i0;K_{0d})h_{2d}\| = O(|d|^{-(\sigma_1+\nu)})$$

with $\nu = 1/2 - \sigma_1 - \kappa > 0$.

Proof. Let $D_1 = \{(x,y) : x \in \Pi_{1d},\ \ y \in \operatorname{supp} h_{2d}\}$. We consider the integral

$$I = \int\int_{D_1} \langle x\rangle^{2(\rho-2)}|G_d(x,y;E)|^2\, dydx,$$

where $G_d(x,y;E)$ is the kernel of $R(E+i0;K_{0d})$. If $(x,y) \in D_1$, then $|x-y| > c(|x|+|d|)$ for some $c > 0$. Hence it follows from (4.17) that I is evaluated as

$$\begin{aligned} I &= O(|d|^{2\kappa})\int_{\Pi_{1d}} \langle x\rangle^{2(\rho-2)}\left(|x|+|d|\right)^{-1}\, dx \\ &= O(|d|^{2\kappa})\, O(|d|^{-1})\int_0^\infty (1+r)^{2(\rho-2)}r\, dr = O(|d|^{-2(1/2-\kappa)}). \end{aligned}$$

Thus we have $I = O(|d|^{-2(\sigma_1+\nu)})$ with ν in the lemma. This proves the lemma. □

Lemma 5.2 *If* g *is a bounded function with support in* $\{x : x \in \Pi_{1d},\ \ |x| > |d|^{\sigma_1}\}$, *then one has*

$$\|\langle x\rangle^\rho g\,(\nabla\gamma\cdot\nabla)\, R(E+i0;K_{0d})h_{2d}\| = O(|d|^{-(\sigma_1/2+\nu)}), \quad \gamma = \gamma(x;-\hat d),$$

with $\nu = \sigma_1/4 - (\rho - 1/2) - \kappa$.

Proof. Let $D_2 = \{(x,y) : x \in \Pi_{1d},\ \ |x| > |d|^{\sigma_1},\ \ y \in \operatorname{supp} h_{2d}\}$. We calculate

$$I(x,y) = (\nabla\gamma\cdot\nabla)\exp(i\sqrt{E}|x-y|)$$

for $(x,y) \in D_2$. A direct calculation yields

$$I(x,y) = i\sqrt{E}\,|x|^{-1}|x-y|^{-1}|y|\,(\hat x_2\hat y_1 - \hat x_1\hat y_2)\exp(i\sqrt{E}|x-y|),$$

where $\hat x = (\hat x_1, \hat x_2)$. If $(x,y) \in D_2$, then $\hat x = -\hat d + O(|d|^{-\sigma_1/2})$ and $\hat y = \hat d + O(|d|^{-1+\kappa})$, so that

$$\hat x_2\hat y_1 - \hat x_1\hat y_2 = O(|d|^{-\sigma_1/2}).$$

Thus we have

$$I(x,y) = O(|d|^{1-\sigma_1/2})|x|^{-1}|x-y|^{-1}$$

uniformly in $(x,y) \in D_2$. Hence the integral obeys the bound

$$\begin{aligned} I &= \iint_{D_2} |x|^{2\rho}|I(x,y)|^2|x-y|^{-1}\, dydx = O(|d|^{2-\sigma_1+2\kappa})\int_{\Pi_{1d}} |x|^{2\rho-2}(|x|+|d|)^{-3}\, dx \\ &= O(|d|^{2-\sigma_1+2\kappa})\, O(|d|^{-\sigma_1/2})\int_0^\infty r^{2\rho-1}(r+|d|)^{-3}\, dr = O(|d|^{-(\sigma_1+2\nu)}) \end{aligned}$$

for ν as in the lemma. The lemma is obtained from this estimate. □

Proof of Proposition 5.1. The proposition follows immediately from the two lemmas above. □

Lemma 5.3 *Let ψ_1 be as in Proposition 5.1. Define V_{1d} and U_{1d} by (4.16) and (4.12) respectively. Then*

$$\begin{aligned} \|\langle x\rangle^{\rho} V_{1d} R(E+i0; K_{0d}) h_{2d}\| &= O(|d|^{-\nu}), \\ \|\langle x\rangle^{\rho} U_{1d} R(E+i0; K_{0d}) h_{2d}\| &= O(|d|^{-\nu}), \end{aligned}$$

where $\nu = \sigma_1/4 - (\rho - 1/2) - \kappa$.

Proof. By definition, $\tilde{W}_{1d} = V_{1d}$ on $\{x : |x| > 2|d|^{\sigma_1}\}$. The coefficients of K_{0d} and V_{1d} are smooth and bounded uniformly in d. If we denote by $h_{1d}(x)$ the characteristic function of the set $\{x : |x| < 2|d|^{\sigma_1}\}$, then it follows from (4.17) that

$$\|\langle x\rangle^{\rho} h_{1d} R(E+i0; K_{0d}) h_{2d}\| = O(|d|^{-\mu})$$

with $\mu = 1/2 - (\rho+1)\sigma_1 - \kappa > 0$, so that

$$\|\langle x\rangle^{\rho} h_{1d} V_{1d} R(E+i0; K_{0d}) h_{2d}\| = O(|d|^{-\mu})$$

by elliptic estimate. It is obvious that $\mu > \nu$ for σ_1 small enough. Hence the first bound follows from Proposition 5.1. The second one is verified in exactly the same way. □

Lemma 5.4 *One has*

$$\|h_{2d} R(E+i0; K_{1d}) \pi_{1d} \langle x\rangle^{-\rho}\| = O(|d|^{-\nu})$$

with $\nu = \sigma_1/4 - (\rho - 1/2) - \kappa$.

Proof. Let ψ_1 be as in Proposition 5.1. Note that $h_{2d}\psi_1 = h_{2d}$. By (4.11), we have

$$\begin{aligned} h_{2d} R(E+i0; K_{1d}) \pi_{1d} \langle x\rangle^{-\rho} &= h_{2d} R(E+i0; K_{0d}) \psi_1 \pi_{1d} \langle x\rangle^{-\rho} \\ &- \; h_{2d} R(E+i0; K_{0d}) U_{1d} R(E+i0; K_{1d}) \pi_{1d} \langle x\rangle^{-\rho}. \end{aligned}$$

It follows from (4.17) that the first operator on the right side obeys

$$\|h_{2d} R(E+i0; K_{0d}) \pi_{1d} \langle x\rangle^{-\rho}\| = O(|d|^{-(\sigma_1/4 + (\rho - 1/2) - \kappa)}).$$

To evaluate the second operator, we decompose U_{1d} into $U_{1d} = U_{1d}\langle x\rangle^{\rho}\langle x\rangle^{-\rho}$. Since

$$\langle x\rangle^{-\rho} R(E+i0; K_{1d}) \langle x\rangle^{-\rho} : L^2 \to L^2$$

is bounded uniformly in d, the lemma is obtained from Lemma 5.3. □

Lemma 5.5 *Let V_{1d} be as in Lemma 5.3 and let σ_2 be as in (5.2). If $\kappa = \sigma_2$, then*

$$\|\langle x\rangle^\rho V_{1d} R(E + i0; K_{2d}) h_{2d}\| \simeq O(|d|^{-\nu})$$

with $\nu = \sigma_1/4 - (\rho - 1/2) - 2\sigma_2 > 0$.

Proof. The proof uses an argument similar to that in the proof of Lemma 4.3. We use (4.11) with $\psi_2(x) = 1 - \chi(|x - d|/\delta|d|^{\sigma_2})$. Then we have

$$\begin{aligned}\langle x\rangle^\rho V_{1d} R(E + i0; K_{2d}) h_{2d} &= \langle x\rangle^\rho V_{1d} R(E + i0; K_{0d}) \psi_2 h_{2d} \\ &- \langle x\rangle^\rho V_{1d} R(E + i0; K_{0d}) U_{2d} R(E + i0; K_{2d}) h_{2d}.\end{aligned}$$

By Lemma 5.3, the first operator on the right side is majorized by $O(|d|^{-\mu})$ with $\mu = \sigma_1/4 - (\rho - 1/2) - \sigma_2$. To estimate the second operator, we decompose U_{2d} into the sum of four operators

$$U_{2d} = g_{2d}^2 U_{2d} + U_\infty(x, D_x) + U_-(x, D_x) + U_+(x, D_x)$$

as in (4.21), where $g_{2d}(x) = \chi(|x - d|/M|d|^{\sigma_2})$ for $M \gg 1$, and

$$U_\infty(x, D_x) = (1 - g_{2d}^2(x)) U_{2d} \beta_\infty(D_x), \quad U_\pm(x, D_x) = (1 - g_{2d}^2(x)) U_{2d} \beta_\pm(D_x).$$

By Lemma 5.3 again, we have

$$\|\langle x\rangle^\rho V_{1d} R(E + i0; K_{0d}) g_{2d}^2 U_{2d} R(E + i0; K_{2d}) h_{2d}\| \simeq O(|d|^{-\nu})$$

for ν as in the lemma, because

$$\|g_{2d} U_{2d} R(E + i0; K_{2d}) h_{2d}\| \simeq O(|d|^{\sigma_2})$$

by the principle of limiting absorption. If we make use of Lemma 4.2, the other operators with $U_\infty(x, D_x)$ and $U_\pm(x, D_x)$ can be shown to obey the bound $O(|d|^{-N})$ for any $N \gg 1$. This proves the lemma. □

Lemma 5.6 *Let*

$$V_+ = V_+(x, D_x) = (1 - g_{2d}^2) V_{2d} \beta_+(D_x), \quad g_{2d}(x) = \chi(|x - d|/M|d|^{\sigma_2}),$$

be as in (4.21). Then

$$\|\langle x\rangle^\rho V_{1d} R(E + i0; K_{2d}) V_+ \langle x\rangle^\rho\| = O(|d|^{-L})$$

for any $L \gg 1$.

Proof. We construct an outgoing approximation for $R(E+i0;K_{2d})V_+\langle x\rangle^\rho$. If the particle starts from $x\in\{x\in\Pi_{2d}:|x-d|>M|d|^{\sigma_2}\}$ with momentum $\xi\in\operatorname{supp}\beta_+$ at time $t=0$, then it does not pass over Π_{1d} for $t>0$. This enables us to construct the approximation in the form

$$\langle x\rangle^\rho V_{1d}R(E+i0;K_{2d})V_+\langle x\rangle^\rho=\tilde r_L+\langle x\rangle^\rho V_{1d}R(E+i0;K_{2d})\tilde r_L.$$

Hence the lemma is implied by Lemma 4.4. □

5.2. We are now in a position to prove Propositions 4.1 and 4.2.

Proof of Proposition 4.1. We prove only the first statement. A similar argument applies to the second one. Throughout the proof, we take $\sigma_1=\sigma$ and use the relations (4.14) and (4.15) with

$$\psi_1(x)=1-\chi(|x|/\delta|d|^\sigma),\quad \psi_2(x)=1-\chi(|x-d|/\delta|d|^{\sigma_2})$$

for $0<\delta\ll 1$ small enough, where σ_2 is specified by (5.2) with $\sigma_1=\sigma$.

We write

$$X=r_LR(E+i0;H_d)\pi_{1d}\langle x\rangle^{-\rho}$$

for the operator in the proposition. Since $\pi_{1d}\psi_2=\pi_{1d}$, it follows from (4.14) that

$$X=r_L\psi_2R(E+i0;K_{1d})\pi_{1d}\langle x\rangle^{-\rho}-r_LR(E+i0;H_d)V_{2d}R(E+i0;K_{1d})\pi_{1d}\langle x\rangle^{-\rho}.$$

The first operator on the right side satisfies

$$\|r_L\psi_2R(E+i0;K_{1d})\pi_{1d}\langle x\rangle^{-\rho}\|=O(|d|^{-L/2}).$$

To estimate the second operator, we decompose V_{2d} into the sum of four operators

$$V_{2d}=g_{2d}^2V_{2d}+V_\infty(x,D_x)+V_+(x,D_x)+V_-(x,D_x)$$

as in (4.21), where $g_{2d}(x)=\chi(|x-d|/M|d|^{\sigma_2})$ for $M\gg 1$, and

$$V_\pm(x,D_x)=(1-g_{2d}^2)V_{2d}\beta_\pm(D_x),\quad V_\infty(x,D_x)=(1-g_{2d}^2)V_{2d}\beta_\infty(D_x).$$

We set

$$\begin{aligned}X_0&=r_LR(E+i0;H_d)g_{2d}^2V_{2d}R(E+i0;K_{1d})\pi_{1d}\langle x\rangle^{-\rho},\\ X_\infty&=r_LR(E+i0;H_d)V_\infty R(E+i0;K_{1d})\pi_{1d}\langle x\rangle^{-\rho},\\ X_\pm&=r_LR(E+i0;H_d)V_\pm R(E+i0;K_{1d})\pi_{1d}\langle x\rangle^{-\rho}.\end{aligned}$$

Then the operator X in question satisfies

$$\|X\|\le C_L|d|^{-L/2}+\|X_0\|+\|X_\infty\|+\|X_-\|+\|X_+\|.$$

Note that $\psi_1 V_\pm = V_\pm$ and $\psi_1 V_\infty = V_\infty$. We can show

$$\|X_\infty\| + \|X_-\| \le C_L \|r_L R(E+i0;H_d)\psi_1 r_L\|$$

as in the proof of Lemma 3.2. To evaluate the operator $r_L R(E+i0;H_d)\psi_1 r_L$, we represent it as

$$r_L \psi_1 R(E+i0;K_{2d}) r_L - r_L R(E+i0;H_d) V_{1d} R(E+i0;K_{2d}) r_L$$

by (4.15). If we decompose V_{1d} into $V_{1d} = \pi_{1d}\langle x\rangle^{-\rho}\langle x\rangle^{\rho} V_{1d}$, then it follows from Lemma 4.4 that

$$\|r_L R(E+i0;H_d)\psi_1 r_L\| = O(|d|^{-L}) + O(|d|^{-L/2})\|X\|$$

and hence we have

$$\|X_\infty\| + \|X_-\| \le C_L \Big(|d|^{-L/2} + |d|^{-L/2}\|X\|\Big).$$

We consider the operator X_+. We decompose it into the product

$$X_+ = \Big(r_L R(E+i0;H_d) V_+ \langle x\rangle^{\rho}\Big)\Big(\langle x\rangle^{-\rho} R(E+i0;K_{1d})\pi_{1d}\langle x\rangle^{-\rho}\Big).$$

The second operator is bounded uniformly in d, and the first one is represented as

$$r_L \psi_1 R(E+i0;K_{2d}) V_+ \langle x\rangle^{\rho} - r_L R(E+i0;H_d) V_{1d} R(E+i0;K_{2d}) V_+ \langle x\rangle^{\rho}$$

by use of (4.15) again. The micro-local resolvent estimate of [9] shows that

$$\|r_L \psi_1 R(E+i0;K_{2d}) V_+ \langle x\rangle^{\rho}\| = O(|d|^{-L/2}),$$

which, together with Lemma 5.6, implies that

$$\|r_L R(E+i0;H_d) V_+ \langle x\rangle^{\rho}\| = O(|d|^{-L/2}) + O(|d|^{-L/2})\|X\|.$$

Thus X satisfies

$$\|X\| \le C_L \Big(|d|^{-L/2} + |d|^{-L/2}\|X\|\Big) + \|X_0\|. \tag{5.3}$$

We shall evaluate X_0. This obeys the bound

$$\|X_0\| = o(1)\,\|r_L R(E+i0;H_d) g_{2d}\|$$

by Lemma 5.4 with $\kappa = \sigma_2$, and $r_L R(E+i0;H_d) g_{2d}$ is written as

$$r_L \psi_1 R(E+i0;K_{2d}) g_{2d} - r_L R(E+i0;H_d) V_{1d} R(E+i0;K_{2d}) g_{2d}$$

by (4.15). Hence Lemma 5.5 yields

$$\|X_0\| = O(|d|^{-L/2}) + o(1)\,\|X\|.$$

Thus the desired bound is obtained from (5.3) and the proof is complete. □

We proceed to the proof of Proposition 4.2. As previously stated, we are allowed to use Lemma 3.2 for the proof of the proposition.

Proof of Proposition 4.2. The proof is based on the same idea as in the proof of Proposition 4.1, although we have to modify slightly the argument there. Throughout the proof, σ_2 is fixed as $\sigma_2 = \sigma$, and σ_1 and ρ are chosen to fulfill (5.1) and (5.2). We set

$$Y = b_{2d} R(E + i0; H_d) \pi_{1d} \langle x \rangle^{-\rho}.$$

Since $\sigma_1 > \sigma$, $b_{1d}\pi_{1d} = b_{1d}$. Hence it suffices to show the bound $\|Y\| = O(|d|^{2\sigma})$ in order to prove the proposition.

We use the relations (4.14) and (4.15) with

$$\psi_1(x) = 1 - \chi(|x|/\delta|d|^{\sigma_1}), \qquad \psi_2(x) = 1 - \chi(|x-d|/\delta|d|^{\sigma}).$$

By (4.14), we have

$$Y = b_{2d}\psi_2 R(E + i0; K_{1d})\pi_{1d}\langle x\rangle^{-\rho} - b_{2d}R(E + i0; H_d)V_{2d}R(E + i0; K_{1d})\pi_{1d}\langle x\rangle^{-\rho}.$$

The first operator on the right side satisfies

$$\|b_{2d}\psi_2 R(E + i0; K_{1d})\pi_{1d}\langle x\rangle^{-\rho}\| = o(1)$$

by Lemma 5.4. We decompose V_{2d} as in the proof of Proposition 4.1 and set

$$\begin{aligned} Y_0 &= b_{2d}R(E + i0; H_d)g_{2d}^2 V_{2d}R(E + i0; K_{1d})\pi_{1d}\langle x\rangle^{-\rho}, \\ Y_\infty &= b_{2d}R(E + i0; H_d)V_\infty R(E + i0; K_{1d})\pi_{1d}\langle x\rangle^{-\rho}, \\ Y_\pm &= b_{2d}R(E + i0; H_d)V_\pm R(E + i0; K_{1d})\pi_{1d}\langle x\rangle^{-\rho}, \end{aligned}$$

where $g_{2d}(x) = \chi(|x-d|/M|d|^\sigma)$ for $M \gg 1$. We can show

$$\|Y_\infty\| + \|Y_-\| + \|Y_+\| \le C_L \left(\|b_{2d}R(E + i0; H_d)r_L\| + O(|d|^{-L})\right) = O(|d|^{-L/2})$$

by Lemma 3.2. To estimate the operator Y_+, we construct an outgoing approximation for $b_{2d}R(E + i0; H_d)V_+$, which takes the form

$$b_{2d}R(E + i0; H_d)V_+ = \tilde{r}_L + b_{2d}R(E + i0; H_d)\tilde{r}_L.$$

Thus we have $\|Y\| = o(1) + \|Y_0\|$. The operator Y_0 is also estimated in the same way as X_0. It satisfies

$$\|Y_0\| \le \|b_{2d}R(E + i0; K_{2d})g_{2d}\| + o(1)\,\|Y\| \le C|d|^{2\sigma} + o(1)\,\|Y\|$$

by Lemmas 5.4 and 5.5. Hence the desired bound follows at once and the proof is complete. □

6 Asymptotic behavior of eigenfunction

In this section we prove Proposition 2.1 which has played a basic role in proving the main theorem. As already stated in section 2, the asymptotic behavior of eigenfunction $\varphi_\mp(x;\lambda,\omega)$ has been studied in the physical literatures [3,5,14]. The proof here is based on the idea from [14]. The original idea is due to T. Takabayashi.

Proof of Proposition 2.1. We consider only the case $\alpha \notin \boldsymbol{Z}$. For brevity, we assume that $0<\alpha<1$, and we set $\lambda = 1$. The proof uses the integral representation

$$J_p(r) = \frac{(i)^p}{\pi}\left(\int_0^\pi e^{-ir\cos t}\cos pt\,dt - \sin p\pi\int_0^\infty e^{-pt+ir\cosh t}\,dt\right), \quad r>0, \tag{6.1}$$

for the Bessel function $J_p(r)$ with $p > 0$ ([8]).

(1) We write $\varphi(x;\omega)$ for $\varphi_+(x;\lambda,\omega)$ with $\lambda = 1$ and denote by

$$\varphi_{\mathrm{inc}}(x;\omega) = \exp(i\alpha(\gamma(x;\omega)-\pi))\exp(ix\cdot\omega)$$

the leading term in the asymptotic formula. If we make a change of variable $\sigma = \sigma(x;\omega) = \gamma(x;\omega)-\pi$, then $-\pi \le \sigma < \pi$ and it follows from (2.3) that

$$\varphi_+(x;\omega) = \sum_{l\in Z}(-i)^\nu e^{il\sigma}J_\nu(|x|)$$

with $\nu = |l-\alpha|$. We also have

$$\varphi_{\mathrm{inc}}(x;\omega) = e^{i\alpha\sigma - i|x|\cos\sigma}.$$

By the Fourier expansion,

$$\varphi_{\mathrm{inc}}(x;\omega) = \frac{1}{2\pi}\sum_{l\in Z}e^{il\sigma}\int_{-\pi}^{\pi}e^{i\alpha t - i|x|\cos t}e^{-ilt}\,dt = \frac{1}{\pi}\sum_{l\in Z}e^{il\sigma}\int_0^\pi e^{-i|x|\cos t}\cos\nu t\,dt.$$

On the other hand, we have

$$\varphi_+(x;\omega) = \frac{1}{\pi}\sum_{l\in Z}e^{il\sigma}\left(\int_0^\pi e^{-i|x|\cos t}\cos\nu t\,dt - \sin\nu\pi\int_0^\infty e^{-\nu t + i|x|\cosh t}\,dt\right)$$

by integral representation (6.1). Hence

$$\varphi_+(x;\omega) - \varphi_{\mathrm{inc}}(x;\omega) = -\frac{1}{\pi}\sum_{l\in Z}e^{il\sigma}\sin\nu\pi\int_0^\infty e^{-\nu t + i|x|\cosh t}\,dt.$$

We calculate the sum on the right side. If $\gamma(x;\omega)\ne 0$, then $|\sigma|<\pi$ and $e^{\pm i\sigma}\ne -1$. A simple computation shows that

$$\sum_{l\in Z}e^{il\sigma}e^{-\nu t}\sin\nu\pi = \sin\alpha\pi\left(\frac{e^{\alpha t}}{1+e^{-i\sigma}e^{t}} + \frac{e^{-\alpha t}}{1+e^{-i\sigma}e^{-t}}\right)$$

for $0 < \alpha < 1$. This yields

$$\varphi_+(x;\omega) - \varphi_{\mathrm{inc}}(x;\omega) = -\frac{\sin\alpha\pi}{\pi}\int_{-\infty}^{\infty}\frac{e^{-\alpha t}}{1+e^{-i\sigma}e^{-t}}e^{i|x|\cosh t}\,dt \tag{6.2}$$

for $|\sigma| < \pi$. We apply the stationary phase method to the integral on the right side. If x fulfills the assumption $|x/|x| - \omega| > c > 0$, then $|\sigma| < \pi - c$ for $|x| \gg 1$ and hence

$$|1 + e^{-i\sigma}e^{-t}| > c_1 > 0$$

in a neighborhood of the stationary point $t = 0$. Thus we can obtain the desired asymptotic expansion.

(2) If we write $\varphi_\mp(x;\omega,\alpha)$ for $\varphi_\mp(x;\omega)$, then

$$\varphi_-(x;\omega,\alpha) = \overline{\varphi}_+(-x;\omega,-\alpha).$$

Hence (2) follows (1) at once.

(3) We consider $\varphi_+(x;\omega)$ only. By assumption, $|x/|x|-\omega| < c|x|^{-q}$ for some $q,\ 1/2 < q \le 1$. We set $\delta = (q-1/2)/3 > 0$ and

$$\eta(x) = i\left(e^{i\sigma}+1\right) = i\left(e^{i\sigma(x;\omega)}+1\right)$$

for x as above. We evaluate the integral I on the right side of (6.2). If $|x|^{-1/2+\delta} < |t| < 1$, then $|\partial_t \cosh t| > c_2|t|$ and $|\partial_t(1+e^{-i\sigma}e^{-t})^{-1}| < c_3|t|^{-2}$, so that

$$\int_{|t|>|x|^{-1/2+\delta}}\frac{e^{-\alpha t}}{1+e^{-i\sigma}e^{-t}}e^{i|x|\cosh t}\,dt = O(|x|^{-2\delta})$$

by partial integration. Thus we have

$$\begin{aligned} I &= -e^{i\sigma}e^{i|x|}\int_{-|x|^{-1/2+\delta}}^{|x|^{-1/2+\delta}}\frac{1}{t+i\eta}e^{i|x|t^2/2}\,dt + O(|x|^{-1+4\delta}) + O(|x|^{-2\delta}) \\ &= -e^{i\sigma}e^{i|x|}\int_{-|x|^{\delta}}^{|x|^{\delta}}\frac{1}{s+i|x|^{1/2}\eta}e^{i|s|^2/2}\,ds + O(|x|^{-2\delta}). \end{aligned}$$

We write $\sigma = -\pi+\varepsilon$ or $\sigma = \pi-\varepsilon$. Then $\varepsilon > 0$ and $\varepsilon = O(|x|^{-q})$. If $\sigma = -\pi+\varepsilon$, then $\eta = \varepsilon + O(\varepsilon^2)$ and $|x|^{1/2}\eta = O(|x|^{-q+1/2})$. Hence it follows that

$$\int_{|x|^{-\delta}}^{|x|^{\delta}}\left(\frac{1}{s+i|x|^{1/2}\eta} - \frac{1}{s}\right)e^{i|s|^2/2}\,ds = O(|x|^{-(q-1/2)+\delta}).$$

This yields

$$I = -e^{i\sigma}e^{i|x|}\int_{-|x|^{-\delta}}^{|x|^{-\delta}}\frac{1}{s+i|x|^{1/2}\eta}\,ds + O(|x|^{-(q-1/2)+\delta}) + O(|x|^{-2\delta}),$$

so that

$$I = -i\pi e^{i|x|} + O(|x|^{-\nu}), \quad \nu = 2(q - 1/2)/3,$$

for $\sigma = -\pi + \varepsilon$. Similarly we have $I = i\pi e^{i|x|} + O(|x|^{-\nu})$ for $\sigma = \pi - \varepsilon$. Thus (3) follows immediately from (6.2).

(4) We again evaluate the integral I. If $|x/|x| - \omega| > |x|^{-1/2}$, then

$$I = \int_{|x|^{-1/2}<|t|<1} \frac{1}{1 + e^{-i\sigma}e^{-t}} e^{i|x|\cosh t}\, dt + O(1), \quad |x| \to \infty.$$

Since $|\partial_t \left(1 + e^{-i\sigma}e^{-t}\right)^{-1}| \le c\,|t|^{-2}$ for $|x|^{-1/2} < |t| < 1$, we see by partial integration that the first term on the right side also obeys the bound $O(1)$. If, on the other hand, $0 < |x/|x| - \omega| < |x|^{-1/2}$, then

$$I = -e^{i\sigma} \int_{|t|<1} \frac{1}{t + i\eta} e^{i|x|\cosh t}\, dt + O(1)$$

for $\eta = i(e^{i\sigma} + 1)$ again. Set $\sigma = -\pi + \varepsilon$ with $\varepsilon > 0$. Then $\varepsilon = O(|x|^{-1/2})$ and also $\eta = O(|x|^{-1/2})$. Since

$$\int_{|x|^{-1/2}<|t|<1} \left(\frac{1}{t + i\eta} - \frac{1}{t}\right) e^{i|x|\cosh t}\, dt = O(1),$$

it follows that

$$I = -e^{i\sigma}e^{i|x|} \int_{-|x|^{-1/2}}^{|x|^{-1/2}} \frac{1}{t + i\eta}\, dt + O(1) = O(1).$$

A similar argument applies to the case $\sigma = \pi - \varepsilon$. Thus (4) is verified. □

7 Magnetic Schrödinger operators with δ–like fields

In this supplementary section, we study the spectral problems for magnetic Schrödinger operators with two δ–like fields. The argument here extends to the case of several distinct centers without any essential changes. We consider the Hamiltonian

$$H = H(A_1 + A_2), \quad A_j(x) = \alpha_j \nabla \gamma_j(x),$$

where $\gamma_j(x) = \gamma(x - e_j)$ with $e_1 \ne e_2$. The potential A_j has the δ–like magnetic field $2\pi\alpha_j\delta(x - e_j)$. As previously stated, the Hamiltonian $H_j = H(A_j)$, $1 \le j \le 2$, is known to be self–adjoint with domain

$$\mathcal{D}(H_j) = \{u \in L^2 : H(A_j)u \in L^2, \quad \lim_{|x-e_j|\to 0} |u(x)| < \infty\}.$$

We discuss the problems about the self–adjointness, the absence of bound states, the principle of limiting absorption and the asymptotic completeness of wave operators for H.

Proposition 7.1 *H is self–adjoint with domain*

$$\mathcal{D} = \{u \in L^2 : H(A_1 + A_2)u \in L^2, \ \lim_{|x-e_j|\to 0} |u(x)| < \infty, \ \ j = 1, 2\}.$$

Proof. We consider the equation

$$(H + \lambda)\, u = f, \quad \lambda \gg 1, \tag{7.1}$$

for given $f \in L^2$. Let $\{\chi_1, \chi_2, \chi_\infty\}$ be a smooth nonnegative partition of unity normalized by $\chi_1(x)^2 + \chi_2(x)^2 + \chi_\infty(x)^2 = 1$, where $\chi_j \in C_0^\infty(\boldsymbol{R}^2)$ takes the value $\chi_j(x) = 1$ in a neighborhood of e_j. We may assume that $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$. Let $B_j \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ be a magnetic potential such that $B_j(x) = A_j(x)$ on the support of χ_∞, and define H_∞ as $H_\infty = H(B_1 + B_2)$. This is self–adjoint with domain $\mathcal{D}(H_\infty) = H^2(\boldsymbol{R}^2)$. We look for the solution $u \in \mathcal{D}$ in the form

$$\begin{aligned} u \ &= \ \chi_1 e^{i\alpha_2\gamma_2} \left(H_1 + \lambda\right)^{-1} e^{-i\alpha_2\gamma_2} \chi_1 v \\ &\quad + \ \chi_2 e^{i\alpha_1\gamma_1} \left(H_2 + \lambda\right)^{-1} e^{-i\alpha_1\gamma_1} \chi_2 v + \chi_\infty \left(H_\infty + \lambda\right)^{-1} \chi_\infty v \end{aligned}$$

for some $v \in L^2$. As is easily seen, u belongs to $\mathcal{D}$. Note that

$$e^{i\alpha_2\gamma_2} H_1 e^{-i\alpha_2\gamma_2} \chi_1 = H\chi_1, \quad e^{i\alpha_1\gamma_1} H_2 e^{-i\alpha_1\gamma_1} \chi_2 = H\chi_2$$

and $H_\infty \chi_\infty = H\chi_\infty$. If we make use of these relations, then we see that v must satisfy

$$(Id + K_\lambda)\, v = f$$

for u to solve the equation (7.1), where

$$\begin{aligned} K_\lambda \ &= \ e^{i\alpha_2\gamma_2} [H_1, \chi_1] \left(H_1 + \lambda\right)^{-1} e^{-i\alpha_2\gamma_2} \chi_1 \\ &\quad + \ e^{i\alpha_1\gamma_1} [H_2, \chi_2] \left(H_2 + \lambda\right)^{-1} e^{-i\alpha_1\gamma_1} \chi_2 + [H_\infty, \chi_\infty] \left(H_\infty + \lambda\right)^{-1} \chi_\infty. \end{aligned}$$

The norm obeys the bound $\|K_\lambda\| = O(\lambda^{-1/2})$ for $\lambda \gg 1$. Hence there exists the bounded inverse $(Id + K_\lambda)^{-1} : L^2 \to L^2$, so that equation (7.1) admits a unique solution in $\mathcal{D}$. Thus $(H + \lambda)^{-1} : L^2 \to L^2$ is bounded with range $\operatorname{Ran}(H + \lambda)^{-1} = \mathcal{D}$. It is easy to see that $(H + \lambda)^{-1}$ is symmetric and hence H is self–adjoint with domain $\mathcal{D}$. □

We move to the problem on the absence of bound states.

Proposition 7.2 *H has no bound states.*

Proof. It is easy to see that H does not have non-positive eigenvalue. We consider the eigenvalue problem

$$Hu = \lambda u, \qquad u \in L^2,$$

for $\lambda > 0$. Let $\alpha = \alpha_1 + \alpha_2$ and define

$$g(x) = \exp(i(\alpha\gamma(x) - \alpha_1\gamma_1(x) - \alpha_2\gamma_2(x)))$$

for $|x| > L \gg 1$. It should be noted that $g(x)$ is well defined as a single–valued function. Set $v = gu$. Then v fulfills $H_\alpha v = \lambda v$ on $G = \{x : |x| > L\}$, where

$$H_\alpha = H(A_\alpha), \qquad A_\alpha = \alpha\nabla\gamma(x). \tag{7.2}$$

The operator above admits the partial wave expansion. If $v \in L^2(G)$, then $v = 0$ over G, and hence it follows by unique continuation that $u = 0$ identically on the whole space. Thus H is shown to have no bound states. □

We shall prove the principle of limiting absorption.

Proposition 7.3 *The resolvent $R(z; H) = (H-z)^{-1}$ with $\operatorname{Im} z \neq 0$ has the boundary values to the positive real axis*

$$R(\lambda \pm i0; H) = \lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon; H) : L^2_s(\boldsymbol{R}^2) \to L^2_{-s}(\boldsymbol{R}^2)$$

for $s > 1/2$ in the uniform topology, where the convergence is locally uniform in $\lambda \in (0, \infty)$.

Proof. The proof uses the positive commutator method due to Mourre [13]. Let H_α be defined by (7.2). Define the operator C as $C = -i\,(x \cdot \nabla + \nabla \cdot x)$. Then we have

$$i[H_\alpha, C] = i\,(H_\alpha C - CH_\alpha) = 4H_\alpha$$

by formal computation. Let $\chi_\infty(x)$ be as in the proof of Proposition 7.1. Recall that $\chi_\infty(x)$ vanishes around two centers e_1 and e_2. We take $D = \chi_\infty C \chi_\infty$ as a conjugate operator. Since $h(H+i)^{-1} : L^2 \to L^2$ is compact for $h(x)$ falling off at infinity and since

$$A_\alpha(x) - A_1(x) - A_2(x) = O(|x|^{-2}) \tag{7.3}$$

as $|x| \to \infty$, we obtain the relation

$$f(H)i[H, D]f(H) = 4\, f(H)Hf(H) + f(H)K_0 f(H)$$

for some compact operator $K_0 : L^2 \to L^2$, where $f \in C_0^\infty(0, \infty)$ is supported away from the origin. This enables us to repeat the same argument as in [6,13] and we get the proposition. □

Finally we discuss the existence and completeness of wave operator

$$W_\pm(H, H_0) = s - \lim_{t \to \pm\infty} \exp(itH)\exp(-itH_0) : L^2 \to L^2.$$

Proposition 7.4 *The wave operator $W_\pm(H, H_0)$ exists and is asymptotically complete*

$$\operatorname{Ran} W_+(H, H_0) = \operatorname{Ran} W_-(H, H_0) = L^2.$$

Proof. The existence can be proved in almost the same way as in the case of smooth magnetic fields ([12]). We skip the proof for it. To prove the completeness, it suffices to show that the limit

$$W_\pm(H_0, H) = s - \lim_{t\to\pm\infty} \exp(itH_0)\exp(-itH) \tag{7.4}$$

exists. Let H_α be again defined by (7.2). We know from [17] that $W_\pm(H_\alpha, H_0)$ exists and is asymptotically complete. This implies the existence of limit

$$W_\pm(H_0, H_\alpha) = s - \lim_{t\to\pm\infty} \exp(itH_0)\exp(-itH_\alpha).$$

On the other hand, the difference $H - H_\alpha$ is a perturbation of short–range class by (7.3). Hence we can show the existence

$$W_\pm(H_\alpha, H) = s - \lim_{t\to\pm\infty} \exp(itH_\alpha)\varphi_\infty \exp(-itH)$$

by use of Kato's smoothness property which follows from Proposition 7.3 ([15]), where $\varphi_\infty(x)$ is a smooth real function such that $\varphi_\infty(x) = 1$ for $|x| > L \gg 1$ and $\varphi_\infty(x) = 0$ for $|x| < L/2$. Thus the limit (7.4) in question can be shown to exist and the proof is completed. □

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