Connected, disconnected and strange quark contributions to HVP

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ABSTRACT: We calculate all neutral vector two-point functions in Chiral Perturbation Theory (ChPT) to two-loop order and use these to estimate the ratio of disconnected to connected contributions as well as contributions involving the strange quark. We extend the ratio of $-1/10$ derived earlier in two flavour ChPT at one-loop order to a large part of the higher order contributions and discuss corrections to it. Our final estimate of the ratio disconnected to connected is negative and a few % in magnitude.

Contents

1 Introduction

The muon anomalous magnetic moment is one of the most precisely measured quantities around. The measurement [1] differs from the standard model prediction by about 3 to 4 sigma depending on precisely which theory predictions are taken. A review is [2] and talks on the present situation can be found in [3]. The main part of the theoretical error at present is from the lowest-order hadronic vacuum polarization (HVP). This contribution can be determined from experiment or can be computed using lattice QCD [4]. An overview of the present situation in lattice QCD calculations is given by [5].

The underlying object that needs to be calculated is the two-point function of electromagnetic currents as defined in (2.1). The contribution to $a_\mu = (g-2)/2$ is given by the integral in (2.9). There are a number of different contributions to the two-point function of electromagnetic currents that need to be measured on the lattice. First, if we only consider the light up and down quarks, there are connected and disconnected contributions depicted schematically in figure 1. If we add the strange quark to the electromagnetic currents then there are contributions with the strange electromagnetic current in both points and the mixed up-down and strange case. In this paper we provide estimates of all contributions at low energies using Chiral Perturbation Theory (ChPT).

The disconnected light quark contribution has been studied at one-loop order in ref. [6] using partially quenched (PQChPT). They found that the ratio in the subtracted form
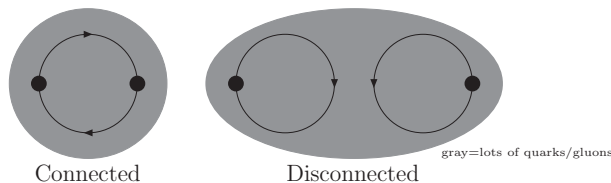
Figure 1. Connected (left) and disconnected (right) diagram for the two-point vector function. The lines are valence quark lines in a sea of quarks and gluons.

factors, as defined in (2.5), is $-1/2$ in the case of valence quarks of a single mass and two degenerate sea quarks. They also found that adding the strange quark did not change the ratio much. Here we give an argument explaining the factor of $-1/2$ and extend their analysis to order p^6. We also present estimates for the contributions from the strange electromagnetic current.

The finite volume, partially quenched and twisted boundary conditions extensions to two loop order will be presented in [7].

In section 2 we give the definitions of the two-point functions and currents we use. Section 3 discusses ChPT and the extra terms and low-energy-constants (LECs) needed for a singlet vector current. Our main analytical results, the two-loop order ChPT expressions for all needed vector two-point functions are in section 4. Section 5 uses the observation given in section 3 of the absence of singlet vector couplings to mesons until ChPT order p^6 to show for which contributions the ratio $-1/2$ is valid. Numerical results need an estimate of the LECs involved, both old and new. This is done in section 6 and applied there to the light connected and disconnected part. Because of the presence of the LECs we find a total disconnected contribution of opposite sign and size a few % of the connected contribution. The same type of estimates are then used for the strange quark contribution in section 7. Here we find a very strong cancellation between p^4 and p^6 contributions, leaving the LEC part dominating strongly. A comparison with a number of lattice results is done in section 8. We find a reasonable agreement in some cases. Our conclusions are summarized in section 9.

2 The vector two-point function

We define the two-point vector function as

$$\Pi_{ab}^{\mu\nu} = i \int d^4x e^{iq\cdot x} \left\langle T(j_a^\mu(x) j_b^{\nu\dagger}(0)) \right\rangle \tag{2.1}$$

where the labels a, b specify the involved currents. We label the currents as

$$
\begin{aligned}
j_{\pi^+}^\mu &= \bar{d}\gamma^\mu u \,, & j_U^\mu &= \bar{u}\gamma^\mu u \,, & j_D^\mu &= \bar{d}\gamma^\mu d \,, \\
j_S^\mu &= \bar{s}\gamma^\mu s \,, & j_{\text{EM}}^\mu &= \frac{2}{3}j_U^\mu - \frac{1}{3}j_D^\mu - \frac{1}{3}j_S^\mu \,, & j_{\text{EM2}}^\mu &= \frac{2}{3}j_U^\mu - \frac{1}{3}j_D^\mu \,, \\
j_{\pi^0}^\mu &= \frac{1}{\sqrt{2}}\left(j_U^\mu - j_D^\mu\right) \,, & j_{\text{I2}}^\mu &= \frac{1}{\sqrt{2}}\left(j_U^\mu + j_D^\mu\right) \,, & j_{\text{I3}}^\mu &= \frac{1}{\sqrt{3}}\left(j_U^\mu + j_D^\mu + j_S^\mu\right) \,.
\end{aligned} \tag{2.2}
$$

The divergence of the vector current is given by

$$\partial_\mu \bar{q}_i \gamma^\mu q_j = i(m_i - m_j)\bar{q}_i q_j \,, \tag{2.3}$$

which means that any current involving equal mass quark and anti-quark is conserved. Assuming isospin for the π^+ current, Lorentz invariance then implies that we can parametrize the vector two-point functions given above as

$$\Pi_{ab}^{\mu\nu}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_{ab}(q^2). \tag{2.4}$$

We also define the subtracted quantity

$$\hat{\Pi}_{ab}(q^2) = \Pi_{ab}(q^2) - \Pi_{ab}(0). \tag{2.5}$$

For simplicity we also use $\Pi_a = \Pi_{aa}$ and $\hat{\Pi}_a = \hat{\Pi}_{aa}$

In this paper we work in the isospin limit. This immediately leads to a number of relations

$$\Pi_{\pi^+} = \Pi_{\pi^0}, \qquad \Pi_U = \Pi_D, \qquad \Pi_{US} = \Pi_{DS}. \tag{2.6}$$

With those one can derive

$$\Pi_{\text{EM}} = \frac{5}{9}\Pi_U + \frac{1}{9}\Pi_S - \frac{4}{9}\Pi_{UD} - \frac{2}{9}\Pi_{US},$$
$$\Pi_{\text{EM2}} = \frac{5}{9}\Pi_U - \frac{4}{9}\Pi_{UD}. \tag{2.7}$$

The two-point functions Π are themselves not directly observable. However, the vector current two-point function in QCD satisfies a once subtracted dispersion relation

$$\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0) = q^2 \int_{threshold}^{\infty} ds \frac{1}{s(s-q^2)} \frac{1}{\pi} \text{Im}\Pi(s). \tag{2.8}$$

The imaginary part can be measured in hadron production if there exists an external vector boson like $W^\pm$ or the photon coupling to the current. Thus $\hat{\Pi}(q^2)$ is an observable, but not $\Pi(0)$. $\Pi(0)$ depends on the precise definitions used in regularizing the product of two currents in the same space-time point. The two-point functions for the electromagnetic current can be determined in e^+e^- collisions and Π_{π^+} in τ-decays.

One main use of $\hat{\Pi}$ is the determination of the lowest order HVP part of the muon anomalous magnetic moment via the integral over the electromagnetic two-point function[1]

$$a_\mu^{\text{LOHVP}} = 4\alpha^2 \int_0^\infty dQ^2 \hat{\Pi}_{\text{EM}}(-Q^2)g(Q^2),$$
$$g(Q^2) = \frac{-16m_\mu^4}{Q^6 \left(1 + \sqrt{1 + 4m_\mu^2/Q^2}\right)^4 \sqrt{1 + 4m_\mu^2/Q^2}}. \tag{2.9}$$

3 Chiral perturbation theory and the singlet current

ChPT describes low-energy QCD as an expansion in masses and momenta [10–12]. The dynamical degrees of freedom are the pseudo-Goldstone bosons (GB) from the spontaneous

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[1]The version mentioned here comes from [4] but the result essentially goes back to [8, 9].

breaking of the left- and right-handed flavor symmetry to the vector subgroup, $SU(3)_L \times SU(3)_R \to SU(3)_V$. The GBs can be parameterized in the $SU(3)$ matrix

$$U = e^{i\sqrt{2}M/F_0} \quad \text{with} \quad M = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}. \quad (3.1)$$

or with the 2×2 matrix with only the pions in the case of two-flavours. The Lagrangians, as well as the divergences, are known at order p^2 (LO), p^4 (NLO) and p^6(NNLO) in the ChPT counting [11–14]. However, the vector currents defined in section 2 contain also a singlet component and the Lagrangians including only this extension are not known. There is work when extending the symmetry to including the singlet GB as well as singlet vector and axial-vector currents at p^4 [15] and p^6 [16]. However this contains very many more terms than we need. If we only add the singlet vector current, in addition to simply extending the external vector field to include the singlet part, there are two extra terms relevant at order p^4:

$$H_3 \left(\langle F_{L\mu\nu} \rangle \langle F_L^{\mu\nu} \rangle + \langle F_{R\mu\nu} \rangle \langle F_R^{\mu\nu} \rangle \right) + H_4 \langle F_{R\mu\nu} \rangle \langle F_L^{\mu\nu} \rangle . \quad (3.2)$$

Since we are only interested in two-point functions of vector currents these will always appear in the combination $2H_3 + H_4$. For the two-flavour case we get $H_3 \to h_4$ and $H_4 \to h_5$ but otherwise similar terms.

It should be noted that none of the terms in the extended p^4 Lagrangian contains couplings of the singlet vector-field to the GB. The singlet appearing in commutators vanishes and the terms involving field strengths vanish, except for the combinations above which do not contain GB fields.

At order p^6 there are many more terms, specifically there are terms appearing that contain interactions of the singlet vector field with the GBs. Two examples are

$$\left\langle F_{R\mu\nu} \chi F_L^{\mu\nu} U^\dagger \right\rangle + \left\langle F_{L\mu\nu} \chi^\dagger F_R^{\mu\nu} U \right\rangle , \qquad \langle F_{L\mu\nu} + F_{R\mu\nu} \rangle \left\langle \left(\chi U^\dagger + U \chi^\dagger \right) D^\mu U D^\nu U^\dagger \right\rangle . \quad (3.3)$$

The extra terms that contribute to the vector two-point function at order p^6 always contain two field strengths and the extra p^2 needed can come from either two derivatives or quark masses. Setting all GB fields to zero, the only possible extra terms have a structure with $F_{V\mu\nu}$ the vector-field field strength and $\bar{\chi}$ the quark mass part of χ. This leads to the possible terms

$$D_1 \langle F_{V\mu\nu} \rangle \langle F_V^{\mu\nu} \bar{\chi} \rangle + D_2 \langle F_{V\mu\nu} \rangle \langle F_V^{\mu\nu} \rangle \langle \bar{\chi} \rangle + D_3 \langle \partial_\rho F_{V\mu\nu} \rangle \langle \partial^\rho F_V^{\mu\nu} \rangle \quad (3.4)$$

The D_i are linear combinations of a number of LECs in the Lagrangian and one can check that they are all independent by writing down a few fully chiral invariant terms. A similar set with $D_i \to d_i$ exists for the two-flavour case.

There is a coupling of the singlet vector current to the GBs already at order p^4 via the Wess-Zumino-Witten (WZW) term. However, due to the presence of $\epsilon^{\mu\nu\alpha\beta}$ we need an even number of insertions of the WZW term or higher order terms from the odd-intrinsic-parity sector to get a contribution to the vector two-point functions.

4 ChPT results up to two-loop order

The vector two-point functions for neutral non-singlet currents were calculated in [17, 18]. We have reproduced their results and added the parts coming from the singlet currents.

The expressions for the two-point functions are most simply expressed in terms of the function

$$\mathcal{G}(m^2, q^2) \equiv \frac{1}{q^2} \left(\overline{B}_{22}(m^2, m^2, q^2) - \frac{1}{2}\overline{A}(m^2) \right) \tag{4.1}$$

The one-loop integrals here are defined in many places, see e.g. [18]. The explicit expression is

$$\mathcal{G}(m^2, q^2) = \frac{1}{16\pi^2} \left[\frac{1}{36} + \frac{1}{12} \log \frac{m^2}{\mu^2} + \frac{q^2 - 4m^2}{12} \int_0^1 dx \log \left(1 - x(1-x)\frac{q^2}{m^2} \right) \right]$$
$$= \frac{1}{16\pi^2} \left(\frac{1}{12} + \frac{1}{12} \log \frac{m^2}{\mu^2} - \frac{q^2}{12m^2} - \frac{q^4}{1680m^4} + \cdots \right) \tag{4.2}$$

We also need

$$\overline{A}(m^2) = -\frac{m^2}{16\pi^2} \log \frac{m^2}{\mu^2} \, . \tag{4.3}$$

μ is the ChPT subtraction scale. We always work in the isospin limit. The expressions we give are in the three flavour case with physical masses. We will quote the corresponding results with lowest order masses in [7].

The two-point functions only start at p^4. We therefore write the result as

$$\Pi = \Pi^{(4)} + \Pi^{(6)} + \cdots \tag{4.4}$$

in the chiral expansion. The p^4 results are

$$\Pi_{\pi^+}^{(4)} = -8\mathcal{G}(m_\pi^2, q^2) - 4\mathcal{G}(m_K^2, q^2) - 4(L_{10}^r + 2H_1^r) \, ,$$

$$\Pi_U^{(4)} = -4\mathcal{G}(m_\pi^2, q^2) - 4\mathcal{G}(m_K^2, q^2) - 4(L_{10}^r + 2H_1^r + 2H_3^r + H_4^r) \, ,$$

$$\Pi_S^{(4)} = -8\mathcal{G}(m_K^2, q^2) - 4(L_{10}^r + 2H_1^r + 2H_3^r + H_4^r) \, ,$$

$$\Pi_{UD}^{(4)} = 4\mathcal{G}(m_\pi^2, q^2) - 4(2H_3^r + H_4^r) \, ,$$

$$\Pi_{US}^{(4)} = 4\mathcal{G}(m_K^2, q^2) - 4(2H_3^r + H_4^r) \, ,$$

$$\Pi_{\text{EM}}^{(4)} = -4\mathcal{G}(m_\pi^2, q^2) - 4\mathcal{G}(m_K^2, q^2) - \frac{8}{3}(L_{10}^r + 2H_1^r) \, . \tag{4.5}$$

The obvious relations visible for the $\mathcal{G}$ terms will be discussed in section 5. This result agrees with [6] when the appropriate limits are taken.

The results at p^6 are somewhat longer but still fairly short.

$$
\begin{aligned}
F_\pi^2\Pi_{\pi^+}^{(6)} = {} & 4q^2\left(2\mathcal{G}(m_\pi^2,q^2)+\mathcal{G}(m_K^2,q^2)\right)^2 - 16q^2 L_9^r\left(2\mathcal{G}(m_\pi^2,q^2)+\mathcal{G}(m_K^2,q^2)\right) \\
& - 8(L_9^r+L_{10}^r)\left(2\overline{A}(m_\pi^2)+\overline{A}(m_K^2)\right) - 32m_\pi^2 C_{61}^r \\
& - 32(m_\pi^2+2m_K^2)C_{62}^r - 8q^2 C_{93}^r\,, \\
F_\pi^2\Pi_U^{(6)} = {} & 8q^2\mathcal{G}(m_\pi^2,q^2)^2 + 8q^2\mathcal{G}(m_\pi^2,q^2)\mathcal{G}(m_K^2,q^2) + 8q^2\mathcal{G}(m_K^2,q^2)^2 \\
& - 16q^2 L_9^r\left(\mathcal{G}(m_\pi^2,q^2)+\mathcal{G}(m_K^2,q^2)\right) - 8(L_9^r+L_{10}^r)\left(\overline{A}(m_\pi^2)+\overline{A}(m_K^2)\right) \\
& - 32m_\pi^2 C_{61}^r - 32(m_\pi^2+2m_K^2)C_{62}^r - 8q^2 C_{93}^r - 4m_\pi^2 D_1^r \\
& - 4(m_\pi^2+2m_K^2)D_2^r - 4q^2 D_3^r\,, \\
F_\pi^2\Pi_S^{(6)} = {} & 24q^2\mathcal{G}(m_K^2,q^2)^2 - 32q^2 L_9^r\mathcal{G}(m_K^2,q^2) - 16(L_9^r+L_{10}^r)\overline{A}(m_K^2) \\
& - 32(2m_K^2-m_\pi^2)C_{61}^r - 32(m_\pi^2+2m_K^2)C_{62}^r - 8q^2 C_{93}^r \\
& - 4(2m_K^2-m_\pi^2)D_1^r - 4(m_\pi^2+2m_K^2)D_2^r - 4q^2 D_3^r\,, \\
F_\pi^2\Pi_{UD}^{(6)} = {} & -8q^2\mathcal{G}(m_\pi^2,q^2)^2 - 8q^2\mathcal{G}(m_\pi^2,q^2)\mathcal{G}(m_K^2,q^2) + 4q^2\mathcal{G}(m_K^2,q^2)^2 \\
& + 16q^2 L_9^r\mathcal{G}(m_\pi^2,q^2) + 8(L_9^r+L_{10}^r)\overline{A}(m_\pi^2) \\
& - 4m_\pi^2 D_1^r - 4(m_\pi^2+2m_K^2)D_2^r - 4q^2 D_3^r\,, \\
F_\pi^2\Pi_{US}^{(6)} = {} & -12q^2\mathcal{G}(m_K^2,q^2)^2 + 16q^2 L_9^r\mathcal{G}(m_K^2,q^2) + 8(L_9^r+L_{10}^r)\overline{A}(m_K^2) \\
& - 4m_K^2 D_1^r - 4(m_\pi^2+2m_K^2)D_2^r - 4q^2 D_3^r\,.
\end{aligned}
\tag{4.6}
$$

For the two-flavour case the results can be derived from the above. First, only keep the integral terms with m_π^2, second replace L_9 by $-(1/2)l_6^r$, $L_{10}^r+2H_1^r$ by $-4h_2^r$ and L_{10}^r by l_5^r. In addition there are also extra counterterms for the singlet current appearing. The results are

$$
\begin{aligned}
\Pi_{\pi^+}^{(4)} = {} & -8\mathcal{G}(m_\pi^2,q^2) + 16h_2^r\,, \\
\Pi_U^{(4)} = {} & -4\mathcal{G}(m_\pi^2,q^2) + 16h_2^r - 4(2h_4^r+h_5^r)\,, \\
\Pi_{UD}^{(4)} = {} & 4\mathcal{G}(m_\pi^2,q^2) - 4(2h_4^r+h_5^r)\,, \\
\Pi_{\mathrm{EM}}^{(4)} = {} & -4\mathcal{G}(m_\pi^2,q^2) + \frac{32}{3}h_2^r - \frac{4}{9}(2h_4^r+h_5^r)\,, \\
F_\pi^2\Pi_{\pi^+}^{(6)} = {} & 16q^2\mathcal{G}(m_\pi^2,q^2)^2 + 16q^2 l_6^r\mathcal{G}(m_\pi^2,q^2) - 8(2l_5^r-l_6^r)\overline{A}(m_\pi^2) - 32m_\pi^2 c_{34}^r - 8q^2 c_{56}^r\,, \\
F_\pi^2\Pi_U^{(6)} = {} & 8q^2\mathcal{G}(m_\pi^2,q^2)^2 + 8q^2 l_6^r\mathcal{G}(m_\pi^2,q^2) - 4(2l_5^r-l_6^r)\overline{A}(m_\pi^2) \\
& - 32m_\pi^2 c_{34}^r - 8q^2 c_{56}^r - 4m_\pi^2(d_1^r+2d_2^r) - 4q^2 d_3^r\,, \\
F_\pi^2\Pi_{UD}^{(6)} = {} & -8q^2\mathcal{G}(m_\pi^2,q^2)^2 - 8q^2 l_6^r\mathcal{G}(m_\pi^2,q^2) + 4(2l_5^r-l_6^r)\overline{A}(m_\pi^2) \\
& - 4m_\pi^2(d_1^r+2d_2^r) - 4q^2 d_3^r\,.
\end{aligned}
\tag{4.7}
$$

5 Connected versus disconnected contributions

If we look at the flavour content of the two-point functions in the isospin limit, it is clear that Π_{π^+} only contains connected contributions while Π_{UD} only contains disconnected

contributions. This is derived by thinking of which quark contractions can contribute as shown in figure 1. In the same way Π_U contains both with

$$\Pi_U = \Pi_{\pi^+} + \Pi_{UD}. \tag{5.1}$$

Inspection of all the results in section 4 shows that (5.1) is satisfied. From (2.7) we thus obtain

$$\Pi_{\text{EM2}} = \frac{5}{9}\Pi_{\pi^+} + \frac{1}{9}\Pi_{UD}, \tag{5.2}$$

and

$$\Pi_{\text{EM}} = \frac{5}{9}\Pi_{\pi^+} + \frac{1}{9}\Pi_{UD} - \frac{2}{9}\Pi_{US} + \frac{1}{9}\Pi_S. \tag{5.3}$$

Π_{US} is fully disconnected while Π_S has both connected and disconnected parts.

5.1 Two-flavour and isospin arguments

In [6], they found, using NLO two-flavour ChPT in the isospin limit, that

$$\frac{\hat{\Pi}_{\text{EM2}}^{\text{Disc}}}{\hat{\Pi}_{\text{EM2}}^{\text{Conn}}} = -\frac{1}{10}. \tag{5.4}$$

They also calculated corrections to this ratio due to the inclusion of strange quarks. Their result is in our terms expressed via

$$\frac{\hat{\Pi}_{UD}^{(4)}}{\hat{\Pi}_{\pi^+}^{(4)}} = -\frac{1}{2} \tag{5.5}$$

which is clearly satisfied for the results shown in (4.7). Note that $\Pi(q^2)$, via the part coming from the LECs, does not satisfy a similar relation due to the extra terms possible for the singlet current. Inspection of (4.7) shows that the loop part at order p^6 also satisfies (5.5) but due to the part of the LECs, the relation is no longer satisfied even for the subtracted functions $\hat{\Pi}$.

The relation (5.5) can be derived in a more general way. As noted in section 3 the singlet current j_{12}^μ only couples to GBs at order p^6 or at order p^4 via the WZW term and we need at least two of the latter for the vector two-point function. For the contributions where those couplings are not present, denoted by a tilde, we get

$$\widetilde{\Pi}_{U(U+D)} = \widetilde{\Pi}_U + \widetilde{\Pi}_{UD} = 0, \tag{5.6}$$

The relation (5.6) if written for $\hat{\Pi}$ has corrections at order p^8. Eq. (5.6) together with (5.1) immediately leads to (5.5) but for many more contributions. The ratio of disconnected to connected is $-1/2$ for all loop-diagrams only involving vertices from the lowest-order Lagrangian or from the normal NLO Lagrangian. So the ratio is true for a large part of all higher order loop diagrams and corrections start appearing only in loop diagrams at order p^8 with one insertion from the p^6-Lagrangian or at p^{10} with two insertions of a WZW vertex. The argument includes diagrams with four or more pions.

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Using the isospin relations we can derive that

$$\Pi_{UD} = \frac{1}{2} \left(\Pi_{\mathrm{I2}} - \Pi_{\pi^0} \right) \tag{5.7}$$

Looking at (5.7), one can see that the ratio $(-1/2)$ is exact for all contributions with isospin $I = 1$ and only broken due to $I = 0$ contributions. This can be used as well to estimate the size of the ratio, see below and [19, 20]. A corollary is that two-pion intermediate state contributions obey (5.5) to all orders.

The contributions to order p^6 for $\hat{\Pi}$ satisfy the relation (5.6) up to the LEC contributions. Using resonance saturation, the LECs can be estimated from ρ and ω exchange. In the large N_c limit that combination will only contribute to the connected contribution. Since the ρ-ω mass splitting and coupling differences are rather small, we expect that the disconnected contribution from this source will be rather small. This will lower the ratio of disconnected to connected contributions compared to (5.5).

In [19] it was also noticed that the ratio of $-1/2$ is valid for all two-pion intermediate states in the isospin limit. They used the slow turn-on of the three-pion channel where the singlet starts contributing to argue for the validity of the one-loop estimate. That slow turn-on follows from the three-pion contribution being p^{10} in our way of looking at it. In [20] the difference between ρ and ω measured masses and couplings were used to obtain an estimate of the disconnected contribution of about -1%. We consider that contribution to be within the error of our estimate given in section 6.

5.2 Three flavour arguments

It was already noted in [6] that kaon loops violate the relation (5.5) in NLO three-flavour ChPT and the same is rather visible in the results (4.5) and (4.6).

The argument for the singlet current coupling to mesons is just as true in three- as in two-flavour ChPT. However here one needs to use the three-flavour singlet current, j_{I3}^{μ}, instead. Again denoting with a tilde the contributions from loop diagrams involving only lowest order vertices or NLO vertices not from the WZW term, we have (after using isospin) two relations similar to (5.6)

$$\begin{aligned}
\widetilde{\Pi}_{U(U+D+S)} &= \widetilde{\Pi}_U + \widetilde{\Pi}_{UD} + \widetilde{\Pi}_{US} = 0 \,, \\
\widetilde{\Pi}_{S(U+D+S)} &= 2\widetilde{\Pi}_{US} + \widetilde{\Pi}_S = 0 \,.
\end{aligned} \tag{5.8}$$

Note that in this subsection we talk about the three-flavour ChPT expressions. Inspection of the expressions in (4.5) and (4.6) show that the relations (5.8) are satisfied. Note that the relation (5.8) if written for $\hat{\Pi}$ has corrections at order p^8.

In general we can write using (5.8)

$$\frac{\widetilde{\Pi}_{UD}}{\widetilde{\Pi}_{\pi^+}} = -\frac{1}{2} - \frac{\widetilde{\Pi}_{US}}{2\widetilde{\Pi}_{\pi^+}} \,. \tag{5.9}$$

This indicates that corrections to the $-1/2$ are expected to be small due to the strange quark being much heavier than the up and down quarks.

The second relation in (5.8) allows a relation involving two-point functions with the strange quark current.

Note that a consequence of (5.8) in the equal mass limit is

$$m_u = m_d = m_s \implies \frac{\widetilde{\Pi}_{UD}}{\widetilde{\Pi}_{\pi^+}} = -\frac{1}{3}. \tag{5.10}$$

In this case the disconnected contribution to the electromagnetic two-point function vanishes identically since the charge matrix is traceless.

6 Estimate of the ratio of disconnected to connected

In order to estimate the ratio of disconnected to connected contributions in ChPT the inputs that appear must be determined. For the plots shown below we use

$$F_\pi = 92.2 \text{ MeV} \qquad m_\pi = 135 \text{ MeV} \qquad m_K = 495 \text{ MeV}$$
$$L_9^r = 0.00593 \qquad \mu = 770 \text{ MeV} \tag{6.1}$$

The values for the decay constant and masses are standard ones. The values for the L_i^r were recently reviewed in [21] and we have taken the values for L_9^r from [22] quoted in [21].

If we only consider $\hat{\Pi}$, the only other LECs we need are C_{93}^r and D_3^r. As first suggested in [23] LECs are expected to be saturated by resonances. For C_{93}^r and D_3^r the main contribution will be from the vector resonance multiplet. Here a nonet approach typically works well and that would suggest that $D_3^r \approx 0$. We will set it to zero in our estimates. The value for C_{93}^r was first determined using resonance saturation in [18] with a value of

$$C_{93}^r = -1.4 \ 10^{-4} \tag{6.2}$$

If we use resonance saturation for the nonet and the constraints from short-distance as used in [24] we obtain for the two-point function

$$\Pi_{\pi^+}^{\text{VMD}}(q^2) = \frac{4F_\pi^2}{m_V^2 - q^2}. \tag{6.3}$$

Assuming that the pure LEC parts reproduce (6.3), leads to the value

$$C_{93}^r = -1.02 \ 10^{-4} \tag{6.4}$$

with $m_V = 770 \text{ MeV}$. Finally fitting the expression for Π_{π^+} to a phenomenological form of the two-point function [25] gives

$$C_{93}^r = -1.33 \ 10^{-4} \tag{6.5}$$

The three values are in reasonable agreement. The size can be compared to other vector meson dominated combinations of LECs, e.g. $C_{88}^r - C_{90}^r = -0.55 \ 10^{-4}$ [22], which is of the same magnitude. In the numerical results we will use the full expression (6.3) for the contribution from higher order LECs rather than just the terms with C_{93}^r.
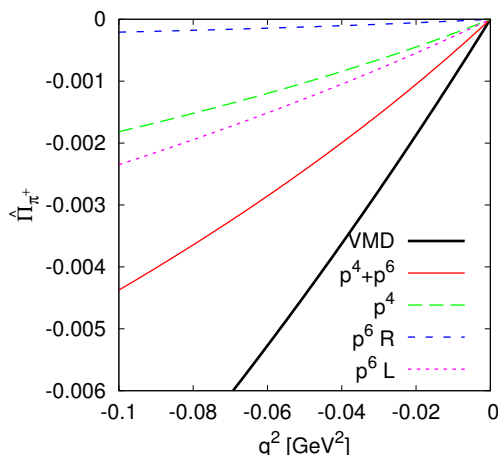
Figure 2. The subtracted two-point function $\hat{\Pi}_{\pi^+}(q^2)$ or the connected part. Plotted are the p^4 contribution of (4.5) labeled p^4 and the three parts of the higher order contribution: the pure two-loop contribution labeled p^6 R, the p^6 contribution from one-loop graphs labeled p^6 L and the pure LEC contribution as modeled by (6.3) labeled VMD.
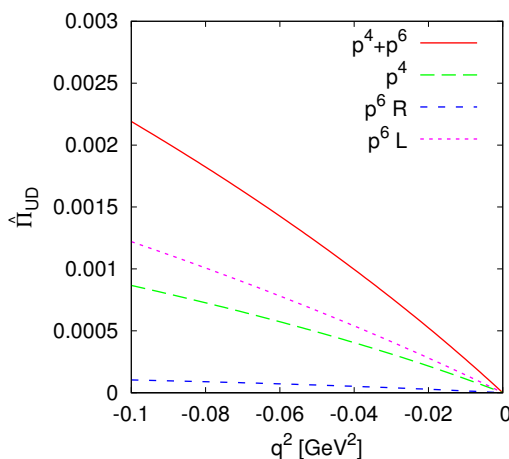

Figure 3. The subtracted two-point function $\hat{\Pi}_{UD}(q^2)$ or the disconnected part. Plotted are the p^4 contribution of (4.5) labeled p^4 and the two non-zero parts of the higher order contribution: the pure two-loop contribution labeled p^6 R and the p^6 contribution from one-loop graphs labeled p^6 L. The the pure LEC contribution is estimated to be zero here.

In figure 2 we have plotted the different contributions to $\hat{\Pi}_{\pi^+}$. This is what is usually called the connected contribution. As we see, the contribution from higher order LECs, as modeled by (6.3), is, as expected, dominant. The full result for $\hat{\Pi}$ is the sum of the VMD and the $p^4 + p^6$ lines. We see that the pure two-loop contribution is small compared to the one-loop contribution but there is a large contribution at order p^6 from the one-loop diagrams involving L_i^r.

In figure 3 we have plotted the same contributions but now for $\hat{\Pi}_{UD}$ or the contribution from disconnected diagrams. Note that the scale is exactly half that of figure 2. The contributions are very close to $-1/2$ times those of figure 2 except for the pure LEC contribution which is here estimated to be zero.
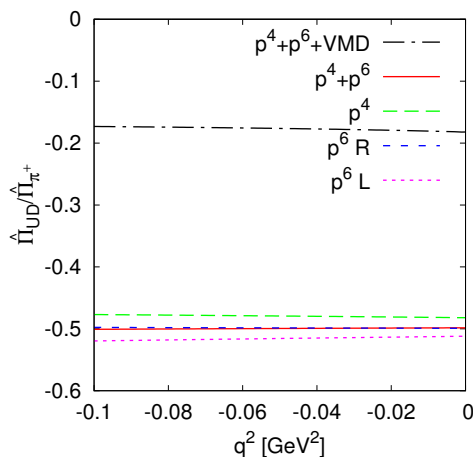
Figure 4. The ratio of the subtracted two-point functions $\hat{\Pi}_{UD}(q^2)/\hat{\Pi}_{\pi^+}(q^2)$ or ratio of the disconnected to the connected part. Plotted are the p^4 contribution of (4.5) labeled p^4, the parts of the higher order contribution: the pure two-loop contribution labeled $p^6\,R$ and the p^6 contribution from one-loop graphs labeled $p^6\,L$ as well as their sum. The ratio of the pure LEC contribution is estimated to be zero. The ratio for all contributions summed is the dash-dotted line.

How well do the estimates of the ratio now hold up. The ratio of disconnected to connected is plotted in figure 4. We see that the contribution at order p^4 has a ratio very close to $-1/2$ and the same goes for all loop contributions at order p^6. The effects of kaon loops is thus rather small. The deviation from $-1/2$ is driven by the estimate of the pure LEC contribution. Using the VMD estimate (6.3) we end up with a ratio of about -0.18 for the range plotted. Taking into account (5.2) we get an expected ratio for the disconnected to connected contribution to the light quark electromagnetic two-point function $\hat{\Pi}_{\mathrm{EM2}}$ of about -3.5%. If we had used the other estimates for C_{93}^r (and assumed a similar ratio for higher orders) the number would have been about -3%.

An analysis using only the pion contributions, so no contribution from intermediate kaon states, would give essentially the same result.

7 Estimate of the strange quark contributions

The numerical results in the previous section included the contribution from kaons but only via the electromagnetic couplings to up and down quarks. In this section we provide an estimate for the contribution when including the photon coupling to strange quarks, i.e. we add the terms coming from Π_{US} and Π_S in (5.3).

The loop contributions satisfy the relations shown in (5.8) with corrections starting earliest at p^8. Alternatively we can write the first relation as

$$\widetilde{\Pi}_{\pi^+} + 2\widetilde{\Pi}_{UD} + \widetilde{\Pi}_{US} = 0\,, \tag{7.1}$$

this, together with the ratios shown in figure 4 and the second relation in (5.8), shows that we can expect the extra contributions to be quite small with the possible exception of the pure LEC contribution.
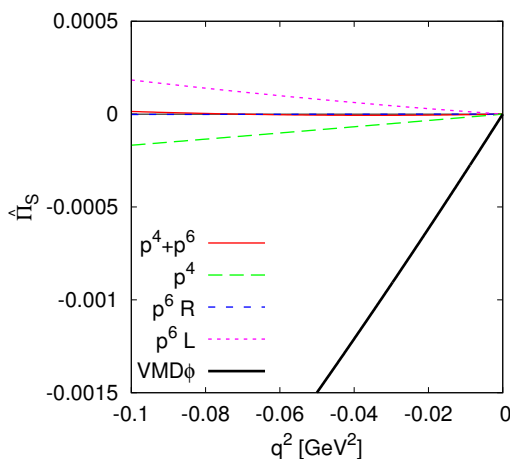
Figure 5. The subtracted two-point function $\hat{\Pi}_S(q^2)$. Plotted are the p^4 contribution of (4.5) labeled p^4, the parts of the higher order contribution: the pure two-loop contribution labeled p^6 R and the p^6 contribution from one-loop graphs labeled p^6 L as well as their sum. The pure LEC contribution is estimated by (6.3) with the mass of the ϕ.

The pure LEC contribution is estimated to only apply to the connected part and so contributes only to Π_S. Given that the ϕ mass is significantly larger than the ρ-mass we will for that part need to include this difference. A first estimate is simply by using (6.3) with m_V now the ϕ-mass of $m_\phi = 1020\,\text{MeV}$. We will call this VMD$\phi$ in the remainder.

The estimate we include for Π_S includes both connected and disconnected contributions. We would need to go to partially quenched ChPT to obtain that split-up generalizing the methods of [6].

Figure 5 shows the different contributions to $\hat{\Pi}_S$. We did not plot $\hat{\Pi}_{US}$ since the relations (5.8) imply that the p^4, p^6L and p^6R are exactly $-1/2$ the contributions for $\hat{\Pi}_S$ and in our estimate the pure LEC part for $\hat{\Pi}_{US}$ vanishes. The contributions are much smaller than those of the connected light quark contribution shown in figure 2. One remarkable effect is that the very strong cancellations between the p^4 and p^6 effects give an almost zero loop contribution. This means that vector meson dominance in the coupling is even more clear in this case than for the lighter quarks.

8   Comparison with lattice and other data

For comparing with lattice and phenomenological data we can use the Taylor expansion around $q^2 = 0$ from our expressions and the same coefficients evaluated from experimental data or via the time moment analysis on the lattice [26].

We expand the functions as

$$\hat{\Pi}(q^2) = \Pi_1 q^2 - \Pi_2 q^4 + \cdots \tag{8.1}$$

The signs follow from the fact that the lattice expansion is defined in terms of $Q^2 = -q^2$ and the usual lattice convention for Π has the opposite sign of ours. The coefficients,

Reference	Π_A	Π_1 (GeV^{-2})	Π_2 (GeV^{-4})
Π^{VMD}	$\hat{\Pi}_{\pi+}$	0.0967	-0.163
p^4	$\hat{\Pi}_{\pi+}$	0.0240	-0.091
$p^6\ R$	$\hat{\Pi}_{\pi+}$	0.0031	-0.014
$p^6\ L$	$\hat{\Pi}_{\pi+}$	0.0286	-0.067
sum	$\hat{\Pi}_{\pi+}$	0.152	-0.336
[28]	$\hat{\Pi}_{\pi+}$	0.1657(16)(18)	$-0.297(10)(05)$
[29]	$\hat{\Pi}_{\pi+}$	0.1460(22)	$-0.2228(65)$
p^4	$\hat{\Pi}_{UD}$	-0.0116	0.045
$p^6\ R$	$\hat{\Pi}_{UD}$	-0.0015	0.007
$p^6\ L$	$\hat{\Pi}_{UD}$	-0.0146	0.032
sum	$\hat{\Pi}_{UD}$	-0.0278	0.085
[28]	$\hat{\Pi}_{UD}$	$-0.015(2)(1)$	0.046(10)(04)
$\Pi^{VMD\phi}$	$\hat{\Pi}_S$	0.0314	-0.030
p^4	$\hat{\Pi}_S$	0.0017	-0.001
$p^6\ R$	$\hat{\Pi}_S$	0.0000	0.000
$p^6\ L$	$\hat{\Pi}_S$	-0.0013	-0.005
sum	$\hat{\Pi}_S$	0.0318	-0.035
[28]	$\hat{\Pi}_S$	0.0657(1)(2)	$-0.0532(1)(3)$
[26]	$\hat{\Pi}_S$	0.06625(74)	$-0.0526(11)$
our result	$\hat{\Pi}_{\rm EM}$	0.0852	-0.182
[27]	$\hat{\Pi}_{\rm EM}$	0.0990(7)	$-0.206(2)$
[28]	$\hat{\Pi}_{\rm EM}$	0.0972(2)(1)	$-0.166(6)(3)$

Table 1. The Taylor expansion coefficients of $\hat{\Pi}$ of [26–29] and a comparison with our estimates.

obtained by fitting an eight-order polynomial to the ranges shown in the plots, are given in table 1. Ref. [27] is from an analysis of experimental data. Ref. [28] are preliminary numbers from the BMW collaboration and we have removed the charm quark contribution from their numbers. These numbers are not corrected for finite volume. For [26, 29] we have taken the numbers from their configuration 8, which has physical pion masses and multiplied by 9/5 for the latter to obtain $\Pi_{\pi+}$. Our estimates are in reasonable agreement for the connected contribution. For the disconnected contribution, our results are higher but of a similar order.

There have been many more studies of the muon $g-2$ on the lattice and in particular a number of studies of the disconnected part. However, their results are often not presented in a form that we can easily compare to. From our numbers above we expect the disconnected contribution to be a few % and of the opposite sign of the connected contribution. Ref. [20] finds $-0.15(5)\%$, much smaller than we expect, [30] finds about -1.5% which is below but of the same order as our estimate.

The same comment applies to studies of the strange contribution, e.g. [31] finds a contribution of about 7% of the light connected contribution which is in reasonable agreement with our estimate.

9 Summary and conclusions

We have calculated in two- and three-flavour ChPT all the neutral two-point functions in the isospin limit including the singlet vector current. We have extended the ratio of $-1/2$ (or $-1/10$ for the electromagnetic current) of [6] to a large part of the higher order loop corrections. We used the nonet estimates of LECs to set the new constants for the singlet current equal to zero and then provided numerical estimates for the disconnected and strange quark contributions.

We find that the disconnected contribution is negative and a few % of the connected contribution, the main uncertainty being the new LECs which we estimated to be zero. A similar estimate for the strange quark contribution has a large cancellation between p^4 and p^6 leaving our rather uncertain estimate of the LECs involved as the main contribution.

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