

Supersymmetric solutions of the cosmological, gauged, \mathbb{C} magic model

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ABSTRACT: We construct supersymmetric solutions of theories of gauged $\mathcal{N} = 1, d = 5$ supergravity coupled to vector multiplets with a $U(1)_R$ Abelian (Fayet-Iliopoulos) gauging and an independent $SU(2)$ gauging associated to an $SU(2)$ isometry group of the Real Special scalar manifold. These theories provide minimal supersymmetrizations of 5-dimensional $SU(2)$ Einstein-Yang-Mills theories with negative cosmological constant. We consider a minimal model with these gauge groups and the “magic model” based on the Jordan algebra $\mathbf{J}_3^{\mathbb{C}}$ with gauge group $SU(3) \times U(1)_R$, which is a consistent truncation of maximal $SO(6)$ -gauged supergravity in $d = 5$ and whose solutions can be embedded in Type IIB Superstring Theory. We find several solutions containing selfdual $SU(2)$ instantons, some of which asymptote to AdS_5 and some of which are very small, supersymmetric, deformations of AdS_5 . We also show how some of those solutions can be embedded in Romans’ $SU(2) \times U(1)$ -gauged half-maximal supergravity, which was obtained by Lu, Pope and Tran by compactification of the Type IIB Superstring effective action. This provides another way of uplifting those solutions to 10 dimensions.

KEYWORDS: Supergravity Models, Black Holes in String Theory, Gauge Symmetry, Solitons Monopoles and Instantons

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1 Introduction

Over the last 25 years, since the first dilaton black-hole and p -brane solutions were found, there has been a continuous effort in finding and studying solutions of supergravity theories in diverse dimensions, specially if the supergravity theories describe the low-energy effective field theory limit of a superstring theory. This continuous effort has been rewarded with the discovery of many interesting solutions, some of which have revolutionized the field.

To a large extent, however, solutions with non-trivial non-Abelian fields have been left out of this research effort. This was probably due to several different reasons: the vast number of interesting Abelian solutions one could work with, the expected complexity of the non-Abelian ones (all the solutions of Einstein-Yang-Mills (EYM) theories which are not Abelian embeddings are only known numerically), the expected violation of the no-hair theorems in non-Abelian black holes, the loss of nice properties such as the *attractor mechanism* in extremal black holes [1–5], and our general lack of understanding of this kind of solutions.

From our viewpoint, the only way to increase our knowledge on the properties of solutions (black holes, black strings, solitons. . .) with non-trivial, non-Abelian Yang-Mills fields (non-Abelian solutions, in short), is to find first many more. Fortunately, although this task may look extremely difficult *a priori*, it turns out that, just as in the Abelian

case, one can use supersymmetry to derive very powerful solution-generating techniques. Typically, these techniques reduce the problem of finding solutions of supergravity theories to the problem of solving a reduced number of differential equations for functions, 1-forms etc. that play the rôle of building blocks of the full solutions.¹

Thus, using the solution-generating techniques derived in refs. [8, 26], a large number of asymptotically-flat non-Abelian solutions of different kinds (black holes, strings and rings, global monopoles and instantons, multi-center black-hole solutions, microstate geometries etc.) have been constructed over the last few years in 4- and 5-dimensional non-Abelian-gauged supergravities with 8 supercharges which can be called Super-Einstein-Yang-Mills (SEYM) theories because they are minimal supersymmetrizations of the EYM theories [7–17]. All these solutions were obtained in fully analytic form, which allows showing, for instance, how the attractor mechanism works in a covariant fashion in the non-Abelian context [18] and, more recently, how a puzzle involving an apparent violation of the no-hair conjecture is solved when the integration constants of the solution are expressed correctly in terms of the charges of string-theory objects [15].

Extending this work to the asymptotically-AdS case requires important modifications of the gaugings considered because the scalar potentials that arise in the simplest gauging of non-Abelian isometries of the scalar manifolds are, necessarily, either positive-definite (in the $d = 4$ case) or identically zero (in the $d = 5$ case). The only way to produce the scalar potential needed is to gauge a subgroup of the R-symmetry group ($U(2)_R$ in $d = 4$ and $SU(2)_R$ in $d = 5$) via the introduction of Fayet-Iliopoulos (FI) terms. Both in the $d = 4$ and $d = 5$ cases the FI terms can be used to gauge either a $U(1)_R$ or a $SU(2)_R$ subgroup. The Abelian $U(1)_R$ has been studied extensively, but always in absence of any other non-Abelian gauging. The non-Abelian $SU(2)_R$ case has been studied in refs. [19, 39] and turns out to be, technically, much more complicated because the gauging of $SU(2)_R$ requires the simultaneous gauging of a $SU(2)$ subgroup of the isometry group of the Special-Kähler ($d = 4$) or Real-Special ($d = 5$) scalar manifold. In contrast, the Abelian $U(1)_R$ gauging never involves the gauging of a single $U(1)$ isometry in these theories.²

In this paper we work in the framework of the 5-dimensional theories ($\mathcal{N} = 1, d = 5$ supergravity coupled to vector multiplets) and we are going to consider the first of these possibilities: an Abelian $U(1)_R$ gauging that will produce a scalar potential with AdS vacua and, at the same time, an independent non-Abelian gauging of a subgroup of the isometry group of the scalar manifold. The resulting theories can be understood as the natural non-Abelian extension of those with an Abelian gauging and additional vector multiplets. They can also be thought of as the simplest supersymmetrization of the cosmological EYM theories (EYM plus a cosmological constant). Thus, they may be expected to give us a handle in the search for solutions of this system via the use of the supersymmetric solution-generating techniques developed over the years in refs. [20–30]. In particular, we will be able to use the techniques of ref. [38] to construct self-dual $SU(2)$ instantons on Kähler spaces admitting a holomorphic isometry, which are one of the main ingredients in those solution-generating techniques.

¹A complete review of these techniques with many references can be found in ref. [6].

²These Abelian gaugings are, actually, not possible in these theories.

There are many possible models of $\mathcal{N} = 1, d = 5$ supergravity coupled to vector supermultiplets and a good number of them admit the kind of gauging we want to consider here. We have decided to consider, as a toy model, the simplest of them admitting the gauging $SU(2) \times U(1)_R$ and, searching for a possible embedding of the solutions in String Theory, the so-called “ \mathbb{C} magic model”, that admits a $SU(3) \times U(1)_R$ gauging. In its ungauged form, this model is one of the few $\mathcal{N} = 1, d = 5$ models that can be obtained by consistent truncation of the maximal supergravity in $d = 5$ and, therefore, can be uplifted to any of the two maximal supergravities in $d = 10$, $\mathcal{N} = 2A$ and $2B$ but, precisely with that gauging, it can be obtained by a consistent truncation of the $\mathcal{SO}(6)$ -gauged maximal supergravity in $d = 5$, which, in its turn, can be obtained by compactification of the $\mathcal{N} = 2B, d = 10$ on S^5 . Thus, in principle, all the solutions of this theory, that we are going to call “cosmological, gauged, \mathbb{C} magic model”, are also solutions of $\mathcal{N} = 2B, d = 10$ supergravity, the low-energy effective field theory of the Type IIB Superstring, and, in particular, the AdS_5 vacuum of the cosmological, gauged, \mathbb{C} magic model corresponds to the maximally supersymmetric $AdS_5 \times S^5$ near-horizon limit of the D3-brane.

We have found that some solutions of the cosmological, gauged, \mathbb{C} magic model can also be embedded in $\mathcal{N} = 2B, d = 10$ supergravity via the $SU(2) \times U(1)$ -gauge half-maximal supergravity obtained by Romans [36] following the recipe given by Lu, Pope and Tran in ref. [37]: there are two consistent truncations (one of the cosmological, gauged, \mathbb{C} magic model and another of the gauged half-maximal supergravity) that lead to exactly the same theory. This provides two different ways of uplifting these solutions to $\mathcal{N} = 2B, d = 10$ supergravity and an embedding into the Type IIB Superstring effective action to zeroth order in α' .

This paper is organized as follows: in section 2 we describe the framework we are going to work in, introducing the formalism of gauged $\mathcal{N} = 1, d = 5$ supergravities coupled to vector multiplets in section 2.1 and the two particular models we are going to consider in sections 2.2 and 2.3. In section 3 we describe the general technique we use to construct timelike supersymmetric solutions of generic gauged $\mathcal{N} = 1, d = 5$ supergravities coupled to vector multiplets and, in section 3.1 we particularize this technique to the kind of gaugings considered here. Then, in sections 3.2 and 3.3 we apply the technique to the two models we have chosen and construct the simplest solutions that have a non-trivial non-Abelian field. Finally, in section 4 we study the embedding of the solutions of the second model in the $SU(2) \times U(1)$ -gauge half-maximal supergravity showing in section 4.1 the relation between the two consistent truncations mentioned above. Section 5 contains our conclusions.

2 The setup

In this section we describe the two theories we are going to work with. They are two different models of gauged $\mathcal{N} = 1, d = 5$ supergravity coupled to vector supermultiplets with gauge groups consisting in a $U(1)$ factor associated to a Fayet-Iliopoulos term and second, non-Abelian factor ($SU(2)$ and $SU(3)$) associated to the gauging of the isometry group of the (Real Special) scalar manifold. $\mathcal{N} = 1, d = 5$ supergravity coupled to vector

supermultiplets with a non-Abelian gauging provides the minimal supersymmetrization of 5-dimensional Einstein-Yang-Mills theory.³

Since the structure of these gaugings is somewhat complicated, but essential to our goals, we start by reviewing gauged $\mathcal{N} = 1, d = 5$ supergravity coupled to vector supermultiplets in general and, next, we describe in detail the two models.

2.1 Gauged $\mathcal{N} = 1, d = 5$ supergravity coupled to vector multiplets

A model of ungauged $\mathcal{N} = 1, d = 5$ supergravity coupled to n vector multiplets⁴ is fully characterized by the constant, completely symmetric tensor C_{IJK} , $I, J, \dots = 0, 1, \dots, n$ and its bosonic content is: the spacetime metric $g_{\mu\nu}$, $n + 1$ vector fields A^I_μ and n scalars ϕ^x , $x, y, \dots = 1, \dots, n$. The latter parametrize a n -dimensional space that can be seen as a codimension-1 hypersurface in a $(n + 1)$ -dimensional space with coordinates h^I and Riemannian metric

$$a_{IJ} = -2C_{IJK}h^K + 3h_Ih_J, \quad \text{where} \quad h_I \equiv C_{IJK}h^Jh^K, \quad \Rightarrow \quad h^Ih_I = 1. \quad (2.1)$$

The codimension-1 hypersurface is defined by the cubic equation

$$C_{IJK}h^Ih^Jh^K = 1, \quad (2.2)$$

which will be solved by some parametrization in terms of the physical scalars $h^I(\phi)$. The metric induced in this hypersurface (up to a normalization factor) is the σ -model metric for the physical scalars

$$g_{xy} \equiv 3a_{IJ} \frac{\partial h^I}{\partial \phi^x} \frac{\partial h^J}{\partial \phi^y}. \quad (2.3)$$

It is customary to define

$$h^I_x \equiv -\sqrt{3}h^I_{,x} \equiv -\sqrt{3} \frac{\partial h^I}{\partial \phi^x}, \quad h_{Ix} \equiv +\sqrt{3}h_{I,x}, \quad \Rightarrow \quad h_Ih^I_x = h^Ih_{Ix} = 0, \quad (2.4)$$

which satisfy⁵

$$h_I = a_{IJ}h^J, \quad h_{Ix} = a_{IJ}h^J_x, \quad (2.5)$$

³The minimal, $\mathcal{N} = 1$ supersymmetrization of a 5-dimensional Einstein-Yang-Mills (EYM) theory requires (apart from the introduction of fermions, which we set to zero here) the introduction of scalars to have complete vector supermultiplets. The scalars have to parametrize a Real Special manifold whose isometry group contains the gauge group. This may not be possible for arbitrary groups because, at the same time, the scalars must transform under the isometry group in a very precise way, which may demand the introduction of more vector fields. As we are going to see, the supersymmetrization of the SU(3) EYM theory corresponds to a highly non-trivial “magical model” and has one extra vector field, the graviphoton. Besides the mere introduction of scalar fields through a σ -model, the supersymmetrized EYM theory (or Super-EYM (SEYM) theory) contains couplings between the scalar and vector fields and Chern-Simons terms for the vector fields which typically are absent in EYM theories. It is the contribution of the Chern-Simons terms gives that rise to very interesting and characteristic supersymmetric solutions of these theories.

⁴Our conventions are those in refs. [25, 26] and the more recent refs. [11, 29] which are those of ref. [31] with minor modifications and adaptations.

⁵These two properties can be seen as the definition of the metric a_{IJ} .

and the completeness relation

$$a_{IJ} = h_I h_J + g_{xy} h_I^x h_J^y. \tag{2.6}$$

The geometry defined by these objects is known as *Real Special Geometry*.

There are two kinds of global symmetries in these theories: the isometries of the Real Special manifold and R-symmetry group, which is SU(2). In absence of hypermultiplets, they can be considered (but not gauged!) independently. The necessary and sufficient conditions for the gauging of a subgroup of the global isometry group are:

1. The subgroup of the isometry group must act on the vector fields A^I_μ in the adjoint representation. This means that we can use the same indices I, J, \dots for the vector multiplets and for the gauge group's generators, some of which could be trivial because the isometry group does not need to act on all the vector fields. It also means that these isometries must act linearly on the functions $h^I(\phi)$.
2. It must be a symmetry of the C_{IJK} tensor that defines the theory. This condition can be expressed in the form

$$-3f_{I(J}{}^M C_{KL)M} = 0, \tag{2.7}$$

where $f_{IJ}{}^K$ are the gauge group's structure constants,⁶ and it automatically implies the invariance of the Riemannian metric a_{IJ} under the linear transformations.

3. The functions $h^I(\phi)$ must be invariant under those linear transformations up to a reparametrization (a field redefinition of the scalars). Combined with the above condition, it implies that these reparametrizations are isometries of the induced metric $g_{xy}(\phi)$ and the vectors that generate them are Killing vectors and must necessarily be of the form

$$k_I^x = -\sqrt{3}f_{IJ}{}^K h_K^x h^J. \tag{2.8}$$

This condition eliminates the possibility of gauging Abelian subgroups of the isometry group and it is the reason why Abelian gauging is a synonym of gauging via Abelian Fayet-Iliopoulos terms in these theories. One can immediately check using the properties of Real Special geometry that these vectors satisfy the Lie algebra

$$[k_I, k_J] = -f_{IJ}{}^K k_K. \tag{2.9}$$

This kind of symmetries can be gauged immediately by the standard procedure, giving rise to what have been called $\mathcal{N} = 1, d = 5$ Super-Einstein-Yang-Mills (SEYM) models, which are the simplest $\mathcal{N} = 1$ supersymmetrization of the $d = 5$ Einstein-Yang-Mills system [11].

An important property of these theories is that their scalar potential vanishes identically. Thus, they cannot be used as supersymmetrizations of EYM-AdS theories. For this purpose one must gauge (a subgroup of) R-symmetry.

⁶These structure constants will be trivial in the direction in which the subgroup to be gauged does not act.

R-symmetry (SU(2) in these theories) is always present in any $\mathcal{N} = 1, d = 5$ supergravity theory and only acts on the fermions. In order to gauge the full R-symmetry group, though, we need a triplet of vector fields transforming in the adjoint of SU(2) and their transformation under this SU(2) must also be a symmetry of the theory. Since vectors come in vector multiplets, it is clear that there must be an SU(2) subgroup of the isometry group that satisfies the above criteria for *gaugeability*. Gauging the full R-symmetry group, then, involves a deformation of a SEYM model in which new couplings to the fermions are introduced in the action, as well as fermion shifts in the supersymmetry transformation rules and a non-vanishing scalar potential (see eq. (2.12) below). Only the latter occurs in the bosonic action. These new couplings are determined by an object, $P_I{}^r$, ($r, s, \dots = 1, 2, 3$ are $\mathfrak{su}(2)$ indices) with only three of the I components non-vanishing⁷ satisfying, for some constant ξ , the property⁸

$$P_I{}^r = \xi e_I{}^r, \quad \varepsilon^{rst} e_I{}^s e_J{}^t = f_{IJ}{}^K e_K{}^r. \quad (2.10)$$

This object plays the rôle of an embedding tensor, selecting the three gauge vectors among the set of all vectors of the theory. It can also be seen as a constant triholomorphic momentum map.

The theories obtained by gauging the whole SU(2) R-symmetry group can be seen as the supersymmetrizations of SU(2)-EYM-AdS theories, but, how about other gauge groups? The only possibility would be to combine a Fayet-Iliopoulos gauging with the gauging of the desired subgroup of the global isometry group G of a theory. The resulting theory would have the gauge group $SU(2) \times G$, but there is a simpler possibility: combining the gauging of the desired subgroup of the global isometry group G of a theory with the gauging of a U(1) subgroup of the R-symmetry group using Fayet-Iliopoulos terms. Gauging a U(1) subgroup of the R-symmetry group is much simpler, since any vector of the theory can be used as gauge vector. It will be associated to a $P_I{}^r$ with only one I -component different from zero. The resulting theory would have the gauge group $U(1) \times G$ and a scalar potential that, potentially, can give rise an AdS cosmological constant. This is the kind of gauging that we are going to study in this paper.⁹

It goes without saying that, being completely independent, each of the factors of the gauge group has its own coupling constant, which we will denote by g for the non-Abelian factor and g_0 for the Abelian one. The latter will not appear explicitly in the action that we are about to write because we have absorbed it into the $P_I{}^r$. This is very convenient in the case we have at hands.

⁷There is always a basis in which this is true.

⁸Here the only non-vanishing components of the structure constants $f_{IJ}{}^K$ are those of the R-symmetry group SU(2).

⁹Supersymmetric solutions of theories in which the whole R-symmetry group has been gauged have been studied in ref. [39].

The bosonic action of a theory of $\mathcal{N} = 1, d = 5$ supergravity coupled to vector multiplets with the two kinds of gaugings that we have discussed above is given by

$$S = \int d^5x \sqrt{g} \left\{ R + \frac{1}{2} g_{xy} \mathfrak{D}_\mu \phi^x \mathfrak{D}^\mu \phi^y - V(\phi) - \frac{1}{4} a_{IJ} F^{I\mu\nu} F^J{}_{\mu\nu} \right. \\ \left. + \frac{1}{12\sqrt{3}} C_{IJK} \frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} \left[F^I{}_{\mu\nu} F^J{}_{\rho\sigma} A^K{}_\alpha - \frac{1}{2} g f_{LM}{}^I F^J{}_{\mu\nu} A^K{}_\rho A^L{}_\sigma A^M{}_\alpha \right. \right. \\ \left. \left. + \frac{1}{10} g^2 f_{LM}{}^I f_{NP}{}^J A^K{}_\mu A^L{}_\nu A^M{}_\rho A^N{}_\sigma A^P{}_\alpha \right] \right\}, \quad (2.11)$$

where $V(\phi)$, the scalar potential, is given by

$$V(\phi) = - (4h^I h^J - 2g^{xy} h_x^I h_y^J) \mathbf{P}_I{}^r \mathbf{P}_J{}^r, \quad (2.12)$$

$\mathfrak{D}_\mu \phi^x$ are the gauge-covariant derivatives of the scalars

$$\mathfrak{D}_\mu \phi^x = \partial_\mu \phi^x + g A^I{}_\mu k_I{}^x, \quad (2.13)$$

and $F^I{}_{\mu\nu}$ are the gauge-covariant vector field strengths

$$F^I{}_{\mu\nu} = 2\partial_{[\mu} A^I{}_{\nu]} + g f_{JK}{}^I A^J{}_\mu A^K{}_\nu. \quad (2.14)$$

The equations of motion are

$$G_{\mu\nu} - \frac{1}{2} a_{IJ} \left(F^I{}_\mu{}^\rho F^J{}_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^{I\rho\sigma} F^J{}_{\rho\sigma} \right) \\ + \frac{1}{2} g_{xy} \left(\mathfrak{D}_\mu \phi^x \mathfrak{D}_\nu \phi^y - \frac{1}{2} g_{\mu\nu} \mathfrak{D}_\rho \phi^x \mathfrak{D}^\rho \phi^y \right) + \frac{1}{2} g_{\mu\nu} V = 0, \quad (2.15)$$

$$\mathfrak{D}_\mu \mathfrak{D}^\mu \phi^x + \frac{1}{4} g^{xy} \partial_y a_{IJ} F^{I\rho\sigma} F^J{}_{\rho\sigma} + g^{xy} \partial_y V = 0, \quad (2.16)$$

$$\mathfrak{D}_\nu (a_{IJ} F^{J\nu\mu}) + \frac{1}{4\sqrt{3}} \frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} C_{IJK} F^J{}_{\nu\rho} F^K{}_{\sigma\alpha} + g k_{Ix} \mathfrak{D}^\mu \phi^x = 0. \quad (2.17)$$

In what remains of this section we are going to describe the two models that we are going to work with and their gaugings.

2.2 A simple model with $SU(2) \times U(1)_R$ gauge symmetry

As a warm-up exercise one can consider the simplest model that admits a gauging of the kind we want to consider. It contains a triplet of vector multiplets labeled by $x, y, z = 1, 2, 3$ and it is defined by the C_{IJK} tensor with components

$$C_{000} = 1, \quad C_{0xy} = -\frac{1}{2} \delta_{xy}. \quad (2.18)$$

The tensor C_{IJK} ¹⁰ is obviously invariant under $SU(2)$ rotations which act in the adjoint representation on the triplet of vector multiplets. Therefore, this group of symmetries can

¹⁰And, as a consequence, the whole Real Special structure. For example, using

$$(h^0)^3 - \frac{3}{2} h^0 h^x h^x = 1, \quad h_0 = (h^0)^2 - \frac{1}{2} h^x h^x = \frac{2}{3} (h^0)^2 + \frac{1}{3h^0}, \quad h_x = -h^0 h^x, \quad (2.19)$$

be gauged using the matter vectors fields $A^x{}_\mu$ as gauge fields. The remaining vector field, the graviphoton $A^0{}_\mu$ can be used to gauge $U(1)_R \subset SU(2)_R$, which, as we have said, is always possible. More explicitly, we choose

$$P_I{}^r \equiv g_0 \delta_I^0 \delta^{r1}, \tag{2.24}$$

which includes a choice of the particular specific $U_R(1) \subset SU(2)_R$ to be gauged.

The only manifestation of this gauging in the bosonic action eq. (2.11) is the presence of the scalar potential, whose explicit form we will not be concerned with. Furthermore,

$$F^0{}_{\mu\nu} = 2\partial_{[\mu} A^0{}_{\nu]}. \tag{2.25}$$

The covariant derivatives of the scalars and the vector field strengths refer to the $SU(2)$ gauging and are explicitly given by¹¹

$$\begin{aligned} \mathfrak{D}_\mu \phi^x &= \partial_\mu \phi^x + g \varepsilon^{xyz} A^z{}_\mu \phi^z, \\ F^x{}_{\mu\nu} &= 2\partial_{[\mu} A^x{}_{\nu]} + g \varepsilon^{xyz} A^z{}_\mu A^y{}_\nu. \end{aligned} \tag{2.27}$$

2.3 The \mathbb{C} magic model with $SU(3) \times U(1)_R$ gauge symmetry

The second model that we are going to consider is the so-called “ \mathbb{C} magic model”, associated with the “magic” Jordan algebra $\mathbf{J}_3^{\mathbb{C}}$ [32]. This model is one of the possible truncations of maximal $d = 5$ supergravity and is one of the symmetric Real Special geometries [33]. Furthermore, in ref. [34] it was shown that the maximal $d = 5$ supergravity with $SO(6)$ gauging can be consistently truncated to this model with an $SU(3) \times U(1)_R$ gauging (a model previously constructed in ref. [35]), which belongs to the class we want to consider in this paper.

the components of the kinetic matrix for the vector fields are given by

$$a_{00} = \frac{4}{3}(h^0)^4 - \frac{2}{3}h^0 + \frac{1}{3(h^0)^2}, \quad a_{0x} = h^x[1 - 2(h^0)^3], \quad a_{xy} = h^0 \delta_{xy} + 3(h^0)^2 h^x h^y. \tag{2.20}$$

Using the coordinates

$$\phi^x \equiv \sqrt{\frac{3}{2}} h^x / h^0, \quad \Rightarrow \quad h^0 = (1 - \phi^2)^{-1/3}, \quad \text{where } \phi^2 \equiv \phi^x \phi^x, \tag{2.21}$$

these take the form

$$a_{00} = \frac{4}{3}(h^0)^4 - \frac{2}{3}h^0 + \frac{1}{3(h^0)^2}, \quad a_{0x} = \sqrt{\frac{2}{3}} \phi^x h^0 [1 - 2(h^0)^3], \quad a_{xy} = h^0 \delta_{xy} + 2(h^0)^4 \phi^x \phi^y, \tag{2.22}$$

and the σ -model metric is given by

$$g_{xy} = \frac{2}{1 - \phi^2} \left[\delta_{xy} + \frac{8(3 - 2\phi^2)}{9(1 - \phi^2)} \phi^x \phi^y \right]. \tag{2.23}$$

¹¹The structure constants and Killing vectors are given by

$$f_{xy}{}^z = \varepsilon_{xyz}, \quad k_x{}^y = \varepsilon_{xyz} \phi^z. \tag{2.26}$$

The \mathbb{C} magic model is determined by the constant symmetric tensor C_{IJK} of non-vanishing components

$$C_{000} = 1, \quad C_{0xy} = -\frac{1}{2}\delta_{xy}, \quad C_{xyz} = \sqrt{\frac{3}{8}} d_{xyz}, \quad (2.28)$$

where $x, y, z = 1, \dots, 8$ and d_{xyz} is the fully symmetric constant tensor associated with $SU(3)$, given in terms of the Gell-Mann matrices λ_x as

$$d_{xyz} = \frac{1}{2} \text{Tr} [\lambda_x \{\lambda_y, \lambda_z\}], \quad (2.29)$$

and having non-vanishing components

$$\begin{aligned} d_{146} &= d_{157} = d_{256} = d_{344} = d_{355} = 1, \\ d_{247} &= d_{366} = d_{377} = -1, \\ d_{118} &= d_{228} = d_{338} = \frac{2}{\sqrt{3}}, \\ d_{448} &= d_{558} = d_{668} = d_{778} = -\frac{1}{\sqrt{3}}, \\ d_{888} &= -\frac{2}{\sqrt{3}}. \end{aligned} \quad (2.30)$$

It can be seen that the scalar fields parametrize the symmetric space $SL(3, \mathbb{C})/SU(3)$. The gauge fields A^x transform in the adjoint representation of $SU(3)$, the maximal compact subgroup of the scalar group manifold, as do the scalar functions h^x and, therefore, they can be used as $SU(3)$ gauge fields. A^0 gauges $U(1)_R \subset SU_R(2)$. Without any loss of generality, we select this subgroup as in eq. (2.24).

Observe that, being a symmetric model, with the normalization chosen here,

$$C^{IJK} = C_{IJK}. \quad (2.31)$$

We will be interested in solutions in which only a subgroup $SU(2) \subset SU(3)$ is active. However, it turns out that an additional $U(1)$ must also remain active.

3 Timelike supersymmetric solutions

The supersymmetric solutions of matter-coupled $\mathcal{N} = 1, d = 5$ supergravity theories with arbitrary gaugings have been fully characterized in a series of papers in which couplings of increasing complexity were considered [20–27]. Using these characterizations one can define procedures to construct, step by step, supersymmetric solutions. These procedures have become extremely useful solution-generating techniques.

We are going to search for timelike supersymmetric solutions of the two models reviewed in sections 2.2 and 2.3. For this case it turns out that we can simply reuse the procedure described in ref. [30] for Abelian gaugings, conveniently covariantized to include the non-Abelian gauging. The solution-generating recipe is in full agreement with the general recipe obtained in the above-mentioned references and, before we specify the choice of momentum maps, it can be summarized as follows:

First of all, the building blocks of the timelike supersymmetric solutions are

1. The 4-dimensional spatial metric $h_{\underline{mn}}$, where $\underline{m}, \underline{n}, \underline{p} = 1, \dots, 4$.¹² It does not depend on the time coordinate and defines a 4-dimensional spatial manifold usually called “base space” which plays an auxiliary rôle and has no direct physical relevance. All the building blocks and operators used in what follows are naturally defined in this 4-dimensional space and, hence, they are time-independent. We use hats to denote them.
2. The antiselfdual almost hypercomplex structure $\hat{\Phi}^{(r)}_{mn}$, $r, s, t = 1, 2, 3$. By definition, the 2-forms satisfy the properties

$$\hat{\Phi}^{(r)mn} = -\frac{1}{2}\varepsilon^{mnpq}\hat{\Phi}^{(r)}_{pq}, \quad \text{or} \quad \hat{\Phi}^{(r)} = -\hat{\star}\hat{\Phi}^{(r)}, \quad (3.1)$$

$$\hat{\Phi}^{(r)m}{}_n\hat{\Phi}^{(s)n}{}_p = -\delta^{rs}\delta^m{}_p + \varepsilon^{rst}\hat{\Phi}^{(t)m}{}_p. \quad (3.2)$$

3. The scalar function \hat{f} .
4. The 1-form $\hat{\omega}_{\underline{m}}$.
5. The 1-form potentials $\hat{A}^I_{\underline{m}}$.
6. The functions of the physical scalars $h^I(\phi)$. They are time-independent as well.

These building blocks must fulfill the following conditions:

1. The antiselfdual almost hypercomplex structure $\hat{\Phi}^{(r)}_{mn}$, the 1-form potentials $\hat{A}^I_{\underline{m}}$ and the base-space metric $h_{\underline{mn}}$ (through its Levi-Civita connection) must solve the equation¹³

$$\hat{\nabla}_m\hat{\Phi}^{(r)}_{np} + \varepsilon^{rst}\hat{A}^I{}_m\mathbf{P}_I{}^s\hat{\Phi}^{(t)}_{np} = 0. \quad (3.3)$$

2. The selfdual part of the spatial vector field strengths $\hat{F}^I \equiv d\hat{A}^I + \frac{1}{2}gf_{JK}{}^I\hat{A}^J \wedge \hat{A}^K$ is given by

$$h_I\hat{F}^{I+} = \frac{2}{\sqrt{3}}(\hat{f}d\hat{\omega})^+. \quad (3.4)$$

3. The antiselfdual part of \hat{F}^I is given by¹⁴

$$\hat{F}^{I-} = -2\hat{f}^{-1}C^{IJK}h_J\mathbf{P}_K{}^r\hat{\Phi}^{(r)}. \quad (3.7)$$

¹²In our conventions, underlined indices are world indices. Tangent-space indices will not be underlined.

¹³The local SU(2) symmetry of this differential equation is formally that of the full SU_R(2) until the values of the momentum maps $\mathbf{P}_I{}^s$ are specified. After the choice eq. (2.24) this differential equation splits into eqs. (3.13)–(3.15). We are going to discuss the specifics of the models we are considering next.

¹⁴In this equation the indices of C^{IJK} have been raised using the inverse metric a^{IJ} . This object satisfies the relations

$$C^{IJK}h_K = h^Ih^J - \frac{1}{2}g^{xy}h_x^Ih_y^J = \frac{3}{2}h^Ih^J - \frac{1}{2}a^{IJ}, \quad (3.5)$$

the first of which allow us to rewrite the scalar potential in eq. (2.12) in the form

$$V(\phi) = -4C^{KIJ}h_K\mathbf{P}_I{}^r\mathbf{P}_J{}^r. \quad (3.6)$$

4. Finally, the following equation relating all the building blocks, where the dots indicate standard contraction of all the indices of the tensors, has to be satisfied

$$\hat{\mathfrak{D}}^2 \left(h_I / \hat{f} \right) - \frac{1}{6} C_{IJK} \hat{F}^J \cdot \hat{\star} \hat{F}^K + \frac{1}{2\sqrt{3}} (a_{IK} - 2C_{IJK} h^J) \hat{F}^K \cdot (\hat{f} d\hat{\omega})^- = 0. \quad (3.8)$$

Having found building blocks that satisfy the above conditions, the physical 5-dimensional fields are reconstructed as follows:

1. The 5-dimensional metric is given by

$$ds^2 = \hat{f}^2 (dt + \hat{\omega})^2 - \hat{f}^{-1} h_{\underline{mn}} dx^m dx^n. \quad (3.9)$$

2. The complete 5-dimensional vector fields are given by

$$A^I = -\sqrt{3} h^I e^0 + \hat{A}^I, \quad \text{where } e^0 \equiv \hat{f} (dt + \hat{\omega}). \quad (3.10)$$

The complete 5-dimensional field strength is given by

$$F^I = -\sqrt{3} \hat{\mathfrak{D}} (h^I e^0) + \hat{F}^I. \quad (3.11)$$

3. The scalar fields ϕ^x can be obtained by inverting the functions $h_I(\phi)$ or $h^I(\phi)$ if the form of these functions is known. One can always use a parametrization of these functions such that the scalars are given by

$$\phi^x = h_x / h_0 = (h_x / \hat{f}) / (h_0 / \hat{f}). \quad (3.12)$$

When we specify the $U(1)_R \subset SU(2)_R$ that we are going to gauge and corresponding gauge vector as in eq. (2.24) it is possible to extract more information from the equations satisfied by the building blocks of timelike supersymmetric solutions. We analyze them next.

3.1 Supersymmetric solutions of cosmological gauged models

With the choice eq. (2.24), eq. (3.3) splits into the following three equations [21, 24]

$$\hat{\nabla}_m \hat{\Phi}^{(1)}{}_{np} = 0, \quad (3.13)$$

$$\hat{\nabla}_m \hat{\Phi}^{(2)}{}_{np} = g_0 \hat{A}^0{}_m \hat{\Phi}^{(3)}{}_{np}, \quad (3.14)$$

$$\hat{\nabla}_m \hat{\Phi}^{(3)}{}_{np} = -g_0 \hat{A}^0{}_m \hat{\Phi}^{(2)}{}_{np}, \quad (3.15)$$

the first of which implies that the “base space” metric $h_{\underline{mn}}$ is Kähler with respect to the complex structure $\hat{J}_{mn} \equiv \hat{\Phi}^{(1)}{}_{mn}$. Then, the integrability condition of the other two equations leads to a relation between the $U_R(1)$ gauge potential and the base space metric

$$\hat{\mathfrak{R}}_{mn} = -g_0 \hat{F}^0{}_{mn}, \quad (3.16)$$

where $\hat{\mathfrak{R}}_{mn}$ is the Ricci 2-form of the Kähler base space.

Eq. (3.5) is not simplified by our choice of gauging, but eq. (3.7) becomes

$$\hat{F}^{I-} = -2g_0 \hat{f}^{-1} C^{0IJ} h_J \hat{J}, \quad (3.17)$$

which implies

$$\hat{F}^{0-} = \frac{1}{2g_0} \hat{f}^{-1} V(\phi) \hat{J}, \quad (3.18)$$

$$h_I \hat{F}^{I-} = -2g_0 \hat{f}^{-1} h^0 \hat{J}. \quad (3.19)$$

Then, the trace of eq. (3.16) and eq. (3.18) with \hat{J}^{mn} together lead to

$$\hat{R} = -2V(\phi)/\hat{f}. \quad (3.20)$$

Finally, substituting eq. (3.17) into eq. (3.8) and using in it eqs. (3.5) and (2.6), and taking into account that h_0 is a singlet under the non-Abelian factor of the gauge group, we get the following two equations

$$\hat{\nabla}^2(h_0/\hat{f}) - \frac{1}{6} C_{0JK} \hat{F}^J \cdot [\star \hat{F}^K + 4\sqrt{3} h^K (\hat{f} d\hat{\omega})^-] - \sqrt{3} g_0 h_0 h^0 \hat{J} \cdot d\hat{\omega} = 0, \quad (3.21)$$

$$\hat{\mathcal{D}}^2(h_x/\hat{f}) - \frac{1}{6} C_{xJK} \hat{F}^J \cdot [\star \hat{F}^K + 4\sqrt{3} h^K (\hat{f} d\hat{\omega})^-] - \sqrt{3} g_0 h_x h^0 \hat{J} \cdot d\hat{\omega} = 0. \quad (3.22)$$

In order to simplify the construction of solutions of this class, which should start by judicious choice of the 4-dimensional Kähler metric, we are going to assume that this Kähler metric admits a holomorphic isometry. Then, it can always be written as¹⁵

$$d\hat{s}_4^2 = h_{mn} dx^m dx^n = H^{-1} (dz + \chi)^2 + H \{ (dx^2)^2 + W^2(\vec{x}) [(dx^1)^2 + (dx^3)^2] \}, \quad (3.23)$$

with the functions H and W , and the 1-form χ , independent of the coordinate z , which is adapted to the holomorphic isometry, and satisfying the constraint

$$\check{\star}_3 d\chi = dH + H \partial_2 \log W^2 dx^2, \quad (3.24)$$

where $\check{\star}_3$ is the Hodge dual in the 3-dimensional manifold

$$d\check{s}_3^2 = (dx^2)^2 + W^2(\vec{x}) [(dx^1)^2 + (dx^3)^2]. \quad (3.25)$$

The integrability condition of the constraint eq. (3.24) is the equation

$$\partial_1 \partial_1 H + \partial_2 \partial_2 (W^2 H) + \partial_3 \partial_3 H = 0. \quad (3.26)$$

Therefore, the simplifying assumption of the existence of a holomorphic isometry allows us to construct any Kähler metric within this class by choosing an arbitrary function W , solving the integrability condition eq. (3.26) for H and then solving the constraint eq. (3.24) for χ .

In order to make progress it is necessary to specify the model under consideration. We start by the simple model described in section 2.2.

¹⁵See ref. [28] and references therein.

3.2 Supersymmetric solutions of the simplest $SU(2) \times U(1)_R$ model

Here we are going to label the three vector multiplets with $A, B, \dots = 1, \dots, 3$ to simplify the comparison with the \mathbb{C} magic model, which will have an $SU(2)$ triplet of vectors active but has more vector multiplets labeled, according with the general notation, by $x, y \dots = 1, \dots, n_V$.

For the sake of simplicity, we are going to impose

$$\hat{F}^{A-} = 0, \quad \text{and} \quad h_A = 0. \quad (3.27)$$

Then, the $SU(2)$ gauge field is a selfdual instanton on the Kähler base space and we can use the results of ref. [38] to construct it. Also, because of eqs. (2.19) we have that $h^A = 0$ and $h_0 = h^0 = 1$ which, because of eq. (2.20), imply in their turn that $a_{IJ} = a^{IJ} = \delta_{IJ}$ and $C^{IJK} = C_{IJK}$.

Then, under these assumptions, eq. (3.17) takes the form

$$\hat{F}^{0-} = -2g_0 \hat{f}^{-1} \hat{J}, \quad (3.28)$$

while eq. (3.4) gives

$$\hat{F}^{0+} = \frac{2}{\sqrt{3}} (\hat{f} d\hat{\omega})^+. \quad (3.29)$$

Finally, eqs. (3.21) and (3.22) take the form

$$\hat{\nabla}^2 \hat{f}^{-1} - \frac{1}{6} \hat{F}^{0-} \cdot \hat{\star} \hat{F}^{0-} + \frac{1}{12} \hat{F}^A \cdot \hat{F}^A + \frac{1}{\sqrt{3}} g_0 \hat{J} \cdot d\hat{\omega} = 0, \quad (3.30)$$

$$\hat{F}^A \cdot \hat{\star} \hat{F}^0 = 0 \quad \Rightarrow \quad \hat{F}^{A+} \cdot (d\hat{\omega})^+ = 0, \quad (3.31)$$

where we have used the previous equations in both equations. The simplest way to solve the last equation is to require¹⁶

$$(d\hat{\omega})^+ = \hat{F}^{0+} = 0. \quad (3.32)$$

Given that $d\hat{J} = d\hat{F}^{0-} = 0$, eq. (3.28) implies that \hat{f} is constant, and we can substitute eq. (3.28) in eq. (3.30) obtaining $(\hat{J} \cdot \hat{J} = 4)$

$$\frac{8}{3} g_0^2 \hat{f}^{-2} + \frac{1}{12} \hat{F}^A \cdot \hat{F}^A + \frac{1}{\sqrt{3}} g_0 \hat{J} \cdot d\hat{\omega} = 0, \quad (3.33)$$

and also in eq. (3.16), which using the results in appendix B of ref. [29] leads to the equations

$$\partial_{\underline{1,3}} (H^{-1} \partial_\varrho \log W^2) = 0, \quad (3.34)$$

$$\partial_\varrho (H^{-1} \partial_\varrho \log W^2) = 4g_0^2 \hat{f}^{-1}, \quad (3.35)$$

$$\hat{\nabla}^2 \log W^2 = 8g_0^2 \hat{f}^{-1}, \quad (3.36)$$

for the functions H and W that appear in the Kähler metric eq. (3.23).

¹⁶Actually, following the treatment in ref. [29] one can show that if one chooses a Kähler metric admitting a holomorphic isometry, which can be put in the form explained in ref. [28] with $H = H(\varrho)$, $W^2 = \Psi(\varrho)\Phi(x^1, x^3)$ and $\hat{f} = \hat{f}(\varrho)$, as we are going to assume here, then $\hat{F}^{0+} \propto \hat{V}^\sharp \wedge \hat{V}^2 + \hat{V}^3 \wedge \hat{V}^1$. It follows that for eq. (3.31) to be satisfied, either $\hat{F}^{0+} = 0$ or $\hat{F}_{\sharp 2}^{A+} = 0 \forall A$.

The first of these equations is automatically solved if we consider the usual ansatz $H = H(\varrho)$ ($\varrho \equiv x^2$) and $W^2 = \Psi(\varrho)\Phi_{(k)}(x^1, x^3)$. The integrability condition eq. (3.26) is then satisfied if

$$H(\varrho) = \frac{\rho^\epsilon}{\Psi(\varrho)}, \quad \epsilon = 0, 1, \tag{3.37}$$

where we have used the freedom to shift ϱ and to rescale in opposite way the functions Ψ and $\Phi_{(k)}$. From now on we consider the $\epsilon = 1$ case, which will be the one that will give an interesting solution (a supersymmetric 1-parameter deformation of AdS₅).

The remaining equations are solved if $\Psi(\varrho)$ is of the form

$$\Psi(\varrho) = \frac{4}{3}g_0^2\hat{f}^{-1}\varrho^3 + k\varrho^2 + \alpha, \tag{3.38}$$

and if $\Phi_{(k)}$ is a solution of Liouville's equation

$$(\partial_{\underline{1}}^2 + \partial_{\underline{2}}^2) \log \Phi_{(k)} = -2k\Phi_{(k)}, \tag{3.39}$$

with k constant and α is an arbitrary integration constant.

For $k = +1$ and $\alpha = 0$ the base space is the Bergman space $\overline{\mathbb{C}\mathbb{P}^2}$.

The only equations left to solve are (3.33) plus the selfduality condition of the non-Abelian field strength $\hat{F}^{A-} = 0$. We need to solve the latter first, but we make the following observation: if we find a selfdual SU(2) instanton such that $\hat{F}^{A+} \cdot \hat{F}^{A+} \equiv 32g_0^2\hat{f}^{-2}\lambda$ where λ is a constant, then eq. (3.33) can be solved by taking $d\hat{\omega} = -\frac{2}{\sqrt{3}}g_0(1+\lambda)\hat{f}^{-2}\hat{J}$, or, up to a closed form,

$$\hat{\omega} = \frac{2g_0}{\sqrt{3}}(1+\lambda)\hat{f}^{-2}\varrho(dz + \chi_{(k)}). \tag{3.40}$$

If this solution exists, then it is not difficult to see that the full 5-dimensional metric is invariant under the rescaling $t \rightarrow t/\sigma$, $\varrho \rightarrow \sigma\varrho$, $\hat{f} \rightarrow \sigma\hat{f}$, $\alpha \rightarrow \sigma^2\alpha$, which we can use to set $\hat{f} = 1$.

Then, we now focus on finding a selfdual SU(2) instanton on the Kähler base space that we have just determined through H and W with constant instanton number density $\hat{F}^{A+} \cdot \hat{F}^{A+}$.

Selfdual SU(2) instantons in 4-dimensional Kähler spaces with one holomorphic isometry have recently been studied in ref. [38], where a Kronheimer-type relation between those instantons and monopoles satisfying a generalization of the Bogomol'nyi equation was found and a subsequent generalization of the *hedgehog ansatz* was used to solve the latter in the spherically-symmetric case $k = +1$.

Let us summarize this result:

1. Decomposing the gauge field with respect to the action of the holomorphic isometry as

$$\hat{A}^A = -H^{-1}\Phi^A(dz + \chi_{(1)}) + \check{A}^A, \tag{3.41}$$

where Φ^A and \check{A}^A are independent of z and are defined in the 3-dimensional space with metric eq. (3.25), and $H(\varrho)$ is one of the functions that occur in the generic Kähler metric eq. (3.23) and where it is assumed that $W = \Psi(\varrho)\Phi_{(1)}$.

2. Assuming in addition that they have the *hedgehog* form

$$\Phi^A = F(\varrho) \frac{y^A}{\varrho}, \quad \text{and} \quad \check{A}^A = L(\varrho) \varepsilon^A{}_{BC} \frac{y^B}{\varrho} d\left(\frac{y^C}{\varrho}\right), \quad (3.42)$$

where y^A are Cartesian coordinates related to ϱ by $y^A y^A = \varrho^2$ and $F(\varrho)$ and $L(\varrho)$ are two functions to be determined,

the field strength F^A will be selfdual in the 4-dimensional Kähler space with metric eq. (3.23) and $H = H(\varrho)$, $W = \Psi(\varrho)\Phi_{(1)}$ if the following two equations are satisfied

$$\begin{cases} K' = G - 1, \\ \Psi G' = 2KG, \end{cases} \quad \text{where} \quad K \equiv g\Psi F, \quad \text{and} \quad G \equiv (1 + gL)^2. \quad (3.43)$$

These equations depend explicitly on the function $\Psi(\varrho)$, which in our case is given by eq. (3.38) and, precisely for the $k = +1$ and $\alpha = 0$, when the base space is the Bergman space $\overline{\mathbb{CP}}^2$, one of the solutions found in ref. [38] has constant instanton number density, as we were looking for. This solution is

$$K = \frac{2}{3}g_0^2\varrho^2, \quad G = 1 + \frac{4}{3}g_0^2\varrho, \quad (3.44)$$

and its instanton number density is given by

$$\hat{F}^A \cdot \hat{F}^A = \frac{16}{3} \left(\frac{g_0^2}{g}\right)^2, \quad \Rightarrow \quad \lambda = \frac{g_0^2}{6g^2}. \quad (3.45)$$

Summarizing: we have found a simple solution whose only non-vanishing fields are a selfdual SU(2) instanton living on the base space $\overline{\mathbb{CP}}^2$ plus an Abelian vector field and the metric. The last two fields take the form

$$\begin{aligned} ds^2 &= \left[dt + \frac{2}{\sqrt{3}}g_0(1 + \lambda)\varrho(dz + \cos\theta d\varphi) \right]^2 \\ &\quad - \varrho \left[1 + \frac{4}{3}g_0^2\varrho \right] (dz + \cos\theta d\varphi)^2 - \frac{d\varrho^2}{\varrho \left[1 + \frac{4}{3}g_0^2\varrho \right]} - \varrho d\Omega_{(2,1)}^2, \\ F^0 &= 2\lambda g_0 \hat{J}, \end{aligned} \quad (3.46)$$

where $d\Omega_{(2,1)}^2$ is the metric of the round 2-sphere. In the $g \rightarrow \infty$ limit, $\lambda \rightarrow 0$, the Abelian and non-Abelian gauge fields vanish and the metric is that of AdS₅.

3.3 Supersymmetric solutions of the SU(3) × U(1)_R-gauged C magic model

Let us now consider the model presented in section 2.3. We start by assuming, for the sake of simplicity,

$$A^x{}_\mu = 0, \quad \text{and} \quad h_x = 0, \quad 3 < x < 8, \quad (3.47)$$

so that we are effectively considering a theory with only four vector multiplets and gauge group SU(2) × U(1)_R with an extra U(1) which is ungauged (nothing is charged under it). It

should be stressed that this is not a truncation, but an Ansatz that produces an important simplification to be tried in the equations. As in the previous case, we will use indices $A = 1, 2, 3$ for the first three vector multiplets that gauge the $SU(2)$ factor. The $U(1)_R$ factor will be gauged by A^0_μ and the other surviving vector multiplet corresponds to A^8_μ .

We are also going to look for solutions containing a selfdual $SU(2)$ instanton on the 4-dimensional Kähler space and, therefore, we impose

$$\hat{F}^{A-} = 0. \tag{3.48}$$

The Ansatz, together with eq. (3.17) and eq. (2.31) implies

$$h_A = 0, \tag{3.49}$$

which in turn implies that

$$h^x = 0, \quad \forall x = 1, \dots, 7, \tag{3.50}$$

so that the only non-vanishing scalar functions h^I are h^0, h^8 and are related to h_0, h_8 by

$$h^0 = (h_0)^2 - \frac{1}{2}(h_8)^2, \quad h^8 = -h_8 \left(h_0 + \frac{1}{\sqrt{2}}h_8 \right). \tag{3.51}$$

Furthermore, they satisfy the constraint

$$\left(h_0 - \sqrt{2}h_8 \right) \left(h_0 + \frac{1}{\sqrt{2}}h_8 \right)^2 = \left(h^0 - \sqrt{2}h^8 \right) \left(h^0 + \frac{1}{\sqrt{2}}h^8 \right)^2 = 1. \tag{3.52}$$

It follows that the non-vanishing components of the metric a_{IJ} are

$$\begin{aligned} a_{00} &= (h_0)^2 + (h_8)^2, & a_{08} &= h_8 \left(2h_0 - \frac{1}{\sqrt{2}}h_8 \right), & a_{88} &= (h_0)^2 - \sqrt{2}h_0h_8 + \frac{3}{2}(h_8)^2 \\ a_{AB} &= \delta_{AB} \left(h_0 + \frac{1}{\sqrt{2}}h_8 \right)^2, & & \forall A, B = 1, 2, 3, \\ a_{xy} &= \delta_{xy} \left[(h_0)^2 - \frac{1}{\sqrt{2}}h_0h_8 - (h_8)^2 \right], & & \forall x, y = 4, \dots, 7. \end{aligned} \tag{3.53}$$

From the same equations one has

$$\hat{F}^{0-} = -2g_0(h_0/\hat{f})\hat{J}, \quad \hat{F}^{8-} = g_0(h_8/\hat{f})\hat{J}. \tag{3.54}$$

while equation (3.4) gives

$$h_0\hat{F}^{0+} + h_8\hat{F}^{8+} = \frac{2}{\sqrt{3}}(\hat{f}d\hat{\omega})^+. \tag{3.55}$$

After using eqs. (3.52) and (3.54), eq. (3.21) takes the form

$$\begin{aligned} \hat{\nabla}^2(h_0/\hat{f}) - \frac{1}{6}(\hat{F}^{0+})^2 + \frac{1}{12}(\hat{F}^{8+})^2 + \frac{1}{12}(\hat{F}^{A+})^2 \\ + \frac{1}{3}g_0^2[8(h_0/\hat{f})^2 - (h_8/\hat{f})^2] + \frac{1}{\sqrt{3}}g_0\hat{J} \cdot d\hat{\omega} = 0, \end{aligned} \tag{3.56}$$

while the only non-trivial components of eq. (3.22) ($x = A, 8$) take the form

$$C_{AJK} \hat{F}^J \cdot \hat{\star} \hat{F}^K \propto F^{A+} \cdot (\hat{F}^{0+} - \sqrt{2} \hat{F}^{8+}) = 0, \quad (3.57)$$

$$\begin{aligned} \hat{\nabla}^2(h_8/\hat{f}) + \frac{1}{6} \hat{F}^{0+} \cdot \hat{F}^{8+} + \frac{1}{6\sqrt{2}} (\hat{F}^{8+})^2 - \frac{1}{6\sqrt{2}} (\hat{F}^{A+})^2 \\ + \frac{1}{3} g_0^2 \left[4(h_0/\hat{f})(h_8/\hat{f}) - \sqrt{2}(h_8/\hat{f})^2 \right] = 0. \end{aligned} \quad (3.58)$$

If one does not want to put additional constraints on the non-Abelian field strengths \hat{F}^A , eq. (3.57) implies

$$\hat{F}^{8+} = \frac{1}{\sqrt{2}} \hat{F}^{0+}, \quad (3.59)$$

and the closure of \hat{F}^0 , \hat{F}^8 , and \hat{J} together with eq. (3.54), leads to

$$d \left(\sqrt{2} h_0/\hat{f} + h_8/\hat{f} \right) \wedge \hat{J} = 0, \quad \Rightarrow \quad h_8 = \sqrt{2} \left(\alpha \hat{f} - h_0 \right), \quad (3.60)$$

for some constant α . Substituting in eq. (3.52) we can solve this constraint, finding these expressions for h_0/\hat{f} and h_8/\hat{f} in terms of \hat{f} :

$$h_0/\hat{f} = \frac{1}{3} \alpha \left(2 + \frac{1}{(\alpha \hat{f})^3} \right), \quad h_8/\hat{f} = \frac{\sqrt{2}}{3} \alpha \left(1 - \frac{1}{(\alpha \hat{f})^3} \right), \quad (3.61)$$

and using all these results in eq. (3.55), we get

$$(d\hat{\omega})^+ = \frac{\sqrt{3}}{2} \alpha \hat{F}^{0+}. \quad (3.62)$$

On the other hand, adding eqs. (3.56) and (3.58) divided by $\sqrt{2}$ gives

$$\hat{J} \cdot d\hat{\omega} = -\frac{4}{\sqrt{3}} g_0 \alpha^2 \left(1 + \frac{1}{(\alpha \hat{f})^3} \right), \quad (3.63)$$

and expanding the anti-selfdual part of $d\hat{\omega}$ in the basis of anti-selfdual 2-forms $\hat{\Phi}^{1,2,3}$ ($\hat{J} = \hat{\Phi}^1$) we find that

$$(d\hat{\omega})^- = -\frac{1}{\sqrt{3}} g_0 \alpha^2 \left(1 + \frac{1}{(\alpha \hat{f})^3} \right) \hat{J} + \Omega_{(2)} \hat{\Phi}^2 + \Omega_{(3)} \hat{\Phi}^3 \equiv -\frac{1}{\sqrt{3}} g_0 \alpha^2 \left(1 + \frac{1}{(\alpha \hat{f})^3} \right) \hat{J} + d\tilde{\omega}. \quad (3.64)$$

In order to make progress, we assume again that the Kähler base space admits a holomorphic isometry and therefore it can be put in the canonical form eq. (3.23). We also assume that $H = H(\varrho)$ ($\varrho = x^2$) and $W^2 = \Psi(\varrho) \Phi_{(k)}(x^1, x^3)$, which leads to the relation eq. (3.37) between $H(\varrho)$ and $\Psi(\varrho)$. Here we are going to consider the two possible values of $\epsilon = 0, 1$.

From eq. (3.16), and using the results in appendix B of ref. [29], one gets

$$\hat{F}^{0+} = -\frac{1}{4g_0 \varrho^\epsilon} \left(\Psi'' - 2\epsilon \frac{\Psi'}{\varrho} + 2k \right) \left[(dz + \chi_{(k)}) \wedge d\varrho + \varrho^\epsilon \Phi_{(k)} dx^3 \wedge dx^1 \right], \quad (3.65)$$

$$\hat{F}^{0-} = -\frac{1}{4g_0 \varrho^\epsilon} (\Psi'' - 2k) \left[(dz + \chi_{(k)}) \wedge d\varrho - \varrho^\epsilon \Phi_{(k)} dx^3 \wedge dx^1 \right] = -\frac{\hat{J}}{4g_0 \varrho^\epsilon} (\Psi'' - 2k), \quad (3.66)$$

and comparing with the expression for \hat{F}^{0-} in eq. (3.54) one has

$$\frac{1}{(\alpha \hat{f})^3} = \frac{3}{8g_0^2 \alpha \varrho^\epsilon} (\Psi'' - 2k) - 2, \quad (3.67)$$

so that

$$h_0/\hat{f} = \frac{\Psi'' - 2k}{8g_0^2 \varrho^\epsilon}, \quad (3.68)$$

$$h_8/\hat{f} = \sqrt{2} \left(\alpha - \frac{\Psi'' - 2k}{8g_0^2 \varrho^\epsilon} \right). \quad (3.69)$$

Finally, substituting everything in eq. (3.58) gives a fourth order differential equation for the function Ψ

$$3\epsilon(\Psi')^2 + 6k\varrho^2\Psi'' - 3\varrho\Psi'(4\epsilon k + \varrho\Psi''') + 3\Psi(4\epsilon k - 2\epsilon\Psi'' + 2\epsilon\varrho\Psi''' - \varrho^2\Psi'''' - 2g_0^2\varrho^{2+\epsilon} [\varrho^\epsilon(\hat{F}^A \cdot \hat{F}^A + 8\alpha^2 g_0^2) - 4\alpha(\Psi'' - 2k)]) = 0, \quad (3.70)$$

which can only be solved if we first find a selfdual SU(2) instanton on the Kähler base space $\hat{F}^{A-} = 0$. Since we only know solutions of this kind for $k = 1$ (see ref. [38] and the discussion in the previous section), we will now carry a case by case analysis of the possible solutions for different values of ϵ setting $k = 1$ and taking into account that, for any given $\Psi(\varrho)$, there are in general two selfdual SU(2) instanton solutions: a “universal” solution and a “constrained” solution.

Let us start by considering the $\epsilon = 1$ case.

3.3.1 The $\epsilon = 1$ case

Following ref. [38], for this case the “universal” instanton solution is, irrespectively of the form of $\Psi(\varrho)$, given by

$$\hat{A}^A = -\frac{1}{g\varrho^2} \left[\left(\frac{1}{2}g\beta - \varrho \right) y^A (dz + \chi_{(1)}) + \varepsilon^A{}_{BC} y^B dy^C \right], \quad \Rightarrow \quad \hat{F}^A \cdot \hat{F}^A = \beta^2/\varrho^4, \quad (3.71)$$

where β is an arbitrary constant and y^A are Cartesian coordinates related to ϱ by $y^A y^A = \varrho^2$.

Then, using this instanton solution in eq. (3.70) and assuming $\Psi(\varrho)$ to be a polynomial, we get two distinct solutions, both for Ψ of order 3, $\Psi = \sum_{i=0}^3 \psi_i \varrho^i$:

$$\mathbf{(u1)} \quad \psi_3 = \frac{4}{3}\alpha g_0^2, \quad \text{and} \quad \beta^2 = \frac{3}{2g_0^2} [\psi_1^2 - 4\psi_0(\psi_2 - 1)], \quad (3.72)$$

$$\mathbf{(u2)} \quad \psi_3 = \frac{4}{9}\alpha g_0^2, \quad \psi_2 = 1, \quad \text{and} \quad \beta^2 = \frac{3}{2} \frac{\psi_1^2}{g_0^2}. \quad (3.73)$$

The “constrained” instanton solution is obtained by assuming from the start that $\Psi = \sum_{i=0}^3 \psi_i \varrho^i$, and is characterized by the function

$$K = \frac{\psi_3}{2} \varrho^2 + \frac{\psi_2 - 1}{3} \varrho + \frac{1}{18\psi_3} [9\psi_1\psi_3 - 2(\psi_2 + 2)(\psi_2 - 1)], \quad (3.74)$$

with the constraint

$$\psi_0 = \frac{\psi_2 + 2}{27\psi_3^2} [9\psi_1\psi_3 - 2(\psi_2 + 2)(\psi_2 - 1)] , \quad (3.75)$$

and leads to the instanton number density

$$\hat{F}^A \cdot \hat{F}^A = \frac{4}{g^2 \varrho^2} \left[\left(K' - \frac{K}{\varrho} \right)^2 + \frac{2}{\Psi} K^2 (K' + 1) \right] . \quad (3.76)$$

Substituting this expression into eq. (3.70) and taking into account the above constraints between the coefficients of the polynomial Ψ we find that, depending on the relative value of the two coupling constants which we denote with the parameter ξ

$$\xi \equiv g/g_0 . \quad (3.77)$$

the differential equation admits two different solutions:

$$\text{(c1)} \quad \xi^2 \neq 2/3, \quad \psi_3 = \frac{4\alpha g_0^2}{3} \frac{6 \pm \sqrt{9 - 6\xi^{-2}}}{9 + 2\xi^{-2}}, \quad \psi_2 = 1, \quad \psi_1 = \psi_0 = 0, \quad (3.78)$$

$$\text{(c2)} \quad \xi^2 = 2/3, \quad \psi_3 = \frac{2\alpha g_0^2}{3}, \quad \psi_2 = 1, \quad \psi_0 = \frac{3\psi_1}{2\alpha g_0^2}. \quad (3.79)$$

In this last case ψ_1 remains undetermined.

In the four cases **u1**, **u2**, **c1**, **c2**,

$$\hat{\omega} = \frac{\sqrt{3}\alpha}{4g_0} \left\{ 2(\psi_2 - 1)\chi_{(1)} + \left[\frac{\psi_1}{\varrho} + \left(3\psi_3 - \frac{4}{3}g_0^2\alpha \right) \varrho \right] (dz + \chi_{(1)}) \right\} + \tilde{\omega}, \quad (3.80)$$

where we remind the reader the definition $d\tilde{\omega} \equiv \Omega_2 \hat{\Phi}^{(2)} + \Omega_3 \hat{\Phi}^{(3)}$.¹⁷ With a constant shift in the time coordinate t it is possible to bring $\hat{\omega}$ to the simpler form

$$\hat{\omega} = \frac{\sqrt{3}\alpha}{4g_0\varrho} \left[\psi_1 + 2(\psi_2 - 1)\varrho + \left(3\psi_3 - \frac{4}{3}g_0^2\alpha \right) \varrho^2 \right] (dz + \chi_{(1)}) + \tilde{\omega}. \quad (3.82)$$

Also, in the four cases the function \hat{f} is given by

$$\left(\alpha \hat{f} \right)^{-3} = \frac{3}{4g_0^2\alpha\varrho} (3\psi_3\varrho + \psi_2 - 1) - 2, \quad (3.83)$$

and this ends the determination of all the building blocks of the solutions, which we now have to analyze.

From now on we take $\tilde{\omega} = 0$ for the sake of simplicity. Then, it is possible to set the constant α to an arbitrary value α/δ with the rescaling $\varrho \rightarrow \varrho/\delta$, $t \rightarrow \delta t$, $\psi_3 \rightarrow \delta\psi_3$, $\psi_1 \rightarrow \psi_1/\delta$, $\psi_0 \rightarrow \psi_0/\delta^2$. This will allow us later to normalize the solution in the most convenient way.

¹⁷Under the assumption that the components of $\hat{\omega}$ are independent of z , closure implies

$$d\tilde{\omega} = (dz + \chi_{(k)}) \wedge \Im[\mathcal{H}(\zeta)d\zeta] + \frac{\varrho d\varrho}{\Psi} \wedge \Re[\mathcal{H}(\zeta)d\zeta], \quad (3.81)$$

where \mathcal{H} is an arbitrary holomorphic function and $\zeta = x^1 + ix^3$.

Every solution of this form presents a (naked) curvature singularity in $\varrho = 0$, except for $\psi_2 = 1, \psi_1 = \psi_0 = 0$, in which case the curvature scalars $R, R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$ are constant. This is the case for the solution **c1** in (3.78). The solutions **u2** and **c2** give a metric with the wrong signature. This leaves us with the only meaningful possibilities:

u1 singular at $\varrho = 0$ except for $\psi_2 = 1, \psi_1 = \psi_0 = 0$, in which case $\beta = 0$, the matter fields are trivial and the solution is just AdS₅.

c1 regular. Defining the parameter

$$\gamma^{-1} = 2 \mp \sqrt{1 - \frac{2}{3}\xi^{-2}}, \quad \Rightarrow \quad \psi_3 = \frac{4}{3}\alpha\gamma g_0^2, \quad \text{and} \quad \hat{f}^{-1} = \alpha(3\gamma - 2)^{1/3}, \quad (3.84)$$

it's easy to see that for the metric to have the right signature one has to take the upper sign in the definition of γ and to impose $\gamma > 2/3$, or equivalently $\xi^2 > 8/9$.

Then it is possible to use the rescaling mentioned above to adjust the integration constant α so that $\hat{f} = 1$ ($\alpha = (3\gamma - 2)^{-1/3}$) and to define \tilde{g} and λ by

$$\tilde{g}_0^2 \equiv \alpha\gamma g_0^2 = \frac{\gamma}{(3\gamma - 2)^{1/3}} g_0^2, \quad \text{and} \quad 1 + \lambda \equiv \frac{3\gamma - 1}{2(3\gamma^2 - 2\gamma)^{1/2}}, \quad (3.85)$$

so that the metric takes the form of that in the solution eq. (3.46) with the replacement $g_0 \rightarrow \tilde{g}_0$. This happens because the scalar potential for these solutions also takes the same value with the replacement of $g_0 \rightarrow \tilde{g}_0$, $V(\phi) = -4\tilde{g}_0$. The remaining non-vanishing fields of the solution are

$$F^0 = \frac{1}{\sqrt{2}} F^8 = -g_0 \frac{\gamma - 1}{(3\gamma - 2)^{4/3}} \hat{J}, \quad (3.86)$$

$$\phi \equiv h_8/h_0 = \sqrt{2}(\gamma^{-1} - 1), \quad (3.87)$$

$$A^A = \hat{A}^A = -\frac{2\tilde{g}_0^2}{3g} y^A (dz + \cos\theta d\varphi) + \frac{(1 + \frac{4}{3}\tilde{g}_0^2\varrho)^{1/2} - 1}{g\varrho^2} \epsilon^A{}_{BC} y^B dy^C. \quad (3.88)$$

The instanton number density is given by

$$\hat{F}^A \cdot \hat{F}^A = \frac{16}{3} \frac{\tilde{g}_0^4}{g^2}. \quad (3.89)$$

In the limit $g \rightarrow \infty$ for fixed g_0 then γ goes to 1 and the above solution reduces to AdS₅.

3.3.2 The $\epsilon = 0$ case

For $\epsilon = 0$ eq. (3.70) takes the much simpler form

$$\Psi' \Psi''' + \Psi \Psi'''' - 2\Psi'' + \frac{2}{3} g_0^2 \left[\hat{F}^A \cdot \hat{F}^A + 8\alpha^2 g_0^2 - 4\alpha(\Psi'' - 2) \right] = 0. \quad (3.90)$$

This equation admits no solution for the constrained instanton solution. Let us then consider the universal solution, for which the instanton number density is always given by

$$\hat{F}^A \cdot \hat{F}^A = \frac{4}{g^2}. \quad (3.91)$$

and eq. (3.90) becomes

$$\Psi'\Psi''' + \Psi\Psi'''' - 2\Psi'' - \frac{8}{3}\alpha g_0^2(\Psi'' - 2 - 2\alpha g_0^2) + \frac{8}{3}(g_0/g)^2 = 0. \quad (3.92)$$

If, as usual, we assume Ψ to be a polynomial in ϱ , from eq. (3.92) we find that it is at most of second order, $\Psi = \sum_{i=0}^2 \psi_i \varrho^i$. There are two possibilities to solve the differential equation (3.92):

1. **u1** If $a \equiv \alpha g_0^2 \neq -\frac{3}{4}$, eq. (3.92) is satisfied for

$$\psi_2 = \frac{2\xi^{-2} + 4a(1+a)}{3+4a}, \quad (3.93)$$

where ξ , ψ_0 and ψ_1 are left unconstrained.

2. **u2** If $a = -\frac{3}{4}$ eq. (3.92) only admits a solution for a specific value of $\xi \equiv g/g_0$:

$$\xi^{-2} = \frac{3}{8}. \quad (3.94)$$

In this case the polynomial Ψ is not constrained by these equations.

In both cases we can, again, impose $\tilde{\omega} = 0$ for simplicity, use eqs. (3.62), (3.63), (3.65), integrate to obtain $\hat{\omega}$, and write the five-dimensional metric as

$$ds^2 = \hat{f}^2 (dt - c_1 \varrho dz - c_2 \cos \theta d\varphi)^2 - \hat{f}^{-1} \left[\Psi dz^2 + \frac{d\varrho^2}{\Psi} + d\Omega_{(2,1)}^2 \right], \quad (3.95)$$

where the constants $c_{1,2}$ are given by

$$c_1 = \frac{a}{\sqrt{3}g_0^3} \left(a - \frac{3}{2}\psi_2 \right), \quad c_2 = \frac{a}{\sqrt{3}g_0^3} \left(a + \frac{3}{2} \right), \quad (3.96)$$

and \hat{f} is determined from eq. (3.67) to be constant:

$$\hat{f}^{-3} = \frac{3a^2(\psi_2 - 1) - 8a^3}{4g_0^6}. \quad (3.97)$$

The general structure of the metric is that of a U(1) fibration over the product of 2 2-dimensional spaces: the 2-sphere and the space parametrized by (ϱ, z) , which we are going to study in more detail below.

The complete non-Abelian 1-form field and its 2-form field strength are given by

$$A^A = \hat{A}^A = \frac{1}{g} \left(y^A dz - \frac{1}{\varrho^2} \epsilon^A_{BC} y^B dy^C \right), \quad F^A = \hat{F}^A = \frac{y^A}{g\varrho} (d\varrho \wedge dz + \sin \theta d\theta \wedge d\varphi), \quad (3.98)$$

thus, the field strength is $1/g$ times the unit vector y^A/ϱ times the sum of the volume forms of the 2-dimensional spaces that enter in the base space.

The remaining fields take the form

$$A^0 = \frac{-2a/g_0}{3(\psi_2 - 1) - 8a} [(1 + 2a)\varrho dz + (2a - \psi_2) \cos \theta d\varphi] , \quad (3.99)$$

$$A^8 = -\sqrt{2}A^0 , \quad (3.100)$$

$$\phi = \frac{g_0^2}{a\hat{f}} . \quad (3.101)$$

By shifting and rescaling the coordinates ϱ and z we can reduce the number of independent parameters and study in more detail the possible 2- and 5-dimensional metrics that arise

- If $\psi_2 \neq 0$, we can bring the base space metric to the form

$$d\hat{s}^2 = \frac{1}{\psi_2} \left[(\varrho^2 - \varepsilon) dz^2 + \frac{d\varrho^2}{\varrho^2 - \varepsilon} \right] + d\Omega_{(2,1)}^2 , \quad (3.102)$$

where $\varepsilon = 0, \pm 1$. The Ricci scalar of the two-dimensional space parametrized by (ϱ, z) is constant and negative for any of the three values of ε and so, it is maximally symmetric. Therefore this is the metric of the hyperbolic plane that we will denote by $d\Omega_{(2,-1)}^2$,¹⁸ and the base-space metric is that of the product of \mathbb{H}_2 with radius squared $1/\psi_2$ and S^2 with radius 1

Denoting by $\chi_{(1)} \equiv \cos \theta d\varphi$ the Kähler 1-form of the 2-sphere and by $\chi_{(-1)} \equiv \varrho dz$ the Kähler 1-form of the hyperbolic plane, the full five-dimensional metric (after shifting the time coordinate) and the rest of the fields can be written as

$$ds^2 = \hat{f}^2 \left(dt - \frac{c_1}{\psi_2} \chi_{(-1)} - c_2 \chi_{(1)} \right)^2 - \hat{f}^{-1} \left[\frac{1}{\psi_2} d\Omega_{(2,-1)}^2 + d\Omega_{(2,1)}^2 \right] , \quad (3.103)$$

$$A^A = \frac{1}{g\psi_2} \left(y^A dz - \frac{\psi_2}{\varrho^2} \epsilon^A{}_{BC} y^B dy^C \right) , \quad (3.104)$$

$$F^A = \frac{y^A}{g\varrho} \left(\frac{1}{\psi_2} d\varrho \wedge dz + \sin \theta d\theta \wedge d\varphi \right) , \quad (3.105)$$

$$A^0 = \frac{-2a/g_0}{3(\psi_2 - 1) - 8a} \left[\frac{(1 + 2a)}{\psi_2} \chi_{(-1)} + (2a - \psi_2) \chi_{(1)} \right] , \quad (3.106)$$

$$A^8 = -\sqrt{2}A^0 , \quad (3.107)$$

$$\phi = \frac{g_0^2}{a\hat{f}} . \quad (3.108)$$

This metric has the same form as the Gödel solutions found in [29]. It is well known that, generically, these metrics have closed timelike curves (CTCs), but one can wonder if it is possible to tune the parameter a in such a way as to avoid them. This would demand setting $c_2 = 0$ ($a = -3/2$) to avoid Misner string singularities

¹⁸Actually, it is easy to see that a simple coordinate change brings the metric to the standard form of the hyperbolic plane in polar, Lobachevsky and Poincaré half-plane coordinates respectively for $\varepsilon = 1, -1, 0$.

or having to compactify the time coordinate to avoid them. It is not possible to set $c_1 = 0$ at the same time (ψ_2 must be strictly positive). Then, the condition for absence of CTCs is

$$(\hat{f}c_1)^2 < \hat{f}^{-1}/\psi_2, \tag{3.109}$$

and with $a = -3/2$, this condition cannot be satisfied for any value of ψ_2 .

Note also that the metric only has the correct signature if \hat{f} and ψ_2 are both positive, which means $a \in (-\frac{3}{4}, 0)$.

- If $\psi_2 = 0$ and $\psi_1 \neq 0$, we get a 5-dimensional metric of the form

$$ds^2 = \hat{f}^2 [dt - c_1 \varrho dz - c_2 \chi_{(1)}]^2 - \hat{f}^{-1} \left[\varrho dz^2 + \frac{d\varrho^2}{\varrho} + d\Omega_{(2,1)}^2 \right]. \tag{3.110}$$

- If $\psi_2 = \psi_1 = 0$ and $\psi_0 > 0$,¹⁹ we get a 5-dimensional metric of the form

$$ds^2 = \hat{f}^2 [dt - c_1 \varrho dz - c_2 \chi_{(1)}]^2 - \hat{f}^{-1} [dz^2 + d\varrho^2 + d\Omega_{(2,1)}^2]. \tag{3.111}$$

The metric for the last two cases is actually the same one written in different coordinates, and can also be written as

$$ds^2 = \hat{f}^2 [dt - c_1 \chi_{(0)} - c_2 \chi_{(1)}]^2 - \hat{f}^{-1} [d\Omega_{(2,0)}^2 + d\Omega_{(2,1)}^2], \tag{3.112}$$

where $d\Omega_{(2,0)}^2$ is the metric of the 2-dimensional Euclidean space and $\chi_{(0)}$ its Kähler 1-form and with

$$c_1 = \frac{a^2}{\sqrt{3}g_0^3}, \quad c_2 = \frac{a}{\sqrt{3}g_0^3} \left(a + \frac{3}{2} \right), \quad \hat{f}^{-3} = -\frac{a^2(3+8a)}{4g_0^6}. \tag{3.113}$$

The non-Abelian fields are given by the general expressions eqs. (3.98) and the Abelian ones by

$$A^0 = \frac{2a/g_0}{3+8a} [(1+2a)\chi_{(0)} + 2a\chi_{(1)}], \tag{3.114}$$

$$A^8 = -\sqrt{2}A^0, \tag{3.115}$$

while the constant scalar field is still given by eq. (3.108).

Observe that, in this case, a is not a free parameter, since $\psi_2 = 0$ implies from eqs. (3.93) and (3.94)

$$a = \frac{-1 \pm \sqrt{1 - 2\xi^{-2}}}{2}. \tag{3.116}$$

For this condition to make sense one must of course impose $\xi^2 \geq 2$.

¹⁹For $\psi_0 < 0$ one would get the wrong signature for the metric.

4 Embedding in half-maximal $d = 5$, $SU(2) \times U(1)$ -gauged supergravity

Uplifting the solutions of the cosmological gauged \mathbb{C} magic model to 10 dimensions presents severe practical difficulties, starting with the embedding of the solutions we have obtained in $SO(6)$ -gauged $d = 5$ maximal supergravity which requires a definite relation between the $U(1)_R$ and $SU(2)$ coupling constants which is not readily available in the literature. Then, one would have to face the problem of uplifting the solution to 10 dimensions.

There are not many reduction ansatzs that lead from 10-dimensional supergravities to gauged 5-dimensional supergravities and which permit the automatic uplifting of the 5-dimensional solutions, especially if one is interested in a particular gauge group. There is, however, a reduction ansatz from $\mathcal{N} = 2B, d = 10$ supergravity to gauged, half-maximal $d = 5$ supergravity with, precisely, the gauge group $SU(2) \times U(1)$ [37].²⁰ Since the ansatz corresponds to compactification on a 5-sphere it is natural to expect that the two gauge coupling constants are not independent and, as we shall see, in fact one gets, in our conventions, $\xi^2 = (g/g_0)^2 = 2/3$.

We would like to embed our solutions in this 5-dimensional theory in order to be able to uplift them to 10 dimensions but the relation $\xi^2 = 2/3$ will only allow us to uplift some of them. It is, by no means, guaranteed that such an embedding is possible but we are going to show that indeed it is for some solutions of the cosmological \mathbb{C} gauged magic model that include several of those we have constructed in the previous section. More specifically, we are going to show that the consistently truncated equations of motion of the cosmological \mathbb{C} gauged magic model (for a truncation that includes the solutions we have constructed) coincide with the consistently truncated equations of motion of $SU(2) \times U(1)$ -gauged, half-maximal $d = 5$ supergravity.

4.1 Truncated equations of motion

Let us consider the equations of motion of the cosmological \mathbb{C} gauged magic model (2.15)–(2.17) (where we have replaced the generic objects $a_{IJ}, C_{IJK}, g_{xy}, k_I^x$ by their values for this particular model, evidently). If we set $h_{1,\dots,7} = h^{1,\dots,7} = 0$ and $A^{4,\dots,7} = 0$, and define

$$X \equiv \left(h_0 + \frac{1}{\sqrt{2}} h_8 \right)^{-1} \quad \Rightarrow \quad h_0 - \sqrt{2} h_8 = X^2, \quad \mathcal{H} \equiv F^0 + \frac{1}{\sqrt{2}} F^8, \quad \mathcal{G} \equiv F^0 - \sqrt{2} F^8, \quad (4.1)$$

the equations of motion reduce to²¹

$$R_{\mu\nu} - \frac{1}{6} X^4 \left(\mathcal{G}_\mu{}^\rho \mathcal{G}_{\nu\rho} - \frac{1}{6} g_{\mu\nu} \mathcal{G}^{\rho\sigma} \mathcal{G}_{\rho\sigma} \right) - \frac{1}{3} X^{-2} \left(\mathcal{H}_\mu{}^\rho \mathcal{H}_{\nu\rho} - \frac{1}{6} g_{\mu\nu} \mathcal{H}^{\rho\sigma} \mathcal{H}_{\rho\sigma} \right) - \frac{1}{2} X^{-2} \left(F^A{}_\mu{}^\rho F^A{}_{\nu\rho} - \frac{1}{6} g_{\mu\nu} F^{A\rho\sigma} F^A{}_{\rho\sigma} \right) + 3 \partial_\mu \log X \partial_\nu \log X + \frac{4}{9} g_0^2 g_{\mu\nu} (X^2 + 2X^{-1}) = 0, \quad (4.2)$$

$$\nabla^2 \log X - \frac{1}{12} X^{-2} F^A \cdot F^A - \frac{1}{18} X^{-2} \mathcal{H} \cdot \mathcal{H} + \frac{1}{18} X^4 \mathcal{G} \cdot \mathcal{G} - \frac{4}{9} g_0^2 (X^2 - X^{-1}) = 0, \quad (4.3)$$

²⁰We thank O. Varela for pointing this reference to us.

²¹We have subtracted the trace of the Einstein equation.

$$F^A \cdot \mathcal{H} = 0, \quad (4.4)$$

$$\nabla_\nu (X^{-2} \mathcal{H}^{\nu\mu}) + \frac{1}{4\sqrt{3}} \frac{\epsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} \mathcal{H}_{\nu\rho} \mathcal{G}_{\sigma\alpha} = 0, \quad (4.5)$$

$$\nabla_\nu (X^4 \mathcal{G}^{\nu\mu}) + \frac{1}{4\sqrt{3}} \frac{\epsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} \mathcal{H}_{\nu\rho} \mathcal{H}_{\sigma\alpha} - \frac{\sqrt{3}}{8} \frac{\epsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} F^A{}_{\nu\rho} F^A{}_{\sigma\alpha} = 0, \quad (4.6)$$

$$\mathfrak{D}_\nu (X^{-2} F^{A\nu\mu}) - \frac{1}{4\sqrt{3}} \frac{\epsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} F^A{}_{\nu\rho} \mathcal{G}_{\sigma\alpha} = 0. \quad (4.7)$$

The constraint eq. (4.4) can be solved preserving the non-Abelian gauge fields by setting $\mathcal{H} = 0$. This also solves the equation for \mathcal{H} , leaving no further constraints. The resulting equations are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{6} X^4 \left(\mathcal{G}_\mu{}^\rho \mathcal{G}_{\nu\rho} - \frac{1}{6} g_{\mu\nu} \mathcal{G}^{\rho\sigma} \mathcal{G}_{\rho\sigma} \right) \\ - \frac{1}{2} X^{-2} \left(F^A{}_\mu{}^\rho F^A{}_{\nu\rho} - \frac{1}{6} g_{\mu\nu} F^A{}_{\rho\sigma} F^A{}_{\rho\sigma} \right) \\ + 3 \partial_\mu \log X \partial_\nu \log X + \frac{4}{9} g_0^2 g_{\mu\nu} (X^2 + 2X^{-1}) = 0, \end{aligned} \quad (4.8)$$

$$\nabla^2 \log X - \frac{1}{12} X^{-2} F^A \cdot F^A + \frac{1}{18} X^4 \mathcal{G} \cdot \mathcal{G} - \frac{4}{9} g_0^2 (X^2 - X^{-1}) = 0, \quad (4.9)$$

$$\nabla_\nu (X^4 \mathcal{G}^{\nu\mu}) - \frac{\sqrt{3}}{8} \frac{\epsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} F^A{}_{\nu\rho} F^A{}_{\sigma\alpha} = 0, \quad (4.10)$$

$$\mathfrak{D}_\nu (X^{-2} F^{A\nu\mu}) - \frac{1}{4\sqrt{3}} \frac{\epsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} F^A{}_{\nu\rho} \mathcal{G}_{\sigma\alpha} = 0, \quad (4.11)$$

which, if $\xi^2 = (g/g_0)^2 = 2/3$ and after a rescaling of \mathcal{G} , are identical to those obtained in ref. [37] when the tensor fields in the latter are set to zero.

Since we have used the constraint $\mathcal{H} = 0$ in the construction of our solutions, we can, in principle, embed all of them in $SU(2) \times U(1)$ -gauged, half-maximal $d = 5$ supergravity and, then, using the dimensional reduction ansatz in ref. [37], uplift them to solutions of $\mathcal{N} = 2B, d = 10$ supergravity. However as we have seen only some of them are compatible with the constraint $\xi^2 = 2/3$, namely the solutions we have called **u1** in the two cases $\epsilon = 1, 0$, and for $\epsilon = 0$ only the subcase $\psi_2 \neq 0$, since otherwise it would require $\xi^2 \geq 2$. These present some undesirable characteristics (a naked singularity for $\epsilon = 1$ and closed timelike curves for $\epsilon = 0$) which are also present in the uplifted 10-dimensional solutions.

5 Conclusions

By exploiting the supersymmetric solution-generating techniques developed over the years we have managed to find some of the simplest non-Abelian solutions of two models of “cosmological, $SU(2)$ -gauged,” $\mathcal{N} = 1, d = 5$ supergravity coupled to vector multiplets where the term “cosmological” refers to an additional $U(1)_R$ gauging that gives rise to a non-vanishing scalar potential. The non-Abelian gauge field configurations in these solutions is, by construction, that of a self-dual instanton over a 4-dimensional Kähler manifold admitting a holomorphic isometry.

We have found a solution that occurs with the same metric eq. (3.46) and slightly different matter fields in both models. This is an interesting supersymmetric 1-parameter deformation of AdS_5 which, as opposed to the deformation found in ref. [21] and studied in ref. [40], is not asymptotically- AdS_5 . It does not have a holographic screen in the $\varrho \rightarrow \infty$ either because in this limit it is not conformal to any regular metric. In the most obvious frame, all the components of the Riemann tensor of this metric are constant, which implies that all its curvature invariants are constant. It might be a homogeneous Riemannian space, though, although we have not checked completely this possibility.

The rest of the solutions that we have found fall in two types: those which may asymptote to AdS_5 but have naked singularities at $\varrho = 0$, and those which are generalizations of the Gödel-like solutions found in ref. [29] in the context of pure cosmological supergravity and whose metrics are timelike $U(1)$ fibrations over products of 2-dimensional maximally symmetric spaces. All of them seem to have closed timelike curves.

Our second goal was to study the possible embedding of the solutions of the cosmological, gauged \mathbb{C} magical model in String Theory via maximal or half-maximal gauged $d = 5$ supergravity. The embedding in maximal supergravity is only possible for the relation between the $U(1)_R$ and $SU(3)$ gauge coupling constants g_0 and g , which follows from the breaking of the $SO(6)$ gauge group. Finding this relation is a very complicated problem whose solution needs a precise knowledge of the relation between the fields used in the formalism of $\mathcal{N} = 1, d = 5$ theories and those of the maximal supergravity, which is not available. This knowledge is also needed for actual embedding and, therefore, although it is guaranteed that some of the solutions found can be embedded and uplifted to 10 dimensions, the embedding and uplifting cannot be realized in practice.

The embedding in the $SU(2) \times U(1)$ -gauged $d = 5$ half-maximal supergravity of ref. [37] also requires a precise relation between the coupling constants, but in this case this relation is known and also satisfied by some of the solutions, although they are the singular ones and they remain singular after uplifting them to 10 dimensions.

The difficulties in uplifting the solutions to 10 dimensions leave us without an interpretation of the fields in terms of branes although the regular solution eq. (3.46) seems to be a generalization of the gravitating Yang-Mills instanton of ref. [14] since, also in this case, in the two models we have studied, the graviphoton field is sourced by the instanton number density 4-form. The model studied in ref. [14] can be obtained by a toroidal compactification and truncation of 10-dimensional Heterotic Supergravity and the graviphoton is related to the Kalb-Ramond 2-form. The solution is, therefore, a compactification of the gauge 5-brane. In the theories that we have considered the graviphoton gauges $U(1)_R$ via a Fayet-Iliopoulos term and the 10-dimensional interpretation is much less transparent.

Although the balance of this work in terms of interesting solutions (especially from the holographic point of view) may look slightly disappointing, it is fair to say this is just the beginning of the exploration of a large, unknown, and very complicated territory. We have put to work all the techniques developed in the field and showed that they work in these very complicated systems. Just as in the asymptotically-flat case, more interesting supersymmetric non-Abelian solutions must exist and we expect to be able to find some of them in forthcoming work.

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References

- [1] S. Ferrara, R. Kallosh and A. Strominger, $N = 2$ extremal black holes, *Phys. Rev. D* **52** (1995) R5412 [[hep-th/9508072](#)] [[INSPIRE](#)].
- [2] A. Strominger, *Macroscopic entropy of $N = 2$ extremal black holes*, *Phys. Lett. B* **383** (1996) 39 [[hep-th/9602111](#)] [[INSPIRE](#)].
- [3] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev. D* **54** (1996) 1514 [[hep-th/9602136](#)] [[INSPIRE](#)].
- [4] S. Ferrara and R. Kallosh, *Universality of supersymmetric attractors*, *Phys. Rev. D* **54** (1996) 1525 [[hep-th/9603090](#)] [[INSPIRE](#)].
- [5] S. Ferrara, G.W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, *Nucl. Phys. B* **500** (1997) 75 [[hep-th/9702103](#)] [[INSPIRE](#)].
- [6] T. Ortín, *Gravity and Strings*, second edition, Cambridge University Press, (2015).
- [7] P. Meessen, *Supersymmetric coloured/hairy black holes*, *Phys. Lett. B* **665** (2008) 388 [[arXiv:0803.0684](#)] [[INSPIRE](#)].
- [8] M. Huebscher, P. Meessen, T. Ortín and S. Vaulà, $N = 2$ Einstein-Yang-Mills's BPS solutions, *JHEP* **09** (2008) 099 [[arXiv:0806.1477](#)] [[INSPIRE](#)].
- [9] P. Bueno, P. Meessen, T. Ortín and P.F. Ramírez, $\mathcal{N} = 2$ Einstein-Yang-Mills' static two-center solutions, *JHEP* **12** (2014) 093 [[arXiv:1410.4160](#)] [[INSPIRE](#)].
- [10] P. Meessen and T. Ortín, $\mathcal{N} = 2$ super-EYM coloured black holes from defective Lax matrices, *JHEP* **04** (2015) 100 [[arXiv:1501.02078](#)] [[INSPIRE](#)].
- [11] P. Meessen, T. Ortín and P.F. Ramírez, *Non-Abelian, supersymmetric black holes and strings in 5 dimensions*, *JHEP* **03** (2016) 112 [[arXiv:1512.07131](#)] [[INSPIRE](#)].
- [12] T. Ortín and P.F. Ramírez, *A non-Abelian Black Ring*, *Phys. Lett. B* **760** (2016) 475 [[arXiv:1605.00005](#)] [[INSPIRE](#)].
- [13] P.F. Ramírez, *Non-Abelian bubbles in microstate geometries*, *JHEP* **11** (2016) 152 [[arXiv:1608.01330](#)] [[INSPIRE](#)].
- [14] P.A. Cano, T. Ortín and P.F. Ramírez, *A gravitating Yang-Mills instanton*, *JHEP* **07** (2017) 011 [[arXiv:1704.00504](#)] [[INSPIRE](#)].

- [15] P.A. Cano, P. Meessen, T. Ortín and P.F. Ramírez, *Non-Abelian black holes in string theory*, *JHEP* **12** (2017) 092 [[arXiv:1704.01134](#)] [[INSPIRE](#)].
- [16] P. Meessen, T. Ortín and P.F. Ramírez, *Dyonic black holes at arbitrary locations*, *JHEP* **10** (2017) 066 [[arXiv:1707.03846](#)] [[INSPIRE](#)].
- [17] J. Ávila, P.F. Ramírez and A. Ruipérez, *One Thousand and One Bubbles*, *JHEP* **01** (2018) 041 [[arXiv:1709.03985](#)] [[INSPIRE](#)].
- [18] M. Huebscher, P. Meessen, T. Ortín and S. Vaulà, *Supersymmetric $N = 2$ Einstein-Yang-Mills monopoles and covariant attractors*, *Phys. Rev. D* **78** (2008) 065031 [[arXiv:0712.1530](#)] [[INSPIRE](#)].
- [19] M. Cariglia and O.A.P. Mac Conamhna, *The general form of supersymmetric solutions of $N = (1, 0)$ U(1) and SU(2) gauged supergravities in six-dimensions*, *Class. Quant. Grav.* **21** (2004) 3171 [[hep-th/0402055](#)] [[INSPIRE](#)].
- [20] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five-dimensions*, *Class. Quant. Grav.* **20** (2003) 4587 [[hep-th/0209114](#)] [[INSPIRE](#)].
- [21] J.P. Gauntlett and J.B. Gutowski, *All supersymmetric solutions of minimal gauged supergravity in five-dimensions*, *Phys. Rev. D* **68** (2003) 105009 [Erratum *ibid.* **D 70** (2004) 089901] [[hep-th/0304064](#)] [[INSPIRE](#)].
- [22] J.B. Gutowski and H.S. Reall, *General supersymmetric AdS_5 black holes*, *JHEP* **04** (2004) 048 [[hep-th/0401129](#)] [[INSPIRE](#)].
- [23] J.P. Gauntlett and J.B. Gutowski, *General concentric black rings*, *Phys. Rev. D* **71** (2005) 045002 [[hep-th/0408122](#)] [[INSPIRE](#)].
- [24] J.B. Gutowski and W. Sabra, *General supersymmetric solutions of five-dimensional supergravity*, *JHEP* **10** (2005) 039 [[hep-th/0505185](#)] [[INSPIRE](#)].
- [25] J. Bellorín, P. Meessen and T. Ortín, *All the supersymmetric solutions of $N = 1, d = 5$ ungauged supergravity*, *JHEP* **01** (2007) 020 [[hep-th/0610196](#)] [[INSPIRE](#)].
- [26] J. Bellorín and T. Ortín, *Characterization of all the supersymmetric solutions of gauged $N = 1, d = 5$ supergravity*, *JHEP* **08** (2007) 096 [[arXiv:0705.2567](#)] [[INSPIRE](#)].
- [27] J. Bellorín, *Supersymmetric solutions of gauged five-dimensional supergravity with general matter couplings*, *Class. Quant. Grav.* **26** (2009) 195012 [[arXiv:0810.0527](#)] [[INSPIRE](#)].
- [28] S. Chimento and T. Ortín, *On 2-dimensional Kähler metrics with one holomorphic isometry*, [arXiv:1610.02078](#) [[INSPIRE](#)].
- [29] S. Chimento and T. Ortín, *On timelike supersymmetric solutions of gauged minimal 5-dimensional supergravity*, *JHEP* **04** (2017) 017 [[arXiv:1611.09383](#)] [[INSPIRE](#)].
- [30] S. Chimento, *On timelike supersymmetric solutions of Abelian gauged 5-dimensional supergravity*, *JHEP* **07** (2017) 059 [[arXiv:1705.01903](#)] [[INSPIRE](#)].
- [31] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *$N = 2$ supergravity in five-dimensions revisited*, *Class. Quant. Grav.* **21** (2004) 3015 [[hep-th/0403045](#)] [[INSPIRE](#)].
- [32] M. Günaydin, G. Sierra and P.K. Townsend, *The Geometry of $N = 2$ Maxwell-Einstein Supergravity and Jordan Algebras*, *Nucl. Phys. B* **242** (1984) 244 [[INSPIRE](#)].

- [33] M. Günaydin, G. Sierra and P.K. Townsend, *Exceptional Supergravity Theories and the MAGIC Square*, *Phys. Lett. B* **133** (1983) 72 [[INSPIRE](#)].
- [34] M. Günaydin, L.J. Romans and N.P. Warner, *Compact and Noncompact Gauged Supergravity Theories in Five-Dimensions*, *Nucl. Phys. B* **272** (1986) 598 [[INSPIRE](#)].
- [35] M. Günaydin, G. Sierra and P.K. Townsend, *Gauging the $d = 5$ Maxwell-Einstein Supergravity Theories: More on Jordan Algebras*, *Nucl. Phys. B* **253** (1985) 573 [[INSPIRE](#)].
- [36] L.J. Romans, *Gauged $N = 4$ Supergravities in Five-dimensions and Their Magnetovac Backgrounds*, *Nucl. Phys. B* **267** (1986) 433 [[INSPIRE](#)].
- [37] H. Lü, C.N. Pope and T.A. Tran, *Five-dimensional $N = 4$, $SU(2) \times U(1)$ gauged supergravity from type IIB*, *Phys. Lett. B* **475** (2000) 261 [[hep-th/9909203](#)] [[INSPIRE](#)].
- [38] S. Chimento, T. Ortín and A. Ruipérez, *Yang-Mills instantons in Kähler spaces with one holomorphic isometry*, *Phys. Lett. B* **778** (2018) 371 [[arXiv:1710.00764](#)] [[INSPIRE](#)].
- [39] T. Ortín and C. Santoli, *Supersymmetric solutions of $SU(2)$ -Fayet-Iliopoulos-gauged $\mathcal{N} = 2$, $d = 4$ supergravity*, *Nucl. Phys. B* **916** (2017) 37 [[arXiv:1609.08694](#)] [[INSPIRE](#)].
- [40] J.P. Gauntlett, J.B. Gutowski and N.V. Suryanarayana, *A Deformation of $AdS_5 \times S^5$* , *Class. Quant. Grav.* **21** (2004) 5021 [[hep-th/0406188](#)] [[INSPIRE](#)].