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VEV of Baxter's Q-operator in $N = 2$ gauge theory and the BPZ differential equation

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Abstract: In this short note using AGT correspondence we express the simplest fully degenerate primary fields of Toda field theory in terms of an analogue of Baxter's Q-operator naturally emerging on the $\mathcal{N} = 2$ gauge theory side. This quantity can be considered as a generating function of certain chiral operators constructed from the scalars of the $\mathcal{N} = 2$ vector multiplets. In the special case of Liouville theory, exploring the second order differential equation satisfied by conformal blocks including a primary field which is degenerate at the second level (BPZ equation) we derive a mixed difference-differential relation for Q-operator. Thus we generalize the T-Q difference equation known in Nekrasov-Shatashvili limit of the Ω-background to the generic case.

Keywords: Conformal and W Symmetry, Gauge Symmetry, Nonperturbative Effects, Supersymmetric gauge theory

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Contents

1 Introduction

Instanton [1] partition function of $\mathcal{N} = 2$ supersymmetric gauge theory in Ω-background admits exact investigation by localization methods [2–6]. In the limit when the background parameters ϵ_1, ϵ_2 vanish, the famous Seiberg-Witten solution [7, 8] is recovered. The case of non-trivial Ω-background has surprisingly rich area of applications. In particular when one of parameters is set to zero (Nekrasov-Shatashvili limit [9]), deep relations to quantum integrable system emerge (see e.g. [10–17] to quote a few from many important works). These are quantum versions of classical integrable systems, which played central role already in Seiberg-Witten theory on trivial background [18, 19]. The remaining non-zero Ω-background parameter just plays the role of Planck's constant. Many familiar concepts of exactly integrable models of statistical mechanics and quantum field theory such as Bethe-ansatz or Baxter's $T-Q$ equations [20, 21] naturally emerge in this context [13]. In the case of generic Ω-background instanton partition function is directly related to the conformal blocks of a 2d CFT (AGT correspondence) [22–26]. In this context the NS limit corresponds to the semi-classical limit of the related CFT [12, 17, 27–31].

In [31] one of present authors (R.P.) has investigated the link between Deformed Seiberg-Witten curve equation and underlying Baxter's $T-Q$ equation in gauge theory side and the null-vector decoupling equation [32] of 2d CFT in quite general setting of linear quiver gauge theories with $\mathrm{U}(n)$ gauge groups and 2d A_{n-1} Toda field theory multi-point conformal blocks in semi-classical limit (see also [30, 33–36] for earlier discussions on the role of degenerate fields in AGT correspondence).

In this short note we'll extend some of the results of [31] to the case of generic Ω-background corresponding to the genuine quantum conformal blocks. For technical reasons we'll restrict ourselves to the case of $\mathrm{U}(2)$ gauge groups corresponding to the Liouville theory leaving Toda field theory case for future work.

In section 2 we show that an appropriate choice of parameters [36] in A_{r+1} linear quiver theory with $\mathrm{U}(n)$ gauge groups is equivalent to insertion of the analoge of Baxters Q-operator into the partition function of a theory with one gauge node less A_r theory with generic parameters. In the 2d CFT side such special choice corresponds to insertion of a degenerate primary field in the conformal block [36]. In section 3, restricting to the case of Liouville theory, starting from the second order differential equation satisfied by the multi-points conformal blocks including a degenerate field $V_{-b/2}$ [32] we derive the analogue equation satisfied by the gauge theory partition function with Q operator insertion. Then we show that this equation leads to a mixed linear difference-differential equation for Q operators which is a direct generalization of the $T-Q$ equation from NS limit to the case of generic Ω-Background. Finally we summarize our results and discuss a couple of further directions which we think are worth pursuing.

2 A special choice of parameters, leading to $Q_{\tilde{Y}}$ insertion

Consider the instanton partition function of the linear quiver theory A_{r+1} with gauge groups $\mathrm{U}(n)$ with parameters specified as in figure 1a. Note that the parameters of the first gauge factor (depicted as a dashed circle) are chosen to be $a_{\tilde{0},u} = a_{0,u} - \epsilon_1 \delta_{1,u}$, where $a_{0,u}$ are the parameters of the "frozen node" corresponding to the n antifundamental hypermultiplets. It has been shown in [36] that under such choice of parameters all n-tuples of Young diagrams $Y_{\tilde{0},u}$ corresponding to the special node $\tilde{0}$ (the dashed circle) give no contribution to the partition function unless the first diagram $Y_{\tilde{0},1}$ consists of a single column while the remaining $n-1$ diagrams are empty. Taking into account this huge simplification we'll be able to separate the contribution of the special node explicitly. According to the rules of construction of the partition function for this contribution we have

$$\prod_{u,v=1}^{n} \frac{Z_{bf}\left(a_{0,u}, \varnothing | a_{\tilde{0},v}, Y_{\tilde{0},v}\right) Z_{bf}\left(a_{\tilde{0},u}, Y_{\tilde{0},u} | a_{1,v}, Y_{1,v}\right)}{Z_{bf}\left(a_{\tilde{0},u}, Y_{\tilde{0},u} | a_{\tilde{0},v}, Y_{\tilde{0},v}\right)} \tag{2.1}$$

where for a pair of Young diagrams λ, μ the bifundamental contribution is given by

$$Z_{bf}(a,\lambda|b,\mu) = \prod_{s\in\lambda}(a-b-\epsilon_1 L_\mu(s)+\epsilon_2(1+A_\lambda(s)))\prod_{s\in\mu}(a-b+\epsilon_1(1+L_\lambda(s))-\epsilon_2 A_\mu(s)), \tag{2.2}$$

the arm length $A_\lambda(s)$ and leg length $L_\lambda(s)$ of a box s with respect to a Young diagram λ are defined as

$$A_\lambda(s) = \lambda_i - j\,; \qquad L_\lambda(s) = \lambda'_j - i\,, \tag{2.3}$$

where (i,j) are coordinates of the box s with respect to the center of the corner box and λ_i (λ'_j) is the i-th column length (j-th row length) of λ as shown in figure 2.

Using (2.2) It is not difficult to compute the factors Z_{bf} present in (2.1). In particular

$$Z_{bf}\left(a, \varnothing | b, \lambda\right) = \prod_{s\in\lambda}(a-b-\varphi(s)) \tag{2.4}$$

where

$$\varphi(s) = \epsilon_1(i_s-1)+\epsilon_2(j_s-1) \tag{2.5}$$

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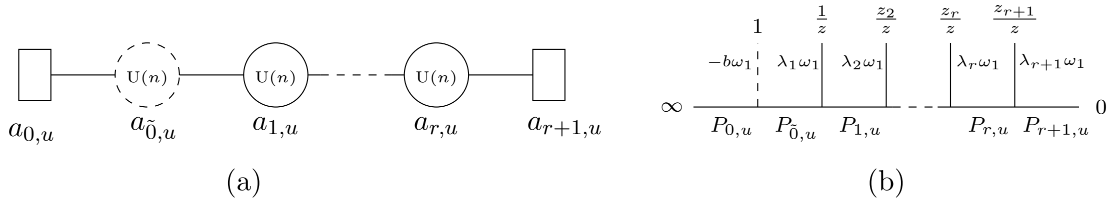

Figure 1. (a) The quiver diagram for the conformal linear quiver U(n) gauge theory: $r+1$ circles stand for gauge multiplets; the two boxes represent n anti-fundamental (on the left edge) and n fundamental (the right edge) hypermultiplets; the lines connecting adjacent circles are the bi-fundamentals. (b) The AGT dual $(r+4)$-point conformal block of the A_{n-1} Toda field theory with (dimensionless) coupling b. The fields/intermediate states corresponding to the horizontal (vertical) lines are specified by momentum (charge) parameters; ω_1 is the highest weight of the defining representation of A_{n-1}. In both diagrams the index u takes values from 1 to n.

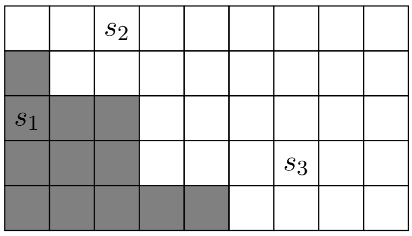

Figure 2. Arm and leg length with respect to a Young diagram $\lambda = \{4,3,3,1,1\}$ (the gray area): $A_\lambda(s_1) = 1$, $L_\lambda(s_1) = 2$, $A_\lambda(s_2) = -2$, $L_\lambda(s_2) = -3$, $A_\lambda(s_3) = -2$, $L_\lambda(s_3) = -4$.

(e.g. in figure 2 $\varphi(s_3) = 6\epsilon_1 + \epsilon_2$). To present the final result for the contribution (2.1) it is convenient to introduce the notation

$$\mathbf{Q}(v|\lambda) = \frac{(-\epsilon_2)^{\frac{v}{\epsilon_2}}}{\Gamma(-\frac{v}{\epsilon_2})} \prod_{s\in\lambda} \frac{v - \varphi(s) + \epsilon_1}{v - \varphi(s)} \tag{2.6}$$

The analogue quantity was instrumental in construction of Baxter's T-Q relation in the context of Nekrasov-Shatashvili limit of $\mathcal{N} = 2$ gauge theories [13]. Recently the importance of this quantity in the case generic Ω-background was emphasized in [37]. A careful examination shows that the contribution (2.1) can be conveniently represented as

$$\prod_{u=1}^{n} \frac{\mathbf{Q}\left(a_{0,1} - a_{1,u} + \epsilon_2 k | Y_{1,u}\right)}{\epsilon_2^k \left(\frac{a_{0,1}-a_{0,u}+\epsilon_2}{\epsilon_2}\right)_k \mathbf{Q}\left(a_{0,1} - a_{1,u} | Y_{1,u}\right)} \prod_{u,v=1}^{n} Z_{bf}\left(a_{\tilde{0},u}, \varnothing | a_{1,v}, Y_{1,v}\right) \tag{2.7}$$

where (and further on) k is the only nonzero column length of the diagram $Y_{\tilde{0},1}$ and

$$(x)_k \stackrel{\mathsf{def}}{=} x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)} \tag{2.8}$$

is the standard Pochammer's symbol. Using (2.4) we can see that the Young diagram dependent part of factor Q in the denominator can be absorbed in the double product.

The net effect is a simple replacement of parameters $a_{\tilde{0},u}$ by $a_{0,u}$ in arguments of the functions Z_{bf}:

$$\prod_{u=1}^{n} \frac{\Gamma\left(-\frac{a_{0,1}-a_{1,u}}{\epsilon_2}\right) \mathbf{Q}\left(a_{0,1}-a_{1,u}+\epsilon_2 k|Y_{1,u}\right)}{\epsilon_2^k \left(-\epsilon_2\right)^{\frac{a_{0,1}-a_{1,u}}{\epsilon_2}} \left(\frac{a_{0,1}-a_{0,u}+\epsilon_2}{\epsilon_2}\right)_k} \prod_{u,v=1}^{n} Z_{bf}\left(a_{0,u}, \varnothing | a_{1,v}, Y_{1,v}\right) \tag{2.9}$$

Thus we conclude that k-instanton sector of the dashed circle in A_{r+1} linear quiver theory can be treated as insertion of the operator

$$\mathbf{Q}_{\vec{Y}_1}\left(a_{0,1}+k\epsilon_2\right) = \prod_{u=1}^{n} \mathbf{Q}\left(a_{0,1}-a_{1,u}+\epsilon_2 k|Y_{1,u}\right) \tag{2.10}$$

in a generic A_r theory. It was already known [36], that the special choice of parameters $a_{\tilde{0},u} = a_{0,u} - \epsilon_1 \delta_{u,1}$ corresponds to the insertion of the completely degenerate field $V_{-b\omega_1(z)}$ in AGT dual Toda CFT conformal block. Thus (2.10) gives an explicit realization of this field in terms of $\mathcal{N}=2$ gauge theory notions.

Until now we were discussing arbitrary gauge $\mathrm{U}(n)$ gauge factors. In what follows, we'll restrict ourselves with the case $n=2$, corresponding to the Liouville theory in AGT dual side. The reason is that in Liouville theory conformal blocks including this degenerate field, satisfy second order differential equation.[1] In remaining part of the paper we'll translate this differential equation in gauge theory terms, finding a linear difference-differential equation, satisfied by the expectation values of the operators $\mathbf{Q}(v)$. Since the equation is valid for infinitely many discrete values of the spectral parameter $v = a_{0,1} + k\epsilon_2$, $k = 0, 1, 2, \ldots$, it can be argued that it is valid for generic values of v as well. The last statement we have checked also by explicit low order instanton computations.

3 Degenerate field decoupling equation in Liouville theory

Let us briefly remind that the Liouville theory (see e.g. [38]) is characterized by the central charge c of Virasoro algebra parameterized as

$$c = 1 + 6Q^2 \qquad Q = b + \frac{1}{b} \tag{3.1}$$

where b is the Liouville's dimensionless coupling constant related to the Ω-background parameters via

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \tag{3.2}$$

The conformal dimension of a primary field V_λ is given by

$$h(\lambda) = \lambda(Q - \lambda)\,. \tag{3.3}$$

The parameter λ is usually referred to as the charge. One alternatively uses the Liouville momenta $p = Q/2 - \lambda$. In figure 1b we found it convenient to specify the fields associated

[1] In generic Toda theory, the analogue null vector decoupling equation has not been investigated in full detail yet. Instead there is a recent progress in the case of quasi-classical limit [31].

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to the horizontal lines by their momenta, while those of vertical lines by charges. The relations between this parameters and the gauge theory VEV's are very simple[2]

$$p_\alpha = \frac{1}{\sqrt{\epsilon_1\epsilon_2}}\frac{a_{\alpha,1}-a_{\alpha,2}}{2}\,; \qquad \lambda_\beta = \frac{1}{\sqrt{\epsilon_1\epsilon_2}}\left(\frac{a_{\beta,1}+a_{\beta,2}}{2}-\frac{a_{\beta-1,1}+a_{\beta-1,2}}{2}\right) \tag{3.4}$$

for $\alpha = 1,2,\ldots,r+1$, $\beta = 2,3,\ldots,r+1$. With the same logic we have

$$\begin{aligned} p_0 &= \frac{1}{\sqrt{\epsilon_1\epsilon_2}}\frac{a_{0,1}-a_{0,2}}{2}\,; \qquad p_{\tilde 0} = \frac{1}{\sqrt{\epsilon_1\epsilon_2}}\frac{a_{0,1}-\epsilon_1-a_{0,2}}{2} \\ \lambda_{\tilde 0} &= -\frac{\epsilon_1}{2\sqrt{\epsilon_1\epsilon_2}} = -\frac{b}{2}\,; \qquad \lambda_1 = \frac{\epsilon_1}{\sqrt{\epsilon_1\epsilon_2}}\left(\frac{a_{1,1}+a_{1,2}}{2}-\frac{a_{0,1}-\epsilon_1+a_{0,2}}{2}\right) \end{aligned} \tag{3.5}$$

Notice that the field $V_{\lambda_{\tilde 0}} = V_{-b/2}$ is indeed a degenerate field satisfying second order differential equation due to the null vector decoupling condition (below L_m are the Virasoro generators)

$$(b^{-2}L_{-1}^2 + L_{-2})V_{-b/2} = 0 \tag{3.6}$$

The differential equation satisfied by our $r+4$-point conformal block

$$G(z|z_\alpha) = \langle p_0|V_{-b/2}(z)V_{\lambda_1}(1)V_{\lambda_2}(z_2)\cdots V_{\lambda_{r+1}}(z_{r+1})|p_{r+1}\rangle_{\{p_{\tilde 0},\ldots,p_r\}} \tag{3.7}$$

reads [32]

$$\left(b^{-2}\partial_z^2 - \frac{2z-1}{z(z-1)}\,\partial_z + \frac{\delta}{z(z-1)} + \sum_{\alpha=2}^{r+1}\frac{z_\alpha\,(z_\alpha-1)}{z(z-1)\,(z-z_\alpha)}\,\partial_{z_\alpha} + \sum_{\alpha=1}^{r+2}\frac{h(\lambda_\alpha)}{(z-z_\alpha)^2}\right)G(z|z_\alpha) = 0 \tag{3.8}$$

where

$$\delta = h\,(Q/2-p_0) - h\,(-b/2) - \sum_{\alpha=1}^{r+2} h(\lambda_\alpha) \quad \text{and} \quad \lambda_{r+2} = Q/2 - p_{r+1}\,. \tag{3.9}$$

According to AGT correspondence the instanton part of the partition function of the $\mathcal{N}=2$ theory considered in previous section with U(2) gauge group factors is related to the conformal block (3.7) as

$$\begin{aligned} G(z|z_\alpha) &= Z_{\text{inst}}\; z^{h(Q/2-p_0)-h(-b/2)-b\sum_{\alpha=1}^{r+1}(Q-\lambda_\alpha)} \prod_{\alpha=1}^{r+1}(z-z_\alpha)^{b(Q-\lambda_\alpha)} \\ &\times \prod_{1\le\alpha<\beta\le r+1}(z_\alpha - z_\beta)^{-2\lambda_\alpha(Q-\lambda_\beta)} \prod_{\alpha=2}^{r+1} z_\alpha^{p_\alpha^2 - p_{\alpha-1}^2 - h(\lambda_\alpha) + 2\lambda_\alpha\sum_{\beta=\alpha+1}^{r+1}(Q-\lambda_\beta)}\,. \end{aligned} \tag{3.10}$$

To complete the map (3.4), (3.5) between the two sides, it remains to mention that the exponentiated gauge couplings (instanton counting parameters) are related to the insertion points as [22]:

$$\begin{aligned} q_\alpha &= z_{\alpha+1}/z_\alpha\,; \quad \text{for} \quad \alpha = 1,\ldots,r \\ q_{\tilde 0} &= 1/z, \end{aligned} \tag{3.11}$$

with $z_1 = 1$.

[2]A Liouville momentum p (charge λ) is simply identified with the $u=1$ component P_1 ($(\lambda\omega_1)_1 = \lambda/2$) of the A_1 Toda theory.

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In (3.11), besides the standard AGT U(1) factors, an extra power of z facilitating a scale transformation with scaling factor z is included. This transformation is needed to map the insertion points shown in figure 1b to the respective insertion points of the conformal block (3.7). Inserting (3.11) into (3.8) and replacing CFT parameters by their gauge theory counterparts we'll find a differential equation satisfied by the partition function. After tedious but straightforward transformations it is possible to represent this equation as (for more details on calculations of this kind see [31])

$$\sum_{\alpha=0}^{r+1}(-)^{\alpha}\chi_{\alpha}(-\epsilon_2\, z\partial_z;\hat{u}_1,\ldots,\hat{u}_{r+1})z^{-\alpha-a_{0,1}/\epsilon_2}Z_{\text{inst}}=0 \tag{3.12}$$

where

$$\hat{u}_1=-\epsilon_1\epsilon_2\sum_{\alpha=2}^{r+1}z_\alpha\partial_{z_\alpha}\,;\qquad \hat{u}_\alpha=\epsilon_1\epsilon_2 z_\alpha\partial_{z_\alpha}\qquad\text{for}\qquad \alpha=2,\ldots,r+1 \tag{3.13}$$

and $\chi_\alpha(v;u_1,\ldots,u_{r+1})$ are quadratic in v and linear in $u_1,\ldots,u_{r+1}$ polynomials (we use notation $\epsilon=\epsilon_1+\epsilon_2$)

$$\begin{aligned}\chi_\alpha(v;u_1,\ldots,u_{r+1})=&\sum_{1\le k_1<\cdots<k_\alpha\le r+1}\left(\prod_{\beta=1}^{\alpha}z_{k_\beta}\right)\Bigg(y_0(v+\alpha\epsilon+(\alpha-\delta_{k_1,1})\epsilon_1)\\&-\sum_{\beta=1}^{\alpha}\Big(y_{k_\beta-1}(v+(\alpha-\beta+1)\epsilon+(\alpha-\delta_{k_1,1})\epsilon_1)\\&-y_{k_\beta}(v+(\alpha-\beta)\epsilon+(\alpha-\delta_{k_1,1})\epsilon_1)\\&+u_{k_\beta}+(c_{0,1}-c_{k_\beta-1,1})(c_{k_\beta-1,1}-c_{k_\beta,1})\Big)\\&+\sum_{1\le\beta<\gamma\le\alpha}(c_{k_\beta-1,1}-c_{k_\beta,1})(c_{k_\gamma-1,1}-c_{k_\gamma,1})\Bigg),\end{aligned} \tag{3.14}$$

where for $\alpha=0,1,\ldots,r+1$

$$y_\alpha(v)=(v-a_{\alpha,1})(v-a_{\alpha,2})\stackrel{\text{def}}{=}v^2-c_{\alpha,1}v+c_{\alpha,2}\,. \tag{3.15}$$

We set by definition

$$\chi_0(v)=y_0(v) \tag{3.16}$$

and for the other extreme value $\alpha=r+1$ it is easy to see that

$$\chi_{r+1}(v)=y_{r+1}(v)\prod_{\beta=1}^{r}z_\beta\,. \tag{3.17}$$

Representing Z_{inst} as a power series in $1/z$,

$$Z_{\text{inst}}=\sum_{v\in a_{0,1}+\epsilon_2\mathbb{Z}}Q(v)z^{-(v-a_{0,1})/\epsilon_2} \tag{3.18}$$

from eq. (3.12) for the coefficients $Q(v)$ we get the relation

$$\sum_{\alpha=0}^{r+1}(-)^{\alpha}\chi_{\alpha}(v;\hat{u}_1,\ldots,\hat{u}_{r+1})Q(v-\alpha\epsilon_2)=0\,, \tag{3.19}$$

which is valid for infinitely many values $v\in a_{0,1}+\epsilon_2\mathbb{Z}$. Since Z_{inst} is regular at $z=\infty$, in fact we have nontrivial equations only for $v_k=a_{0,1}+\epsilon_2 k$, with $k\geq 0$.

Recall now that as discussed in previous section, due to eqs. (2.9), (2.10), Z_{inst} of the A_{r+1} theory up to a simple factor is the same as VEV of the quantity $\mathbf{Q}_{\vec{Y}_1}$ (2.10) calculated in the framework of A_r gauge theory (i.e. in theory without the dashed circle in figure 1a). Explicitly

$$Q(v_k)=C\prod_{u=1}^{2}\frac{\epsilon_2^{(a_{0,1}-v_k)/\epsilon_2}}{\Gamma\left(\frac{v_k-a_{0,u}}{\epsilon_2}+1\right)}\,\langle\mathbf{Q}_{\vec{Y}_1}(v_k)\rangle_{A_r}\,, \tag{3.20}$$

where the constant C takes the value

$$C=\prod_{u=1}^{2}\frac{\Gamma\left(\frac{a_{1,u}-a_{0,1}}{\epsilon_2}\right)\Gamma\left(\frac{a_{0,1}-a_{0,u}}{\epsilon_2}+1\right)}{(-\epsilon_2)^{\frac{a_{0,1}-a_{0,u}}{\epsilon_2}}}\,,$$

if one adopts conventional unit normalization for both partition function and the conformal block. The right hand side of the eq. (3.20) can be calculated by means of gauge theory for arbitrary $v\in\mathbb{C}$. There are all reasons to believe that also for generic values of v the equation (3.19) still holds. Indeed, for a given instanton order, the equation (3.19) states, that some combination of rational functions[3] of v vanish for all values $v=v_k$, but this is possible only if this combination vanishes identically.

A simple inspection ensures that the equation (3.19) in Nekrasov-Shatashvili limit completely agrees with the analogous difference equation investigated in details in [31].

4 Summary

We made an explicit link between the insertion of the $\mathbf{Q}$ operator in $\mathcal{N}=2$ gauge theory and insertion of simplest degenerate field in AGT dual 2d CFT.

In the special case of the gauge groups U(2) we found analog of the Baxter's $T-Q$ equation, previously known only in the Nekrasov-Shatashvili limit of the Ω-background [13–17].

To conclude let us mention that a "microscopic" proof of this statement e.g. along the line presented in [39] to prove qq-character identities of [37] would be highly desirable.

Another important contribution would be generalization of our analysis to the case of arbitrary U(n) or other choices of gauge groups.

[3]Evidently, by multiplying with suitable gamma and exponential functions it is easy to get rid of non-rational prefactors of (2.6), (3.20).

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