Ann. Henri Poincaré 1 (2000) 59 – 100

1424-0637/00/010059-42 $ 1.50+0.20/0

Annales Henri Poincaré

Correlation Asymptotics of Classical Lattice Spin Systems with Nonconvex Hamilton Function at Low Temperature

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Abstract. The present paper continues Sjöstrand's study [14] of correlation functions of lattice field theories by means of Witten's deformed Laplacian. Under the assumptions specified in the paper and for sufficiently low temperature, we derive an estimate for the spectral gap of a certain Witten Laplacian which enables us to prove the exponential decay of the two-point correlation function and, further, to derive its asymptotics, as the distance between the spin sites becomes large. Typically, our assumptions do not require uniform strict convexity and apply to Hamiltonian functions which have a single, nondegenerate minimum and no other extremal point.

Keywords. Correlation Function, Lattice Spin Systems, Exponential Decay, Witten Laplacian.

I Introduction and results

The present paper can be viewed as a continuation of works by Helffer-Sjöstrand [10] and Sjöstrand [14] on Laplace integrals

$$\int_{\mathbb{R}^m} e^{-2\beta H(x)} u(x)\, dx \tag{I.1}$$

in the limit $m \to \infty$ and for large $\beta > 0$. In particular, we are interested in the two-point correlation functions

$$\mathbb{E}_\beta^T(x_j\,; x_k) \;:=\; \mathbb{E}_\beta(x_j \cdot x_k) \;-\; \mathbb{E}_\beta(x_j)\, \mathbb{E}_\beta(x_k) \tag{I.2}$$

when $|j-k| \to \infty$ (and more precise assumptions will be given below), where

$$\mathbb{E}_\beta(x_j) \;:=\; \Big(\int_{\mathbb{R}^m} e^{-2\beta H(x)}\, dx\Big)^{-1} \int_{\mathbb{R}^m} x_j\, e^{-2\beta H(x)}\, dx \tag{I.3}$$

is the expectation of x_j. In [10], the authors studied exponential decay of the correlations under assumptions on the function H containing that of uniform strict convexity. They exhibited a certain matrix Schrödinger operator for gradients and studied it by means of a maximum principle. The global convexity was quite crucial for the maximum principle to apply.

In [14], one of us identified the matrix operator (up to a conjugation) with the Witten Laplacian in degree 1, i.e., with the Hodge Laplacian associated to the conjugated de Rahm complex

$$\beta^{-1}\, e^{-\beta H}\, d\, e^{\beta H}\ ,\ d = \text{exterior derivative}\,, \tag{I.4}$$

and an explicit identity ((I.27) below) was given for the correlations. A more systematic use of L^2-methods, together with the use of Grushin-Feshbach reductions led (essentially) to an asymptotic formula for the correlations when $|j-k| \to \infty$. Unfortunately, some use of the maximum principle remained and consequently it was still necessary to impose uniform strict convexity, as well as some other unnatural assumptions.
The purpose of the present paper is to completely eliminate the maximum principle and to work entirely with L^2-methods. This allows us to weaken the assumptions on H considerably. The present assumptions (see below) imply that H has a non-degenerate minimum and that this is the only critical point. Away from the minimum, however, H is allowed to be non-convex. The novelty (see Section III) is the use of certain weighted estimates on quantities related to H (like, for instance, $H''(x) - H''(0)$) in terms of the derivatives $\partial H/\partial x_j$. Similar ideas have recently been developed by Helffer [6, 7, 8, 9], who, for a wide range of parameters, derives exponentially decaying upper bounds for the two-point correlation function in case that the interaction is strictly convex or quadratic (Gaussian), while H may be non-convex. In other parts of the paper, we roughly follow [14].

Physically, H is the Hamiltonian (energy) function for a continuous spin system on the lattice $\Lambda_L \subseteq \mathbb{Z}^d$ which one may either derive directly from first principles or from a discrete spin system by a Sine-Gordon transformation. Even though we have weakened the assumptions on H compared to [14], our results imply that the system represented by this Hamiltonian function does not exhibit phase transitions, and the extension of our method to include the description of multiple phases, our ultimate goal, is not obvious. We remark that continuous spin systems with multiple phases have been successfully studied by other methods, e.g., the Pirogov-Sinai theory and contour methods [5, 18].

We consider a system of real-valued spins on the sequence $\{\Lambda_L\}_{L\in\mathbb{N}}$ of finite, n-dimensional lattices $\Lambda_L := (\mathbb{Z}/L\mathbb{Z})^n$. Given $L \in \mathbb{N}$, the corresponding spin configuration space is $\mathbb{R}^{|\Lambda_L|}$, and the energy of a spin configuration is determined by a Hamilton function $H_L \in C^2(\mathbb{R}^{|\Lambda_L|}, \mathbb{R})$. To ensure the existence of the thermodynamic limit, we shall generally assume the following hypothesis.

Hypothesis 1. *There exist constants $C_{(H1)} > 0$, $\delta' = \delta'_{(H1)} \geq \delta = \delta_{(H1)} > 0$, independent of $L \in \mathbb{N}$, such that, for all $x = (x_j)_{j\in\Lambda_L} \in \mathbb{R}^{|\Lambda_L|}$,*

$$\sum_{j\in\Lambda_L} \Big(|x_j|^{\delta} - C_{(H1)} \Big) \ \leq\ H_L(x)\ \leq\ \sum_{j\in\Lambda_L} \Big(|x_j|^{\delta'} + C_{(H1)} \Big)\,. \tag{I.5}$$

Under this assumption there exists a constant $m = m(\beta, \delta_{(H1)}, \delta'_{(H1)})$ such that, for any *inverse temperature* $2\beta > 0$,

$$\exp\big(-2\beta m|\Lambda_L|\big) \;\le\; \Xi(2\beta) \;:=\; \int e^{-2\beta H_L(x)}\, dx \;\le\; \exp\big(2\beta m|\Lambda_L|\big) \,. \tag{I.6}$$

Thus, replacing H_L and $C_{(H1)}$ by $H_{L,\beta} := H_L(x) + (2\beta)^{-1}\log \Xi(2\beta)$ and $C_{(H1)}(\beta, \delta_{(H1)}, \delta'_{(H1)}) := C_{(H1)} + m(\beta, \delta_{(H1)}, \delta'_{(H1)})$, respectively, we obtain that $H_{L,\beta}$ fulfills Hypothesis 1, as well, and

$$\int e^{-2\beta H_{L,\beta}(x)}\, dx \;=\; 1 \,. \tag{I.7}$$

We note that this replacement does not affect the derivatives of $H_{L,\beta}$. Henceforth, we often neither display the dependence of $H_{L,\beta}$ on L nor β and simply write $H = H_{L,\beta}$. Thus we have that $e^{-\beta H} \in \mathcal{H}^{(0)} := L^2(\mathbb{R}^{|\Lambda_L|})$ and $\|e^{-\beta H}\| = 1$. Equivalently, $d\mu(x) := e^{-2\beta H(x)} dx$ defines a probability measure, the *Gibbs measure*, on $\mathbb{R}^{|\Lambda_L|}$.

Given a polynomially bounded observable u, i.e., a polynomially bounded, measurable function $\mathbb{R}^{|\Lambda_L|} \to \mathbb{R}$, we define its expectation by

$$\mathbb{E}_{L,\beta}(u) \;:=\; \int u(x)\, e^{-2\beta H_{L,\beta}(x)}\, dx \,. \tag{I.8}$$

By (I.7), $\mathbb{E}_{L,\beta}(1) = 1$. The truncated correlation of two polynomially bounded observables u, v is defined by

$$\mathbb{E}^T_{L,\beta}(u\,; v) \;:=\; \mathbb{E}_{L,\beta}(u \cdot v) \;-\; \mathbb{E}_{L,\beta}(u) \cdot \mathbb{E}_{L,\beta}(v) \,. \tag{I.9}$$

To formulate our first main result, we assume the following specific hypothesis on $H_{L,\beta}$, remarking that below and henceforth we use the notation

$$F'_i(x) \;:=\; \partial_i F(x) \;=\; \frac{\partial F}{\partial x_i}(x) \,, \quad F''_{i,j}(x) \;:=\; \partial_i \partial_j F(x) \;=\; \frac{\partial^2 F}{\partial x_i \partial x_j}(x) \,, \tag{I.10}$$

for any $F \in C^2(\mathbb{R}^{|\Lambda_L|}; \mathbb{C})$.

Hypothesis 2. *$H_{L,\beta} \in C^2(\mathbb{R}^{|\Lambda_L|}; \mathbb{R})$ has a unique minimum at $x = 0$, and for any other critical point $x_c \in \mathbb{R}^{|\Lambda_L|} \setminus \{0\}$ of $H_{L,\beta}$, we have $H_{L,\beta}(x_c) \ge H_{L,\beta}(0) + C_{(H2)}$, for some constant $C_{(H2)} > 0$. Furthermore, the Hessian $H''_{L,\beta}(0)$ of $H_{L,\beta}$ at $x = 0$ is bounded by*

$$0 \;<\; \lambda_{\min} \cdot \mathbf{1} \;\le\; H''_{L,\beta}(0) \;\le\; \lambda_{\max} \cdot \mathbf{1} \,, \tag{I.11}$$

for two constants $\lambda_{\max} \ge \lambda_{\min} > 0$. These constants $C_{(H2)}$, $\lambda_{\min}$, and $\lambda_{\max}$ neither depend on L nor β.

Hypotheses 1–2 guarantee that, for fixed L, the Gibbs measure $\mathbb{E}_{L,\beta}(\,\cdot\,)$ is concentrated about $x = 0$, as $\beta \to \infty$. We do not expect that the system described by H_L undergoes a phase transition. Rather, we expect to have exponentially decaying correlations,

$$\big|\, \mathbb{E}^T_{L,\beta}(x_j\,;\,x_k) \,\big| \;\le\; C_\beta \, \exp\Big(-\mu_\beta \, d(j-k)\Big)\,, \quad 1 \ll d(j-k) \ll L\,, \tag{I.12}$$

for some $C_\beta \ge 0$, $\mu_\beta > 0$. Indeed, we give such an upper bound in Theorem I.6, and we derive the precise asymptotics of $\mathbb{E}^T_{L,\beta}(x_j\,;\,x_k)$ in Theorem I.9 below. In Eqn. (I.12), we use the natural euclidean distance function $d : (\mathbb{R}/L\mathbb{Z})^n \to \mathbb{R}$ on the torus, given by

$$d(k) \;:=\; \min\Big\{ |\tilde{k}|_{\mathbb{R}^n} \;=\; \sqrt{\tilde{k}_1^2 + \ldots + \tilde{k}_d^2} \;\Big|\; \tilde{k} \in \pi^{-1}(k) \Big\}\,, \tag{I.13}$$

where $\pi : \mathbb{R}^n \to \Lambda_L = (\mathbb{R}/L\mathbb{Z})^n$ is the canonical projection. In other words, if we identify Λ_L with the fundamental domain $\Lambda_L^{(box)} := [-L/2\,,\,L/2)^n \cap \mathbb{Z}^n$ then

$$d(k) \;:=\; \big|J(k)\big|_{\mathbb{R}^n} \tag{I.14}$$

is the euclidean length of $J(k)$, where $J : \Lambda_L \to \Lambda_L^{(box)}$ is the natural bijection given by $J^{-1} = \pi|_{\Lambda_L^{(box)}}$.

For the derivation of the asymptotics of $\mathbb{E}^T_{L,\beta}(x_j\,;\,x_k)$, for large $d(j-k)$, the following summability hypothesis, which depends on a weight function $G : \Lambda_L \to [0, +\infty)$, is an important requirement.

Hypothesis 3. [**G**] *For an even function $G : \Lambda_L \to [0, +\infty)$, there exist weights $a_{ij}(k) = a_{ji}(k) \ge 0$, $b_{ij}(k) = b_{ji}(k) \ge 0$, where $i, j, k \in \Lambda_L$, such that, for all $i, j \in \Lambda_L$ and $x \in \mathbb{R}^{|\Lambda_L|}$,*

$$\Big| H''_{i,j}(x) \;-\; H''_{i,j}(0) \Big| \;\le\; \sum_{k \in \Lambda_L} \Big\{ a_{ij}(k) |H'_k(x)| \;+\; b_{ij}(k) |H'_k(x)|^2 \Big\}\,. \tag{I.15}$$

These weights fulfill the summability condition that

$$\max_{j \in \Lambda_L} \Big\{ \sum_{i,k \in \Lambda_L} e^{G(i-j)} \big(a_{ij}(k) + b_{ij}(k) \big) \Big\}\,,\; \max_{k \in \Lambda_L} \Big\{ \sum_{i,j \in \Lambda_L} \big(a_{ij}(k) + b_{ij}(k) \big) \Big\} \tag{I.16}$$

is bounded above by some constant $C_{(H3)}(G)$, which neither depends on L nor β.

In Theorem I.4 we only require Hypothesis 3 with the trivial weight $G \equiv 0$ to prove a spectral estimate which implies the existence of a spectral gap for the relevant operator. To turn this gap estimate into an exponential decay estimate similar to (I.12), we need to make a slightly stronger assumption, namely, that Hypothesis 3 holds for $G = \mu d$, for some $\mu > 0$, where d is the euclidean distance function on Λ_L specified in (I.13). Additionally, we require Hypothesis 4[νd] below, for some $\nu > 0$, which is an estimate on the Hessian of H at $x = 0$.

Hypothesis 4. [**G**] *Assume Hypothesis 2. For an even function $G : \Lambda_L \to [0, +\infty)$, there exists a constant $1 > C_{(H4)}(G) > 0$, neither depending on L nor β, such that the Hessian of H at 0 satisfies*

$$\forall L, \forall i \in \Lambda_L \ : \quad \sum_{j \in \Lambda_L \setminus \{i\}} e^{G(i-j)} \big| H''_{i,j}(0) \big| \ \leq \ \frac{\lambda_{\min}}{\lambda_{\max}} \big(1 - C_{(H4)}(G) \big) \big| H''_{i,i}(0) \big| \, . \tag{I.17}$$

For the derivation of the precise asymptotics of $\mathbb{E}^T_{L,\beta}(x_j\, ; x_k)$, our requirement for G in Hypothesis 3 is even stronger. Indeed, starting from the norm $S^v_r : \mathbb{R}^n \to [0, +\infty)$ given by the support function S^v_r defined in (I.40), for some $r > 1$, we assume that Hypothesis 3[G] holds with $G \equiv \widetilde{\theta}^S_r$, where

$$\widetilde{\theta}^S_r(k) \ := \ \inf \Big\{ S^v_r(\tilde{k}) \ \Big| \ \tilde{k} \in \pi^{-1}(k) \Big\} \, . \tag{I.18}$$

We note here that in general, if $S : \mathbb{R}^n \to [0, +\infty)$ is a semi-norm then $d_S : (\mathbb{R}/L\mathbb{Z})^n \to [0, +\infty)$ defined similarly to (I.18) by $d_S(x) := \inf_{\tilde{x} \in \pi^{-1}(x)} \{ S(\tilde{x}) \}$ defines a semi-metric which obeys the triangle inequality, $d_S(x+y) \leq d_S(x) + d_S(y)$. The choice of G in the three cases decribed above can be expressed in terms of the underlying semi-norm on $\mathbb{R}^n$, namely, $G \equiv d_S \equiv 0$, for $S \equiv 0$, $G \equiv d_S = \nu d$, for $S = \nu | \cdot |_{\mathbb{R}^n}$, and $G \equiv d_S = \widetilde{\theta}^S_r$, for $S \equiv S^v_r$.

We remark that Hypotheses 1 and 3 partially strengthens Hypothesis 2, as they imply that there is only one critical point, namely at the minimum. To see this, we observe that Hypothesis 3 imposes that any critical point is a strictly relative minimum. If there were two different critical points, there would exist a saddle point by the Mountain Pass Lemma (see [3, 15]) in contradiction to Hypothesis 3.

Nevertheless, by using Hypothesis 3[G], we avoid Sjöstrand's requirement [14] of uniformly strict convexity of H, i.e., $H''(x) \geq c \cdot \mathbf{1} > 0$, for all $x \in \mathbb{R}^{|\Lambda_L|}$.

The main example we have in mind is a pair interaction Hamilton function of the following form, for some $\nu > 0$.

Example I.1. [ν] *There exist $0 < g \leq 1$, $f \in C^2(\mathbb{R}; \mathbb{R})$, obeying $|t|^\delta - c \leq f(t) \leq |t|^{\delta'} + c$, for some $\delta, \delta', c > 0$, and $w_{ij} \in C^2(\mathbb{R}^2; \mathbb{R})$, for all $i, j \in \Lambda_L$, $w_{ii} \equiv 0$, such that*

$$H_L(x) \ = \ \sum_{j \in \Lambda_L} f(x_j) \ + \ g \sum_{i,j \in \Lambda_L} e^{-\nu\, d(i-j)} \, w_{ij}(x_i, x_j) \, . \tag{I.19}$$

Furthermore $f''(0) > 0$, and f and $\{w_{ij}\}_{i,j \in \Lambda_L}$ obey

$$\begin{aligned} |\, \partial_s w_{ij}(s,t) \,| \, , \ \ |\, \partial_t w_{ij}(s,t) \,| \ \ &\leq \ \ |f'(s)| + |f'(t)| \, , \\ |\, f''(t) - f''(0) \,| \ \ &\leq \ \ |f'(t)| + |f'(t)|^2 \, , \\ \left. \begin{array}{r} |\partial_s^2 w_{ij}(s,t) - \partial_s^2 w_{ij}(0,0)| \, , \\ |\partial_t^2 w_{ij}(s,t) - \partial_t^2 w_{ij}(0,0)| \, , \\ |\partial_s \partial_t w_{ij}(s,t) - \partial_s \partial_t w_{ij}(0,0)| \, , \end{array} \right\} \ \ &\leq \ \ |f'(s)| + |f'(s)|^2 \ + \ |f'(t)| + |f'(t)|^2 \, , \\ |\partial_s^2 w_{ij}(0,0)| \ \ , \ \ |\partial_t^2 w_{ij}(0,0)| \quad , \ \ \ |\partial_s \partial_t w_{ij}(0,0)| \ \ &\leq \ \ 1 \, . \end{aligned} \tag{I.20}$$

The function given by (I.19) satisfies Hypotheses 1, 2, and 4[μd], for some $0 < \mu < \nu$ small enough. Note that in our example, we require that $f''(x_c) = f''(0) > 0$, for any critical point x_c. Thus, f has its minimum at $t = 0$ and no other critical point. Furthermore, we remark that the restriction to small values of $g \geq 0$ should ensure that $e^{-2\beta H_L(x)} \prod_j dx_j$ is close to a product measure of the form $\prod_j e^{-2\beta f(x_j)} dx_j$. For such a Hamilton function, we prove the following lemma in Section A.

Lemma I.2. *Assume that H_L is a Hamilton function as in Example I.1[ν], and let $M_\alpha := 2^n(1 - e^{-\alpha/\sqrt{n}})^{-n}$, for $\alpha > 0$. Then, for $0 \leq g < M_\nu^{-3}/24$ and any $0 \leq \mu < \nu$, the Hamilton function H_L fulfills Hypothesis 3[μd], with $C_{(H3)} = 2 + 12gM_{\nu-\mu}$ and*

$$a_{ij}(k) \;:=\; b_{ij}(k) \;:=\; \sum_{l \in \Lambda_L} \widetilde{c}_{ij}(l) R_{lk} \,, \tag{I.21}$$

where $\widetilde{c}_{ij}(l)$ and R_{lk} are defined in (A.8) and (A.12) in Section A below.

To state our first main result, we introduce some more notation. We define two operators,

$$\Delta_H^{(1)} \;:=\; \Delta_H^{(0)} \otimes \mathbf{1} \;+\; \frac{2}{\beta} H''(x) \,, \tag{I.22}$$

$$A^{(1)} \;:=\; \Delta_H^{(0)} \otimes \mathbf{1} \;+\; \frac{2}{\beta} H''(0) \,, \tag{I.23}$$

on $\mathcal{H}^{(1)} := L^2(\mathbb{R}^{|\Lambda_L|}) \otimes \mathbb{C}^{|\Lambda_L|}$, the space of square-integrable one-forms on $\mathbb{R}^{|\Lambda_L|}$, where $H''(x)$ is the multiplication by the Hessian matrix of H at x, and

$$\begin{aligned} \Delta_H^{(0)} \;&:=\; \sum_{j \in \Lambda_L} \Big\{ -\frac{1}{\beta^2} \frac{\partial^2}{\partial x_j^2} \;+\; \big| H_j'(x) \big|^2 \;-\; \frac{1}{\beta} H_{j,j}''(x) \Big\} \\ &=\; \sum_{j \in \Lambda_L} Z_j(H)^* \, Z_j(H) \,, \end{aligned} \tag{I.24}$$

with

$$Z_j(H) \;:=\; e^{-\beta H} \big(\beta^{-1} \partial_j \big) e^{\beta H} \;=\; \beta^{-1} \partial_j \;+\; H_j'(x) \,, \tag{I.25}$$

$$Z_j(H)^* \;:=\; e^{\beta H} \big(-\beta^{-1} \partial_j \big) e^{-\beta H} \;=\; -\beta^{-1} \partial_j \;+\; H_j'(x) \,. \tag{I.26}$$

Under the assumption of Hypotheses 1 and 2, both $\Delta_H^{(1)}$ and $A^{(1)}$ are strictly positive, invertible operators on $\mathcal{H}^{(1)}$. While for $A^{(1)}$, this follows simply from $A^{(1)} \geq 2\beta^{-1} \lambda_{\min} \mathbf{1}$, which is implied by the positivity of $\Delta_H^{(0)}$, the strict positivity of $\Delta_H^{(1)}$ is less obvious. It origins from the fact that $\Delta_H^{(0)}$ and $\Delta_H^{(1)}$ can be viewed as restrictions to the space of square-integrable zero- and one-forms on $\mathbb{R}^{|\Lambda_L|}$, respectively, of the *Witten Laplacian*, Δ_H, the Hodge Laplacian conjugated with $e^{-\beta H}$, which acts on forms of all degrees [17, 4]. We outline the argument in Section II.

It is convenient to introduce the set $\mathcal{O}^{(1)}$ of observables $u \in C^1(\mathbb{R}^{|\Lambda_L|};\mathbb{R})$, for which both u and ∇u are polynomially bounded. We remark that $e^{-\beta H}\nabla u \in \mathcal{H}^{(1)}$, for any $u \in \mathcal{O}^{(1)}$. The importance of the Laplacian $\Delta_H^{(1)}$ lies in the following identity, used implicitly by Helffer and Sjöstrand [10] and stated explicitly in [14], and for which we give a new derivation in Section II.

Lemma I.3. *Assume Hypotheses 1 and 2. Then $\Delta_H^{(1)}$ is strictly positive on $\mathcal{H}^{(1)}$, and, for any two observables $u, v \in \mathcal{O}^{(1)}$, the following identity holds:*

$$\mathbb{E}^T_{L,\beta}(u\,;v) \;=\; \frac{1}{\beta^2}\Big\langle e^{-\beta H}\nabla u \,\Big|\, \big(\Delta_H^{(1)}\big)^{-1} e^{-\beta H}\nabla v\Big\rangle_{\mathcal{H}^{(1)}} . \tag{I.27}$$

Lemma I.3 allows us to express the truncated correlations by matrix elements of the resolvent of $\Delta_H^{(1)}$. Thus, the analysis of the truncated correlation traces back to the spectral analysis of $\Delta_H^{(1)}$. The latter is not entirely trivial, a priori, as the Hessian $H''(x)$ may become small or even negative, for some $x \in \mathbb{R}^{|\Lambda_L|}$. Our first main result, Theorem I.4 below, shows that, under the additional assumption of Hypothesis 3 without exponential weights, i.e., $G \equiv 0$, the values of the Hessian $H''(x)$, for x away from the origin, are irrelevant.

Theorem I.4. *Assume Hypotheses 1, 2, and 3*[0]*. Then there exist constants $C \geq 0$ and $\beta_0 \geq 0$, both independent of L, such that, for all $\beta \geq \beta_0$,*

$$\Big(1 - \frac{C}{\beta^{1/2}}\Big) A^{(1)} \;\leq\; \Delta_H^{(1)} \;\leq\; \Big(1 + \frac{C}{\beta^{1/2}}\Big) A^{(1)} \tag{I.28}$$

holds in the sense of quadratic forms on $\mathcal{Q}^{(1)} \subseteq \mathcal{H}^{(1)}$, the form domain of $A^{(1)}$ and $\Delta_H^{(1)}$.

We begin the discussion of Theorem I.4 by deriving a corollary which immediately follows from $A^{(1)} \geq 2\beta^{-1}H''(0)$.

Corollary I.5. *Assume Hypotheses 1, 2, and 3*[0]*. Then there exist constants $C \geq 0$ and $\beta_0 \geq 0$, both independent of L, such that, for any observable $u \in C^1(\mathbb{R}^{|\Lambda_L|};\mathbb{R})$, for which u and ∇u are polynomially bounded, and any $\beta \geq \beta_0$, we have*

$$\begin{aligned}\mathbb{E}^T_{L,\beta}(u\,;u) \;&\leq\; \frac{2}{\beta}\Big(1 + \frac{C}{\beta^{1/2}}\Big)\Big\langle e^{-\beta H}\nabla u \,\Big|\, \big(H''(0)\big)^{-1} e^{-\beta H}\nabla u\Big\rangle_{\mathcal{H}^{(1)}} \\ &\leq\; \frac{2}{\beta}\Big(1 + \frac{C}{\beta^{1/2}}\Big)\frac{1}{\lambda_{\min}}\,\mathbb{E}_{L,\beta}\Big(\,|\nabla u|^2\,\Big) .\end{aligned} \tag{I.29}$$

We compare this result to the Brascamp-Lieb inequality [2, 14, 7, 12], which states that

$$\mathbb{E}^T_{L,\beta}(u\,;u) \;\leq\; \frac{2}{\beta}\Big\langle e^{-\beta H}\nabla u \,\Big|\, \big(H''(x)\big)^{-1} e^{-\beta H}\nabla u\Big\rangle_{\mathcal{H}^{(1)}} , \tag{I.30}$$

for strictly convex H, i.e., $H''(x) \geq \lambda_{\min}(x) > 0$, for all $x \in \mathbb{R}^{|\Lambda_L|}$, where $\lambda_{\min}(x)$ may become very small, for certain values of x. Our result in Corollary I.5 is stronger in the sense that it only requires $H''(0) \geq \lambda_{\min}$ and a certain control of $H''(x) - H''(0)$ by $|H'(x)|$, specified in Hypothesis 3[0].

Our second main result concerns the low-temperature asymptotics of the two-point correlation function $\mathbb{E}^T_{L,\beta}(x_j\,;x_k)$. An application of Lemma I.3 with $u := x_j$ and $v := x_k$ yields

$$\mathbb{E}^T_{L,\beta}(x_j\,;x_k) \;=\; \frac{1}{\beta^2}\Big\langle e^{-\beta H} \otimes e_j \Big|\, \big(\Delta^{(1)}_H\big)^{-1}\, e^{-\beta H} \otimes e_k \Big\rangle_{\mathcal{H}^{(1)}}, \tag{I.31}$$

where $\{e_i\}_{i\in\Lambda_L}$ denotes the standard basis in $\mathbb{C}^{|\Lambda_L|}$. On the other hand, we trivially have

$$\frac{1}{2\beta}\Big(\big(H''(0)\big)^{-1}\Big)_{j,k} \;=\; \Big\langle e^{-\beta H} \otimes e_j \Big|\, (A^{(1)})^{-1}\, e^{-\beta H} \otimes e_k \Big\rangle_{\mathcal{H}^{(1)}}, \tag{I.32}$$

and Theorem I.4 asserts that $\Delta^{(1)}_H$ agrees with $A^{(1)}$ up to a relative error which becomes small, as $\beta \to \infty$. It is thus reasonable to believe that

$$\mathbb{E}^T_{L,\beta}(x_j\,;x_k) \;\approx\; \frac{1}{2\beta}\Big(\big(H''(0)\big)^{-1}\Big)_{j,k}, \tag{I.33}$$

as $\beta \to \infty$, in a suitable sense made precise in Lemma I.8 and Theorem I.9 below. In fact, under the additional requirement of Hypotheses 3 and 4, it is fairly straightforward to turn the spectral estimates of Theorem I.4 into an upper bound for $|\mathbb{E}^T_{L,\beta}(x_j\,;x_k)|$ with an exponential decay in $d(j-k)$, as the following theorem makes explicit.

Theorem I.6. *Assume Hypotheses 1, 2, 3[μd], and 4[νd], for some $\mu, \nu > 0$, i.e., $G = \mu d$ in Hypothesis 3 and $G = \nu d$ in Hypothesis 4. Then there exist constants $C \geq 0$ and $\beta_0 \geq 0$, both independent of L, such that*

$$\big|\,\mathbb{E}^T_{L,\beta}(x_j\,;\,x_k)\,\big| \;\leq\; \frac{C}{\beta}\,\exp\Big(-\min(\mu\,;\nu)\;d(j-k)\Big), \tag{I.34}$$

for all $\beta \geq \beta_0$.

To prove and quantify the relation (I.33), we assume the translation invariance of the Hamilton function.

Hypothesis 5. *The Hamilton function is* translation invariant. *That is, $H_L(\tau_m x) = H_L(x)$, for any $m \in \Lambda_L$, where $(\tau_m x)_j := x_{j-m}$ denotes the shift on the lattice by m.*

Note that translation invariance of H_L implies the translation invariance of the Hessian of H_L at $x = 0$. Indeed, since

$$H''_{j,k}(0) \;=\; H''_{j-k,0}(0), \tag{I.35}$$

the Hessian $H''(0)$ operates on $\mathbb{C}^{|\Lambda_L|}$ as convolution with $H''_{\cdot,0}(0)$. We remark that $H''_{0,0}(0) \geq \lambda_{\min} > 0$, assuming Hypothesis 2. Furthermore, we assume the Hessian $H''(0)$ to be *ferromagnetic, of finite range*, and independent of L, for L sufficiently large. More precisely, we require the following additional hypothesis.

Hypothesis 6. *Assume Hypotheses 2 and 5, and define a function v_L by setting $v_L(k) := -H''_{k,0}(0)/H''_{0,0}(0)$, for $k \neq 0$, and $v_L(0) := 0$. There exists an even, nonnegative function $v : \mathbb{Z}^n \to [0, +\infty)$, $v(k) = v(-k) \geq 0$, of bounded support such that $v_L = v \circ J$ (where J is defined in (I.14)), for all L larger than 2 times the diameter of the support of v. Moreover, the subgroup of $\mathbb{Z}^n$ generated by the support of v is $\mathbb{Z}^n$,*

$$\mathrm{Gr}\Big(\mathrm{supp}\{v\} \Big) \ = \ \mathbb{Z}^n \, , \tag{I.36}$$

i.e., the smallest nontrivial subgroup of $\mathbb{Z}^n$, which contains $\mathrm{supp}\{v\}$, *is $\mathbb{Z}^n$ itself.*

We list two important consequences of Hypotheses 5 and 6. First, they imply that there exists a set of linearly independent vectors $\{k_1, \ldots, k_n\} \subseteq \mathbb{Z}^n$ and a constant $\delta > 0$, such that $v(k_\nu) \geq \delta$, for all $1 \leq \nu \leq n$. Secondly, Hypotheses 5 and 6 and a Perron-Frobenius argument imply that the lowest eigenvalue of $H''(0)$ is given by

$$0 \ < \ \lambda_{\min} \ = \ H''_{0,0}(0) \Big(1 - \sum_{k \in \mathbb{Z}^n} v(k) \Big) \, . \tag{I.37}$$

Moreover, this eigenvalue is nondegenerate, and the corresponding eigenvector has constant entries. Note further that, under Hypothesis 6 and the additional assumption that $\sum_{k \in \mathbb{Z}^n} v(k) < 1/2$, we can find some $\nu > 0$ such that Hypothesis 4[νd] is satisfied.

Using v, we define a function $F_v : \mathbb{R}^n \to [0, +\infty)$ by the following finite sum,

$$F_v(\eta) \ := \ \sum_{k \in \mathbb{Z}^n} e^{\eta \cdot k} \, v(k) \, , \tag{I.38}$$

for all $\eta \in \mathbb{R}^n$. We point out that $F_v(\eta) = \hat{v}(i\eta)$, where $\hat{v}$ is the Fourier transform of v. Moreover, under Hypothesis 2, Eqns. (I.37) and (I.38) imply that $1 - F_v(0) > 0$. Next, for $r > 1 - F_v(0)$, we introduce the open level sets and their boundaries

$$D_v(r) \ := \ \big\{ \eta \in \mathbb{R}^n \ \big| \ F_v(\eta) \ < \ r \, \big\} \, , \quad \Sigma_v(r) := \partial D_v(r) \, , \tag{I.39}$$

and by means of these we define the support function $S_r^v : \mathbb{R}^n \to [0, +\infty)$ as

$$S_r^v(x) \ := \ \sup\big\{ \eta \cdot x \ \big| \ \eta \in D_v(r) \, \big\} \, . \tag{I.40}$$

Finally, we need to make use of the following closed subset of $\mathbb{R}^n$,

$$\mathcal{A}_r \ := \ \big\{ x \in \mathbb{R}^n \ \big| \ S_r^v(x) \ = \ \min_{q \in \mathbb{Z}^n} \{ S_r^v(x + qL) \} \, \big\} \, , \tag{I.41}$$

implicitly using that the minimum is attained. The definitions of F_v, $D_v(1)$, $\Sigma_v(1)$, and S_1^v, in this context, go back to [14] (although no finite range condition (see Hypothesis 6) is imposed there), and most of the properties collected in Lemma I.7 below can already be found there. We give a proof of Lemma I.7 at the end of Section VI.

Lemma I.7. *Assume Hypotheses 5 and 6.*
Let $0 < \delta_0 < \min\{F_v(0)\,,\, 1 - F_v(0)\}$. *Then*

- **(i)** *the function* F_v *is strictly convex, and there exist constants* $C, C' \geq 0$ *such that, for any* $\varepsilon > 0$ *and any* $\eta \in \mathbb{R}^n$,

$$F_v(\eta) + C|\eta|^2\varepsilon \;\leq\; F_v\big((1+\varepsilon)\eta\big) \;\leq\; e^{C'|\eta|\varepsilon}\, F_v(\eta) \;\; ; \tag{I.42}$$

- **(ii)** *for every* $1 - \delta_0 \leq r \leq 1 + \delta_0$, $D_v(r)$ *is a strictly convex, bounded, open set with smooth boundary* $\Sigma_v(r) := \partial D_v(r)$. *More specifically,* $r \mapsto D_v(r)$ *is monotonically increasing, and there exist two constants* $R_1, R_2 > 0$ *such that*

$$B(R_1, 0) \;\subseteq\; D_v(r) \;\subseteq\; B(R_2, 0) \;\; ; \tag{I.43}$$

- **(iii)** *the support function* $S_r^v : \mathbb{R}^n \to R_{+0}$ *defines a norm on* $\mathbb{R}^n$, *for each* $1 - \delta_0 \leq r \leq 1 + \delta_0$. *Furthermore,* $S_r^v(x) = \eta_v(x) \cdot x$, *where* $\eta_v(x) \in \Sigma_v(r)$ *is uniquely determined by* $\nabla_\eta F_v(\eta_v(x)) = \mu x$, *for some* $\mu > 0$, *and we have* $\nabla_x S_r^v(x) = \eta_v(x)$. *Moreover, there exist constants* $C, C' \geq 0$ *such that, for any* $0 < \varepsilon < 1$,

$$(1+\varepsilon) S^v_{(1-C\varepsilon)r} \;\leq\; S^v_r \;\leq\; (1+\varepsilon) S^v_{r\exp(-C'\varepsilon)} \;\; ; \tag{I.44}$$

- **(iv)** *the set* $\mathcal{A}_r$ *is star-shaped. There exist two constants* $R_1', R_2' > 0$ *such that*

$$B(R_1' L, 0) \;\subseteq\; \mathcal{A}_1 \;\subseteq\; B(R_2' L, 0)\,. \tag{I.45}$$

  Moreover, there is a fundamental domain $(\mathcal{A}_r)^\circ \subset \widetilde{\mathcal{A}}_r \subseteq \mathcal{A}_r$ *for the canonical projection* $\pi : \mathbb{R}^n \to (\mathbb{R}/L\mathbb{Z})^n$. *That is,* $\mathbb{R}^n = \bigcup_{q \in \mathbb{Z}^n} \widetilde{\mathcal{A}}_r + qL$, *and* $\widetilde{\mathcal{A}}_r + qL \cap \widetilde{\mathcal{A}}_r + q'L = \emptyset$, *for* $q \neq q'$, $q, q' \in \mathbb{Z}^n$.

In [14], these definitions are used to prove the following asymptotics for the inverse of the Hessian $H''(0)$.

Lemma I.8. *Assume Hypotheses 2, 5, and 6. Then there exists a constant* $0 < \delta \leq 1/2$ *such that, for* $j \in \Lambda_L$ *with* $1 \ll d(j) = |J(j)| \leq \delta L$,

$$\Big(\big(H''(0)\big)^{-1}\Big)_{j,0} = \tag{I.46}$$

$$\frac{1 + \mathcal{O}\big(1/|J(j)|\big)}{H''_{0,0}(0)\big(2\pi\,|J(j)|\big)^{\frac{d-1}{2}}}\;\frac{\Big|\partial_\parallel F_v\big(\eta_v(J(j))\big)\Big|^{\frac{d-3}{2}}}{\Big(\det\big[\partial_\perp^2 F_v\big(\eta_v(J(j))\big)\big]\Big)^{1/2}}\;\exp\Big[-S_1^v(J(j))\Big]\,,$$

where $\big|\mathcal{O}\big(1/d(k)\big)\big| \leq C/d(k)$*, for some constant* $C \geq 0$ *which is uniform in* $L \to \infty$ *and* $\beta \to \infty$.

Here $\partial_{\|}$ *(resp.* $\partial_{\perp}$*) represents the derivative along the direction of* $J(j)$ *(resp. along the directions orthogonal to* $J(j)$*).*

The next theorem quantifies the relation (I.33), as it asserts a formula for the low-temperature asymptotics of the two-point correlation function $\mathbb{E}^T_{L,\beta}(x_j\,;x_k)$ very similar to (I.46).

Theorem I.9. *Assume Hypotheses 1, 2, 5, 6, and Hypothesis 3*$[S^v_r]$*, for some* $1 < r < 2 - F_v(0)$*. Denote by* $J_{\mathcal{A}} : \Lambda_L \to \widetilde{\mathcal{A}}_1$ *the natural bijection given by* $J^{-1}_{\mathcal{A}} = \pi|_{\widetilde{\mathcal{A}}_1}$ *(see Lemma I.7(iv)) and fix some* $0 < \lambda < 1$*. Then, for* β *sufficiently large, for* $j \in \Lambda_L$ *such that* $d(j)$ *is sufficiently large and that* $J_{\mathcal{A}}(j) \in \lambda\widetilde{\mathcal{A}}_1$*, we have*

$$\mathbb{E}^T_{L,\beta}(x_j\,;x_0) \;=\; \frac{1+\mathcal{O}\big(1/|J_{\mathcal{A}}(j)|\big)}{H''_{0,0}(0)\big(2\pi\,|J_{\mathcal{A}}(j)|\big)^{\frac{d-1}{2}}}\;\frac{\Big|\partial_{\|}F_w\big(\eta_w(J_{\mathcal{A}}(j))\big)\Big|^{\frac{d-3}{2}}}{\Big(\det\big[\partial^2_{\perp}F_w\big(\eta_w(J_{\mathcal{A}}(j))\big)\big]\Big)^{1/2}}\;\exp\Big[-S^w_1(J_{\mathcal{A}}(j))\Big]\,, \tag{I.47}$$

where the functions F_w*,* η_w*, and* S^w_1 *are defined below in Sections VI–VII and fulfill* $\partial^{\alpha}_{\eta}F_w(\eta_w(k)) = \partial^{\alpha}_{\eta}F_v(\eta_v(k)) + \mathcal{O}(\beta^{-1/2})$ *and* $S^w_1 = S^v_1 + \mathcal{O}(\beta^{-1/2})$*, so that we furthermore have*

$$\mathbb{E}^T_{L,\beta}(x_j\,;x_0) \;=\; \frac{1+\mathcal{O}\big(|J_{\mathcal{A}}(j)|^{-1}\big)+\mathcal{O}\big(\beta^{-1/2}\big)}{H''_{0,0}(0)\big(2\pi\,|J_{\mathcal{A}}(j)|\big)^{\frac{d-1}{2}}} \cdot\frac{\Big|\nabla_{\eta}F_v\big(\eta_v(J_{\mathcal{A}}(j))\big)\Big|^{\frac{d-3}{2}}}{\Big(\det\big[\partial^2_{\perp}F_v\big(\eta_v(J_{\mathcal{A}}(j))\big)\big]\Big)^{1/2}}\;\exp\Big[-\big(1+\mathcal{O}(\beta^{-1/2})\big)S^v_1(J_{\mathcal{A}}(j))\Big]\,, \tag{I.48}$$

where the $\mathcal{O}$*-symbols in (I.48) are uniform in* $L \to \infty$ *and* $\beta \to \infty$.

Comparing (I.48) to (I.46), we finally notice that the decay rate of the two-point correlation function $\mathbb{E}^T_{L,\beta}(x_k\,;x_0)$ agrees with the decay rate of the resolvent matrix elements of the Hessian $H''(0)$, given by the support function S^v_1 of $D_v(1)$, modulo a factor of $1+\mathcal{O}(\beta^{-1/2})$.

Furthermore, we note that in Theorem I.9 it is natural to use the bijection provided by $J_{\mathcal{A}} : \Lambda_L \to \widetilde{\mathcal{A}}_1$ rather than J. Using $J_{\mathcal{A}}$ instead of J amounts to projecting onto the fundamental domain $\widetilde{\mathcal{A}}_1$ rather than $\Lambda^{(box)}_L$. The difference between these two maps becomes important when studying the two-point function $\mathbb{E}^T_{L,\beta}(x_j, x_0)$ for $J(j)$ which are close to the boundary of $\Lambda^{(box)}_L$.

We conclude this introduction with a brief survey on the organization of the following sections. In the next section, we introduce a deformed Dirac operator

D_H, by means of which we rederive Lemma I.3. In Section III, we prove our first main result, the operator inequalities (I.28) in Theorem I.4. They derive from a form bound proven in slightly greater generality, as to accommodate for the case of distorted operators, which we have to deal with in Sections VI–VII when deriving the asymptotics of the two-point function. In Section IV we give a short proof of exponential decay of the two-point function under a least hypothesis. Section VI is devoted to the analysis of the support function S_r^w which determines the precise rate of exponential decay of the two-point function. The estimates derived in Section VI are then used in Section VII to prove the asymptotics as claimed in Lemma I.8 and Theorem I.9.

Finally, this paper has three appendices. Appendix A contains the verification of the admissibility of Example I.1[ν], asserted in Lemma I.2. In Appendix B, the self-adjointness and other basic spectral properties of the Dirac operator D_H are proven, and Appendix C contains an example of a non-ferromagnetic (i.e., non-positivity preserving) Hessian of H at $x = 0$, for which yet the decay of the matrix elements of the resolvent can be made precise.

II Dirac operator and Witten Laplacian

In the present section, **we prove Lemma I.3**, i.e., we show that, for any two observables $u, v \in \mathcal{O}^{(1)}$, the following relation (I.27) holds

$$\mathbb{E}^T_{L,\beta}(u\,; v) \;=\; \frac{1}{\beta^2}\big\langle e^{-\beta H}\nabla u \,\big|\, (\Delta_H^{(1)})^{-1}\, e^{-\beta H}\nabla v\big\rangle_{\mathcal{H}^{(1)}}\,, \tag{II.1}$$

where $\Delta_H^{(1)}$ was defined in (I.22).

Before turning to the proof, we introduce some more notation. The fermion Fock space $\mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]$ over $\mathbb{C}^{|\Lambda_L|}$ is defined to be the orthogonal sum of the N-fermion sectors $\mathcal{F}_f^{(N)}$, for $N \in \{0, 1, \dots, |\Lambda_L|\}$,

$$\mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}] \;=\; \bigoplus_{N=0}^{|\Lambda_L|} \mathcal{F}_f^{(N)}\,. \tag{II.2}$$

For each $N \in \{0, 1, \dots, |\Lambda_L|\}$, an orthonormal basis in $\mathcal{F}_f^{(N)}$ is given by

$$\big\{\, c^*_{j_1} c^*_{j_2} \cdots c^*_{j_N}\,\Omega \;\big|\; j_1, j_2, \dots, j_N \in \Lambda_L,\ j_1 < j_2 < \dots < j_N \,\big\}\,, \tag{II.3}$$

where "<" is the order on Λ_L induced by some bijection $\Lambda_L \to \{1, \dots, |\Lambda_L|\}$. Here c_j^* and c_j are the standard fermion creation and annihilation operators on $\mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]$ obeying the canonical anticommutation relations (CAR),

$$\big\{\, c_i\,,\, c_j^* \,\big\} \;=\; \delta_{i,j}\,, \qquad \big\{\, c_i\,,\, c_j \,\big\} \;=\; \big\{\, c_i^*\,,\, c_j^* \,\big\} \;=\; 0\,, \tag{II.4}$$

where $\{A, B\} := AB + BA$, and Ω is the unique (up to a phase) normalized vector in $\mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]$ such that $c_j\Omega = 0$, for all $j \in \Lambda_L$. Note that $\mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]$ is nothing but the space of antisymmetric N-forms over $\mathbb{R}^{|\Lambda_L|}$, frequently denoted $\bigwedge^{(N)}(\mathbb{R}^{|\Lambda_L|})$ (see, e.g., [4]). In particular, $\mathcal{F}_f^{(0)} = \mathbb{C}\cdot\Omega$, and $\mathcal{F}_f^{(1)}$ may be naturally identified with $\mathbb{C}^{|\Lambda_L|}$. We may therefore identify $L^2(\mathbb{R}^{|\Lambda_L|})$ with $\mathcal{H}^{(0)}$ and $L^2(\mathbb{R}^{|\Lambda_L|};\mathbb{C}^{|\Lambda_L|})$ with $\mathcal{H}^{(1)}$, where

$$\mathcal{H}^{(N)} \;:=\; L^2(\mathbb{R}^{|\Lambda_L|})\otimes\mathcal{F}_f^{(N)} \tag{II.5}$$

is the space of square-integrable N-forms, by means of the natural identification maps

$$\mathcal{I}^{(0)}:\qquad L^2(\mathbb{R}^{|\Lambda_L|}) \;\to\; \mathcal{H}^{(0)}\,,\qquad \psi \;\mapsto\; \psi\otimes\Omega\,, \tag{II.6}$$

$$\mathcal{I}^{(1)}:\quad L^2(\mathbb{R}^{|\Lambda_L|};\mathbb{C}^{|\Lambda_L|}) \;\to\; \mathcal{H}^{(0)}\,,\quad (\psi_j)_{j\in\Lambda_L} \;\mapsto\; \sum_{j\in\Lambda_L}\psi_j\otimes c_j^*\Omega\,. \tag{II.7}$$

Equipped with this notation, we return to the proof of Lemma I.3. We introduce the rank-one projection $P_0 = |e^{-\beta H}\rangle\langle e^{-\beta H}| = P_0^2 = P_0^*$ and $\overline{P}_0 = 1 - P_0$, and we observe that

$$\begin{aligned}\mathbb{E}^T_{L,\beta}(u\,;v) \;&=\; \big\langle e^{-\beta H}u\,\big|\,e^{-\beta H}v\big\rangle_{L^2} \;-\; \big\langle e^{-\beta H}u\,\big|\,e^{-\beta H}\big\rangle_{L^2}\big\langle e^{-\beta H}\,\big|\,e^{-\beta H}v\big\rangle_{L^2}\\ &=\; \big\langle e^{-\beta H}u\,\big|\,\overline{P}_0\,e^{-\beta H}v\big\rangle_{L^2}\,,\end{aligned} \tag{II.8}$$

where L^2 is a shorthand notation for $L^2(\mathbb{R}^{|\Lambda_L|})$. Using the isomorphism $\mathcal{I}^{(0)}$ defined in (II.6) and $\overline{P_0\otimes P_\Omega} := \mathbf{1} - P_0\otimes P_\Omega$, we find that

$$\mathbb{E}^T_{L,\beta}(u\,;v) \;=\; \big\langle e^{-\beta H}u\otimes\Omega\,\big|\,\big(\overline{P_0\otimes P_\Omega}\big)\,e^{-\beta H}v\otimes\Omega\big\rangle_{\mathcal{H}^{(0)}}\,. \tag{II.9}$$

Next, we use $Z_j(H) := \beta^{-1}\partial_j + H_j'(x)$ and $Z_j(H)^* = -\beta^{-1}\partial_j + H_j'(x)$, as defined in (I.25)–(I.26), to introduce a deformed exterior differential d_H and its adjoint d_H^* on $\mathcal{H}$ by

$$d_H \;:=\; \sum_{j\in\Lambda_L} Z_j(H)\otimes c_j^*\;\;,\qquad d_H^* \;:=\; \sum_{j\in\Lambda_L} Z_j(H)^*\otimes c_j\,. \tag{II.10}$$

Adding these two, we obtain the Dirac operator,

$$D_H \;:=\; d_H \;+\; d_H^* \;=\; \sum_{j\in\Lambda_L} Z_j(H)\otimes c_j^* \;+\; Z_j(H)^*\otimes c_j\,. \tag{II.11}$$

Squaring D_H yields the Witten Laplacian Δ_H, and using $d_H^2 = (d_H^*)^2 = 0$ and the CAR (II.4), one easily sees that

$$\Delta_H \;:=\; D_H^2 \;=\; \Big(\sum_{j\in\Lambda_L} Z_j(H)^*Z_j(H)\Big)\otimes\mathbf{1} \;+\; \frac{2}{\beta}\sum_{i,j\in\Lambda_L} H_{i,j}''(x)\otimes c_i^*c_j\,. \tag{II.12}$$

In Lemma B.1 in the Appendix, we prove that D_H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^{|\Lambda_L|})\otimes\mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]$. Moreover, we show in Lemma B.3 that

$$\mathrm{Ker}\{D_H\} \;=\; \mathrm{Ker}\{\Delta_H\} \;=\; \mathbb{C}\cdot\big(e^{-\beta H}\otimes\Omega\big) \;=\; \mathrm{Ran}\{P_0\otimes P_\Omega\}\,. \tag{II.13}$$

We observe that Δ_H leaves the N-fermion sectors invariant, $\Delta_H : \mathcal{H}^{(N)} \to \mathcal{H}^{(N)}$, and we denote the restriction of Δ_H to $\mathcal{H}^{(N)}$ by $\widetilde{\Delta}_H^{(N)}$. Recalling the definitions (I.22) and (I.24) of $\Delta_H^{(1)}$ and $\Delta_H^{(0)}$, we find that

$$\widetilde{\Delta}_H^{(0)} \;=\; \mathcal{I}^{(0)} \circ \Delta_H^{(0)} \circ (\mathcal{I}^{(0)})^{-1} \,, \qquad \widetilde{\Delta}_H^{(1)} \;=\; \mathcal{I}^{(1)} \circ \Delta_H^{(1)} \circ (\mathcal{I}^{(1)})^{-1} \,, \tag{II.14}$$

and henceforth we do not distinguish between $\widetilde{\Delta}_H^{(0)}$, $\widetilde{\Delta}_H^{(1)}$ and $\Delta_H^{(0)}$, $\Delta_H^{(1)}$ anymore.

Equation (II.13) implies that D_H is invertible on $\mathrm{Ran}\{\overline{P_0 \otimes P_\Omega}\}$. Thus we have $\mathbf{1} = D_H(\Delta_H)^{-1} D_H$ on $\mathrm{Ran}\{\overline{P_0 \otimes P_\Omega}\}$, and using this identity in Eqn. (II.9), we obtain

$$\mathbb{E}^T_{L,\beta}(u\,;v) \;=\; \big\langle D_H \left(\overline{P_0 \otimes P_\Omega}\right) e^{-\beta H} u \otimes \Omega \,\big|\, \left(\Delta_H\right)^{-1} D_H \left(\overline{P_0 \otimes P_\Omega}\right) e^{-\beta H} v \otimes \Omega \big\rangle_{\mathcal{H}} \,. \tag{II.15}$$

Furthermore, an easy computation shows that

$$D_H \left(\overline{P_0 \otimes P_\Omega}\right) e^{-\beta H} u \otimes \Omega \;=\; \frac{1}{\beta} \sum_{j \in \Lambda_L} e^{-\beta H} \partial_j u \otimes c_j^* \Omega \;\in\; \mathcal{H}^{(1)} \,. \tag{II.16}$$

We may hence restrict Δ_H to $\mathcal{H}^{(1)}$ in (II.15), and inserting (II.16), we arrive at

$$\mathbb{E}^T_{L,\beta}(u\,;v) \;=\; \frac{1}{\beta^2} \Big\langle \sum_{j \in \Lambda_L} e^{-\beta H} \partial_j u \otimes c_j^* \Omega \;\Big|\; \big(\Delta_H^{(1)}\big)^{-1} \sum_{j \in \Lambda_L} e^{-\beta H} \partial_j v \otimes c_j^* \Omega \Big\rangle_{\mathcal{H}^{(1)}} \,, \tag{II.17}$$

which is equivalent to (I.27). □

III Summable weights

In this section, we show that the operators

$$\Delta_H \;:=\; \Delta_H^{(0)} \otimes \mathbf{1} \;+\; \frac{2}{\beta} \sum_{i,j \in \Lambda_L} H''_{i,j}(x) \otimes c_i^* c_j \quad \text{and} \tag{III.1}$$

$$A \;:=\; \Delta_H^{(0)} \otimes \mathbf{1} \;+\; \frac{2}{\beta} \sum_{i,j \in \Lambda_L} H''_{i,j}(0) \otimes c_i^* c_j \tag{III.2}$$

are close to each other, even after deformation by suitable weights. This will be important in Sections IV, VI, and Appendix B. In particular, the main result of this section, Theorem III.1 below, implies Theorem I.4 by restricting the asserted bound to the invariant subspace $\mathcal{H}^{(1)} \subseteq \mathcal{H}$.

Theorem III.1. *Assume Hypotheses 1, 2, and 3*[0]*. Then there exist constants $C \geq 0$ and $\beta_0 \geq 0$, both independent of L, such that, for all $\beta \geq \beta_0$,*

$$\Big(1 - \frac{C}{\beta^{1/2}}\Big) A \;\leq\; \Delta_H \;\leq\; \Big(1 + \frac{C}{\beta^{1/2}}\Big) A \tag{III.3}$$

holds in the sense of quadratic forms on $\mathcal{Q} \subseteq \mathcal{H}$, the form domain of A and Δ_H.

Theorem III.1 is a simple consequence of Theorem III.3 below, in which we derive a bound on the difference between Δ_H and A relative to A. We observe that this difference, $\Delta_H - A$, is given by $2\beta^{-1}W$, where W is the operator of multiplication by the x-dependent matrix

$$W \;=\; \frac{\beta}{2}\big(\Delta_H^{(1)} - A_0\big) \;=\; \sum_{i,j\in\Lambda_L} \big\{H''_{i,j}(x) - H''_{i,j}(0)\big\} \otimes c_i^* c_j \,. \tag{III.4}$$

Theorem III.3 below gives an estimate on W conjugated by a diagonal x-independent matrix of weights, and Theorem III.1 follows in the special case where the weights are all equal to 1.

As a preparation, we develop some estimates of independent interest. Recall by (I.24) that, for all $j \in \Lambda_L$,

$$Z_j(H)^*\, Z_j(H) \;=\; -\frac{1}{\beta^2}\frac{\partial^2}{\partial x_j^2} + \big|H'_j(x)\big|^2 - \frac{1}{\beta}H''_{j,j}(x) \;\geq\; \big|H'_j(x)\big|^2 - \frac{1}{\beta}H''_{j,j}(x). \tag{III.5}$$

We improve this inequality in the following lemma.

Lemma III.2. *Under Hypotheses 1, 2, and 3[0], for any $\beta_0 > 3C_{(H3)}(0)$ and $\beta \geq \beta_0$, we can find a β-dependent matrix $(C_{j,k}(\beta))_{j,k\in\Lambda_L}$ with nonnegative entries, satisfying*

$$\max_{j\in\Lambda_L}\Big\{\sum_{k\in\Lambda_L} C_{j,k}(\beta)\Big\} \;+\; \max_{k\in\Lambda_L}\Big\{\sum_{j\in\Lambda_L} C_{j,k}(\beta)\Big\} \;\leq\; 3C_{(H3)}(0)\left(1 - \frac{3C_{(H3)}(0)}{\beta}\right)^{-1}, \tag{III.6}$$

uniformly in L, such that, for $\beta \geq \beta_0$, $j \in \Lambda_L$, and $x \in \mathbb{R}^{|\Lambda_L|}$,

$$\big|H'_j(x)\big|^2 \;\leq\; \sum_{k\in\Lambda_L} \big(\delta_{jk} \,+\, \beta^{-1}C_{j,k}(\beta)\big)\big(Z_k(H)^*\, Z_k(H) \,+\, C\beta^{-1}\big)\,, \tag{III.7}$$

where $C := \lambda_{\max} + C_{(H3)}(0)/2$.

Proof. By means of (III.5) and (I.15), we obtain the estimate

$$\begin{aligned} \big|H'_j(x)\big|^2 \;&\leq\; \big|H'_j(x)\big|^2 - \frac{1}{\beta}H''_{j,j}(x) + \frac{1}{\beta}H''_{j,j}(0) + \frac{1}{\beta}\big|H''_{j,j}(x) - H''_{j,j}(0)\big| \\ &\leq\; Z_j(H)^*\, Z_j(H) \,+\, \frac{1}{\beta}H''_{j,j}(0) \\ &\quad + \frac{1}{\beta}\sum_{k\in\Lambda_L} a_{jj}(k)\big|H'_k(x)\big| \,+\, \frac{1}{\beta}\sum_{k\in\Lambda_L} b_{jj}(k)\big|H'_k(x)\big|^2 \,. \end{aligned} \tag{III.8}$$

Next, using

$$\big|H'_k(x)\big| \;\leq\; \frac{1}{2\epsilon}\,\big|H'_k(x)\big|^2 \,+\frac{\epsilon}{2}, \quad 0 < \epsilon < \infty\,, \tag{III.9}$$

with $\epsilon = 1$, and thanks to Hypothesis 2, we get

$$\begin{aligned} \left|H_j'(x)\right|^2 \;\; &\leq \;\; Z_j(H)^* \, Z_j(H) \;+\; \frac{1}{\beta}\lambda_{\max} \;+\; \frac{1}{2\beta} \sum_{k\in\Lambda_L} a_{jj}(k) \\ &\qquad + \frac{1}{\beta} \sum_{k\in\Lambda_L} \big(\frac{1}{2} a_{jj}(k) \;+\; b_{jj}(k)\big) \left|H_j'(x)\right|^2 , \end{aligned} \tag{III.10}$$

which is equivalent to

$$\sum_{k\in\Lambda_L} \big(\delta_{j,k} - M_{j,k}\big) \left|H_j'(x)\right|^2 \;\leq\; Z_j(H)^* \, Z_j(H) \;+\; \frac{1}{\beta}\lambda_{\max} \;+\; \frac{1}{2\beta} \sum_{k\in\Lambda_L} a_{jj}(k) \, , \tag{III.11}$$

where $M_{j,k} := a_{jj}(k)/2 \,+\, b_{jj}(k)$. In view of Hypothesis 3[0], $\sum_{k\in\Lambda_L} a_{jj}(k)$ is bounded by $C_{(H3)}(0)$ and

$$\max_{j\in\Lambda_L} \Big\{ \sum_{k\in\Lambda_L} M_{j,k} \Big\} \;+\; \max_{k\in\Lambda_L} \Big\{ \sum_{j\in\Lambda_L} M_{j,k} \Big\} \;\leq\; 3\, C_{(H3)}(0) \, , \tag{III.12}$$

so that the $\mathcal{L}(l^p; l^p)$-norm of the matrix $M \equiv (M_{j,k})_{j,k\in\Lambda_L}$ is bounded by $3C_{(H3)}(0)$, uniformly in L and $p \in [1;\infty]$. Then, for $\beta_0 > 3C_{(H3)}(0)$ and $\beta \geq \beta_0$, setting $\mathbf{1} := (\delta_{j,k})_{j,k\in\Lambda_L}$,

$$\Big(\big(\mathbf{1} \;-\; \beta^{-1} M\big)^{-1} \Big)_{j,k} \;=\; \delta_{j,k} \;+\; \beta^{-1}\, C_{j,k}(\beta) \, , \tag{III.13}$$

where the matrix $C(\beta) := (C_{j,k}(\beta))_{j,k\in\Lambda_L}$ satisfies (III.6). Multiplying (III.11) by the nonnegative numbers $\delta_{jk} + \beta^{-1} C_{j,k}(\beta)$ and summing over $j \in \Lambda_L$, we arrive at (III.7), with $C = \lambda_{\max} + C_{(H3)}(0)/2$. □

Theorem III.3. *Let S be a semi-norm on $\mathbb{R}^d$ and assume Hypothesis 3[d_S] in addition to Hypotheses 1 and 2, where d_S is the corresponding semi-metric on Λ_L. For weights $\theta : \Lambda_L \longrightarrow \mathbb{R}$ satisfying*

$$\forall j,k \in \Lambda_L : \quad |\theta(j) - \theta(k)| \;\leq\; d_S(j-k) \, , \tag{III.14}$$

we denote

$$W(\theta) \;:=\; W(x,\theta) \;:=\; \sum_{i,j\in\Lambda_L} e^{\theta(i)-\theta(j)} \left\{ H_{i,j}''(x) - H_{i,j}''(0) \right\} \otimes c_i^* c_j \, . \tag{III.15}$$

Then, there exist $C, \beta_0 > 0$, such that, uniformly in L, θ, and $\beta \geq \beta_0$, for all $u, v \in C_0^\infty(\mathbb{R}^{|\Lambda_L|}; \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}])$,

$$\left| \langle u |\, W(\theta)\, v \rangle_{\mathcal{H}} \right| \;\leq\; C\, \beta^{1/2} \left\| (A^{(1)})^{1/2} u \right\|_{\mathcal{H}} \left\| (A^{(1)})^{1/2} v \right\|_{\mathcal{H}} , \tag{III.16}$$

where $\mathcal{H} = L^2(\mathbb{R}^{|\Lambda_L|}; \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}])$.

We remark that on $\mathcal{H}^{(1)}$, $W(\theta) := e^{\theta} W e^{-\theta}$ is the conjugation of W by the diagonal matrix $e^{\theta} := (\delta_{jk} e^{\theta(j)})_{j,k\in\Lambda_L}$ (i.e., $e^{-\theta} := (e^{\theta})^{-1}$).

Proof of Theorem III.3. First, we control the perturbation $W(\theta)$ by a diagonal perturbation. To this end, we fix $x \in \mathbb{R}^{|\Lambda_L|}$, denote $X_{i,j} = X_{j,i} := |H''_{i,j}(x) - H''_{i,j}(0)|$, $\mathcal{F} := \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]$, and we observe that, pointwise in $x \in \mathbb{R}^{|\Lambda_L|}$,

$$\begin{aligned} \big|\langle u(x)|\, W(x,\theta)\, v(x)\rangle_{\mathcal{F}}\big| \;&\leq\; \sum_{i,j\in\Lambda_L} e^{\theta(i)-\theta(j)}\, X_{i,j}\, \|c_i\, u(x)\|_{\mathcal{F}}\, \|c_j v(x)\|_{\mathcal{F}} \\ &\leq\; \langle u(x)|\, \widetilde{W}(x,\theta)\, u(x)\rangle_{\mathcal{F}}^{1/2}\, \langle v(x)|\, \widetilde{W}(x,\theta)\, v(x)\rangle_{\mathcal{F}}^{1/2} , \end{aligned} \tag{III.17}$$

where we use the Cauchy-Schwartz inequality, Eqn. (III.14), and

$$\widetilde{W}(x,\theta) \;:=\; \sum_{i\in\Lambda_L}\Big(\sum_{j\in\Lambda_L} \exp[d_S(i-j)]\, X_{i,j}\Big) \otimes c_i^* c_i \;. \tag{III.18}$$

Due to Hypothesis 3$[d_S]$, we have $X_{i,j} \leq \sum_k \big\{a_{ij}(k)|H'_k(x)| + b_{ij}(k)|H'_k(x)|^2$ and thus, for any $0 < \varepsilon \leq 1/2$,

$$\begin{aligned} &\widetilde{W}(x,\theta) \;\leq \\ &\sum_{i\in\Lambda_L}\Big(\sum_{j\in\Lambda_L} e^{d_S(i-j)}\Big\{\big(\frac{1}{2\varepsilon}a_{ij}(k) + b_{ij}(k)\big)\,|H'_k(x)|^2 \;+\; \frac{\varepsilon}{2}a_{ij}(k)\Big\}\Big) \otimes c_i^* c_i \;. \end{aligned} \tag{III.19}$$

Next, we pass from $\mathcal{F}$ to $\mathcal{H} = L^2(\mathbb{R}^{|\Lambda_L|}) \otimes \mathcal{F}$. We abbreviate $Q_{ij} := \delta_{ij} + \beta^{-1} C(\beta)_{i,j}$, and we insert (III.7) and $0 \leq c_i^* c_i \leq 1$ into (III.19). This yields

$$\begin{aligned} \widetilde{W}(\theta) \;&\leq\; \sum_{i\in\Lambda_L}\Big(\sum_{j,k,l\in\Lambda_L} e^{d_S(i-j)}\Big\{\frac{Q_{kl}}{2\varepsilon}\,\big(a_{ij}(k) + b_{ij}(k)\big)\,\big(Z_l(H)^* Z_l(H) + C\beta^{-1}\big) \\ &\qquad\qquad + \frac{\varepsilon}{2}\,\delta_{kl}\, a_{ij}(k)\Big\}\Big) \otimes c_i^* c_i \\ &\leq\; \frac{1}{2\varepsilon}\sum_{l\in\Lambda_L}\Big(\sum_{i,j,k\in\Lambda_L} e^{d_S(i-j)}\, Q_{kl}\,\big(a_{ij}(k) + b_{ij}(k)\big)\Big)\, Z_l(H)^* Z_l(H) \otimes \mathbf{1} \\ &\quad + \Big(\frac{\varepsilon}{2} + \frac{C}{\beta\,\varepsilon}\Big)\sum_{i\in\Lambda_L}\Big(\sum_{j,k,l\in\Lambda_L} e^{d_S(i-j)}\, Q_{kl}\,\big(a_{ij}(k) + b_{ij}(k)\big)\Big) \otimes c_i^* c_i \;. \end{aligned} \tag{III.20}$$

Then (III.6), (I.16), and the choice $\varepsilon := \beta^{-1/2}$ give

$$\widetilde{W}(\theta) \;\leq\; C'\,\beta^{1/2}\big(\Delta_H^{(0)} \otimes \mathbf{1} \;+\; \beta^{-1}\,\mathbf{1} \otimes \mathbb{N}_L\big) , \tag{III.21}$$

where $C' := (1 + C_{(H3)}(0) + \lambda_{\max})\, C_{(H3)}(d_S)\,\big(1 + C_{(H3)}(0)(\beta - C_{(H3)}(0))^{-1}\big)$, and $\mathbb{N}_L := \sum_{i\in\Lambda_L} c_i^* c_i$ is the number operator. On the other hand,

$$A \;=\; \Delta_H^{(0)} \otimes \mathbf{1} \;+\; \frac{2}{\beta}\sum_{i,j\in\Lambda_L} H''_{i,j}(0) \otimes c_i^* c_j \;\geq\; \Delta_H^{(0)} \otimes \mathbf{1} \;+\; \frac{2\,\lambda_{\min}}{\beta}\mathbf{1} \otimes \mathbb{N}_L \,, \tag{III.22}$$

and we arrive at the claim. □

IV Exponential decay of the two-point function

In this section, we give a direct proof of *exponential decay* of the two-point function $\mathbb{E}^T_{L,\beta}(x_j\,;\,x_k)$ under least requirements. For the derivation of its asymptotics in Section VII, however, our assumptions will be somewhat stronger.

As in [8, 10, 14], we use a Combes-Thomas type argument to prove Theorem I.6, i.e., the estimate (I.34) for some $C > 0$,

$$\big|\,\mathbb{E}^T_{L,\beta}(x_j\,;\,x_k)\,\big| \;\le\; \frac{C}{\beta}\,\exp\Big(-\min\{\mu,\nu\}\,d(j-k)\Big)\,, \tag{IV.1}$$

for β large enough.

Proof of Theorem I.6. Starting from (I.31), we introduce diagonal weights of the form used in Section III. Using the notation introduced in Theorem III.3, we write

$$\mathbb{E}^T_{L,\beta}(x_j\,;x_k) \;=\; \frac{1}{\beta^2}\Big\langle e^{-\beta H}\otimes e^{-\theta}e_j\Big|\,\big(\Delta^{(1)}_H(\theta)\big)^{-1}\,e^{-\beta H}\otimes e^{\theta}e_k\Big\rangle_{\mathcal{H}^{(1)}}\,, \tag{IV.2}$$

where θ satisfies (III.14), for $d_S = \min(\mu;\nu)d$, and

$$\Delta^{(1)}_H(\theta) \;:=\; (\mathbf{1}\otimes e^{\theta})\Delta^{(1)}_H(\mathbf{1}\otimes e^{-\theta}) \;=\; \Delta^{(0)}_H\otimes\mathbf{1} \,+\, \frac{2}{\beta}\,(\mathbf{1}\otimes e^{\theta})H''(x)(\mathbf{1}\otimes e^{-\theta})\,, \tag{IV.3}$$

temporarily assuming its invertibility. Using (III.1)–(III.2) and the definition of $W(\theta)$ given in Theorem III.3, we can write

$$\Delta^{(1)}_H(\theta) \;=\; A^{(1)} \,+\, \frac{2}{\beta}\,W^{(1)}(\theta) \,+\, \frac{2}{\beta}\,\mathbf{1}\otimes B(\theta)\,. \tag{IV.4}$$

where $W^{(1)}(\theta)$ denotes the restriction of $W(\theta)$ to $\mathcal{H}^{(1)}$ and

$$B(\theta) \;:=\; e^{\theta}\,H''(0)\,e^{-\theta} \;-\; H''(0)\,. \tag{IV.5}$$

By Theorem III.3, we can bound $W^{(1)}(\theta)$ by (III.16). Furthermore,

$$\begin{aligned}\sup_{i\in\Lambda_L}\sum_{j\in\Lambda_L}\big|B(\theta)_{i,j}\big| \;&=\; \sup_{i\in\Lambda_L}\sum_{j\in\Lambda_L\setminus\{i\}}\Big|e^{\theta(j)-\theta(i)}-1\Big|\,\big|H''_{i,j}(0)\big| \\ &\le\; \sup_{i\in\Lambda_L}\sum_{j\in\Lambda_L\setminus\{i\}} e^{|\theta(j)-\theta(i)|}\,\big|H''_{i,j}(0)\big|\,,\end{aligned} \tag{IV.6}$$

and interchanging i and j, we convince ourselves that a similar estimate holds for $\sup_j\sum_i\big|B(\theta)_{i,j}\big|$. Since θ satisfies (III.14) for $d_S = \min(\mu;\nu)d$ and H satisfies Hypothesis 4[$\min(\mu;\nu)d$], we see that, setting $C := C_{(H4)}(\min(\mu;\nu)d)$,

$$\|B(\theta)\| \;\le\; \frac{\lambda_{\min}}{\lambda_{\max}}(1-C)\sup_{i\in\Lambda_L}\big|H''_{i,i}(0)\big| \;\le\; \lambda_{\min}(1-C)\,. \tag{IV.7}$$

Since $A^{(1)} \geq \frac{2}{\beta}\lambda_{\min}$, we obtain

$$\| \, (A^{(1)})^{-1/2} \, \frac{2}{\beta} \big(\mathbf{1} \otimes B(\theta) \big) \, (A^{(1)})^{-1/2} \, \| \;\leq\; (1 - C) \, , \tag{IV.8}$$

uniformly in L. Thus, for β_0 large enough and $\beta \geq \beta_0$, we can explicitly construct the resolvent of $\Delta_H^{(1)}(\theta)$ as

$$\big(\Delta_H^{(1)}(\theta) \big)^{-1} \;=\; \big(A^{(1)} \big)^{-1/2} \left(\sum_{n=0}^{\infty} Q(\theta)^n \right) \big(A^{(1)} \big)^{-1/2} \, , \tag{IV.9}$$

where

$$Q(\theta) \;=\; \frac{2}{\beta} \, \big(A^{(1)} \big)^{-1/2} \big\{ W(\theta) \;+\; \mathbf{1} \otimes B(\theta) \big\} \, \big(A^{(1)} \big)^{-1/2} \tag{IV.10}$$

is a bounded operator of norm less than $1 - C + \mathcal{O}(\beta^{-1/2})$. Hence, for $\beta_0 > 0$ sufficiently large, the operator norm of $\big(\Delta_H^{(1)}(\theta)\big)^{-1}$ is bounded by $\frac{2}{1+C} \| (A^{(1)})^{-1/2} \|^2$, which, in turn, is bounded by $\beta / \lambda_{\min}(1 + C)$. Inserting this bound in (IV.2), we obtain

$$\big| \, \mathbb{E}_{L,\beta}^{T}(x_j \, ; x_k) \, \big| \;\leq\; \frac{1}{\beta \lambda_{\min}(1 + C)} \, e^{\theta(k) - \theta(j)} \, , \tag{IV.11}$$

uniformly w.r.t. L and $\beta \geq \beta_0$, and for θ satisfying (III.14) with $d_S = \min(\mu; \nu) d$. Now we choose $\theta = \theta_k$ defined by

$$\forall l \in \Lambda_L, \quad \theta_k(l) \;=\; \min(\mu; \nu) \, d(l - k) \, . \tag{IV.12}$$

Thanks to the triangle inequality, θ_j satisfies (III.14) for $d_S = \min(\mu; \nu) d$, and (IV.11) for $\theta = \theta_k$ yields (IV.1), i.e., (I.13). □

V The Feshbach operator

In this section, we introduce a suitable Feshbach operator. The Feshbach map is a key tool in Sections VI and VII and some properties derived in this section and concerning this Feshbach operator are used in Section VI. Let

$$P \;:=\; | \, e^{-\beta H} \rangle \langle e^{-\beta H} \, | \otimes \mathbf{1} \;\; \text{and} \;\; \overline{P} \;=\; 1 - P \, . \tag{V.1}$$

Note that P is the orthogonal projection onto the ground state of $\Delta_H^{(0)} \otimes \mathbf{1}$. So we expect that

$$\overline{\Delta_H^{(1)}} \;:=\; \overline{P} \, \Delta_H^{(1)} \, \overline{P} \, , \tag{V.2}$$

restricted to $\mathrm{Ran}\overline{P}$, has a much larger spectral gap above zero than $2\lambda_{\min}/\beta$, which is the spectral gap of $\Delta_H^{(1)}$. Indeed, in the following proposition, we show that the spectral gap of $\overline{\Delta_H^{(1)}}$ is almost twice as big as the one of $\Delta_H^{(1)}$.

Proposition V.1. *Under Hypotheses 1,2, and 3*[0] *and for* β *large enough,*

$$\overline{\Delta_H^{(1)}} \;\geq\; \frac{4}{\beta}\,\lambda_{\min}\big(1-\mathcal{O}(\beta^{-1/2})\big)\,\overline{P}\,. \tag{V.3}$$

Proof. According to Lemma B.5 (see also [14, 12]), we have that

$$\overline{P}\,\big(\Delta_H^{(0)}\otimes\mathbf{1}\big)\,\overline{P} \;\geq\; \inf\sigma(\Delta_H^{(1)})\,\overline{P}\,, \tag{V.4}$$

where $\sigma(\Delta_H^{(1)})$ denote the spectrum of $\Delta_H^{(1)}$. Using this and (I.28), we therefore obtain

$$\begin{aligned}\overline{P}\,\Delta_H^{(1)}\,\overline{P} \;&\geq\; \big(1-\frac{C}{\beta^{1/2}}\big)\,\overline{P}\,A^{(1)}\,\overline{P} \\ &\geq\; \big(1-\frac{C}{\beta^{1/2}}\big)\,\big(\inf\sigma(\Delta_H^{(1)})\;+\;\frac{2}{\beta}\lambda_{\min}\big)\,\overline{P}\,,\end{aligned} \tag{V.5}$$

by (I.23) and Hypothesis 2. Using (I.28) again, this leads to

$$\overline{P}\,\Delta_H^{(1)}\,\overline{P} \;\geq\; \big(1-\frac{C}{\beta^{1/2}}\big)\left\{\big(1-\frac{C}{\beta^{1/2}}\big)\;+\;1\right\}\frac{2}{\beta}\lambda_{\min}\,\overline{P}\,, \tag{V.6}$$

which proves (V.3). □

The **Feshbach operator** of $\Delta_H^{(1)}$ associated to P is defined by

$$\begin{aligned}\mathcal{F}_P \;&:=\; P\,\Delta_H^{(1)}\,P \;-\; P\,\Delta_H^{(1)}\,\overline{P}\left(\overline{\Delta_H^{(1)}}\right)^{-1}\overline{P}\,\Delta_H^{(1)}\,P \\ &=\; \frac{2}{\beta}H''(0)\,P \;+\; \frac{2}{\beta}P\,W^{(1)}\,P \;-\; \frac{4}{\beta^2}\,P\,W^{(1)}\,\overline{P}\left(\overline{\Delta_H^{(1)}}\right)^{-1}\overline{P}\,W^{(1)}\,P\,,\end{aligned} \tag{V.7}$$

since $\Delta_H^{(0)}P=0$. Thanks to the invertibility of $\overline{\Delta_H^{(1)}}$ on $\mathrm{Ran}\overline{P}$, the inverse of $\Delta_H^{(1)}$ is given by

$$\begin{aligned}\big(\Delta_H^{(1)}\big)^{-1} \;=\; &\left(P\;-\;\overline{P}\left(\overline{\Delta_H^{(1)}}\right)^{-1}\overline{P}\Delta_H^{(1)}P\right)\mathcal{F}_P^{-1}\left(P\;-\;P\,\Delta_H^{(1)}\,\overline{P}\left(\overline{\Delta_H^{(1)}}\right)^{-1}\overline{P}\right) \\ &+\;\overline{P}\left(\overline{\Delta_H^{(1)}}\right)^{-1}\overline{P}\end{aligned} \tag{V.8}$$

(see [1]). If we insert (V.8) in (I.31), we obtain

$$\mathbb{E}_{L,\beta}^{T}(x_j\,;x_k) \;=\; \left\langle e^{-\beta H}\otimes e_j\,\middle|\,\mathcal{F}_P^{-1}\,e^{-\beta H}\otimes e_k\right\rangle_{\mathcal{H}^{(1)}}\,, \tag{V.9}$$

since $\overline{P}\,e^{-\beta H}\otimes e_l\;=\;0$, for $l\;=\;j,k$. So we do not loose any information if we replace $\Delta_H^{(1)}$ by $\mathcal{F}_P$ in (I.31). Furthermore, $\mathcal{F}_P$ is related to the Hessian at 0 in the following way.

Proposition V.2. *Assume Hypotheses 1,2, 3*[0]*. For β large enough, we have*

$$\left\| \mathcal{F}_P \; - \; \frac{2}{\beta} \, H''(0) \, P \right\| \;\; = \;\; \mathcal{O}(\beta^{-3/2}) \, , \tag{V.10}$$

where the $\mathcal{O}$-symbol is uniform in $L \to \infty$ and $\beta \to \infty$.

Proof. Using the fact that $P \, = \, (A^{(1)})^{-1/2} \, P \, (\frac{2}{\beta} \, H''(0))^{1/2}$, we obtain the estimate

$$\frac{2}{\beta} \, \| P \, W^{(1)} \, P \| \;\; \le \;\; \frac{2}{\beta} \, \lambda_{\max} \, \cdot \, C \, \beta^{-1/2} \, , \tag{V.11}$$

by Hypothesis 2 and Theorem III.3. Thanks to (V.3), we also have

$$\frac{4}{\beta^2} \left\| P \, W^{(1)} \, \overline{P} \left(\overline{\Delta_H^{(1)}} \right)^{-1} \overline{P} \, W^{(1)} \, P \right\| \;\; \le \;\; \frac{2 \, C^2}{\beta^2} \, \lambda_{\max} \, \frac{\beta}{4 \lambda_{\min}} \left(1 - \mathcal{O}(\beta^{-1/2}) \right) . \tag{V.12}$$

Using (V.7), the estimates (V.11) and (V.12) imply (V.10). □

VI Analysis of the support function

In this section, we analyze the support function S_r^w which we identify in Section VII to be the rate of exponential decay of the two-point function $\mathbb{E}_{L,\beta}^T(x_j \, , x_k)$. Throughout this section we require Hypotheses 1, 2, 5, 6 and Hypothesis 3[S_r^v], for some $1 < r < 2 - F_v(0)$.

We note that $1 - F_v(0) > 0$, thanks to (I.37)–(I.38). We recall that $\pi : \mathbb{R}^n \to (\mathbb{R}/L\mathbb{Z})^n$ is the canonical projection and that $J : (\mathbb{R}/L\mathbb{Z})^n \to [-L/2 \, , \, L/2)^n$ is the natural identification map from the torus to the fundamental domain. In particular, J is a bijection from Λ_L to $\Lambda_L^{(box)} := \mathbb{Z}^n \cap [-L/2 \, , \, L/2)^n$.

Another natural bijection is provided by $J_{\mathcal{A}} : \Lambda_L \to \widetilde{\mathcal{A}}_1$, where $J_{\mathcal{A}}$ is determined by $J_{\mathcal{A}}^{-1} = \pi|_{\widetilde{\mathcal{A}}_1}$ and $\widetilde{\mathcal{A}}_1$ is described in Lemma I.7(iv). Using $J_{\mathcal{A}}$ instead of J amounts to projecting onto the fundamental domain $\widetilde{\mathcal{A}}_1$ rather than $[-L/2 \, , \, L/2)^n$. The difference between these two maps becomes important when studying the two-point function $\mathbb{E}_{L,\beta}^T(x_j, x_0)$ for $J(j)$ close to the boundary of $\Lambda_L^{(box)}$.

Next, we recall from Eqns. (V.7)–(V.9) that

$$\mathbb{E}_{L,\beta}^T(x_j \, , x_k) \;\; = \;\; \beta^{-2} \Big\langle e^{-\beta H} \otimes e_j \Big| \, \mathcal{F}_P^{-1} \, e^{-\beta H} \otimes e_k \Big\rangle \, , \tag{VI.1}$$

where

$$\mathcal{F}_P \;\; = \;\; 2\beta^{-1} \, P \otimes \mathbb{E}_{L,\beta}(H'') \; - \; 4\beta^{-2} \, P W^{(1)} \overline{P} \big(\, \overline{\Delta_H^{(1)}} \, \big)^{-1} \overline{P} W^{(1)} P \, , \tag{VI.2}$$

and $P = |e^{-\beta H}\rangle\langle e^{-\beta H}| \otimes \mathbf{1}$. Thus $\mathbb{E}^T_{L,\beta}(x_j\,;\,x_k) = (2\beta)^{-1}\big(Q^{-1}\big)_{j,k}$, where Q is the real, self-adjoint matrix

$$Q_{j.k} \;=\; \mathbb{E}_{L,\beta}(H''_{j,k}) \;-\; 2\beta^{-1}\left\langle e^{-\beta H}\otimes e_j \,\middle|\, W^{(1)}\overline{P}\big(\,\overline{\Delta^{(1)}_H}\,\big)^{-1}\overline{P}W^{(1)}\; e^{-\beta H}\otimes e_k\right\rangle . \tag{VI.3}$$

Since H is assumed to be translation invariant, $Q_{j,k} = Q_{j-k,0}$. We define a real, even function $w : \mathbb{Z}^n \to \mathbb{R}$ by

$$w(k) \;:=\; \begin{cases} -Q_{\pi(k),0}/H''_{0,0}(0) & \text{if } k \in \widetilde{\mathcal{A}}_1 \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases} \tag{VI.4}$$

In other words, $w \equiv 0$ outside $\widetilde{\mathcal{A}}_1$ and solves $Q_{\pi(k),0} = H''_{0,0}(0)\big(\delta_{k,0} - w(k)\big)$ inside $\widetilde{\mathcal{A}}_1$. We remark that w is similar to v, which also vanishes outside $\widetilde{\mathcal{A}}_1$ and solves $H''_{\pi(k),0}(0) = H''_{0,0}(0)\big(\delta_{k,0} - v(k)\big)$ inside $\widetilde{\mathcal{A}}_1$. Comparing w to v, we observe that

$$\begin{aligned} w(k) \;&=\; v(k) - \big(H''_{0,0}(0)\big)^{-1}\mathbb{E}_{L,\beta}\big(H''_{\pi(k),0} - H''_{\pi(k),0}(0)\big) \\ &\quad + 2\big(H''_{0,0}(0)\beta\big)^{-1}\left\langle e^{-\beta H}\otimes e_{\pi(k)} \,\middle|\, W^{(1)}\overline{P}\big(\,\overline{\Delta^{(1)}_H}\,\big)^{-1}\overline{P}W^{(1)}\; e^{-\beta H}\otimes e_0\right\rangle , \end{aligned} \tag{VI.5}$$

for any $k \in \widetilde{\mathcal{A}}_1$.

Given $\eta \in \mathbb{R}^n$ and an even function $u : \mathbb{Z}^n \to \mathbb{R}$ of compact support, we define a smooth (in fact, analytic) function $F_u \in C^\infty(\mathbb{R}^n)$ by

$$F_u(\eta) \;:=\; \sum_{k\in\mathbb{Z}^n} e^{\eta\cdot k}\, u(k) \;=\; \sum_{k\in\mathbb{Z}^n} \cosh(\eta\cdot k)\, u(k)\,, \tag{VI.6}$$

and, for any $r \in \mathbb{R}$, we set

$$D_u(r) \;:=\; \big\{\eta \in \mathbb{R}^n \;\big|\; F_u(\eta) < r\,\big\}\,. \tag{VI.7}$$

Then the support function $S^v_r : \mathbb{R}^n \to [0, +\infty)$ is defined by

$$S^u_r(z) \;:=\; \sup\big\{\eta\cdot z \;\big|\; \eta \in D_u(r)\,\big\}\,. \tag{VI.8}$$

Recall from Lemma I.7, which we shall prove at the end of this section, that $F_v(\eta)$ is strictly convex and that $B(C,0) \subseteq D_v(r) \subseteq B(C',0)$, for some constants C, C' and all $1-\delta_0 \le r \le 1+\delta_0$, provided $\delta_0 < F_v(0)$. Thus $D_v(r) = -D_v(r)$ is a strictly convex, bounded set with smooth boundary $\Sigma_v(r) = \partial D_v(r) \subseteq \{F_v(\eta) = r\}$. Moreover, for any $z \in \mathbb{R}^n \setminus \{0\}$, there is a unique vector $\eta(z) \in \Sigma_v(r)$ such that $\eta(z) = \lambda\, z$, for some $\lambda > 0$, and this vector realizes the supremum in (VI.8), $S^v_r(z) = \eta(z)\cdot z$. Moreover, an easy computation shows that $\nabla_z S^v_r(z) = \eta(z)$.

Our goal in this section is to prove the following theorem.

Theorem VI.1. *Assume Hypotheses 1, 2, 5, 6 and Hypothesis 3*$[S_r^u]$*, for some* $1 < r < 2 - F_v(0)$*. There exist constants* $C \geq 0$ *and* β_0 *which are independent of* L*, such that, for any* $\beta > \beta_0$ *and* $\eta \in D_v(r)$*,*

$$\left|\partial_\eta^\alpha F_w(\eta) \;-\; \partial_\eta^\alpha F_v(\eta)\right| \;\leq\; C\,\beta^{-1/2}\,, \tag{VI.9}$$

where $\alpha \in \{0,1,2,3\}^n$ *is a multiindex.*

This theorem has several important consequences, obtained from Taylor expansions and (VI.9). We collect these in the following corollary.

Corollary VI.2. *Assume Hypotheses 1, 2, 5, 6 and Hypothesis 3*$[S_r^v]$*, for some* $1 < r < 2 - F_v(0)$*. There exist constants* $C \geq 0$ *and* β_0 *which are independent of* β *and* L*, such that, for any* $\beta > \beta_0$*,* $1 - \delta_0 \leq r \leq 1 + \delta_0$ *and* $0 < \delta_0 < \min\{F_v(0), 1 - F_v(0)\}$*,*

$$D_v\big(r - C\beta^{-1/2}\big) \;\subseteq\; D_w\big(r\big) \;\subseteq\; D_v\big(r + C\beta^{-1/2}\big)\,, \tag{VI.10}$$

$$\big(1 - C\beta^{-1/2}\big)\,S_r^v \;\leq\; S_r^w \;\leq \big(1 + C\beta^{-1/2}\big)S_r^v\,. \tag{VI.11}$$

Moreover, $D_w(r)$ *is a strictly convex, bounded, open set, with a smooth boundary* $\Sigma_w(r) = \partial D_w(r)$*.*

We break up the proof of Theorem VI.1 into several lemmata. Our general strategy is to apply a Combes-Thomas argument. To formulate this, we introduce some suitable exponential weights on $(\mathbb{R}/L\mathbb{Z})^n$.

We construct these weights from a family of uniformly Lipschitz-continuous functions $\widetilde{\theta}_r : (\mathbb{R}/L\mathbb{Z})^n \to [0, +\infty)$, where $1 - \delta_0 \leq r \leq 1 + \delta_0$, for some $\delta_0 > 0$. That is, we require that, for some constant C which neither depends on r nor L, we have

$$\left|\,\widetilde{\theta}_r(x) \;-\; \widetilde{\theta}_r(x)\,\right| \;\leq\; C\,d(x - y)\,, \tag{VI.12}$$

for all $x, y \in (\mathbb{R}/L\mathbb{Z})^n$. Furthermore, we assume that $\widetilde{\theta}_r(0) = 0$ and that, for almost every $x \in (\mathbb{R}/L\mathbb{Z})^n$,

$$\nabla_x \widetilde{\theta}_r(x) \;\in\; D_v(r)\,. \tag{VI.13}$$

The main example we bear in the back of our mind is the following.

Lemma VI.3. *Let* $\widetilde{\theta}_r^S : (\mathbb{R}/L\mathbb{Z})^n \to [0, +\infty)$ *be defined by the seminorm*

$$\widetilde{\theta}_r^S(x) \;:=\; \inf\big\{S_r^v(\tilde{x}) \;\big|\; \tilde{x} \in \pi^{-1}(x)\,\big\}\,, \tag{VI.14}$$

for $1 - \delta_0 \leq r \leq 1 + \delta_0$*, with* $\delta_0 \leq F_v(0)$*. Then* $\{\widetilde{\theta}_r^S\}_{1-\delta_0 \leq r \leq 1+\delta_0}$ *is a family of uniformly Lipschitz-continuous functions obeying (VI.12)–(VI.13).*

Proof. We first recall that $S_r^v : \mathbb{R}^n \to [0, +\infty)$ is a norm, so $\mu_r^{(1)}|\tilde{x}| \le S_r^v(\tilde{x}) \le \mu_r^{(2)}|\tilde{x}|$, for some $0 < \mu_r^{(1)} \le \mu_r^{(2)} < \infty$ and all $\tilde{x} \in \mathbb{R}^n$. Therefore, $S_r^v(J(x)) \le S_r^v(\tilde{x})$, whenever $|\tilde{x}| > 2^{d/2}\mu_r^{(2)} L/\mu_r^{(1)}$, and we conclude that the infimum in (VI.14) is actually a minimum, attained for some $\tilde{x}$ in the finite set $\pi^{-1}(x) \cap B(2^{d/2}\mu_r^{(2)} L/\mu_r^{(1)}, 0)$,

$$\widetilde{\theta}_r^S(x) \ := \ \min\big\{ S_r^v(\tilde{x}) \ \big| \ \tilde{x} \in \pi^{-1}(x) \cap B(2^{d/2}\mu_r^{(2)} L/\mu_r^{(1)}, 0) \ \big\} \ . \tag{VI.15}$$

Next, we observe that $S_r^v \in C^1(\mathbb{R}^n \setminus \{0\})$ and $\nabla_{\tilde{x}} S_r^v(\tilde{x}) = \eta(\tilde{x}) \in D_v(r) \subseteq B(C', 0)$, for some $C' \ge 0$ which is independent of r and L. We conclude that $\widetilde{\theta}_r^S$ is the minimum over finitely many C^1 functions and, as such, it is Lipschitz-continuous. Indeed, for almost every $x \in (\mathbb{R}/L\mathbb{Z})^n$, its gradient is given by $\nabla_x \widetilde{\theta}_r^S(x) = \nabla_{\tilde{x}} S_r^v(\tilde{x})$, for some $\tilde{x} \in B(2^{d/2}\mu_r^{(2)} L/\mu_r^{(1)}, 0)$. Thus, $\nabla_x \widetilde{\theta}_r^S(x) \in D_v(r) \subseteq B(C', 0)$, for almost every $x \in (\mathbb{R}/L\mathbb{Z})^n$. □

Next, we pick $\chi \in C_0^\infty(B(1,0)\,;[0,+\infty))$ such that $\int \chi(x) d^n x = 1$, and we define $\chi_R : (\mathbb{R}/L\mathbb{Z})^n \to [0, +\infty)$ by $\chi_R(x) := R^{-d}\chi(x/R)$, for $R < L/2^n$. Convolving $\widetilde{\theta}_r$ and χ_R, we obtain a family of smooth functions $\theta_r : (\mathbb{R}/L\mathbb{Z})^n \to [0, +\infty)$,

$$\theta_r \ := \ \widetilde{\theta}_r * \chi_R \ , \tag{VI.16}$$

having the following important properties:

Lemma VI.4. *Assume $\{\widetilde{\theta}_r\}_{1-\delta_0 \le r \le 1+\delta_0}$ to be a family of uniformly Lipschitz continuous functions described in (VI.12)–(VI.13), and define $\theta_r := \widetilde{\theta}_r * \chi_R$, as in (VI.16). Then,*

$$\nabla_x \theta_r(x) \ \in \ D_v(r) \, , \tag{VI.17}$$

for all $x \in (\mathbb{R}/L\mathbb{Z})^n$. Moreover, there exists a constant $C \ge 0$, neither depending on r nor L, such that

$$\big| \, \theta_r(x) \ - \ \widetilde{\theta}_r(x) \, \big| \ \le \ C\, R \tag{VI.18}$$

and

$$\big| \, \theta_r(x) - \theta_r(y) \ - \ \nabla_x \theta_r(y) \cdot J(x-y) \, \big| \ \le \ C\, d(x-y)^2\, R^{-1} \, , \tag{VI.19}$$

for all $x, y \in (\mathbb{R}/L\mathbb{Z})^n$ with $d(x-y) \le R$.

Proof. Since the convolution with χ_R defines a probability measure and $D_v(r)$ is convex, the assertion (VI.17) follows from (VI.13).

From the Lipschitz-continuity of $\widetilde{\theta}_r$, we obtain that

$$\big|\theta_r(x) - \widetilde{\theta}_r(x)\big| \ \le \ \int \big|\widetilde{\theta}_r(y) - \widetilde{\theta}_r(x)\big| \, \chi_R(x-y) d^n y \ \le \ C\, R \, , \tag{VI.20}$$

and thus (VI.18).

Finally, (VI.19) follows from a Taylor expansion and the observation that

$$\big\|\partial_{x_k}\partial_{x_l}\theta_r\big\|_\infty \;\leq\; \big\|\nabla_x\widetilde{\theta}_r\big\|_\infty \, \big\|\nabla_x\chi_R\big\|_\infty \;\leq\; C\,R^{-1}\,. \tag{VI.21}$$

□

We now come to defining the exponential weights mentioned above. Given a family of functions θ_r with the properties (VI.17)–(VI.19), we denote by $e^{\pm\theta_r}$ the multiplication operators on $\mathbb{C}^{|\Lambda_L|}$, acting as $\big(e^{\pm\theta_r}\varphi\big)(k) := e^{\pm\theta_r(k)}\varphi(k)$. Next, we introduce various operators on $\mathbb{C}^{|\Lambda_L|}$ and $\mathcal{H}^{(1)}$ obtained from conjugating by $e^{\pm\theta_r}$ and $\mathbf{1}\otimes e^{\pm\theta_r}$, respectively:

$$\begin{aligned} W^{(1)}(\theta_r) &\equiv W^{(1)}(\theta_r,x) := \big(\mathbf{1}\otimes e^{\theta_r}\big)\,W^{(1)}(x)\,\big(\mathbf{1}\otimes e^{-\theta_r}\big)\,, && \text{(VI.22)}\\ B(\theta_r) &:= e^{\theta_r}\,H''(0)\,e^{-\theta_r} \;-\; H''(0)\,, && \text{(VI.23)}\\ \Delta_H^{(1)}(\theta_r) &:= \big(\mathbf{1}\otimes e^{\theta_r}\big)\,\Delta_H^{(1)}\,\big(\mathbf{1}\otimes e^{-\theta_r}\big) && \text{(VI.24)}\\ &= A_0 \;+\; 2\beta^{-1}\big(\mathbf{1}\otimes B(\theta_r) \;+\; W^{(1)}(\theta_r)\big)\,. \end{aligned}$$

We derive various norm estimates on these operators that allow us to analyze F_w, $D_w(r)$, and S_r^w. We first estimate the norm of $2\mathrm{Re}\{B(\theta_r)\} = B(\theta_r) + B(-\theta_r)$.

Lemma VI.5. *Let $\{\theta_r : (\mathbb{R}/L\mathbb{Z})^n \to [0,+\infty)\}_{1-\delta_0\leq r\leq 1+\delta_0}$, for some $\delta_0 > 0$, be a family of functions with the properties (VI.17)–(VI.19), and assume Hypotheses 2, 5, and 6. Then, for some constant $C\geq 0$,*

$$\big\|\,\mathrm{Re}\{B(\theta_r)\}\,\big\| \;\leq\; H''_{0,0}(0)\,\Big(e^{C/R}\,r \;-\; F_v(0)\Big)\,. \tag{VI.25}$$

Proof. Since $\mathrm{Re}\{B(\theta_r)\}$ is a self-adjoint matrix, we may estimate its norm by

$$\begin{aligned} \big\|\mathrm{Re}\{B(\theta_r)\}\big\| &\leq \sup_{i\in\Lambda_L}\Big\{\sum_{j\in\Lambda_L}\big|\big(\,\mathrm{Re}\{B(\theta_r)\}\,\big)_{i,j}\big|\Big\} && \text{(VI.26)}\\ &\leq H''_{0,0}(0)\,\sup_{i\in\Lambda_L}\Big\{\sum_{j\in\Lambda_L}\Big\{\cosh\Big[\theta_r(i)-\theta_r(j)\Big] \;-\; 1\Big\}\,v_L(i-j)\Big\}\\ &\leq H''_{0,0}(0)\,\sup_{i\in\Lambda_L}\Big\{\sum_{j\in\Lambda_L}\Big\{\cosh\Big[\nabla_x\theta_r(i)\cdot J(i-j)+\delta(i,j)\Big] \;-\; 1\Big\}\,v_L(i-j)\Big\}\,, \end{aligned}$$

where $|\delta(i,j)| \leq Cd(i-j)^2R^{-1}$, and the last inequality derives from (VI.19), provided R is chosen larger than C', where C' is a constant such that $\mathrm{supp}\{v\}\subseteq B(C',0)$. Therefore, $|\delta(i,j)|\leq \widetilde{C}R^{-1}$ where $\widetilde{C} := C(C')^2$.

Next, we observe that $\cosh(a+b) \le e^{|b|}\cosh(a)$, for arbitrary $a, b \in \mathbb{R}$. Inserting this estimate in (VI.26), we obtain

$$\begin{aligned}
&\big\|\mathrm{Re}\{B(\theta_r)\}\big\| \\
&\quad\le\ H''_{0,0}(0)\left(e^{\widetilde{C}/R}\sup_{i\in\Lambda_L}\left\{\sum_{j\in\Lambda_L}\cosh\big[\nabla_x\theta_r(i)\cdot J(i-j)\big]\,v_L(i-j)\right\} \ -\ F_v(0)\right)\\
&\quad\le\ H''_{0,0}(0)\left(e^{\widetilde{C}/R}\sup_{i\in\Lambda_L}\Big\{F_v\big(\nabla_x\theta_r(i)\big)\Big\}\ -\ F_v(0)\right)\\
&\quad\le\ H''_{0,0}(0)\left(e^{\widetilde{C}/R}\,r\ -\ F_v(0)\right),
\end{aligned}\tag{VI.27}$$

since $\nabla_x\theta_r(i) \in D_v(r)$, for all $i \in \Lambda_L$, by (VI.17). □

Next we need the following estimate on the norm of the resolvent of $\overline{\Delta_H^{(1)}(\theta_r)} := \overline{P}\,\Delta_H^{(1)}(\theta_r)\,\overline{P}$ on $\mathrm{Ran}\overline{P}$, relative to $A^{(1)}$.

Lemma VI.6. *Let $\{\theta_r : (\mathbb{R}/L\mathbb{Z})^n \to [0,+\infty)\}_{1-\delta_0\le r\le 1+\delta_0}$, for some $\delta_0 > 0$, be a family of functions with the properties (VI.17)–(VI.19), and assume Hypotheses 2, 5, and 6. Then, for any $r < 2 - F_v(0)$, there exist constants $C, \beta_0 \ge 0$, independent of L, such that, for any $\beta \ge \beta_0$ and $R \ge C$,*

$$\Big\|\,\big(A^{(1)}\big)^{1/2}\Big(\overline{\Delta_H^{(1)}(\theta_r)}\Big)^{-1}\big(A^{(1)}\big)^{1/2}\overline{P}\,\Big\| \ \le\ C\,. \tag{VI.28}$$

Proof. We recall form (IV.4) and (VI.22)–(VI.24) that

$$\Delta_H^{(1)}(\theta_r)\ =\ A^{(1)}\ +\ \frac{2}{\beta}\,W^{(1)}(\theta_r)\ +\ \frac{2}{\beta}\,\mathbf{1}\otimes B(\theta_r)\,, \tag{VI.29}$$

and hence, using Theorem III.3 and the fact that $A^{(1)}\overline{P} \ge 4\lambda_{\min}\beta^{-1}\overline{P}$ (see Proposition V.1), we obtain

$$\begin{aligned}
&\big(A^{(1)}\big)^{1/2}\,\mathrm{Re}\Big\{\overline{\Delta_H^{(1)}}(\theta_r)\Big\}\,\big(A^{(1)}\big)^{1/2}\,\overline{P}\\
&\qquad\ge\quad \Big(1-\frac{1}{2\lambda_{\min}}\big\|\mathrm{Re}\big\{B(\theta_r)\big\}\big\| - C\beta^{-1/2}\Big)\cdot\overline{P}\,.
\end{aligned}\tag{VI.30}$$

Since, according to (I.37), $\lambda_{\min} = H''_{0,0}(0)(1-F_v(0))$, Lemma VI.5 yields that

$$\frac{1}{2\lambda_{\min}}\big\|\mathrm{Re}\big\{B(\theta_r)\big\}\big\|\ \le\ \frac{1}{2}\big(e^{\widetilde{C}/R}r - F_v(0)\big)\big(1-F_v(0)\big)^{-1}\,. \tag{VI.31}$$

By assumption, $r < 2 - F_v(0)$, and thus also $e^{\widetilde{C}/R}r < 2 - F_v(0)$, provided R is sufficiently large. Hence, for some small positive constants c, c', we have that

$$\mathrm{Re}\Big\{(A^{(1)})^{1/2}\,\overline{\Delta_H^{(1)}}(\theta_r)\,(A^{(1)})^{1/2}\Big\}\overline{P}\ \ge\ \big(c - C\beta^{-1/2}\big)\ \ge\ c'\cdot\overline{P}\,, \tag{VI.32}$$

proving the claim. □

Proof of Theorem VI.1. First, we recall that the family

$$\{\theta_r^S : (\mathbb{R}/L\mathbb{Z})^n \to [0,+\infty)\}_{1-\delta_0 \le r \le 1+\delta_0}, \quad \text{for some } \delta_0 > 0,$$

is a family of functions with the properties (VI.17)–(VI.19), according to Lemmata VI.3 and VI.4. Since we have furthermore assumed Hypotheses 2, 5, 6, and $\delta_0 < \min\{F_v(0),\ 1 - F_v(0)\}$, we may apply Lemma VI.6.

We first recall from Eqn. (VI.5) that

$$\begin{aligned} w(k) - v(k) \ &= \ \frac{2}{\beta}\big(H''_{0,0}(0)\big)^{-1}\Big\langle e^{-\beta H} \otimes e_k \Big|\ W^{(1)}\ e^{-\beta H} \otimes e_0 \Big\rangle \\ &- \frac{4}{\beta^2}\big(H''_{0,0}(0)\beta\big)^{-1}\Big\langle e^{-\beta H} \otimes e_k \Big|\ W^{(1)}\,\overline{P}\,\big(\overline{\Delta_H^{(1)}}\big)^{-1}\,\overline{P}\,W^{(1)}\ e^{-\beta H} \otimes e_0 \Big\rangle , \end{aligned} \tag{VI.33}$$

for any $k \in \Lambda_L$. Using the conjugation by $e^{\theta_r^S}$, we may rewrite the terms in Eqn. (VI.33) as

$$\Big\langle e^{-\beta H} \otimes e_k \Big|\ W^{(1)}\ e^{-\beta H} \otimes e_0 \Big\rangle \ = \ e^{-\theta_r^S(k)}\Big\langle e^{-\beta H} \otimes e_k \Big|\ W^{(1)}(\theta_r)\ e^{-\beta H} \otimes e_0 \Big\rangle , \tag{VI.34}$$

and similarly

$$\begin{aligned} &\Big\langle e^{-\beta H} \otimes e_k \Big|\ W^{(1)}\overline{P}\big(\overline{\Delta_H^{(1)}}\big)^{-1}\overline{P}W^{(1)}\ e^{-\beta H} \otimes e_0 \Big\rangle \ = \\ &\quad e^{-\theta_r^S(k)}\Big\langle e^{-\beta H} \otimes e_k \Big|\ W^{(1)}(\theta_r)\,\overline{P}\,\big(\overline{\Delta_H^{(1)}(\theta_r)}\big)^{-1}\,\overline{P}\,W^{(1)}(\theta_r)\ e^{-\beta H} \otimes e_0 \Big\rangle . \end{aligned} \tag{VI.35}$$

Using these two identities and the fact that

$$\Big\| \big(A^{(1)}\big)^{1/2}\ e^{-\beta H} \otimes e_k \Big\|^2 \ = \ \Big\| \big(A^{(1)}\big)^{1/2}\ e^{-\beta H} \otimes e_k \Big\|^2 \ = \ \frac{2\,H''_{0,0}(0)}{\beta} , \tag{VI.36}$$

we obtain from (VI.33) that

$$\begin{aligned} e^{\theta_r(k)}\big|w(k) - v(k)\big| \ &\le \ 2\beta^{-1}\Big\| \big(A^{(1)}\big)^{-1/2}\,W^{(1)}(\theta_r)\,\big(A^{(1)}\big)^{-1/2}\ \Big\| \\ &+4\beta^{-2}\Big\|\big(A^{(1)}\big)^{-1/2}W^{(1)}(\theta_r)\big(A^{(1)}\big)^{-1/2}\Big\|^2\ \Big\|\big(A^{(1)}\big)^{1/2}\big(\overline{\Delta_H^{(1)}(\theta_r)}\big)^{-1}\big(A^{(1)}\big)^{1/2}\overline{P}\Big\| \\ &\le \ C\beta^{-1/2} + C\beta^{-1} \ \le \ 2C\beta^{-1/2} , \end{aligned} \tag{VI.37}$$

for any $1 - \delta_0 \le r \le 1 + \delta_0$, provided that $\delta_0 < \min\{F_v(0),\ 1 - F_v(0)\}$. Here we use Lemma VI.6 and Theorem III.3 to derive the second inequality.

To make use of (VI.37), we recall that according to Lemma I.7(iii) there exist constants $C, C' \ge 0$ such that, for any $0 < \varepsilon < 1$,

$$(1+\varepsilon)S^v_{(1-C\varepsilon)r} \ \le \ S^v_r \ \le \ (1+\varepsilon)S^v_{r\exp(-C'\varepsilon)} , \tag{VI.38}$$

provided that $1-\delta_0 \le r \le 1+\delta_0$. Thus, for any $\eta \in D_v(r)$, $0<\varepsilon<1$, and any multiindex $\alpha \in \{0,1,2,3\}^n$, we have the estimate

$$\begin{aligned}
\left|\partial_\eta^\alpha F_w(\eta) - \partial_\eta^\alpha F_v(\eta)\right| \ &\le \sum_{k\in\widetilde{\mathcal{A}}_1\cap\mathbb{Z}^n} \exp[\eta\cdot k]\,\left|w(k)-v(k)\right|\,|k|^{|\alpha|} \\
&\le \sum_{k\in\widetilde{\mathcal{A}}_1\cap\mathbb{Z}^n} \exp[S_r^v(k)]\,\left|w(k)-v(k)\right|\,|k|^{|\alpha|} \\
&\le \sum_{k\in\widetilde{\mathcal{A}}_1\cap\mathbb{Z}^n} \exp\big[(1+C(r-1))S_1^v(k)\big]\,\left|w(k)-v(k)\right|\,|k|^{|\alpha|} \\
&\le \left(\sum_{k\in\widetilde{\mathcal{A}}_1\cap\mathbb{Z}^n} \exp\big[-\varepsilon S_1^v(k)\big]\,|k|^{|\alpha|}\right) \qquad \text{(VI.39)} \\
&\quad \sup_{k\in\widetilde{\mathcal{A}}_1\cap\mathbb{Z}^n}\Big\{\exp\big[(1+C(r-1)+\varepsilon)S_1^v(k)\big]\left|w(k)-v(k)\right|\Big\}.
\end{aligned}$$

The important point about using $\widetilde{\mathcal{A}}_1$ rather than $[-L/2,\,L/2)^n$ for the definition of w is that on $\widetilde{\mathcal{A}}_1$ we have $S_1^v = \widetilde{\theta}_1^S$. Moreover, by (VI.38) and (VI.18), there exists a constant $C' \ge 0$ such that

$$\begin{aligned}
(1+C(r-1)+\varepsilon)S_1^v(k) \ &= \ \min_{q\in\mathbb{Z}^n}\big\{(1+C(r-1)+\varepsilon)S_1^v(k+qL)\big\} \\
&\le \ \min_{q\in\mathbb{Z}^n}\big\{S^v_{(1+C'(r-1+\varepsilon))}(k+qL)\big\} \ = \ \widetilde{\theta}^S_{(1+C'(r-1+\varepsilon))}(k) \\
&\le \ \theta^S_{(1+C'(r-1+\varepsilon))}(k) \ + \ C'\,. \qquad \text{(VI.40)}
\end{aligned}$$

Thus, if $r>1$ is sufficiently small such that $1+C'(r-1+\varepsilon) < 1+\delta_0$, we obtain

$$\left|\partial_\eta^\alpha F_w(\eta) - \partial_\eta^\alpha F_v(\eta)\right| \ \le \ C_{\varepsilon,\alpha} \sup_{k\in\widetilde{\mathcal{A}}_1\cap\mathbb{Z}^n}\Big\{\exp\big[\theta^S_{1+\delta_0}(k)\big]\left|w(k)-v(k)\right|\Big\}, \qquad \text{(VI.41)}$$

which is bounded by $C\,\beta^{-1/2}$, due to Eqn. (VI.37). This proves (VI.9). □

We close this section with a

Proof of Lemma I.7.

(i) First, Hölder's inequality implies that, for $\eta, \eta' \in \mathbb{R}^n$ and $\alpha \in (0,1)$,

$$\begin{aligned}
F_v\big(\alpha\eta + (1-\alpha)\eta'\big) \ &= \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \big(e^{\eta\cdot m}v(m)\big)^{\alpha}\cdot\big(e^{\eta'\cdot m}v(m)\big)^{1-\alpha} \\
&\le \ F_v(\eta)^{\alpha}\cdot F_v(\eta')^{1-\alpha}\,, \qquad \text{(VI.42)}
\end{aligned}$$

and hence convexity of $\ln F_v$. To obtain strict convexity, we observe that equality holds in (VI.42) if and only if there is a constant $\mu > 0$ such that $e^{\eta\cdot k}v(k) = \mu\, e^{\eta'\cdot k}v(k)$, for all $k \in \mathbb{Z}^n$. But since $\mathrm{Gr}(\mathrm{supp}\{v\}) = \mathbb{Z}^n$, by Hypothesis 6, this is equivalent to $\eta = \eta'$. Since $\ln F_v$ is strictly convex, so is F_v.

Again by Hypothesis 6, there exists a set of linearly independent vectors $\{k_1, \ldots, k_n\} \subseteq \mathbb{Z}^n$ and a constant $\delta > 0$, such that $v(k_\nu) \geq \delta$, for all $1 \leq \nu \leq n$. Thus, there exist a constant C such that $\max_{1\leq\nu\leq n} |\eta \cdot k_\nu| \geq C|\eta|$, and we observe that, for $\varepsilon > 0$,

$$F_v\big((1+\varepsilon)\eta\big) - F_v\big(\eta\big) \;\geq\; \delta\Big[\cosh\big((1+\varepsilon)C\eta\big) - \cosh\big(C\eta\big)\Big] \;\geq\; \frac{1}{2}\delta C|\eta|\varepsilon\,, \tag{VI.43}$$

implying the first inequality in (I.42). The second inequality in (I.42) follows from $\cosh(a+b) \leq \cosh(a)\, e^{|b|}$ and the boundedness of $\mathrm{supp}\{v\}$.

(ii) Similarly to (VI.43), we obtain that $F_v(\eta) - F_v(0) \geq C|\eta|^2$, for some constant $C > 0$ and thus $D_v(r) \subseteq B(\sqrt{(r - F_v(0))/C}\,, 0)$, for any $r \geq 1 - \delta_0 > F_v(0)$. Conversely, since $F_v(\eta) \leq e^{C|\eta|} F_v(0) \leq r$, for any $r \geq 1 - \delta_0 > F_v(0)$, provided that $|\eta|$ is sufficiently small, we have $B(C, 0) \subseteq D_v(r)$, for some $C > 0$. Smoothness and strict convexity of $D_v(r)$ follows from (i).

(iii) The support function S_r^v obviously defines a semi-norm on $\mathbb{R}^n$, for each $1 - \delta_0 \leq r \leq 1 + \delta_0$, and, by additionally using Hypothesis 6, one checks that $S_r^v(x) = 0$ implies $x = 0$. The vector $\eta(x)$ is the solution of the Euler-Lagrange equation $\nabla_\eta F_v(\eta(x)) = \mu x$ for S_r^v, and $\nabla_x S_r^v(x) = \eta(x)$ follows from differentiating $F_v(\eta(x)) = r$.

The estimate (I.44) follows from Eqn. (I.42) with the additional observation that we assume $|\eta| \geq C$ for some $C > 0$ in this estimate. For example, $F_v\big((1+\varepsilon)\eta\big) \leq e^{C'|\eta|\varepsilon} F_v(\eta)$ implies that $D_v(r) \subseteq (1+\varepsilon) D_v\big(e^{-C''\varepsilon} r\big)$ which, in turn, implies that $S_r^v \leq (1+\varepsilon) S^v_{\exp(-C''\varepsilon)r}$.

(iv) From the definition of $\mathcal{A}_r$ it is immediate that there is a fundamental domain $(\mathcal{A}_r)^\circ \subset \widetilde{\mathcal{A}}_r \subseteq \mathcal{A}_r$ for π. That is, $\mathbb{R}^n = \bigcup_{q\in\mathbb{Z}^n} \widetilde{\mathcal{A}}_r + qL$, and $\widetilde{\mathcal{A}}_r + qL \;\cap\; \widetilde{\mathcal{A}}_r + q'L = \emptyset$, for $q \neq q'$, $q, q' \in \mathbb{Z}^n$.

Next, we note that since $S_r^v(k)$ is a norm on $\mathbb{R}^n$, there exist two constants $0 < \mu_1^r \leq \mu_2^r$ such that $\mu_1^r|k| \leq S_r^v(k) \leq \mu_2^r|k|$. Thus, for given $x \in [-L/2, L/2)^n$ and $q \in \mathbb{Z}^n \setminus \{0\}$, we have the estimate

$$S_r^v(x + qL) \;-\; S_r^v(x) \;\geq\; \mu_1^r\Big(|q|\, L \;-\; \big(2\mu_2^r/\mu_1^r\big)\, |x|\Big)\,. \tag{VI.44}$$

Therefore, $B\big(\mu_1^r L/2\mu_2^r, 0\big) \subseteq \mathcal{A}_r \subseteq B\big(2\mu_2^r L/\mu_1^r, 0\big)$.

Finally, to prove that $\mathcal{A}_r$ is star-shaped, we notice that, for $\lambda > 0$, $x \in \mathbb{R}^n$, and $q \in \mathbb{Z}^n \setminus \{0\}$,

$$\begin{aligned}\lambda\frac{d}{d\lambda}\Big(S_r^v(\lambda x) - S_r^v(\lambda x + qL)\Big) \;&=\; \Big(\eta(\lambda x) - \eta(\lambda x + qL)\Big)\cdot \lambda x \\ &=\; S_r^v(\lambda x) - \eta(\lambda x + qL)\cdot\lambda x \;>\; 0\,.\end{aligned} \tag{VI.45}$$

Since $S_r^v(0) - S_r^v(qL) < 0$, Eqn. (VI.45) proves that $\widetilde{\mathcal{A}}$ is star-shaped. □

VII Asymptotics of the two-point function

This section is devoted to the proof of Theorem I.9. We recall from Sections IV, Eqns. (V.7)–(V.9) and VI, Eqn. (VI.1) that, for $j, k \in \Lambda_L$, the two-point function is given by

$$\mathbb{E}^T_{L,\beta}(x_j\,, x_k) \;=\; \beta^{-2} \Big\langle e^{-\beta H} \otimes e_j \Big| \, \mathcal{F}_P^{-1} \, e^{-\beta H} \otimes e_k \Big\rangle \,. \tag{VII.1}$$

Thus, Fourier's inversion formula gives

$$\mathbb{E}^T_{L,\beta}(x_i\,, x_j) \;=\; \frac{1}{2\beta \, H''_{0,0}(0)\, L^n} \sum_{\xi \in \Lambda_L^*} \frac{\exp[i\langle \xi, i-j\rangle]}{1 \,-\, w\big(J(i-j)\big)} \,, \tag{VII.2}$$

where w is defined in (VI.5), and $\Lambda_L^* = 2\pi(\mathbb{Z}/L\mathbb{Z})^n$ is the dual lattice to Λ_L. A Poisson formula, derived in [14, Eqn. (4.32)], yields

$$\mathbb{E}^T_{L,\beta}(x_i\,, x_j) \;=\; \frac{1}{2\beta \, H''_{0,0}(0)\, (2\pi)^n} \sum_{k \in \pi^{-1}(i-j)} E_w(k) \,, \tag{VII.3}$$

where

$$E_w(k) \;=\; \int_{\mathbb{T}^n} \frac{e^{i\xi\cdot k}\, d^n\xi}{1 \,-\, F_w(i\xi)} \,, \tag{VII.4}$$

and we recall from (VI.6) that $F_u(\eta) = \sum_{k\in\mathbb{Z}^n} \cosh(\eta\cdot k)\, u(k)$, so $F_u(i\,\cdot\,)$ is the Fourier transform of u (u having bounded support),

$$F_u(i\xi) \;=\; \sum_{k\in\mathbb{Z}^n} \cosh(i\xi\cdot k)\, u(k) \;=\; \sum_{k\in\mathbb{Z}^n} \cos(\xi\cdot k)\, u(k) \;=\; \sum_{k\in\mathbb{Z}^n} e^{-i\xi\cdot k}\, u(k) \,. \tag{VII.5}$$

In [14] the following asymptotics was shown to hold for E_v.

Lemma VII.1. *Assume Hypothesis 6. Then there exists $C_0 > 0$ such that, for any $|k| \geq C_0$, we have*

$$E_v(k) \;=\; \Big(1 \,+\, \mathcal{O}\big(|k|^{-1}\big)\Big) \, \frac{\big|\nabla_\eta F_v(\eta_v(k))\big|^{\frac{n-3}{2}} \, \exp\Big[-S_1^v(k)\Big]}{\big(\det[\partial_\perp^2 F_v(\eta_v(k))]\big)^{1/2} \big(2\pi\,|k|\big)^{\frac{n-1}{2}}} \,. \tag{VII.6}$$

Theorem VI.1 ensures that F_v and F_w and their derivatives only differ by terms of order $\mathcal{O}(\beta^{-1/2})$, and using this, we show Theorem VII.2 below.

We note that while both Theorem VI.1 and Lemma VII.1 require Hypothesis 6, particularly $v \geq 0$ and $\mathrm{Gr}\big(\mathrm{supp} v\big) = \mathbb{Z}^n$, it is possible to derive an asymptotic formula for E_v like (VII.6) also for some cases of v without assuming its positivity. We make the requirements that substitute for Hypothesis 6 more precise in Appendix C.

Theorem VII.2. *Assume Hypotheses 1, 2, 5, 6 and Hypothesis 3*$[S_r^v]$*, for some* $1 < r \leq 2 - F_v(0)$*. Then there exist* $\beta_0 > 0$ *and* $C_0 > 0$ *such that, for any* $\beta > \beta_0$ *and* $|k| \geq C_0$*, we have*

$$E_w(k) \;=\; \Big(1 + \mathcal{O}\big(|k|^{-1} + \beta^{-1/2}\big)\Big) \frac{\big|\nabla_\eta F_v(\eta_v(k))\big|^{\frac{n-3}{2}} \exp\Big[-\big(1+\mathcal{O}(\beta^{-1/2})\big) S_1^v(k)\Big]}{\big(\det[\partial_\perp^2 F_v(\eta_v(k))]\big)^{1/2} \big(2\pi\,|k|\big)^{\frac{n-1}{2}}}\,. \tag{VII.7}$$

In view of the Poisson summation formula (VII.3), Theorem VII.2 is the main ingredient for the proof of Theorem I.9, and we demonstrate below how Theorem I.9 derives from it.

Proof of Theorem I.9. First, fixing $0 < \lambda < 1$, the homogeneity of S_r^v implies the existence of a constant $c > 0$ such that $S_1^v(x) \leq S_1^v(x + qL) - c|q|L$, for all $x \in \lambda\widetilde{\mathcal{A}}$ and $q \in \mathbb{Z}^n \setminus \{0\}$. Thus, by Theorem VII.2, for sufficiently large β, we have

$$E_w(x + qL) \;\leq\; \exp\big[-\varepsilon\, L\, |q|\big]\; E_w(x)\,, \tag{VII.8}$$

for some $\varepsilon > 0$. Hence, we obtain

$$\begin{aligned} \mathbb{E}^T_{L,\beta}(x_j\,, x_0) \;&=\; \frac{1}{2\beta\, H''_{0,0}(0)\,(2\pi)^n} \sum_{k \in \pi^{-1}(j)} E(k) \\ &=\; \frac{1}{2\beta\, H''_{0,0}(0)\,(2\pi)^n} E\big(J_{\mathcal{A}}(j)\big)\; \Big(1 + \mathcal{O}\big(e^{-\varepsilon L}\big)\Big)\,, \end{aligned} \tag{VII.9}$$

and thus Theorem I.9. □

Proof of Theorem VII.2. We first recall that F_w is a finite sum of exponentials and thus entire. Moreover, Theorem VI.1 implies that

$$\mathrm{Re}\big\{F_w(\eta + i\xi)\big\} \;\leq\; F_v(\eta) \;+\; C\beta^{-1/2} \;<\; 1\,, \tag{VII.10}$$

for all $\eta + i\xi \in D_v(r) + i\mathbb{T}^n$, $r < 1$, and β sufficiently large. Thus the integrand in (VII.4) is regular, and we may deform the contour $\mathcal{C}_0 := 0 + i\mathbb{T}^n$ into

$$\mathcal{C}_1 \;:=\; \Big\{ (1 - \varepsilon + \tau\xi^2)\eta_w(k) \;+\; i\xi \;\in\; \mathbb{R}^n + i\mathbb{T}^n \;\Big|\; \xi \in \mathbb{T}^n \Big\}\,, \tag{VII.11}$$

where $\eta_w(k) \in \Sigma_w(1)$ is the unique vector such that $\eta_w(k) \cdot k = S_1^w(k)$. Here, $\tau, \varepsilon > 0$ are two sufficiently small numbers chosen below (independently of β, k, and L). Moreover, we identify the Torus $\mathbb{T}^n$ with the rectangular domain $[-\pi,\pi)^n$. Thus we obtain a new representation for $E_w(k)$ by the following integral,

$$E_w(k) \;=\; \int_{[-\pi,\pi)^n} \frac{\exp\big\{(1 - \varepsilon + \tau\xi^2)\eta_w(k)\cdot k \;+\; i\xi\cdot k\big\}\; d^n\xi}{1 \;-\; F_w\big((1 - \varepsilon + \tau\xi^2)\eta_w(k) \;+\; i\xi\big)}\,. \tag{VII.12}$$

We introduce $\chi \in C_0^\infty(\mathbb{R}^n\,;\,[0,1])$, $\chi \equiv 1$ on $B(1/2, 0)$, and $\mathrm{supp}\chi \subseteq B(1,0)$. We denote $\bar{\chi} := 1 - \chi$.

Lemma VII.3. *For any $N > 0$ there exists constants C_N and α_N such that*

$$\lim_{\varepsilon\to 0}\left|\int_{[-\pi,\pi)^n} \bar{\chi}\Big(\frac{\xi}{R}\Big) \frac{\exp\big[(1-\varepsilon+\tau\xi^2)\eta_w(k)\cdot k \,+\, i\xi\cdot k\big]\, d^n\xi}{1 \,-\, F_w\big((1-\varepsilon+\tau\xi^2)\eta_w(k) \,+\, i\xi\big)}\right| \leq C_N \exp[-S_1^w(k)]\,|k|^{-N}, \tag{VII.13}$$

where $R > 0$ is defined by $R^2 := \alpha_N \max\{\beta^{-1/2},\, |k|^{-1}\ln|k|\}$.

Proof. We first recall from Hypothesis 6 that $v(k_\nu) \geq \delta > 0$, for some linearly independent $\{k_1, \dots, k_n\}$. Hence, we have

$$\begin{aligned}\sum_{\nu=1}^n v(k_\nu)\cosh(\eta\cdot k_\nu)\cos(\xi\cdot k_\nu) & \\ \leq\ & \sum_{\nu=1}^n v(k_\nu)\cosh(\eta\cdot k_\nu) \,-\, \frac{\delta}{2}\min_{1\leq\nu\leq n}\Big\{|\xi\cdot k_\nu|^2\Big\} \\ \leq\ & \sum_{\nu=1}^n v(k_\nu)\cosh(\eta\cdot k_\nu) \,-\, \delta'\,\xi^2\,,\end{aligned} \tag{VII.14}$$

for some $\delta' > 0$, and we thus obtain

$$\Re\big\{F_v(\eta+i\xi)\big\} \,\leq\, F_v(\eta) \,-\, \delta'\,\xi^2\,. \tag{VII.15}$$

Next, we observe that

$$\tilde{\eta} \,:=\, (1-\varepsilon-\tau\xi^2)\eta_w(k) \,\in\, D_w\big(1+C\tau\xi^2\big) \,\subseteq\, D_v\big(1+C\tau\xi^2+C'\beta^{1/2}\big)\,, \tag{VII.16}$$

for some constants C and C', by Theorem VI.1 and Corollary VI.2. Furthermore we remark that $\xi^2 \geq R^2/16 \geq (\alpha_N/16)\beta^{-1/2}$. Using this observation, (VII.15), and (VII.16), we get the following estimate,

$$\begin{aligned}\mathrm{Re}\big\{F_w(\tilde{\eta}+i\xi)\big\} \ &\leq\ F_v(\tilde{\eta}+i\xi) \,+\, C\,\beta^{-1/2} \,\leq\, F_v(\eta) \,-\, \delta'\,\xi^2 \,+\, C\,\beta^{-1/2} \\ &\leq\ 1 \,-\, (\delta'-C'\tau-16C'/\alpha_N)\xi^2 \,\leq\, 1 \,-\, \frac{\delta'}{2}\,\xi^2\,,\end{aligned} \tag{VII.17}$$

provided τ and $1/\alpha_N$ are sufficiently small. We insert this estimate and $S_1^w(k) \geq C|k|$ into (VII.13), which yields

$$\begin{aligned}&\left|\int_{[-\pi,\pi)^n} \bar{\chi}\Big(\frac{\xi}{R}\Big) \frac{\exp\big[(1-\varepsilon+\tau\xi^2)\eta_w(k)\cdot k \,+\, i\xi\cdot k\big]\, d^n\xi}{1 \,-\, F_w\big((1-\varepsilon+\tau\xi^2)\eta_w(k) \,+\, i\xi\big)}\right| \\ &\qquad\leq\ \exp[-(1-\varepsilon)S_1^w(k)]\int_{R/2\leq|\xi|\leq\sqrt{n}\pi} \frac{C\,\exp[-C\,|k|\,\xi^2]\,d^n\xi}{\xi^2} \\ &\qquad\leq\ C_N\,\exp[-(1-\varepsilon)S_1^w(k)]\,|k|^{-N}\,,\end{aligned} \tag{VII.18}$$

as one easily verifies using the properties of R and choosing α_N sufficiently large.
□

Proof of Theorem VII.2 (continued). By Lemma VII.3 and (VII.12), we have

$$E_w(k) \;=\; \int_{[-\pi,\pi)^n} \chi\Big(\frac{\xi}{R}\Big)\, \frac{\exp\big[(1-\varepsilon+\tau\xi^2)\eta_w(k)\cdot k \,+\, i\xi\cdot k\big]\; d^n\xi}{1 \,-\, F_w\big((1-\varepsilon+\tau\xi^2)\eta_w(k) \,+\, i\xi\big)} \;+\; \exp[-(1-\varepsilon)S_1^w(k)]\;\mathcal{O}\big(|k|^{-N}\big)\,, \qquad \text{(VII.19)}$$

where $R^2 := \alpha_N \max\{\,\beta^{-1/2}\,,\ |k|^{-1}\ln|k|\,\}$. Since $|\xi| \le 2R \ll 1$ on the support of the integrand in (VII.19), we can now find a smooth change of coordinates $\xi \mapsto \phi(\xi) = \xi + \mathcal{O}(|\xi|^2)$, $\partial_j\phi_k(\xi) = \delta_{jk} + \mathcal{O}(|\xi|)$, and $\partial^\alpha\phi(\xi) = \mathcal{O}(|\xi|)$, for $|\alpha| \ge 2$, such that

$$\begin{aligned} 1 - F_w&\big((1-\varepsilon+\tau|\xi|^2)\eta_w(k) + i\xi\big) \\ &= 1 - F_w\big((1-\varepsilon)\eta_w(k)\big) + i\nabla_\eta F_w\big((1-\varepsilon)\eta_w(k)\big)\cdot\phi(\xi) \\ &\quad -\frac{1}{2}\big\langle \phi(\xi)\,,\ F_w''\big((1-\varepsilon)\eta_w(k)\big)\phi(\xi)\big\rangle\,. \end{aligned} \qquad \text{(VII.20)}$$

Integrating first along the direction of $\nabla_\eta F_w\big((1-\varepsilon)\eta_w(k)\big)$ and using the cancellation due to the sign change of the variable, we convert the integral into an absolutely convergent one. Then, using the stationary phase method as in [14] to compute the asymptotics of the oscillatory integral in (VII.19) and taking $\varepsilon \to 0$, we arrive at

$$E_w(k) \;=\; \Big(1 \,+\, \mathcal{O}\big(|k|^{-1}\big)\Big)\ \exp\Big[-S_1^w(k)\Big] \qquad \text{(VII.21)}$$
$$\left(\frac{\big|\nabla_\eta F_w(\eta_w(k))\big|^{\frac{n-3}{2}}}{\big(\det[\partial^2_{\eta^\perp}F_w(\eta_w(k))]\big)^{1/2}\big(2\pi\,|k|\big)^{\frac{n-1}{2}}} \;+\; \mathcal{O}\big(|k|^{-N}\big)\right).$$

Now, the claim follows by absorbing the errors made by approximating $\partial^\alpha F_w\big(\eta_w(k)\big) = \partial F_v\big(\eta_v(k)\big) + \mathcal{O}(\beta^{-1/2})$ and by choosing $N \ge (n+1)/2$. □

Appendix

A Admissibility of example I.1[ν]

In this appendix, we prove Lemma I.2, i.e., we justify that Example I.1[ν] satisfies Hypothesis 3[μd], for all $\nu > \mu$. We recall that in Example I.1[ν] the Hamilton function H_L is assumed to be of the form

$$H_L(x) \;=\; \sum_{j\in\Lambda_L} f(x_j) \;+\; g \sum_{i,j\in\Lambda_L} e^{-\nu\, d(i-j)}\, w_{ij}(x_i,x_j)\,, \qquad \text{(A.1)}$$

where $\nu > 0$, and $g > 0$ is sufficiently small, $w_{ij} = w_{ji}$, $w_{ii} = 0$, and furthermore, for $m = 1$ or $m = 2$, f and $\{w_{ij}\}_{i,j\in\Lambda_L}$ obey Eqns. (I.20). Lemma I.2 is equivalent to the following one.

Lemma A.1. *Let* $M_\alpha := 2^n(1 - e^{-\alpha/\sqrt{n}})^{-n}$*, for* $\alpha > 0$*. Then, for any* $\mu > \nu$ *and* $0 \leq g < M_\nu^{-3}/24$*, the Hamilton function* H_L *fulfills Hypothesis 3*$[\mu d]$*, with* $C_{(H3)} = 2 + 12gM_{\nu-\mu}$ *and*

$$a_{ij}(k) \;:=\; b_{ij}(k) \;:=\; \sum_{l\in\Lambda_L} \widetilde{c}_{ij}(l) R_{lk} \,, \tag{A.2}$$

where $\widetilde{c}_{ij}(l)$ *and* R_{lk} *are defined in (A.8) and (A.12) below.*

Proof. Denoting by $\partial_1 w$ (resp. $\partial_2 w$) the derivative with respect to the first (resp. second) variable of w, the derivatives of H are given by

$$H_k'(x) \;=\; f'(x_k) \;+\; g\sum_{l\in\Lambda_L} e^{-\nu\, d(k-l)}\Big\{\partial_1 w_{kl}(x_k,x_l) + \partial_2 w_{lk}(x_l,x_k)\Big\}\,, \tag{A.3}$$

$$H_{i,j}''(x) \;=\; g e^{-\nu\, d(i-j)}\Big\{\partial_1\partial_2 w_{ij}(x_i,x_j) + \partial_1\partial_2 w_{ji}(x_j,x_i)\Big\}\,, \tag{A.4}$$

$$H_{i,i}''(x) \;=\; f''(x_i) \;+\; g\sum_{l\in\Lambda_L} e^{-\nu\, d(i-l)}\Big\{\partial_1^2 w_{il}(x_i,x_l) + \partial_2^2 w_{li}(x_l,x_i)\Big\}\,, \tag{A.5}$$

for $k \neq k'$. Furthermore, we note that, for $\alpha > 0$,

$$\sum_{k\in\Lambda_L} e^{-\alpha\, d(k)} \;\leq\; \sum_{\hat{k}\in\mathbb{Z}^n} e^{\left(-\alpha\sum_{\nu=1}^n |\hat{k}_\nu|/\sqrt{n}\right)} \;\leq\; \left(\frac{2}{1 - e^{-\alpha/\sqrt{n}}}\right)^n \;=:\; M_\alpha\,, \tag{A.6}$$

uniformly in L. Thus, using Eqns. (I.20), we obtain that

$$e^{\mu\, d(i-j)}\,\big|\,H_{i,j}''(x) - H_{i,j}''(0)\big| \;\leq\; \sum_{k\in\Lambda_L} \widetilde{c}_{ij}(k)\left(|f'(x_k)| + |f'(x_k)|^2\right)\,, \tag{A.7}$$

where

$$\begin{aligned}\widetilde{c}_{ij}(k) \;:=\;& 2g\, e^{-(\nu-\mu)\, d(i-j)}\,(1-\delta_{ij})\,(\delta_{ik}+\delta_{jk}) \\ &+ \delta_{ij}\Big\{\delta_{ik}\big(1 + gM_\nu\big) \;+\; g\, e^{-\nu\, d(i-k)}\Big\}\,.\end{aligned} \tag{A.8}$$

Next, we derive a bound that enables us to control the derivatives of the form $\big\{|f'(x_k)|^m\big\}_{k\in\Lambda_L}$ in (A.7) by $\big\{|H_k'(x)|^m\big\}_{k\in\Lambda_L}$. By Eqn. (A.3), we may estimate the differences by

$$\big|\,H_k'(x) - f'(x_k)\,\big| \;\leq\; 2g\sum_{l\in\Lambda_L\setminus\{k\}} e^{-\nu\, d(k-l)}\Big\{|f'(x_k)| + |f'(x_l)|\Big\}\,, \tag{A.9}$$

for each $k \in \Lambda_L$. Hence, for $m = 1, 2$ and $k \in \Lambda_L$, we obtain from $M_\nu \geq 1$ that

$$\begin{aligned}|f'(x_k)|^m \;\leq\;& 3\,\big|H_k'(x)\big|^m \;+\; 12gM_\nu^2\,|f'(x_k)|^m \\ &+ 12gM_\nu^2 \sum_{l\in\Lambda_L\setminus\{k\}} e^{-\nu\, d(k-l)}|f'(x_l)|^m\,,\end{aligned} \tag{A.10}$$

where we have used Schwarz' inequality, in case that $m = 2$. Since $M_\nu \geq 1$ in (A.6), this implies that

$$\sum_{l \in \Lambda_L} \Big(\delta_{nl} - Q_{nl} \Big) \left(|f'(x_l)| + |f'(x_l)|^2 \right) \ \leq \ \Big(\big| H_n^{'}(x) \big| + \big| H_n^{'}(x) \big|^2 \Big) \, , \qquad \text{(A.11)}$$

for all $n \in \Lambda_L$, where Q is the self-adjoint matrix on $\mathbb{C}^{|\Lambda_L|}$ with matrix elements given by $Q_{kl} := 12 g M_\nu^2 e^{-\nu \, d(k-l)}$. Since these matrix elements only depend on the difference $k - l$, the norm of Q is bounded by $12 g M_\nu^2 \sum_{k \in \Lambda_L} e^{-\nu \, d(k)} = 12 g M_\nu^3$. Thus, for $g < M_\nu^{-3}/12$, the matrix $\mathbf{1} - Q$ is invertible, and its inverse R is given by the norm-convergent Neumann series

$$R \ := \ \Big(\mathbf{1} - Q \Big)^{-1} \ = \ \sum_{p=0}^{\infty} Q^p \, . \qquad \text{(A.12)}$$

Since $Q_{kn} \geq 0$, also $R_{kn} \geq 0$, and Inequality (A.11) is preserved under left multiplication by R. That is, multiplying (A.11) by R_{kn} and summing over n, we obtain

$$|f'(x_k)|^m \ \leq \ \sum_{l \in \Lambda_L} R_{kn} \, \big| H_n^{'}(x) \big|^m \, . \qquad \text{(A.13)}$$

Inserting (A.13) into (A.7), we arrive at

$$e^{\mu \, d(i-j)} \, \big| \, H''_{i,j}(x) - H''_{i,j}(0) \big| \ \leq \ \sum_{k \in \Lambda_L} c_{ij}(k) \Big(\big| H'_k(x) \big| + \big| H'_k(x) \big|^2 \Big) \, , \qquad \text{(A.14)}$$

with

$$c_{ij}(k) := \sum_{l \in \Lambda_L} \widetilde{c}_{ij}(l) \, R_{lk} \, . \qquad \text{(A.15)}$$

It remains to check the summability condition (I.16). We first observe that, for fixed $i, j \in \Lambda_L$,

$$\begin{aligned} \sum_{k \in \Lambda_L} c_{ij}(k) \ &= \ \sum_{k,l \in \Lambda_L} \widetilde{c}_{ij}(l) \, R_{lk} \ = \ \Big(\sum_{l \in \Lambda_L} \widetilde{c}_{ij}(l) \Big) \Big(\sum_{k \in \Lambda_L} R_{lk} \Big) \\ &= \ |\Lambda_L|^{-1} \Big(\sum_{l \in \Lambda_L} \widetilde{c}_{ij}(l) \Big) \Big(\sum_{k,l \in \Lambda_L} R_{kl} \Big) \, , \end{aligned} \qquad \text{(A.16)}$$

since R_{lk} depends only on the difference $k - l$. Using the normalized vector $\eta \in \mathbb{C}^{|\Lambda_L|}$, $\eta_k := |\Lambda_L|^{-1/2}$, we estimate the sum of the R_{kl} as follows:

$$|\Lambda_L|^{-1} \Big(\sum_{k,l \in \Lambda_L} R_{kl} \Big) \ = \ \langle \eta | \, R \, \eta \rangle \ \leq \ \| \, R \, \| \ \leq \ \frac{1}{1 \ - \ 12 g M_\nu^3} \, . \qquad \text{(A.17)}$$

Inserting (A.17) into (A.16) and summing over i and l, we have

$$\begin{aligned}\sum_{i,k\in\Lambda_L} c_{ij}(k) \quad &\leq \quad \sum_{i\in\Lambda_L}\left\{\delta_{ij}\frac{1+2gM_\nu}{1\,-\,12gM_\nu^3} \;+\; (1-\delta_{ij})\frac{4g\,e^{-(\nu-\mu)\,d(i-j)}}{1\,-\,12gM_\nu^3}\right\} \\ &\leq \quad \frac{1+6gM_{\nu-\mu}}{1\,-\,12gM_\nu^3}\,, \end{aligned} \tag{A.18}$$

for any $j \in \Lambda_L$. Next, we note that $\widetilde{c}_{ii}(l) = \delta_{il}(1+gM) + ge^{-\nu\, d(i-l)}$ only depends on the difference $i-l$. Similar to (A.16)–(A.18), we obtain

$$\sum_{i,j\in\Lambda_L} c_{ij}(k) \;\leq\; \frac{1+6gM_\nu}{1\,-\,12gM_\nu^3}\,, \tag{A.19}$$

for any $k \in \Lambda_L$. □

B Self-adjointness of D_H

In this appendix, we show that D_H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^{|\Lambda_L|}) \otimes \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]$, under Hypotheses 1 and 2. To this end, we follow the strategy in [16], p. 113. Since we assume that the function H is only C^2, we need to adapt the arguments, especially the elliptic regularity. Still assuming Hypotheses 1 and 2, we derive some useful properties of D_H, in particular a kind of integration by parts. This allows us to prove (II.13). Using further some results from [12], we also show (V.4). Here, we work in the representation

$$\mathcal{H} \;:=\; \bigoplus_{N=0}^{|\Lambda_L|} \mathcal{H}^{(N)} \;\cong\; L^2\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]\big)\,. \tag{B.1}$$

Lemma B.1. *The Dirac operator, given by (II.11), is essentially self-adjoint on* $C_0^\infty\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]\big)$.

Proof. It suffices to show that each equation $(D_H \pm i)u = 0$, for $u \in L^2\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f]\big)$, in the distributional sense, implies $u = 0$. (Here and henceforth we abbreviate $\mathcal{F}_f := \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]$.)
First, we note that, for $v \in C_0^\infty\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)$,

$$D_H v \;=\; D_0 v \;+\; M_H v\,, \tag{B.2}$$

where $D_0 = D_H$ for $H = 0$ and where M_H is a multiplication operator by a matrix with C^1 entries. Let $u \in L^2$ with $(D_H \pm i)u = 0$. Since $M_H u \in L^2_{loc}\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)$, we see that $(D_0 \pm i)u \in L^2_{loc}\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)$. Since, for any integer k,

$$\big(D_0^2 \,+\, 1\big)^{-1} : H^k\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big) \;\longrightarrow\; H^{k+2}\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)\,, \tag{B.3}$$

is bounded (H^k being the k^{th} Sobolev space), we have that

$$\big(D_0^2 + 1\big)^{-1}\chi(D_0 \pm i)u \in H^2\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)\,, \tag{B.4}$$

for any $\chi \in C_0^\infty(\mathbb{R}^{|\Lambda_L|}, \mathbb{C})$. Since the commutator $[\chi, D_0]$ is a multiplication by a bounded matrix, we find that

$$\big(D_0^2 + 1\big)^{-1}(D_0 \pm i)\chi u \in H^2\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)\,. \tag{B.5}$$

Thus, the following equality in $H^1\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)$,

$$\begin{aligned}(D_0 \mp i)\big(D_0^2 + 1\big)^{-1}\chi(D_0 \pm i)u &= (D_0 \mp i)\big(D_0^2 + 1\big)^{-1}[\chi, D_0]u \\ &\quad + (D_0 \mp i)\big(D_0^2 + 1\big)^{-1}(D_0 \pm i)\chi u\end{aligned} \tag{B.6}$$

holds, and the last term equals in fact χu. This yields $\chi u \in H^1\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)$ and $u \in H^1_{loc}\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)$. Since $C^1\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big) \subseteq H^1_{loc}\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)$, $M_H u \in H^1_{loc}\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)$. Following the previous lines again, we prove that $u \in H^2_{loc}\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f\big)$.
Now, let $f \in C_0^\infty(\mathbb{R}^{|\Lambda_L|}, \mathbb{R})$ with $f(x) = 1$ for $|x| \le 1$ and $f_n(x) = f(x/n)$. A direct computation shows that

$$\begin{aligned}(D_H \pm i)f_n u &= [D_H, f_n \otimes \mathbf{1}]u + 0 \\ &= \frac{1}{n}\Big(\sum_{j\in\Lambda_L} \beta^{-1}(\partial_{x_j} f)(x/n) \otimes \big(c_j^* - c_j\big)\Big)u\,.\end{aligned} \tag{B.7}$$

Thus

$$\|f_n u\|^2_{L^2} + \big\|D_H f_n u\big\|^2_{L^2} \le \frac{1}{n^2}\sup_x \big\|\nabla f\big\|^2_{l^\infty}\, \|u\|^2_{L^2}\,. \tag{B.8}$$

Since the norm of $f_n u$ converges to the norm of u, as n goes to infinity, we obtain $u = 0$. □

Lemma B.2. *("Integration by parts.") The domain of D_H is $\mathcal{D}(d_H) \cap \mathcal{D}(d_H^*)$, where*

$$\begin{aligned}\mathcal{D}(d_H) &= \Big\{u \in L^2\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]\big)\,\Big|\, d_H u \in L^2\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]\big)\Big\}\,, \\ \mathcal{D}(d_H^*) &= \Big\{u \in L^2\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]\big)\,\Big|\, d_H^* u \in L^2\big(\mathbb{R}^{|\Lambda_L|}, \mathcal{F}_f[\mathbb{C}^{|\Lambda_L|}]\big)\Big\}\,.\end{aligned}$$

Moreover $\mathcal{D}(D_H^2) \subseteq \mathcal{D}(D_H)$ and, for $u \in \mathcal{D}(D_H^2)$ and $v \in \mathcal{D}(D_H)$,

$$\langle D_H^2 u | v\rangle = \langle D_H u | D_H v\rangle = \langle d_H u | d_H v\rangle + \langle d_H^* u | d_H^* v\rangle\,. \tag{B.9}$$

Proof. We follow a standard argument, e.g., given in [11]. Let $u \in \mathcal{D}(D_H)$, $f_m(x) := f(x/m)$, and $F_n(x) := n^{|\Lambda_L|}\, f(nx)$, where $f \in C_0^\infty(\mathbb{R}^{|\Lambda_L|}; [0,1])$, as in the proof of Lemma B.1. Define

$$u_{m,n}(x) := \big(f_m(F_n * u)\big)(x) = f_m(x) \int F_n(y - x)\, u(y)\, d^{|\Lambda_L|}y\,. \tag{B.10}$$

Then $u_{m,n} \in C_0^\infty(\mathbb{R}^{|\Lambda_L|}; \mathcal{F}_f)$, and thus we have that

$$\| d_H u_{m,n} \|^2 + \| d_H^* u_{m,n} \|^2 = \| D_H u_{m,n} \|^2 . \tag{B.11}$$

Since $\lim_{m,n\to\infty} u_{m,n} = u$ (the order of the limits is irrelevant) and D_H is closed, it follows that $\lim_{m,n\to\infty} D_H u_{m,n} = D_H u$. Next, as in the proof of Lemma B.1, we have that

$$\| d_H^{\#} u_{m,n} - f_m d_H^{\#}(F_n * u) \| \le \frac{C_1}{m} , \tag{B.12}$$

for some constant C_1, since $[d_H^{\#}, f_m] = \mathcal{O}(m^{-1})$, where $d_H^{\#} = d_H$ or $d_H^{\#} = d_H^*$. Similarly, we observe that

$$\begin{aligned} & d_H^{\#}(F_n * u)(x) - (F_n * d_H^{\#} u)(x) \\ & \quad = \int F_n(x-y) \sum_{j\in\Lambda_L} \big(H_j'(x) - H_j'(y)\big)(c_j^{\#} u)(y)\, d^{|\Lambda_L|} y . \end{aligned} \tag{B.13}$$

Therefore, for any $\varphi \in \mathcal{H}$, an application of Schwarz' inequality yields

$$\begin{aligned} & \big|\big\langle \varphi \big|\, f_m\big(d_H^{\#}(F_n * u) - (F_n * d_H^{\#} u)\big) \big\rangle\big| \\ & \le \frac{|\Lambda_L|^2}{n} \sup_{|x|\le m+1} \big\{ \max_{i,j\in\Lambda_L} |H_{i,j}''(x)| \big\} \\ & \qquad \int F_n(x-y)\, \|\varphi(x)\|_{\mathcal{F}_f}\, \|u(y)\|_{\mathcal{F}_f}\, d^{|\Lambda_L|}x\, d^{|\Lambda_L|} y . \\ & \le \frac{C_2(\beta, L, m)}{n} \|\varphi\|\, \|u\| , \end{aligned} \tag{B.14}$$

for some constant $C_2(\beta, L, m)$. Choosing the number $m \in \mathbb{N}$ sufficiently large such that $C_1/m \le \varepsilon/2$, and afterwards choosing $n \in \mathbb{N}$ large enough so that $C_2(\beta, L, m)/n \le \varepsilon/2$, we observe that for any given $\varepsilon > 0$ we have

$$\| d_H^{\#} u_{m,n} - f_m(F_n * d_H^{\#} u) \| \le \varepsilon , \tag{B.15}$$

provided that m and n are sufficiently large. Therefore, $d_H u, d_H^* u \in \mathcal{H}$ and

$$\| d_H u \|^2 + \| d_H^* u \|^2 = \| D_H u \|^2 . \tag{B.16}$$

Thus $\mathcal{D}(D_H) \subseteq \mathcal{D}(d_H) \cap \mathcal{D}(d_H^*)$, and since the opposite inclusion is trivial, we have that

$$\mathcal{D}(D_H) = \mathcal{D}(d_H) \cap \mathcal{D}(d_H^*) . \tag{B.17}$$

The remaining parts of Lemma B.2 are now an immediate consequence of this fact.

□

Lemma B.3. *For sufficiently large $\beta > 0$, the kernels of the Dirac operator D_H and the Witten Laplacian $\Delta_H := D_H^2$ are given by (II.13), i.e.,*

$$\mathrm{Ker}\{D_H\} \;=\; \mathrm{Ker}\{\Delta_H\} \;=\; \mathbb{C}\cdot\left(e^{-\beta H}\otimes\Omega\right). \tag{B.18}$$

Proof. First, we note that $\mathrm{Ker}\{D_H\} \subseteq \mathrm{Ker}\{D_H^2\}$, since D_H is self-adjoint (see [13]). Second, we observe that $e^{-\beta H}\otimes\Omega \in \mathrm{Ker}\{D_H\}$, and hence we have

$$\mathrm{span}\{e^{-\beta H}\otimes\Omega\} \;\subseteq\; \mathrm{Ker}\{D_H\} \;=\; \mathrm{Ker}\{D_H^2\} \;=\; \mathrm{Ker}\{\Delta_H\}. \tag{B.19}$$

Theorem III.1 implies that $\mathrm{Ker}\{\Delta_H\} = \mathrm{Ker}\{A\}$, for $\beta > 0$ sufficiently large. Furthermore we recall from Eqn. (III.22) that

$$A \;\geq\; \Delta_H^{(0)}\otimes\mathbf{1} \;+\; \frac{2\,\lambda_{\min}}{\beta}\mathbf{1}\otimes\mathbb{N}_L\,. \tag{B.20}$$

Since the number operator $\mathbb{N}_L$ act as multiplication by the form degree, $(\mathbf{1}\otimes\mathbb{N}_L)|_{\mathcal{H}^{(\ell)}} = \ell(\mathbf{1}\otimes\mathbf{1})|_{\mathcal{H}^{(\ell)}}$, we conclude that

$$\mathrm{Ker}\{A\} \;=\; \mathrm{Ker}\{\Delta_H^{(0)}\}\otimes\Omega \;\subseteq\; \bigcap_{j\in\Lambda_L}\mathrm{Ker}\{Z_j(H)\}\otimes\Omega\,. \tag{B.21}$$

So, if $f \in \bigcap_{j\in\Lambda_L}\mathrm{Ker}\{Z_j(H)\}$ then $f_j'(x) = -H_j'(x)f(x)$, and elliptic regularity tells us that $f \in H^2_{\mathrm{loc}}(\mathbb{R}^{|\Lambda_L|})\cap L^2(\mathbb{R}^{|\Lambda_L|})$, since $H_j' \in C^1(\mathbb{R}^{|\Lambda_L|})$. Moreover, $e^{\beta H}f \in H^2_{\mathrm{loc}}(\mathbb{R}^{|\Lambda_L|})$ and

$$\nabla_x\left(e^{\beta H}f\right) \;=\; 0\,, \tag{B.22}$$

which implies that $e^{\beta H}f = const$, and this in turn gives $f \in \mathrm{span}\{e^{-\beta H}\}$. □

From the proofs of Lemmata B.2 and B.3 we also derive the following corollary

Corollary B.4. *For sufficiently large $\beta > 0$,*

$$\mathcal{D}(D_H) \;=\; \mathcal{Q}(\Delta_H) \;=\; \left\{\psi\in\mathcal{H}\;\Big|\;\sum_{j\in\Lambda_L}\|Z_j(H)\psi\|^2<\infty\right\}. \tag{B.23}$$

Finally we derive the following lemma.

Lemma B.5. *Under Hypothesis 1–2, the inequality (V.4) holds, that is*

$$\overline{P}\left(\Delta_H^{(0)}\otimes\mathbf{1}\right)\overline{P} \;\geq\; \inf\sigma(\Delta_H^{(1)})\,\overline{P}\,, \tag{B.24}$$

where $P = P_0\otimes\mathbf{1}$, P_0 is the orthogonal projection of L^2 onto $e^{-\beta H}$, and where $\sigma(\Delta_H^{(1)})$ denotes the spectrum of $\Delta_H^{(1)}$.

Proof. Since $\Delta_H^{(0)} \geq 0$, (B.24) is true if $\inf\sigma(\Delta_H^{(1)}) = 0$. Let us then assume that it is strictly positive. Note further that P projects onto the kernel of Δ_H, which is also the kernel of the restriction $d_H^{(0)}$ of d_H to $\mathcal{H}^{(0)}$. Thanks to [12], the restriction of $\Delta_H^{(0)}$ to the orthogonal of this kernel is unitarily equivalent to the restriction of $\Delta_H^{(1)}$ to the range of $d_H^{(0)}$. This proves (B.24). □

C About the ferromagnetic condition

In this section of the Appendix, we want to give another situation where we can obtain a similar asymptotics as in Lemma I.8. In Hypothesis 6, we require, as in [14], that the Hessian at 0 is of ferromagnetic type (i.e $v \geq 0$). This is not necessary to obtain the asymptotics of E_v as we shall show it in Lemma C.1 below, using the arguments in [14].
We consider a function v satisfying all the conditions in Hypothesis 6 but the non-negativity requirement. Starting from (VI.4) for v, we follow the reasoning in [14]. Recall that $F_v(i\,\cdot\,)$ is the Fourier transform of v. The key point is to move the contour of integration into $\mathbb{T}^d + i\mathbb{R}^d$ until we meet the singularities set

$$\{z \in \mathbb{T}^d + i\mathbb{R}^d;\ \hat{v}(z) = 1\}. \tag{C.1}$$

To this end, we introduce the function $\widetilde{F}_v$ given by (I.38) for $u = |v|$, which satisfies

$$\text{Re } \hat{v}(k + i\eta) \ \leq \ \sum_{n\in\mathbb{Z}^d} |v(n)|\cosh(n \cdot \eta) \ = \ \widetilde{F}_v(\eta). \tag{C.2}$$

Since we assume that v has compact support, $\widetilde{F}_v$ is well defined and analytic on $\mathbb{R}^d$. By (I.36), $\widetilde{F}_v$ is strictly convex everywhere and

$$\Sigma \ = \ \{\eta \in \mathbb{R}^d;\ \widetilde{F}_v(\eta) = 1\} \tag{C.3}$$

is the boundary of a strictly convex, relatively compact domain in $\mathbb{R}^d$ (see [14] or Lemma I.7 (ii)). Thus we have the same property as in [14] but with the new function $\widetilde{F}_v$. A good situation to derive the rate of the exponential decay appears when equality in (C.2) holds at a unique point $k_0 \in \mathbb{T}^d$. Let us describe this condition. Equality in (C.2) holds if and only if

$$\forall n \in \mathbb{Z}^d,\ v(n)\cos(n \cdot k) = |v(n)|, \tag{C.4}$$

If we choose v such that the group generated by $\{y \in \mathbb{Z}^d,\ v(y) > 0\}$ is $\mathbb{Z}^d$ and such that there exists some $y_0 \in \mathbb{Z}^d$ with $v(y_0) < 0$, then equality in (C.2) holds for no point k. On the other hand, let us construct a function v, with non-constant sign, for which equality in (C.2) holds for exactly one point $k_0 \in \mathbb{T}^d$. We split $\mathbb{Z}^d$ into the direct sum $\mathbb{Z}^{d-1} \oplus \mathbb{Z}$ and write

$$\mathbb{Z}^d \ni n \ = \ (\bar{n}, n_d) \in \mathbb{Z}^{d-1} \oplus \mathbb{Z}.$$

We choose v with

$$(-1)^{n_d} v(n) \geq 0, \tag{C.5}$$

such that its support is bounded and satisfies (I.36). Let $k_0 = (0, \cdots, 0, \pi) \in \mathbb{T}^d$, we have, according to (C.2),

$$\text{Re } \hat{v}(k + k_0 + i\eta) \ = \ \sum_{n\in\mathbb{Z}^d} |v(n)| \cos(n \cdot k)\cosh(n \cdot \eta).$$

Thus, equality in (C.4) holds if and only if $k = 0$ in $\mathbb{T}^d$.

The method in [14] allows us to derive

Lemma C.1. *Let v as above. Then the asymptotics given in Lemma I.8 holds with a new function $\widetilde{F}_v$ defined by* $\operatorname{Re}\hat{v}(k_0+i\eta) = \widetilde{F}_v(\eta)$*, where* $k_0 = (0,\cdots,0,\pi) \in \mathbb{T}^d$.

Acknowledgements

We are especially indebted to B. Helffer for many useful discussions we have had with him during the progress of our work and for making his results accessible to us prior to publication. Furthermore we are grateful to A. Bovier, J.-D. Deuschel, L. Erdős, J. Fröhlich, J. Johnsen, M. Klein, T. Spencer, H. T. Yau for various discussions and for their helpful comments. Finally, we thank V. Ivrii and M. Zworski at the Fields Institute in Toronto, T. Paul at the Université Paris-Dauphine (IX), and A. Laptev at the KTH Stockholm for their hospitality.

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Communicated by J. Bellissard
submitted 04/09/98; revised 22/02/99; accepted 02/03/99