Conformal structure of massless scalar amplitudes beyond tree level

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ABSTRACT: We show that the one-loop on-shell four-point scattering amplitude of massless ϕ^4 scalar field theory in 4D Minkowski space time, when Mellin transformed to the Celestial sphere at infinity, transforms covariantly under the global conformal group $(\mathrm{SL}(2,\mathbb{C}))$ on the sphere. The unitarity of the four-point scalar amplitudes is recast into this Mellin basis. We show that the same conformal structure also appears for the two-loop Mellin amplitude. Finally we comment on some universal structure for all loop four-point Mellin amplitudes specific to this theory.

KEYWORDS: Conformal Field Theory, Global Symmetries, Scattering Amplitudes

Contents

1 Introduction and summary

In a Quantum field theory (QFT), an interesting set of observables are scattering amplitudes. In flat space they are covariant under the Poincare group. In a four-dimensional QFT on the Minkowski space, the amplitude are usually constructed out of the asymptotic plane wave solutions to the free wave equation. The plane waves are eigenstates of the translation operators. As a result energy-momentum conservation is manifest in this basis and the amplitudes are translationally invariant. Whereas Lorentz transformation properties of the plane wave states are complicated and hence the whole of $SL(2, \mathbb{C})$ invariance is more subtle.

Another interesting fact is that the four dimensional Lorentz group $SL(2, \mathbb{C})$ acts as the group of global conformal transformations on the celestial sphere, denoted by $\mathcal{CS}^2$. This sphere is defined at null infinity where the asymptotic states are specified. As a result of this, it might be expected that the scattering amplitude of (massive) massless particles should transform in a representation of the global conformal group, when expressed in terms of right basis states. In particular for massless particle scattering amplitude, this is not difficult to understand. As we describe in the next section, in four dimensions, the null momentum of a massless particle is completely specified by the energy and the direction of the three-momentum. Thus modulo a scale, the null momentum can be completely specified by a point on a two dimensional sphere. This two dimensional sphere is a space-like cross-section of the light-cone in the momentum space. Hence if we think of the light-cone as embedded in Minkowski space, the two dimensional sphere may be regarded as the celestial sphere. Thus, when the amplitude is expressed in terms of the coordinates of this sphere, it is expected to transform covariantly under the global conformal symmetry.

The transformation properties of scattering amplitudes under $\mathrm{SL}(2,\mathbb{C})$ was first considered by Dirac [1]. During last few years, the subject has gotten new interests, since it has shed some light on the holographic structure of flat space gravity. We have already said $\mathrm{SL}(2,\mathbb{C})$ acts as the group of global conformal transformations on $\mathcal{CS}^2$. In particular when gravity is coupled, it has been conjectured that this global conformal group gets enhanced to the infinite dimensional virasoro algebra. Thus the CFT looks a lot like a standard 2-D CFT, although the representation of the conformal group may be different. Related works on this topics can be found in [2–15]. One important ingredient in this study is the construction of the proper basis states. Recently in a series of papers [16–19], Sabrina Pasterski et al. have constructed a very interesting integral transform of the flat space wave functions and scattering amplitudes of massive and massless particles. In particular for massless particle, the transformation takes the form of a Mellin transform [17]. Under this transformation, the momentum-space scattering amplitude maps to a function, namely the Mellin amplitude, on the Celestial sphere. The Mellin amplitude transforms like the correlation function of (quasi-) primary operators of a two dimensional CFT defined on the sphere. The putative CFT has operators with all possible "scaling dimensions" of the form $(1 + i\lambda)$ where λ is a real number. The two dimensional spin of an operator is determined by the helicity of the external particle to which it corresponds to.

So far, the conformal structure of flat space tree level massive scalar amplitudes [18] and gluon amplitudes [16] have been studied in the literature. In this paper we use the similar techniques to further explore the Mellin transform of the one and two loop four-point amplitude of a massless scalar field theory with ϕ^4 interaction. Thus, in a sense, this is the first attempt to understand the conformal structure of flat space scattering amplitudes beyond tree level. To summarize our main results, we do see the conformal structure[1] even at loop level Mellin amplitudes. In particular, the on-shell one loop four point Mellin amplitude looks like,

$$\tilde{\mathcal{T}}_2 = \frac{i\lambda_R^2}{4}\left(\frac{2}{\mu}\right)^{-i\Lambda}\left[6\pi^3\delta'(\Lambda) + \pi^4\delta(\Lambda)\right]\delta(|z - \bar{z}|)\left(\prod_{i<j}^{4}|z_{ij}|^{h/3 - h_i - h_j}|\bar{z}_{ij}|^{\bar{h}/3 - \bar{h}_i - \bar{h}_j}\right)[z(z - 1)]^{2/3},$$

where, λ_R is the renormalized coupling constant of the theory defined at energy scale μ and $z_i, i = 1\ldots, 4$ are the position of each of the four particles on the celestial sphere $\mathcal{CS}^2$. $h_i = \frac{1 + \lambda_i}{2}$ are the conformal dimension of the i-th particle, $\Lambda = \sum \lambda_i$ and z is the conformal cross-ratio function. Similar structure holds for two loop amplitude given in equations (5.2). The unitarity property of the QFT amplitude, or equivalently the Optical theorem can also be recast in terms of the Mellin amplitude as in (4.1). The results are subtle and have some universal structure, that leads us to comment on the form of the amplitude to any arbitrary order in the perturbation theory for massless ϕ^4 theory.

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[1]Four point function in a CFT has the form

$$\langle\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle = f(z, \bar{z})\left(\prod_{i<j}^{4}|z_{ij}|^{h/3 - h_i - h_j}|\bar{z}_{ij}|^{\bar{h}/3 - \bar{h}_i - \bar{h}_j}\right),$$

where $f(z, \bar{z})$ is some arbitrary function of the cross ratio $(z, \bar{z})$ and $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$.

The paper is organized as follows: in section 2, we briefly review the conformal basis states for the massive and massless scalar fields. In the next section 3, we present our main result, i.e. the one-loop four point Mellin amplitude of the massless scalar ϕ^4 theory. The section is self contained. In section 4, we explain how the unitarity of the theory can be implemented on the Mellin space. Finally, in section 5, we extend the analysis to two-loop amplitude and also comment on a generic structure for all loop four point amplitude in this theory. In section 6, we conclude with some possible future directions to follow.

Note: on the day of submission of this draft, another paper [20] appeared on arXiv, that has also addressed the issue of unitarity in Mellin space.

2 Conformal basis for massless scalar fields

In this section we briefly review how four dimensional scattering amplitudes of a QFT on a flat space can be recast with manifest global conformal symmetry. The construction is given by Sabrina Pasterski et al. in [17] and we refer the readers to their paper for details. We have already mentioned that to see the conformal structure of the flat space amplitudes, we need to first find the right basis for the asymptotic states that are defined on the celestial sphere. In [17], the authors have defined a new basis for scalars (massive and massless) and spin one gluons, namely the *conformal primary wave functions*, that manifests the conformal structure of their corresponding 4D amplitudes. These conformal primaries are characterized by their conformal dimensions and positions $(w, \bar{w})$ on a 2-dimensional space, that refer to the boundary of the on-shell three diemnsional momentum hyperboloid (H^3). The conformal basis for the massless scalars is obtained as a limit of the corresponding massive one. More precisely, for a massive particle $(p^0 > 0)$, the on-shell momentum hyperboloid looks like

$$(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = m^2, \tag{2.1}$$

and the corresponding conformal primary wave function defined on the boundary of this hyperboloid is given as:

$$\phi_\Delta(x^\mu; w, \bar{w}) = \int_{H^3} \frac{dy}{y^3} dz d\bar{z} \, G_\Delta(y, z, \bar{z}; w, \bar{w}) e^{ip_\mu(y, z, \bar{z})x^\mu}. \tag{2.2}$$

Here, $(y, z, \bar{z})$ are the coordinates on the on-shell momentum hyperboloid and $(w, \bar{w})$ are the coordinates on the boundary of this hyperboloid. G_Δ is the bulk to boundary propagator on the hyperboloid and is given as,

$$G_\Delta(y, z, \bar{z}; w, \bar{w}) = \left(\frac{y}{y^2 + |z - w|^2} \right)^\Delta. \tag{2.3}$$

This propagator transforms covariantly under the conformal transformation of the boundary coordinates $w \to \frac{aw+b}{cw+d}, \bar{w} \to \frac{a\bar{w}+b}{c\bar{w}+d}; ad - bc = 1$. Here, Δ is a label that defines the conformal dimension of the propagator. Hence, by construction, the conformal primary wave function $\phi_\Delta(x^\mu; w, \bar{w})$ has right transformation properties under conformal transformations of $(w, \bar{w})$.
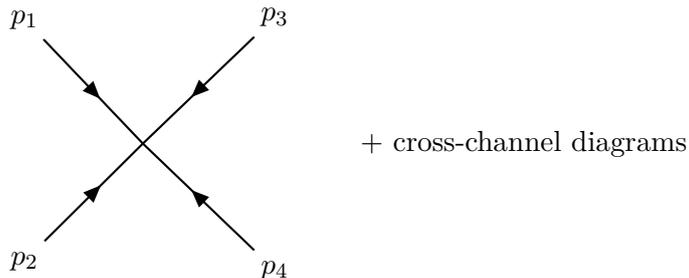
Figure 1. Tree level diagram's. Cross-channel corresponds to two other channel of scattering.

In this paper, we shall be interested in massless scalars. The corresponding conformal primary can be obtained by appropriately taking the mass $m \to 0$ limit of the massive one. For the massless case, the map is direct as the on-shell momenta are null to start with. Hence, they are already at the boundary of the momentum hyperboloid. A null momenta can be parametrized as:

$$p = E\left(1 + |z|^2, z + \bar{z}, -i(z - \bar{z}), 1 - |z|^2\right), \tag{2.4}$$

where, E is an overall scaling of the momenta. As shown in [17], with the appropriate $m \to 0$ limit of the bulk to boundary propagator, the above definition of conformal primary wave function (2.2) simply reduces to a Mellin transform. Thus for a massless scalar, the conformal primary takes the form:

$$\phi(x^{\mu}; z, \bar{z}, \lambda) = \int_0^{\infty} dE E^{i\lambda} e^{ip(E, z, \bar{z})x}. \tag{2.5}$$

Here, λ is a parameter that labels a particular scalar primary and its implication will be clear in the later section.
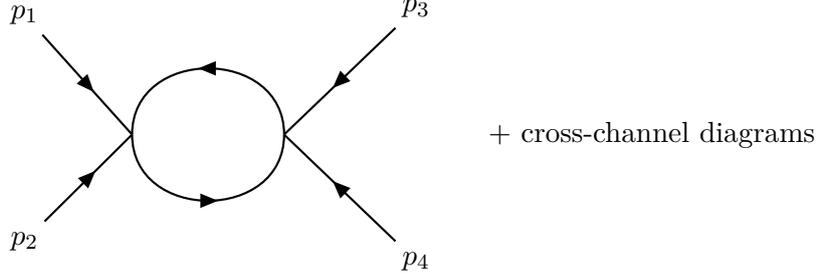
3 4 point one-loop amplitude in massless ϕ^4 theory

In this paper, we are interested in understanding the conformal structure of scattering amplitudes at the loop level, for massless ϕ^4 theory. To set up our normalizations, the theory that we are working with is,

$$S = \int d^4 x \left[\frac{1}{2}(\partial\phi)^2 - \frac{\hat{\lambda}}{4!}\phi^4\right]. \tag{3.1}$$

We want to compute the $(2 \to 2)$ scattering amplitude in this theory. In this section, we shall only be interested up to one loop amplitude, although in later sections we shall comment on higher loop amplitudes. At tree and one loop level, three diagrams (see figure 1, figure 2) describing three channels s, t and u, contribute to the $(2 \to 2)$ scattering amplitude and the non-trivial contribution to the amplitude is given as,

$$A = -i(2\pi)^4 \delta^4(\Sigma p_i)\left[\lambda_R - \frac{i\pi\lambda_R^2}{32\pi^2} + \frac{\lambda_R^2}{32\pi^2}\left(\ln\frac{s}{\mu^2} + \ln\frac{|t|}{\mu^2} + \ln\frac{|u|}{\mu^2}\right)\right]. \tag{3.2}$$

Figure 2. One loop diagram's. Cross-channel corresponds to two other channel of scattering.

Here λ_R is the renormalized coupling constant defined at energy scale μ. p_1, p_2 are incoming momenta and p_3, p_4 are outgoing momenta. We have defined (s, t, u) as,

$$s = (p_1 + p_2)^2, \qquad t = (p_1 + p_3)^2, \qquad u = (p_1 + p_4)^2. \tag{3.3}$$

We want to write the amplitude on the Celestial sphere in an explicit s, t and u channel symmetric from. For this, we first rewrite the above amplitude $A(p_i)$ (3.2) as a function of the on-shell momentum hyperboloid coordinates $(E, z, \bar{z})$ defined in equation (2.4). Finally, using definition (2.5), we Mellin transform this amplitude to the one defined only on the sphere as,

$$\tilde{A}(\lambda_i, z_i, \bar{z}_i) = \prod_{j=1}^{4} \int_0^\infty dE_j E_j^{i\lambda_i} A(E_i, z_i, \bar{z}_i). \tag{3.4}$$

Here λ_i are some labels for the Mellin amplitude. The inverse transform can be readily obtained as,

$$A(E_i, z_i, \bar{z}_i) = \prod_{j=1}^{4} \int_{-\infty}^{\infty} \frac{d\lambda_j}{2\pi} E_i^{-1-i\lambda_j} \tilde{A}(\lambda_j, z_i, \bar{z}_i). \tag{3.5}$$

Our construction closely follows [18]. The convention is that all momenta are incoming with different signs for the energy component. We are now considering the process where 1 and 2 are incoming and 3 and 4 are outgoing. Hence,

$$p_1^0 = -E_1(1 + |z_1|^2), \quad p_2^0 = -E_2(1 + |z_2|^2), \quad p_3^0 = E_3(1 + |z_3|^2), \quad p_4^0 = E_4(1 + |z_4|^2),$$

and $E_i \geq 0$. Thus, we get,

$$s = 4E_1E_2|z_{12}|^2, \qquad t = -4E_1E_3|z_{13}|^2, \qquad u = -4E_1E_4|z_{14}|^2. \tag{3.6}$$

Using energy momentum conservation relation $p_1 + p_2 + p_3 + p_4 = 0$, one can write three other possible expressions for s, t and u as,

$$s = 4E_1E_2|z_{12}|^2, \qquad t = -4E_2E_4|z_{24}|^2, \qquad u = -4E_2E_3|z_{23}|^2,$$
$$s = 4E_3E_4|z_{34}|^2, \qquad t = -4E_1E_3|z_{13}|^2, \qquad u = -4E_2E_3|z_{23}|^2,$$
$$s = 4E_3E_4|z_{34}|^2, \qquad t = -4E_2E_4|z_{24}|^2, \qquad u = -4E_1E_4|z_{14}|^2.$$

Using the above four expressions for s, t and u, we simplify the amplitude in (3.2) as,

$$A'(E_i, z_i, \bar{z}_i) = \ln\frac{s|t||u|}{\mu^6} = \ln\left[64\left(\frac{E_1 E_2 E_3 E_4}{\mu^4}\right)^{\frac{3}{2}}\prod_{i<j}|z_{ij}|\right]. \tag{3.7}$$

In writing (3.7) we have only displayed the momentum dependent piece of the amplitude and the momentum conserving delta function has been omitted. As in [18], it is convenient to change the integration variables to an overall frequency $S \equiv \sum_{i=1}^4 E_i$ and a set of *simplex variables* $\sigma_i = S^{-1} E_i \in [0, 1]$ that satisfies a constraint $\sum_{i=1}^4 \sigma_i = 1$. Using these new variables we finally get,

$$A' = \ln\left[\frac{S^6}{\mu^6}64\left(\prod_{i=1}^4 \sigma_i\right)^{\frac{3}{2}}\prod_{i<j}|z_{ij}|\right]. \tag{3.8}$$

The integral over E_i now gets transformed to integrals over S and σ_i, with a proper delta function insertion. The momentum conserving delta function can be expressed in-terms of these new simplex variables as,

$$\delta^4(\Sigma p_i) = \frac{\delta(|z - \bar{z}|)}{4z_{13}z_{24}\bar{z}_{13}\bar{z}_{24}}\delta(\sigma_1 - \sigma_1^*)\delta(\sigma_2 - \sigma_2^*)\delta(\sigma_3 - \sigma_3^*)\delta(\sigma_4 - \sigma_4^*), \tag{3.9}$$

where, we have defined,

$$\sigma_1^* = \frac{z_{24}\bar{z}_{34}}{Dz_{12}\bar{z}_{13}}, \quad \sigma_2^* = -\frac{z_{34}\bar{z}_{14}}{Dz_{23}\bar{z}_{12}}, \quad \sigma_3^* = -\frac{z_{24}\bar{z}_{14}}{Dz_{23}\bar{z}_{13}}, \quad \sigma_4^* = \frac{1}{D}, \quad D = 2\left(\frac{z_{24}\bar{z}_{34}}{z_{12}\bar{z}_{13}} - \frac{z_{34}\bar{z}_{14}}{z_{23}\bar{z}_{12}}\right).$$

After performing the S integral, we get,

$$\int_0^\infty \frac{dS}{S}S^{i\Lambda}\ln\left(\frac{S^6}{\mu^6}64\left(\prod_{i=1}^4 \sigma_i\right)^{\frac{3}{2}}\left(\prod_{i<j}|z_{ij}|^2\right)^{\frac{1}{2}}\right) = -6i\delta'(\Lambda)\left[2\left(\prod_{i=1}^4 \sigma_i\right)^{\frac{1}{4}}\frac{\left(\prod_{i<j}|z_{ij}|^2\right)^{\frac{1}{12}}}{\mu}\right]^{-i\Lambda}. \tag{3.10}$$

Finally, performing the σ integrals, we get the one-loop Mellin amplitude on the sphere as, $\tilde{A}(\lambda, z, \bar{z}) = \tilde{\mathcal{T}}_1 + \tilde{\mathcal{T}}_2$, where, $\tilde{\mathcal{T}}_1$ is the tree level four point amplitude given as,

$$\tilde{\mathcal{T}}_1 = -\lambda_R(2\pi)^5\delta(\Sigma_j \lambda_j)\frac{\delta(|z - \bar{z}|)}{4z_{13}z_{24}\bar{z}_{13}\bar{z}_{24}}\Pi_{j=1}^4(\sigma_*)_j^{i\lambda_j},$$

and the one-loop contribution $\tilde{\mathcal{T}}_2$ is given as,

$$\tilde{\mathcal{T}}_2 = (2\pi)^5\frac{\lambda_R^2}{32\pi^2}\frac{\delta(|z - \bar{z}|)}{4z_{13}z_{24}\bar{z}_{13}\bar{z}_{24}}\left(\prod_{j=1}^4 \sigma_{*j}^{i\lambda_j}\right)\left\{6i\delta'(\Lambda)\left[2\left(\prod_{i=1}^4 \sigma_{*i}\right)^{\frac{1}{4}}\frac{\left(\prod_{i<j}|z_{ij}|^2\right)^{\frac{1}{12}}}{\mu}\right]^{-i\Lambda} + \delta(\Sigma_j \lambda_j)\pi i\right\}. \tag{3.11}$$

The above results can be written in nice conformal covariant form by using $h_i = \bar{h}_i = \frac{1+i\lambda_i}{2}$ and $h = \sum_i h_i$ as:

$$\tilde{\mathcal{T}}_1 = -8\pi^5 \lambda_R \delta(\Sigma_j \lambda_j) \delta(|z - \bar{z}|) \left(\prod_{i<j}^4 |z_{ij}|^{h/3 - h_i - h_j} |\bar{z}_{ij}|^{\bar{h}/3 - \bar{h}_i - \bar{h}_j} \right) [z(z-1)]^{2/3}, \tag{3.12}$$

$$\tilde{\mathcal{T}}_2 = \frac{i\lambda_R^2}{4} \left(\frac{2}{\mu} \right)^{-i\Lambda} \left[6\pi^3 \delta'(\Lambda) + \pi^4 \delta(\Lambda) \right] \delta(|z - \bar{z}|) \left(\prod_{i<j}^4 |z_{ij}|^{h/3 - h_i - h_j} |\bar{z}_{ij}|^{\bar{h}/3 - \bar{h}_i - \bar{h}_j} \right) [z(z-1)]^{2/3}. \tag{3.13}$$

Here we see that the label λ_i of the Mellin amplitude is actually related to the conformal dimensions of each of the four conformal primaries. The amplitude is channel covariant and transforms properly as a function of its arguments $(z, \bar{z})$. To be precise, the function takes the form in the s channel $12 \to 34$ ($z > 1$),

$$f(z, \bar{z}) = [z(z-1)]^{2/3}. \tag{3.14}$$

In the t channel $13 \to 24$ ($0 < z < 1$)

$$f(z, \bar{z}) = [z(1-z)]^{2/3}. \tag{3.15}$$

In the u channel $14 \to 23$ ($z < 0$)

$$f(z, \bar{z}) = [z(z-1)]^{2/3}.\,[2] \tag{3.16}$$

Thus, we see that the one-loop amplitude of flat space massless ϕ^4 theory retains its conformal structure when expressed on the Celestial sphere. Although this is not a surprise, but, here we also see that the Function of the cross rations $f(z, \bar{z})$ is identical at tree level and loop-level amplitude. We shall comment more on this structure in the later section.

4 Unitarity for Mellin amplitudes

The massless ϕ^4 theory that we are interested in is an unitary field theory. As we know, in usual field theories, unitarity of the scattering amplitude implies that the S-matrix satisfies $SS^\dagger = 1$. Inserting $S = I + i\mathcal{T}$, the condition on the non-trivial contribution to the scattering matrix $\mathcal{T}$ reduces to,

$$-i(\mathcal{T} - \mathcal{T}^\dagger) = \mathcal{T}\mathcal{T}^\dagger.$$

This relation plays an extremely important role in Quantum Field Theory. It computes the imaginary part in a scattering amplitude and simultaneously predicts that one can only get

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[2]One can easily check that the Mellin amplitudes for different channels are related as:

$$\tilde{\mathcal{T}}_{12->34}(z) = \tilde{\mathcal{T}}_{13->24}(1/z) = \tilde{\mathcal{T}}_{14->23}(1-z),$$

with argument > 1. The functional form of amplitude is same in all channels only the ranges of z are different.

the imaginary contribution to scattering when the virtual particles in a feynman diagram go on-shell. In this section, we would find the consequence of unitarity on the Mellin amplitudes $\tilde{\mathcal{T}}$. We shall be presenting the computation for s channel processes $(12 \to 34)$. Using the inverse Mellin transform as defined in (3.5), one can restate the above relation in terms of Mellin amplitudes. Thus we readily get the following relation,

$$
\begin{aligned}
\frac{-i}{(2\pi)^4} & [\tilde{\mathcal{T}}_{12\to34}(\lambda_i, z_i) - \tilde{\mathcal{T}}^*_{34\to12}(-\lambda_i, z_i)] \\
= & \frac{1}{(2\pi)^8} \int \frac{d^3p}{(2\pi)^6} \frac{d^3p'}{4e_p e_{p'}} \int d\lambda_p d\lambda_{p'} d\lambda_P d\lambda_{P'} E_p^{-2-i(\lambda_p+\lambda_P)} E_{p'}^{-2-i(\lambda_{p'}+\lambda_{P'})} \tilde{\mathcal{T}}_{12\to p,p'}(\lambda, z) \tilde{\mathcal{T}}^*_{34\to P,P'}(\lambda, z) \\
= & \frac{4 \times 4\pi^2}{(2\pi)^{14}} \int dz_p d\bar{z}_p \, dz_{p'} d\bar{z}_{p'} \int d\lambda_p d\lambda_{p'} \tilde{\mathcal{T}}(\lambda_1, \lambda_2, \lambda_p, \lambda_{p'}; z_1, z_2, z_p, z_{p'}) \tilde{\mathcal{T}}^*(\lambda_3, \lambda_4, -\lambda_p, -\lambda_{p'}; z_3, z_4, z_p, z_{p'}),
\end{aligned}
$$
(4.1)

Here, $i = 1, 2, 3, 4$ and we have used following two relations for simplifications,

$$
\begin{aligned}
d^3p &= 2E_p e_p \, dE_p \, dz_p d\bar{z}_p, \\
\int E^{-1-i\lambda} dE &= 2\pi\delta(\lambda), \qquad e_p = E_p \left(1 + |z_p|^2\right).
\end{aligned}
$$
(4.2)

Therefore we see that, for a unitary flat space QFT, the corresponding Mellin amplitudes has to satisfy (4.1). This relation is generic and should hold for any unitary QFT. We have explicitly checked that the scattering amplitude for ϕ^4 theory obtained in equations (3.12) and (3.13) satisfies the above relation. For our case the Mellin amplitude[3] gets its first imaginary contribution at oneloop level and hence the above relation (4.1) gets first contribution at order λ_R^2. For the s channel process, using explicit expressions as given in (3.13), the l.h.s. of (4.1) simplifies to,

$$
\frac{-i}{(2\pi)^4} [\tilde{\mathcal{T}}_2(\lambda_i, z_i) - \tilde{\mathcal{T}}^*_2(-\lambda_i, z_i)] = \frac{\lambda_R}{8(2\pi)^5} \tilde{\mathcal{T}}_1(\lambda_i, z_i), \qquad (i = 1, 2, 3, 4),
$$
(4.3)

where as the r.h.s. of equation (4.1) only picks up the tree level amplitude. Let us rewrite the r.h.s. as,
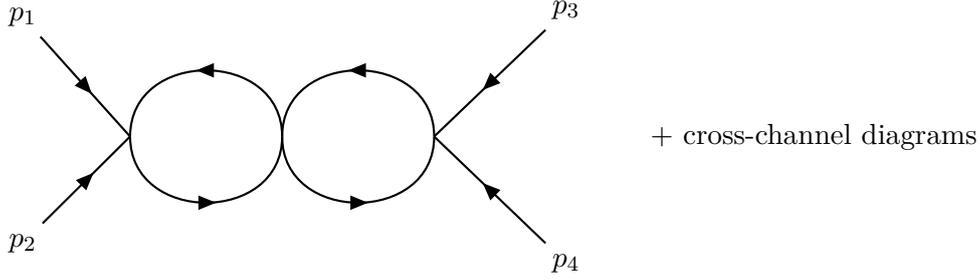
$$
\begin{aligned}
A(z_i) = & \frac{4 \times 4\pi^2}{4(2\pi)^{14}} \int dz_p d\bar{z}_p \, dz_{p'} d\bar{z}_{p'} \\
& \int d\lambda_p d\lambda_{p'} \tilde{\mathcal{T}}_1(\lambda_1, \lambda_2, \lambda_p, \lambda_{p'}; z_1, z_2, z_p, z_{p'}) \tilde{\mathcal{T}}_1(\lambda_3, \lambda_4, -\lambda_p, -\lambda_{p'}; z_3, z_4, z_p, z_{p'}).
\end{aligned}
$$
(4.4)

The last integral is hard to perform in general. Keeping in mind the conformal structure of the Mellin amplitude, we set the points $z_2 = 1, z_4 = 0, z_1 = \infty$. This simplifies the computation and we check that:

$$
\lim_{z_1 \to \infty} \left[\left(\prod_{i<j}^4 |z_{ij}|^{-2h/3+2h_i+2h_j} \right) A(z_1, 1, z, 0) \right] = \frac{-1}{2^5} \delta(|z - \bar{z}|) \delta(\Sigma_j \lambda_j) [z(z-1)]^{2/3}.
$$
(4.5)

Figure 3. Leading log two loop diagram.

The Dirac delta function of Λ is easily obtained from two Dirac delta functions of λ's of A(by integrating over $\lambda_{p'}$). The other Dirac delta functions of cross ratios are also handeled in the straight forward way. The integral over λ_p gives:

$$\int d\lambda_p \left[\frac{(z\bar{z}_p - \bar{z}z_p)^2(\bar{z}(-1+\bar{z}_p)z_p + z(\bar{z}z_p - \bar{z}_p(-1+\bar{z}+z_p)))^2}{z^2\bar{z}^2(\bar{z}_p - z_p)^2(\bar{z} + z(-1+\bar{z}_p) - \bar{z}_p + z_p - \bar{z}z_p)^2} \right]^{i\lambda_p}. \tag{4.6}$$

It gives a Dirac delta function with three solutions; one of it is $z = \bar{z}$. The other two solutions are physically non-significant and hence we discard them. The integrals over remaining variables are not hard, for example they can be converted to real integrals and be evaluated using Mathematica. Thus we see that the Mellin amplitudes nicely satisfies the unitarity constraint of the QFT.

5 Comments on higher order Mellin amplitudes

We can extend the above discussion on the conformal structure of the flat space scattering amplitude to the two and higher loops . First we talk about the two loop amplitude. There are two different types of diagrams that contribute at two loop. The first kind is given by diagram in figure 3. They lead to momentum dependent log contribution to scattering amplitude given by

$$T \sim \lambda_R^3 \left(\log^2 \left(\frac{s}{\mu} \right) + \log^2 \left(\frac{t}{\mu} \right) + \log^2 \left(\frac{u}{\mu} \right) \right). \tag{5.1}$$

This expression when transformed to Mellin space reduces to:

$$\tilde{\mathcal{T}}_3 \sim \lambda_R^3 \delta''(\Lambda)\delta(|z - \bar{z}|) \times \left(\prod_{i<j}^4 |z_{ij}|^{h/3-h_i-h_j}|\bar{z}_{ij}|^{h/3-h_i-h_j} \right)$$

$$\left(\frac{2}{\mu} \right)^{-i\Lambda} [z(z-1)]^{2/3} \left\{ z^{-i\frac{\Lambda}{3}}(z-1)^{i\frac{\Lambda}{6}} + z^{i\frac{\Lambda}{6}}(z-1)^{i\frac{\Lambda}{6}} + (z-1)^{-i\frac{\Lambda}{3}}z^{i\frac{\Lambda}{6}} \right\}. \tag{5.2}$$

Let us pause here to discuss few universal features of the expression in (5.2). First, we note that, even at the two loop order, the Mellin amplitude has the proper conformal factor that makes it transform covariantly under the global conformal transformation of the boundary sphere. Secondly, the similar dependence on the form of the delta function at tree level,

one loop and two loop suggests that, at n-loop level Mellin amplitude, one should get a delta function dependence of the form $\delta^n(\Lambda)$. It is quite interesting that a same cross-ratio dependent function $[z(z-1)]^{2/3}$ appeared at tree, one loop and two loop level. We expect it to arise at any loop level for leading log term. Also note that factor in the curly bracket of (5.2) appears to be universal at all level. To understand why it is universal, let us start with tree level amplitude $\tilde{\mathcal{T}}_1$ in (3.12). If we multiply $\tilde{\mathcal{T}}_1$ in (3.12) by $f(\Lambda)$ defined by:

$$f(\Lambda) = \frac{1}{3}\left\{ z^{-i\frac{\Lambda}{3}}(z-1)^{i\frac{\Lambda}{6}} + z^{i\frac{\Lambda}{6}}(z-1)^{i\frac{\Lambda}{6}} + (z-1)^{-i\frac{\Lambda}{3}}z^{i\frac{\Lambda}{6}} \right\}, \qquad (5.3)$$

we immediately see that the product equates to $\tilde{\mathcal{T}}_1$ since $\delta(\Lambda)f(\Lambda) = 1$. Thus, we can as well write the tree level Mellin amplitude as $\tilde{\mathcal{T}}_1 f(\Lambda)$. Similarly note that the one loop amplitude of (3.13) can also be rewritten as $\tilde{\mathcal{T}}_2 f(\Lambda)$. The term proportional to $\delta(\Lambda)$ in $\tilde{\mathcal{T}}_2$ does not change much like the tree level amplitude. The term proportional to the derivative of $\delta(\Lambda)$, i.e.

$$\tilde{\mathcal{T}}_2 \sim \lambda_R^2 \left(\prod_{i<j}^4 |z_{ij}|^{2h/3 - 2h_i - 2h_j} \right) \delta'(\Lambda)\delta(|z - \bar{z}|) \times \left(\frac{2}{\mu} \right)^{-i\Lambda} [z(z-1)]^{2/3}f(\Lambda), \qquad (5.4)$$

can be shown to be identical to the one in (3.13).[4] Based on this observation, we can readily extend the result to arbitrary n-loop order Mellin amplitude. We see that, for any n-loop order, the momentum space amplitude will have a contribution like: $\mathcal{M}_{n+1} \sim \lambda_R^{n+1}[\log^n(s) + \log^n(t) + \log^n(u)]$. The corresponding Mellin amplitude will behave as,

$$\tilde{\mathcal{T}}_{n+1} \sim \lambda_R^{n+1} \left(\prod_{i<j}^4 |z_{ij}|^{2h/3 - 2h_i - 2h_j} \right) \delta^n(\Lambda)\delta(|z - \bar{z}|)$$

$$\times 2^{-i\Lambda}[z(z-1)]^{2/3} \left\{ z^{-i\frac{\Lambda}{3}}(z-1)^{i\frac{\Lambda}{6}} + z^{i\frac{\Lambda}{6}}(z-1)^{i\frac{\Lambda}{6}} + (z-1)^{-i\frac{\Lambda}{3}}z^{i\frac{\Lambda}{6}} \right\}. \qquad (5.5)$$

The numerical coefficient will depend on the exact computation, but the momentum dependence is fixed as in (5.5).

The other diagram that contribute at the two loop level is figure 4. However, this has no finite momentum dependent contribution to scattering amplitude (see appendix A). It only contributes a coupling constant dependent constant factor. Thus, for the corresponding Mellin amplitude, it contributes similar to (3.12). The same feature seems to be true to all loop order Mellin amplitudes, but we do not have a concrete proof for this yet.

6   Conclusions and future directions

In this paper, we have studied the conformal structure of the flat space QFT four-point amplitude of massless ϕ^4 theory. We have computed exact results up to two loop order in the perturbation theory as given in equations (3.12), (3.13) and (5.2). We have also

---

[4]This can be shown by using $g(\Lambda)\partial_\Lambda \delta(\Lambda) = \partial_\Lambda[\delta(\Lambda)g(0)] - \delta(\Lambda)\partial_\Lambda g(\Lambda)$. Further when derivative hits $f(\Lambda)$, the terms add up to 0 for $\Lambda = 0$.

+ cross-channel diagrams
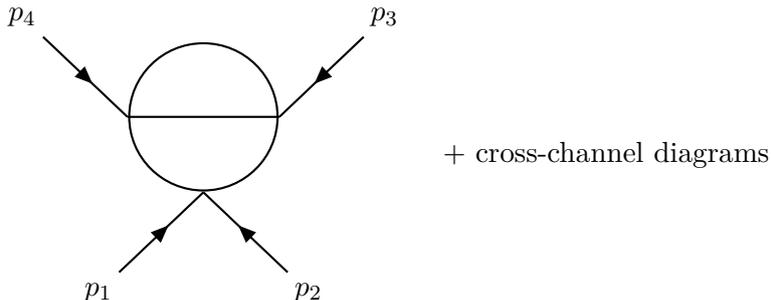
Figure 4. Two loop contribution.

reformulated the role of Unitarity of QFT for the corresponding Mellin amplitudes. Equation (4.1) is the constraint that the Mellin amplitudes of any unitary theory has to satisfy. In particular, we have shown that the four point Mellin amplitude of massless ϕ^4 theory does satisfy the required relation. While the conformal structure of the Mellin amplitude is guaranteed by proper choice of conformal primary wave functions, the interesting aspect of our results is the universality of the Mellin amplitude to all these three orders in the perturbation series. We have seen that the dependence of the Mellin amplitude on the conformal cross ration factor (the only non-trivial dependence of the celestial sphere) to tree, oneloop and two level are identical. Only difference in them appears as in the order of derivative of delta function of $\sum \lambda_i$. It is also certain that similar contribution will be there in all loop Mellin amplitude. Based on this observation, we see that all loop answer for ϕ^4 theory, the leading log part of the Mellin amplitude takes the form:

$$
\tilde{\mathcal{T}}_{\text{all-loop}} \sim \left(\prod_{i<j}^{4} |z_{ij}|^{2h/3 - 2h_i - 2h_j} \right) \left(\sum_{l=0}^{n} \lambda_R^{l+1} a_l \delta^l(\Lambda) \right) \delta(|z - \bar{z}|)
$$
$$
\times \left(\frac{2}{\mu} \right)^{-i\Lambda} [z(z-1)]^{2/3} \left\{ z^{-i\frac{\Lambda}{3}}(z-1)^{i\frac{\Lambda}{6}} + z^{i\frac{\Lambda}{6}}(z-1)^{i\frac{\Lambda}{6}} + (z-1)^{-i\frac{\Lambda}{3}} z^{i\frac{\Lambda}{6}} \right\},
$$
(6.1)

where $\delta^l(\Lambda)$ is l^{th} derivative of the delta function with respect to its argument. For example

$$
\delta^0(\Lambda) = \delta(\Lambda), \qquad \delta^1(\Lambda) = \delta'(\Lambda),
$$
(6.2)

and the coefficients a_l are the only undetermined numbers which one has to compute. It would be remarkable if it turns out that the form in (6.1) is the entire result for the four-point scattering amplitude for ϕ^4 theory at all loop. We have only proved it to be true up to two loop order. There can be other contributions as well at three and higher loop order and we do not yet have any concrete comment on that.

We end the paper with some comment on possible future directions. First of all, it would be nice to find the conformal covariant amplitudes for QED. For that, one needs to write down the conformal primaries for asymptotic fermionic states [21]. This will have importance on the structure of soft photon theorems of QED. Also, this will provide us with another example to study the CFT structure of flat space amplitudes.

On a deeper note, in [22], the authors outlined how the 4D scattering amplitudes can be reformulated in the language of a 2D CFT on the Celestial sphere. To give similar interpretation to our results, one needs to show how the Mellin amplitudes satisfy all the properties of a CFT amplitude. In our work we have only shown that they transform covariantly under the global conformal transformation, but a lot is left to do. It would be nice to make this connection precise. Once the connection is established, the ultimate goal would be to recast all interesting question of a QFT in their dual 2d CFT and to compute them directly in the CFT.

Acknowledgments

A Two-loop computation in momentum space

In this appendix we calculate the contribution of figure 4 to the massless four-point scattering amplitude at two loop order. In the following, we will only keep track of terms that are finite and momentum dependent and not worry about the overall constants. Only the diverging Gamma functions are explicitly written with other finite ones being omitted. Here, $p = p_1 + p_2$. The amplitude behave as,

$$\mathcal{M} \sim \int d^d k \int d^d k' \frac{1}{k^2(k+p)^2} \frac{1}{k'^2(k'+p_3+k)^2}. \tag{A.1}$$

Combining k' dependent factors from the denominator and integrating over them we get,

$$\mathcal{M} \sim \Gamma(2-d/2) \int_0^1 dx \int d^d k \frac{1}{k^2(k+p)^2} \frac{1}{[-x(1-x)(k+p_3)^2]^{2-d/2}}. \tag{A.2}$$

The x-integral is trivial and momentum independent. Combining the three denominators (see for example Kleinert, chapter 8 [23]) the above expression simplifies to,

$$\mathcal{M} \sim \int_0^1 dy dz \int d^d k \frac{z^{1-d/2}}{[(1-y-z)k^2 + y(k+p)^2 + z(k+p_3)^2]^{4-d/2}}. \tag{A.3}$$

Completing the squares and integrating over k we get,

$$\mathcal{M} \sim \Gamma(4-d) \int_0^1 dy dz \frac{z^{1-d/2}}{[yp^2 - y^2 p^2 - 2yz pp_3]^{4-d}}. \tag{A.4}$$

Now, using the relation $p^2 = 2p \cdot p_3$, the integral reduces to,

$$\mathcal{M} \sim \Gamma(\epsilon) \int_0^1 dy dz \, z^{-1}[y - y^2 - yz]^{-\epsilon}(p^2)^{-\epsilon}. \tag{A.5}$$

Finally, we have to take $\epsilon \to 0$ and it gives,

$$\mathcal{M} \sim \int_0^1 dz \, z^{-1} \log(p^2). \tag{A.6}$$

This is the final contribution from figure 4. As we see, the lower limit of the integral is divergent and by usual renormalization techniques, it cancels with diverging pieces of other diagrams. The finite contribution comes from the upper limit of the Integral and it is simply 0 at the upper limit. Thus, we conclude that for massless ϕ^4 theory, the two loop four point scattering amplitude does not get any momentum dependence contribution from figure 4. This feature may not be true at higher loops.

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