$N = 1$ superfield description of vector-tensor couplings in six dimensions

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Abstract: We express supersymmetric couplings among the vector and the tensor multiplets in six dimensions (6D) in terms of $N = 1$ superfields. The superfield description is derived from the invariant action in the projective superspace. The obtained expression is consistent with the known superfield actions of 6D supersymmetric gauge theory and 5D Chern-Simons theory after the dimensional reduction. Our result provides a crutial clue to the $N = 1$ superfield description of 6D supergravity.

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Contents

1 Introduction

The $N = 1$ superfield description[1] of higher dimensional supersymmetric (SUSY) theories is quite useful when we discuss phenomenological SUSY models with extra dimensions. It makes the derivation of 4-dimensional (4D) effective theories easier since the Kaluza-Klein mode expansion can be performed keeping the $N = 1$ off-shell structure. Besides, the action is expressed compactly, and general setups can be treated. Since higher-dimensional SUSY theories have extended SUSY, the full off-shell formulations are complicated and less familiar, or do not even exist for theories higher than 6 dimensions (6D). In contrast, the $N = 1$ superfield description is always possible because it only respects part of the

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[1] "$N = 1$" denotes supersymmetry with four supercharges in this paper.

full off-shell SUSY structure. Hence it is powerful especially when we describe interactions between sectors whose dimensions are different, such as the bulk-boundary couplings in 5D theories compactified on S^1/Z_2. For the above reasons, a lot of works along this direction have been published [1]–[9].

When we discuss a realistic extra-dimensional models, the moduli play important roles. They have to be stabilized to finite values by some mechanism, and are often relevant to the mediation of SUSY-breaking to the visible sector. In order to treat the moduli properly, we need to consider supergravity (SUGRA). The $N = 1$ superfield description of 5-dimensional (5D) SUGRA is already obtained in refs. [5]–[9]. Making use of it, the moduli dependence of the 4D effective action can systematically be derived [10]–[13]. Our aim is to extend the 5D superfield action to 6D. Since the minimal number of SUSY is the same in the 5D and 6D cases, the desired $N = 1$ description is expected to be similar to that of 5D theories. However, there is an obstacle to a straightforward extension of the 5D result. In contrast to the 5D case, the 6D superconformal Weyl multiplet contains an anti-self-dual antisymmetric tensor T_{MNL}^- (M, N, L: 6D Lorentz indices) [14]. This leads to a difficulty for the Lagrangian formulation, similar to that for type IIB SUGRA. This difficulty can be evaded by introducing a tensor multiplet, which contains an antisymmetric tensor B_{MN}^+ whose field strength $F_{MNL}^+ \equiv \partial_{[M} B_{NL]}^+$ is subject to the self-dual constraint [14]. Combining this multiplet with the Weyl multiplet, we obtain a new multiplet[2] that contains an unconstrained antisymmetric tensor B_{MN}, whose field strength is given by the sum of T_{MNL}^- and F_{MNL}^+. Namely, the off-shell formulation of 6D SUGRA requires the existence of the tensor field B_{MN}, which is not a necessary ingredient in 5D SUGRA.[3]

The off-shell action of 6D SUGRA is provided in refs. [14, 15]. In that action, the tensor field B_{MN} is coupled to the vector fields. Thus, in this paper, we clarify how the vector-tensor couplings are expressed in terms of $N = 1$ superfields. Since we focus on these couplings, we do not consider the gravitational couplings in this paper. In this sense, this work is the generalization of ref. [1] including the vector-tensor couplings. For our purpose, the projective superspace formulation [16–18] is useful. In fact, the $N = 1$ superfield description of 5D SUGRA can be derived from the action in the projective superspace [8]. As for 6D SUGRA, the off-shell action in this formulation is provided in ref. [19].[4] We derive the $N = 1$ superfield action from it.

The paper is organized as follows. In section 2, we provide a brief review of $N = 2$ supersymmetric actions in the projective superspace. In section 3, we decompose $N = 2$ superfields into $N = 1$ superfields, and express the vector-tensor couplings in terms of the latter. We also clarify the relation between our result and the known $N = 1$ superfield description of 6D SUSY gauge theory or 5D SUSY Chern-Simons theory through the dimensional reduction. Section 4 is devoted to the summary. In the appendices, we list our notations for spinors, and show explicit derivations of some of the results in the text.

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[2]This is called the "Weyl 2 multiplet" in ref. [15], and the "type-II Weyl multiplet" in ref. [19].

[3]Note also that an antisymmetric rank-2 tensor field is dual to a vector field in 5 dimensions.

[4]The 6D action in the harmonic superspace is provided in ref. [20].

2 Invariant action in projective superspace

2.1 Action formula

An $N = 2$ off-shell action can be constructed by using the projective superspace formulation [16–18]. We consider 6D $(1, 0)$ SUSY theories. The 6D projective superspace is parametrized by the spacetime coodinates x^M ($M = 0, 1, \cdots, 5$), the Grassmannian coordinates $\Theta_{\underline{\alpha}}^i$ ($i = 1, 2$; $\underline{\alpha} = 1, 2, 3, 4$),[5] which form an SU(2)-Majorana-Weyl spinor, and the complex coordinate ζ of $\mathbb{CP}^1$. A projective superfield $\Xi(x, \Theta, \zeta)$ is a holomorphic function in ζ that satisfies[6]

$$\mathcal{D}_{\underline{\alpha}}^{[1]} \Xi \equiv \left(-\zeta \mathcal{D}_{\underline{\alpha}}^1 + \mathcal{D}_{\underline{\alpha}}^2 \right) \Xi = 0, \tag{2.1}$$

where the spinor derivatives are defined in (A.26). It can be expanded as

$$\Xi(x, \Theta, \zeta) = \sum_{n=-\infty}^{\infty} \Xi_n(x, \Theta)\zeta^n, \tag{2.2}$$

where $N = 2$ superfields Ξ_n satisfy

$$\mathcal{D}_{\underline{\alpha}}^1 \Xi_n = \mathcal{D}_{\underline{\alpha}}^2 \Xi_{n+1}. \tag{2.3}$$

The constraint (2.3) fixes the dependence of Ξ_n on half of the Grassmann coordinates $\Theta_{\underline{\alpha}}^i$, and thus Ξ_n can be considered as superfields which effectively live on an $N = 1$ superspace.

The natural conjugate operation in the projective superspace is the combination of the complex conjugate and the antipodal map on $\mathbb{CP}^1$ ($\zeta^* \to -1/\zeta$), which is called the smile conjugate denoted as

$$\breve{\Xi}(x, \Theta, \zeta) = \sum_{n=-\infty}^{\infty} (-1)^n \bar{\Xi}_{-n}(x, \Theta)\zeta^n. \tag{2.4}$$

Then the $N = 2$ SUSY invariant action formula is given by [19, 22]

$$S = \int d^6x \left\{ \oint_C \frac{d\zeta}{2\pi i} \, \zeta \mathcal{D}^{[-4]} L(x, \Theta, \zeta) \Big|_{\Theta=0} \right\}, \tag{2.5}$$

where C is a contour surrounding the origin $\zeta = 0$, the "Lagrangian superfield" $L(x, \Theta, \zeta)$ is a smile-real projective superfield ($\breve{L} = L$), and

$$\mathcal{D}^{[-4]} \equiv -\frac{1}{96} \epsilon^{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} \mathcal{D}_{\underline{\alpha}}^{[-1]} \mathcal{D}_{\underline{\beta}}^{[-1]} \mathcal{D}_{\underline{\gamma}}^{[-1]} \mathcal{D}_{\underline{\delta}}^{[-1]},$$

$$\mathcal{D}_{\underline{\alpha}}^{[-1]} \equiv \frac{1}{1 + \zeta\eta} \left(\mathcal{D}_{\underline{\alpha}}^1 + \eta \mathcal{D}_{\underline{\beta}}^2 \right). \tag{2.6}$$

The complex number η is chosen arbitrarily as long as $1 + \zeta\eta \neq 0$. In fact, the action (2.5) is independent of η.

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[5]In this paper, $\underline{\alpha}, \underline{\beta}, \cdots$ denote the 4-component spinor indices, and $\alpha, \beta, \cdots$ and $\dot{\alpha}, \dot{\beta}, \cdots$ are used as the 2-component indices of 4D SL(2,$\mathbb{C}$) spinors.

[6]The index $[k]$ indicates the weight-k quantity. It coincides with the superconformal weight in the superconformal theories [21].

2.2 Explicit forms of Lagrangians

A 6D hypermultiplet is described by an arctic superfield Υ, which is a projective superfield that is non-singular at the north pole of $\mathbb{CP}^1$ ($\zeta = 0$). Namely, it is expanded as

$$\Upsilon(x, \Theta, \zeta) = \sum_{n=0}^{\infty} \Upsilon_n(x, \Theta)\zeta^n. \tag{2.7}$$

A 6D vector multiplet is described by a tropical superfield $\boldsymbol{V}$, which is a smile-real projective superfield,

$$\check{\boldsymbol{V}}(x, \Theta, \zeta) = \boldsymbol{V}(x, \Theta, \zeta). \tag{2.8}$$

Namely, it is expanded as

$$\boldsymbol{V}(x, \Theta, \zeta) = \sum_{n=-\infty}^{\infty} V_n(x, \Theta)\zeta^n, \quad V_{-n} = (-1)^n \bar{V}_n. \tag{2.9}$$

Using these projective superfields, the Lagrangian superfield L in the hypermultiplet sector is given by

$$L_{\text{hyper}} = \check{\Upsilon} e^{-\boldsymbol{V}} \Upsilon. \tag{2.10}$$

In the following, we consider Abelian gauge theories, for simplicity.

In contrast to the above multiplets, a 6D tensor multiplet is not described by a projective superfield. As first shown in ref. [23], it can be described by a constrained real superfield Φ that satisfies

$$\mathcal{D}_{\underline{\alpha}}^{(i} \mathcal{D}_{\underline{\beta}}^{j)} \Phi = 0. \tag{2.11}$$

or equivalently described by an SU(2)-Majorana-Weyl spinor superfield $T^{i\underline{\alpha}}$ constrained by

$$\mathcal{D}_{\underline{\alpha}}^{(i} T^{j)\underline{\beta}} - \frac{1}{4} \delta_{\underline{\alpha}}^{\underline{\beta}} \mathcal{D}_{\underline{\gamma}}^{(i} T^{j)\underline{\gamma}} = 0. \tag{2.12}$$

where the parentheses denote the symmetrization for the indices. We can identify these superfields as

$$\Phi = \mathcal{D}_{i\underline{\alpha}} T^{i\underline{\alpha}} = \epsilon_{ij} \mathcal{D}_{\underline{\alpha}}^{j} T^{i\underline{\alpha}}, \tag{2.13}$$

but we can also regard them as independent tensor multiplets. From these two superfields, we can construct a projective composite superfield,

$$\mathcal{T}^{[2]} \equiv \frac{i}{\zeta} \left\{ (\mathcal{D}_{\underline{\alpha}}^{[1]} \Phi) T^{[1]\underline{\alpha}} + \frac{1}{4} \Phi \mathcal{D}_{\underline{\alpha}}^{[1]} T^{[1]\underline{\alpha}} \right\}, \tag{2.14}$$

where $T^{[1]\underline{\alpha}} \equiv -\zeta T^{1\underline{\alpha}} + T^{2\underline{\alpha}}$. This certainly satisfies the condition $\mathcal{D}_{\underline{\alpha}}^{[1]} \mathcal{T} = 0$ due to the constraints (2.11) and (2.12). For an SU(2)-Majorana-Weyl spinor $\Psi^{i\underline{\alpha}}$, a quantity $\Psi^{[1]\underline{\alpha}} \equiv -\zeta \Psi^{1\underline{\alpha}} + \Psi^{2\underline{\alpha}}$ is transformed by the smile conjugation as

$$\Psi^{[1]\underline{\alpha}} \to \check{\Psi}^{[1]\underline{\alpha}} \equiv \left(\overline{\Psi^{[1]}} \right)^{\underline{\alpha}} \Big|_{\zeta^* \to -1/\zeta} = -\frac{1}{\zeta} \Psi^{[1]\underline{\alpha}}, \tag{2.15}$$

where the overline denotes the covariant conjugation defined by (A.20), and we have used (A.23). Using this property, it is shown that $\mathcal{T}^{[2]}$ is smile-real ($\check{\mathcal{T}}^{[2]} = \mathcal{T}^{[2]}$), and thus it can be the Lagrangian superfield for the tensor multiplets.

$$L_{\text{tensor}} = \mathcal{T}^{[2]}. \tag{2.16}$$

Besides the description by the tropical superfield, a 6D vector multiplet is also described by a superfield $F^{i\underline{\alpha}}$ subject to the same constraint as (2.12) if it is further constrained by

$$\mathcal{D}_{i\underline{\alpha}}F^{i\underline{\alpha}} = 0. \tag{2.17}$$

As we will see later, this superfield contains the field strength of the gauge field, and thus gauge-invariant. Using $F^{i\underline{\alpha}}$ with Φ and $\boldsymbol{V}$, we can construct the Lagrangian superfield for the vector-tensor couplings as

$$L_{\text{VT}} = \boldsymbol{V}\mathcal{F}^{[2]}, \tag{2.18}$$

where

$$\mathcal{F}^{[2]} \equiv \frac{i}{\zeta}\left\{(\mathcal{D}_{\underline{\alpha}}^{[1]}\Phi)F^{[1]\underline{\alpha}} + \frac{1}{4}\Phi\mathcal{D}_{\underline{\alpha}}^{[1]}F^{[1]\underline{\alpha}}\right\},$$

$$F^{[1]\underline{\alpha}} \equiv -\zeta F^{1\underline{\alpha}} + F^{2\underline{\alpha}}. \tag{2.19}$$

The action constructed from the above Lagrangian superfields (2.10), (2.16) and (2.18) is invariant under the following gauge transformations.

$$\delta_\Lambda \boldsymbol{V} = \boldsymbol{\Lambda} + \check{\boldsymbol{\Lambda}}, \qquad \delta_\Lambda \Upsilon = \boldsymbol{\Lambda}\Upsilon, \qquad \delta_\Lambda \Phi = \delta_\Lambda T^{i\underline{\alpha}} = 0,$$

$$\delta_G T^{i\underline{\alpha}} = \boldsymbol{G}^{i\underline{\alpha}}, \qquad \delta_G \boldsymbol{V} = \delta_G \Upsilon = \delta_G \Phi = 0, \tag{2.20}$$

where the transformation parameters $\boldsymbol{\Lambda}$ and $\boldsymbol{G}^{i\underline{\alpha}}$ are an arctic superfield and a constrained superfield that satisfies the same constraints as (2.12) and (2.17), respectively.

3 $N = 1$ superfield description

In this section, we express the $N = 2$ invariant action in the previous section in terms of $N = 1$ superfields. For this purpose, it is convenient to devide the bosonic coordinates x^M into the 4D part x^μ ($\mu = 0, 1, 2, 3$) and the extra-dimensional part $z \equiv \frac{1}{2}(x^4 + ix^5)$ and $\bar{z}$. As for the fermionic coordinates $\Theta^{i\underline{\alpha}}$, they are decomposed into $(\theta^\alpha, \bar{\theta}_{\dot{\alpha}})$ that describes the $N = 1$ subsuperspace we focus on and the rest part $(\theta'^\alpha, \bar{\theta}'_{\dot{\alpha}})$ as shown in (A.28). We follow the notations of ref. [24] for the 2-component spinor indices.

3.1 Superfield action formula

Since the action (2.5) is independent of the choice of η, we choose it as $\eta = 0$ in the following. Then, $\mathcal{D}^{[-4]}$ becomes

$$\mathcal{D}^{[-4]} = -\frac{1}{96}\epsilon^{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}\mathcal{D}_{\underline{\alpha}}^1\mathcal{D}_{\underline{\beta}}^1\mathcal{D}_{\underline{\gamma}}^1\mathcal{D}_{\underline{\delta}}^1 = \frac{1}{16}D^2\bar{D}'^2, \tag{3.1}$$

where D_α and $\bar{D}'^{\dot\alpha}$ are defined in (A.30), and we have used (A.8) and (A.29). Since the Lagrangian superfield L is a projective superfield, it satisfies $\mathcal{D}_{\underline\alpha}^{[1]} L = 0$. From (A.29), this is rewritten as

$$\left(-\zeta D_\alpha - D'_\alpha\right) L = 0, \qquad \left(-\zeta \bar{D}'^{\dot\alpha} + \bar{D}^{\dot\alpha}\right) L = 0. \tag{3.2}$$

Thus, $\mathcal{D}^{[-4]} L$ is rewritten as

$$\mathcal{D}^{[-4]} L = \frac{1}{16} D^2 \left(\frac{1}{\zeta} \bar{D}' \bar{D} L\right) = \frac{1}{16} D^2 \left(\frac{1}{\zeta^2} \bar{D}^2 - \frac{4}{\zeta} \partial\right) L, \tag{3.3}$$

where $\partial \equiv \partial_z = \partial_4 - i\partial_5$, and we have used (A.32). Therefore, the action (2.5) becomes

$$\begin{aligned}
S &= \int d^6 x \left\{ \oint_C \frac{d\zeta}{2\pi i \zeta} \frac{1}{16} D^2 \bar{D}^2 L \bigg|_{\theta = \theta' = 0} \right\} \\
&= \int d^6 x \left\{ \oint_C \frac{d\zeta}{2\pi i \zeta} \int d^4\theta \, L| \right\} \equiv \int d^6 x \, \mathcal{L}.
\end{aligned} \tag{3.4}$$

where a total derivative term is dropped, and the symbol $|$ denotes the projection $\theta' = 0$.

For a given projective superfield $\Xi(x, \Theta, \zeta)$, its expansion coefficients $\Xi_n(x, \Theta)$ in (2.2) satisfy

$$D_\alpha \Xi_n = -D'_\alpha \Xi_{n+1}, \qquad \bar{D}_{\dot\alpha} \Xi_n = \bar{D}'_{\dot\alpha} \Xi_{n-1}, \tag{3.5}$$

which comes from the constraint (2.1). Note that $\Xi_n(x, \Theta)$ is decomposed into the following $N = 1$ superfields.

$$\begin{aligned}
&\Xi_n|, \quad D'_\alpha \Xi_n|, \quad \bar{D}'_{\dot\alpha} \Xi_n|, \quad D'^2 \Xi_n|, \quad D'_\alpha \bar{D}'_{\dot\alpha} \Xi_n|, \\
&\bar{D}'^2 \Xi_n|, \quad \bar{D}'^2 D'_\alpha \Xi_n|, \quad D'^2 \bar{D}'_{\dot\alpha} \Xi_n|, \quad D'^2 \bar{D}'^2 \Xi_n|.
\end{aligned} \tag{3.6}$$

The condition (3.5) provides constraints on these $N = 1$ superfields. The action formula (3.4) is expressed in terms of them. Although each projective superfield contains infinite number of $N = 1$ superfields, only a finite small number of them survive in the final expression of the action as we will see below.

As a simple example, let us consider a free hypermultiplet. The Lagrangian is given by

$$\mathcal{L}_{\text{hyp}} = \int d^4\theta \, \oint_C \frac{d\zeta}{2\pi i \zeta} \, \breve{\Upsilon} \Upsilon \bigg| = \int d^4\theta \, \left\{ \sum_{n=0}^\infty (-1)^n |\Upsilon_n|^2 \right\} \bigg|. \tag{3.7}$$

Since the arctic superfield Υ does not have terms with negative power in ζ (i.e., $\Upsilon_n = 0$ for $n < 0$), the constraint (3.5) becomes

$$\begin{aligned}
D_\alpha \Upsilon_n &= -D'_\alpha \Upsilon_{n+1} \qquad (n \geq 0), & D'_\alpha \Upsilon_0 &= 0, \\
\bar{D}_{\dot\alpha} \Upsilon_n &= \bar{D}'_{\dot\alpha} \Upsilon_{n-1} \qquad (n \geq 1), & \bar{D}_{\dot\alpha} \Upsilon_0 &= 0.
\end{aligned} \tag{3.8}$$

Thus the constraints on $\Upsilon_0|$ and $\Upsilon_1|$ are isolated from the other $N = 1$ superfields.

$$\bar{D}_{\dot\alpha} \Upsilon_0| = 0, \qquad \bar{D}^2 \Upsilon_1| = 4\partial \Upsilon_0|. \tag{3.9}$$

We have used (A.32). Note that $\Upsilon_n|$ $(n \geq 2)$ are unconstrained superfields.[7] Hence they can be easily integrated out and obtain

$$\mathcal{L}_{\mathrm{hyp}} = \int d^4\theta \ \left\{ |\Upsilon_0|^2 - |\Upsilon_1|^2 \right\}\Big|. \tag{3.10}$$

This is further rewritten as

$$\mathcal{L}_{\mathrm{hyp}} = \int d^4\theta \ \left[\left(|\Phi|^2 - |\xi|^2 \right) + \left\{ \kappa \left(\bar{D}^2\xi - 4\partial\Phi \right) + \mathrm{h.c.} \right\} \right], \tag{3.11}$$

where $\Phi \equiv \Upsilon_0|$ is a chiral superfield, and ξ and κ are unconstrained $N = 1$ superfields. In fact, integrating out κ and $\bar{\kappa}$, this reduces to (3.10) with $\xi = \Upsilon_1|$. On the other hand, if we integrate out ξ and $\bar{\xi}$, we obtain [22]

$$\begin{aligned}
\mathcal{L}_{\mathrm{hyp}} &= \int d^4\theta \ \left\{ |\Phi|^2 + |\bar{D}^2\kappa|^2 - (4\kappa\partial\Phi + \mathrm{h.c.}) \right\} \\
&= \int d^4\theta \left(|\Phi|^2 + |\tilde{\Phi}|^2 \right) + \left\{ \int d^2\theta \ \tilde{\Phi}\partial\Phi + \mathrm{h.c.} \right\},
\end{aligned} \tag{3.12}$$

where $\tilde{\Phi} \equiv \bar{D}^2\kappa$ is another chiral superfield, up to total derivatives. This is consistent with (2.3) in ref. [1].

3.2 Decomposition into $N = 1$ superfields

The constraint (2.12) is rewritten as

$$\mathcal{D}_{\underline{\alpha}}^{[1]} T^{[1]\underline{\beta}} - \frac{1}{4}\delta_{\underline{\alpha}}{}^{\underline{\beta}} \mathcal{D}_{\underline{\gamma}}^{[1]} T^{[1]\underline{\gamma}} = 0. \tag{3.13}$$

Since $\left\{ \mathcal{D}_{\underline{\alpha}}^{[1]}, \mathcal{D}_{\underline{\beta}}^{[1]} \right\} = 0$, the solution of this constraint is expressed as [23]

$$T^{[1]\underline{\alpha}} = \frac{i}{3!\zeta}\epsilon^{\underline{\alpha\beta\gamma\delta}} \mathcal{D}_{\underline{\beta}}^{[1]} \mathcal{D}_{\underline{\gamma}}^{[1]} \mathcal{D}_{\underline{\delta}}^{[1]} \boldsymbol{P}^{[-2]}, \tag{3.14}$$

where the prepotential $\boldsymbol{P}^{[-2]}$ is a ζ-independent $N = 2$ superfield, which is a real scalar. The overall ζ-dependence is determined so that $T^{[1]\underline{\alpha}}$ satisfy

$$\breve{T}^{[1]\underline{\alpha}} = -\frac{1}{\zeta} T^{[1]\underline{\alpha}}. \tag{3.15}$$

(See (2.15).) In the 2-component-spinor notation, (3.14) is rewritten as

$$\begin{aligned}
T^{[1]\alpha} &= \frac{i}{2\zeta} \left(-\zeta\bar{D}' + \bar{D} \right)^2 \left(-\zeta D^\alpha - D'^\alpha \right) \boldsymbol{P}^{[-2]}, \\
T_{\dot{\alpha}}^{[1]} &= \frac{i}{2\zeta} \left(-\zeta D - D' \right)^2 \left(-\zeta\bar{D}'_{\dot{\alpha}} + \bar{D}_{\dot{\alpha}} \right) \boldsymbol{P}^{[-2]}.
\end{aligned} \tag{3.16}$$

We have used (A.8) and (A.29).

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[7]From (3.8), each $\Upsilon_n|$ $(n \geq 2)$ is related to $D'\Upsilon_{n\pm1}|$. However, since the latter does not appear in the action, the former can be regarded as an unconstrained superfield.

Since $T^{[1]\underline{\alpha}}$ is a linear function of ζ, the prepotential $\boldsymbol{P}^{[-2]}$ should satisfy

$$\bar{D}_{\dot{\alpha}} D'^2 \boldsymbol{P}^{[-2]} = \bar{D}^2 D'_\alpha \boldsymbol{P}^{[-2]} = 0. \tag{3.17}$$

From the linear and constant terms in ζ, we can read off the components $T^{i\underline{\alpha}}$ as

$$T^{1\alpha} = \frac{i}{2} \left(D'^\alpha \bar{D}'^2 - 2D^\alpha \bar{D}\bar{D}' + 4\partial D^\alpha \right) \boldsymbol{P}^{[-2]},$$

$$T^{2\alpha} = -\frac{i}{2} \left(\bar{D}^2 D^\alpha - 2\bar{D}\bar{D}' D'^\alpha + 4\partial D'^\alpha \right) \boldsymbol{P}^{[-2]},$$

$$T^1_{\dot{\alpha}} = -(T^2_\alpha)^*, \qquad T^2_{\dot{\alpha}} = (T^1_\alpha)^*. \tag{3.18}$$

Then, Φ constructed by (2.13) is calculated as

$$\begin{aligned}
\Phi &= D_\alpha T^{2\alpha} + \bar{D}'^{\dot{\alpha}} T^2_{\dot{\alpha}} + D'_\alpha T^{1\alpha} - \bar{D}^{\dot{\alpha}} T^1_{\dot{\alpha}} \\
&= D_\alpha T^{2\alpha} + D'_\alpha T^{1\alpha} + \text{h.c.} \\
&= i \left(-2D^\alpha \bar{D}\bar{D}' D'_\alpha + 2\bar{D}_{\dot{\alpha}} DD' \bar{D}'^{\dot{\alpha}} + 4\partial DD' - 4\bar{\partial}\bar{D}\bar{D}' \right) \boldsymbol{P}^{[-2]}. \tag{3.19}
\end{aligned}$$

The $N=2$ superfield $\boldsymbol{P}^{[-2]}$ is decomposed into the following $N=1$ superfields.

$$\begin{aligned}
p_0 &\equiv \boldsymbol{P}^{[-2]}|, & p_1^\alpha &\equiv D'^\alpha \boldsymbol{P}^{[-2]}|, \\
p_2^{\dot{\alpha}\alpha} &\equiv \bar{D}'^{\dot{\alpha}} D'^\alpha \boldsymbol{P}^{[-2]}|, & p_3 &\equiv D'^2 \boldsymbol{P}^{[-2]}|, \\
p_4^\alpha &\equiv D'^\alpha \bar{D}'^2 \boldsymbol{P}^{[-2]}|, & p_5 &\equiv D'^\alpha \bar{D}'^2 D'_\alpha \boldsymbol{P}^{[-2]}|. \tag{3.20}
\end{aligned}$$

Then, (3.17) is translated into the following constraints.[8]

$$\begin{aligned}
\bar{D}^2 p_1^\alpha &= 0, & \bar{D}_{\dot{\alpha}} p_3 &= 0, \\
\bar{D}_{\dot{\alpha}} \bar{p}_{4\dot{\beta}} + 2\epsilon_{\dot{\alpha}\dot{\beta}}\partial p_3 &= 0, & \bar{D}^2 p_2^{\dot{\alpha}\alpha} + 4\partial \bar{D}^{\dot{\alpha}} p_1^\alpha &= 0, \\
\bar{D}_{\dot{\alpha}} p_5 - 4i\sigma^\mu_{\alpha\dot{\beta}} \partial_\mu \bar{D}_{\dot{\alpha}} p_2^{\dot{\beta}\alpha} + 4\partial \bar{p}_{4\dot{\alpha}} &= 0, & & \\
\bar{D}^2 \left(p_4^\alpha - 4i\bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \bar{p}_{1\dot{\alpha}} \right) + 8\partial \bar{D}_{\dot{\alpha}} p_2^{\dot{\alpha}\alpha} - 16\partial^2 p_1^\alpha &= 0. \tag{3.21}
\end{aligned}$$

When the spinor derivatives D_α and $\bar{D}^{\dot{\alpha}}$ act on $N=1$ superfields, they are understood as the 4D $N=1$ ones, i.e., $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar{\theta})_\alpha \partial_\mu$ and $\bar{D}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + i(\bar{\sigma}^\mu\theta)^{\dot{\alpha}} \partial_\mu$. From the second and the third constraints in (3.21), we obtain

$$\bar{D}^2 \bar{p}_4^{\dot{\alpha}} = 0. \tag{3.22}$$

Namely, p_4^α is a complex anti-linear superfield and expressed as

$$p_4^\alpha = D^\alpha q_4, \tag{3.23}$$

where q_4 is a complex scalar superfield. The fourth constraint in (3.21) indicates that

$$\chi^\alpha \equiv \bar{D}_{\dot{\alpha}} p_2^{\dot{\alpha}\alpha} - 2\partial p_1^\alpha \tag{3.24}$$

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[8]The last four constraints are obtained by operating D'_β or $\bar{D}'_{\dot{\beta}}$ on (3.17) and putting $\theta' = 0$.

is a chiral superfield. Thus χ^α can be expressed as $\chi^\alpha = \bar{D}^2 U_\chi^\alpha$, where U_χ^α is a spinor superfield. The sixth constraint in (3.21) is rewritten as

$$
\begin{aligned}
0 &= \bar{D}^2 \left(p_4^\alpha + 2 \left\{ D^\alpha, \bar{D}^{\dot\alpha} \right\} \bar{p}_{1\dot\alpha} \right) + 8 \partial \chi^\alpha \\
&= \bar{D}^2 \left(p_4^\alpha + 2 D^\alpha \bar{D}^{\dot\alpha} \bar{p}_{1\dot\alpha} + 8 \partial U_\chi^\alpha \right),
\end{aligned}
\tag{3.25}
$$

which indicates that

$$
Z^\alpha \equiv \frac{1}{2} p_4^\alpha - D^\alpha \bar{D}_{\dot\alpha} \bar{p}_1^{\dot\alpha} + 4 \partial U_\chi^\alpha
\tag{3.26}
$$

is a complex linear superfield, i.e., $\bar{D}^2 Z^\alpha = 0$.

3.3 $N = 1$ description of tensor multiplet

From (3.18) and (3.20), we obtain

$$
\begin{aligned}
T^{1\alpha}| &= \frac{i}{2} \left(p_4^\alpha - 2 D^\alpha \bar{D}_{\dot\alpha} \bar{p}_1^{\dot\alpha} + 4 \partial D^\alpha p_0 \right) = i D^\alpha X, \\
T^{2\alpha}| &= -\frac{i}{2} \left(\bar{D}^2 D^\alpha p_0 - 2 \bar{D}_{\dot\alpha} \bar{p}_2^{\dot\alpha\alpha} + 4 \partial p_1^\alpha \right) = i \bar{D}^2 Y^\alpha,
\end{aligned}
\tag{3.27}
$$

where

$$
\begin{aligned}
X &\equiv \frac{1}{2} q_4 - \bar{D}_{\dot\alpha} \bar{p}_1^{\dot\alpha} + 2 \partial p_0, \\
Y^\alpha &\equiv U_\chi^\alpha - \frac{1}{2} D^\alpha p_0.
\end{aligned}
\tag{3.28}
$$

Using Z^α defined in (3.26), (3.27) is also expressed as

$$
T^{1\alpha}| = i \left(Z^\alpha - 4 \partial Y^\alpha \right), \qquad T^{2\alpha}| = i \bar{D}^2 Y^\alpha.
\tag{3.29}
$$

From (3.18), we can calculate

$$
\begin{aligned}
\mathcal{D}_{\underline{\alpha}}^{[1]} T^{[1]\underline{\alpha}} &= \left(-\zeta D_\alpha - D_\alpha' \right) T^{[1]\alpha} + \left(-\zeta \bar{D}'^{\dot\alpha} + \bar{D}^{\dot\alpha} \right) T_{\dot\alpha}^{[1]} \\
&= i \Bigg\{ \zeta^2 \left(-4 \partial D^2 + \frac{3}{2} D^2 \bar{D} \bar{D}' - 2 i \sigma_{\alpha\dot\alpha}^\mu \partial_\mu D^\alpha \bar{D}'^{\dot\alpha} + 2 \bar{\partial} \bar{D}'^2 - \frac{3}{2} D D' \bar{D}'^2 \right) \\
&\quad + \zeta \left(-D^\alpha \bar{D}^2 D_\alpha + 16 \partial \bar{\partial} - 8 \partial D D' - 8 \bar{\partial} \bar{D} \bar{D}' \right. \\
&\qquad \left. + 2 D^\alpha \bar{D} \bar{D}' D_\alpha' + 2 \bar{D}_{\dot\alpha} D D' \bar{D}'^{\dot\alpha} - D'^\alpha \bar{D}'^2 D_\alpha' \right) \\
&\quad + \left(4 \bar{\partial} \bar{D}^2 - \frac{3}{2} \bar{D}^2 D D' + 2 i \sigma_{\alpha\dot\alpha}^\mu \partial_\mu \bar{D}^{\dot\alpha} D'^\alpha - 2 \partial D'^2 + \frac{3}{2} \bar{D} \bar{D}' D'^2 \right) \Bigg\} \boldsymbol{P}^{[-2]}.
\end{aligned}
\tag{3.30}
$$

Thus,

$$
\frac{i}{\zeta} \mathcal{D}_{\underline{\alpha}}^{[1]} T^{[1]\underline{\alpha}} \bigg| = \zeta \mathcal{A}_1 + \mathcal{A}_0 - \frac{1}{\zeta} \bar{\mathcal{A}}_1,
\tag{3.31}
$$

where

$$
\begin{aligned}
\mathcal{A}_1 &= 4 \partial D^2 p_0 - \frac{3}{2} D^2 \bar{D}_{\dot\alpha} \bar{p}_1^{\dot\alpha} - \frac{1}{2} \left[D^2, \bar{D}_{\dot\alpha} \right] \bar{p}_1^{\dot\alpha} - 2 \bar{\partial} \bar{p}_3 + \frac{3}{2} D^\alpha p_{4\alpha} \\
&= D^\alpha \left(p_{4\alpha} - 2 D_\alpha \bar{D}_{\dot\alpha} \bar{p}_1^{\dot\alpha} + 4 \partial D_\alpha p_0 \right) \\
&= 2 D^\alpha \left(Z_\alpha - 4 \partial Y_\alpha \right), \\
\mathcal{A}_0 &= \left(D^\alpha \bar{D}^2 D_\alpha - 16 \partial \bar{\partial} \right) p_0 + 8 \partial D^\alpha p_{1\alpha} + 8 \bar{\partial} \bar{D}_{\dot\alpha} \bar{p}_1^{\dot\alpha} \\
&\quad - 2 D^\alpha \bar{D}_{\dot\alpha} (p_2)^{\dot\alpha}{}_\alpha - 2 \bar{D}_{\dot\alpha} D^\alpha (\bar{p}_2)_\alpha{}^{\dot\alpha} + p_5.
\end{aligned}
\tag{3.32}
$$

Although $\mathcal{A}_0$ cannot be expressed in terms of only Y_α and Z_α, $\bar{D}_{\dot\alpha}\mathcal{A}_0$ can. In fact, after some calculations, we obtain

$$\begin{aligned}
\bar{D}_{\dot\alpha}\mathcal{A}_0 &= -8\partial\left(\bar{Z}_{\dot\alpha} - 4\bar\partial\bar{Y}_{\dot\alpha}\right) + 2\bar{D}^2 D^2 \bar{Y}_{\dot\alpha} \\
&= \bar{D}_{\dot\alpha}\left(-8\partial\bar{X} - 4\bar{D}_{\dot\beta}D^2\bar{Y}^{\dot\beta}\right),
\end{aligned} \tag{3.33}$$

where we have used that $Z^\alpha - 4\partial Y^\alpha = D^\alpha X (= -iT^{1\alpha}|)$. Thus, $\mathcal{A}_0$ is expressed as

$$\mathcal{A}_0 = -8\partial\bar{X} - 4\bar{D}_{\dot\alpha}D^2\bar{Y}^{\dot\alpha} + \phi_0, \tag{3.34}$$

where ϕ_0 is a chiral superfield that is determined so that $\mathcal{A}_0$ is real.

From (3.19), we have

$$\begin{aligned}
\Phi| &= i\left(-2D^\alpha\bar{D}_{\dot\alpha}(p_2)^{\dot\alpha}{}_\alpha + 2\bar{D}_{\dot\alpha}D^\alpha(\bar{p}_2)_\alpha{}^{\dot\alpha} + 4\partial D^\alpha p_{1\alpha} - 4\bar\partial\bar{D}_{\dot\alpha}\bar{p}_1^{\dot\alpha}\right) \\
&= -2iD^\alpha\chi_\alpha + 2i\bar{D}_{\dot\alpha}\bar\chi^{\dot\alpha} \\
&= -2iD^\alpha\bar{D}^2 Y_\alpha + 2i\bar{D}_{\dot\alpha}D^2\bar{Y}^{\dot\alpha}, \\
D'_\alpha\Phi| &= -\frac{i}{2}\bar{D}^2 D^2 p_{1\alpha} + \frac{i}{2}\left(D_\alpha\bar{D}_{\dot\alpha} + 2\bar{D}_{\dot\alpha}D_\alpha\right)\bar{p}_4^{\dot\alpha} - 4i\bar\partial\chi_\alpha \\
&= i\left(D_\alpha\bar{D}_{\dot\alpha} + 2\bar{D}_{\dot\alpha}D_\alpha\right)\left(\bar{Z}^{\dot\alpha} - 4\bar\partial\bar{Y}^{\dot\alpha}\right) - 4i\bar\partial\bar{D}^2 Y_\alpha.
\end{aligned} \tag{3.35}$$

In summary, the 6D tensor multiplet is described by the spinor superfields Y_α and Z_α, where the latter is constrained by $\bar{D}^2 Z_\alpha = 0$.

3.4 $N = 1$ description of vector multiplet

As mentioned in section 2.2, a 6D vector multiplet can also be described by the constrained superfield $F^{i\underline{\alpha}}$. This is decomposed into $N = 1$ superfields in a similar way to the tensor multiplet.

$$\begin{aligned}
F^{1\alpha}| &= i\left(Z_F^\alpha - 4\partial Y_F^\alpha\right) = iD^\alpha X_F, \qquad F^{2\alpha}| = i\bar{D}^2 Y_F^\alpha, \\
\frac{i}{\zeta}\mathcal{D}_{\underline{\alpha}}^{[1]} F^{[1]\underline{\alpha}}\bigg| &= \zeta\mathcal{A}_{F1} + \mathcal{A}_{F0} - \frac{1}{\zeta}\mathcal{A}_{1F}^*, \\
\mathcal{A}_{F1} &= 2D^\alpha\left(Z_{F\alpha} - 4\partial Y_{F\alpha}\right), \\
\mathcal{A}_{F0} &= -8\partial\bar{X}_F - 4\bar{D}_{\dot\alpha}D^2\bar{Y}_F^{\dot\alpha} + \phi_{F0}, \\
\mathcal{D}_{i\underline{\alpha}}F^{i\underline{\alpha}}| &= -2iD^\alpha\bar{D}^2 Y_{F\alpha} + 2i\bar{D}_{\dot\alpha}D^2\bar{Y}_F^{\dot\alpha},
\end{aligned} \tag{3.36}$$

where Z_F^α and ϕ_{F0} are a complex linear and a chiral superfields, respectively. In contrast to the tensor multiplet, $F^{i\underline{\alpha}}$ is further constrained by (2.17), which indicates that

$$D^\alpha\bar{D}^2 Y_{F\alpha} = \bar{D}_{\dot\alpha}D^2\bar{Y}_F^{\dot\alpha}. \tag{3.37}$$

This is regarded as the Bianchi identity, and solved as

$$Y_F^\alpha = -\frac{1}{4}D^\alpha V, \tag{3.38}$$

where V is an unconstrained real superfield. Then, Z_F^α is expressed as

$$Z_F^\alpha = 4\partial Y_F^\alpha + D^\alpha X_F = -D^\alpha \Sigma, \tag{3.39}$$

where $\Sigma \equiv \partial V - X_F$. Since $\bar{D}^2 Z_F^\alpha = 0$, Σ is a chiral superfield. Thus, (3.36) becomes

$$F^{1\alpha}| = iD^\alpha(\partial V - \Sigma), \qquad F^{2\alpha} = -\frac{i}{4}\bar{D}^2 D^\alpha V,$$
$$\mathcal{A}_{F1} = 2D^2(\partial V - \Sigma), \qquad \mathcal{A}_{F0} = -8\partial(\bar{\partial}V - \bar{\Sigma}) + \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha} V + \phi_{F0}. \tag{3.40}$$

As mentioned in section 2.2, $F^{i\alpha}$ are invariant under the gauge transformation,

$$V \to V + \Lambda + \bar{\Lambda}, \qquad \Sigma \to \Sigma + \partial\Lambda, \tag{3.41}$$

where Λ is a chiral superfield. Especially, $F^{2\alpha}$ is proportional to the field strength superfield,

$$\mathcal{W}^\alpha \equiv -\frac{1}{4}\bar{D}^2 D^\alpha V. \tag{3.42}$$

Since ϕ_{F0} is a chiral superfield and $\mathcal{A}_{F0}$ is real, we find that $\phi_{F0} = 8\bar{\partial}\Sigma$. Namely,

$$\mathcal{A}_{F0} = 8\left\{-\left(\Box_4 \mathcal{P}_{\rm T} + \partial\bar{\partial}\right)V + \bar{\partial}\Sigma + \partial\bar{\Sigma}\right\}, \tag{3.43}$$

where $\Box_4 \equiv \partial_\mu \partial^\mu$, and

$$\mathcal{P}_{\rm T} \equiv -\frac{\bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha}}{8\Box_4} \tag{3.44}$$

is the projection operator [24].

In summary, the 6D vector multiplet is described by a chiral superfield Σ and a real superfield V, which are independent of each other.

3.5 Vector-tensor couplings

Now we consider the vector-tensor couplings. Note that $\mathcal{F}^{[2]}$ defined in (2.19) is an O(2) multiplet, i.e.,

$$\mathcal{F}^{[2]} = \zeta\mathcal{F}_1^{[2]} + \mathcal{F}_0^{[2]} - \frac{1}{\zeta}(\mathcal{F}_1^{[2]})^*, \tag{3.45}$$

where $\mathcal{F}_0^{[2]}$ is real. Since

$$\frac{i}{\zeta}\left(\mathcal{D}_{\underline{\alpha}}^{[1]}\Phi\right)F^{[1]\underline{\alpha}} = i\zeta\left\{D_\alpha\Phi F^{1\alpha} - (D'_\alpha\Phi F^{2\alpha})^*\right\}$$
$$+ i\left\{D'_\alpha\Phi F^{1\alpha} - D_\alpha\Phi F^{2\alpha} - (D'_\alpha\Phi F^{1\alpha})^* + (D_\alpha\Phi F^{2\alpha})^*\right\}$$
$$- \frac{i}{\zeta}\left\{D'_\alpha\Phi F^{2\alpha} - (D_\alpha\Phi F^{1\alpha})^*\right\}, \tag{3.46}$$

we can calculate $\mathcal{F}_1^{[2]}|$ and $\mathcal{F}_0^{[2]}|$ after some calculations by using the results in the previous subsections as

$$\mathcal{F}_1^{[2]}| = \frac{1}{2}D^2\left\{\Phi_T\left(\partial V - \Sigma\right)\right\} - \bar{\mathcal{W}}_T\bar{\mathcal{W}},$$
$$\mathcal{F}_0^{[2]}| = \left\{-\mathcal{W}_T^\alpha D_\alpha\left(\partial V - \Sigma\right) - D^\alpha\Phi_T\mathcal{W}_\alpha + \text{h.c.}\right\}$$
$$-2\Phi_T\left\{\left(\Box_4\mathcal{P}_\text{T} + \partial\bar{\partial}\right)V - \bar{\partial}\Sigma - \partial\bar{\Sigma}\right\}, \tag{3.47}$$

where

$$\Phi_T \equiv \Phi| = -2iD^\alpha\bar{D}^2Y_\alpha + 2i\bar{D}_{\dot{\alpha}}D^2\bar{Y}^{\dot{\alpha}},$$
$$\mathcal{W}_{T\alpha} \equiv i\bar{D}^2\left(D_\alpha\bar{X} + 4\bar{\partial}Y_\alpha\right)$$
$$= -i\left(D_\alpha\bar{D}_{\dot{\alpha}} + 2\bar{D}_{\dot{\alpha}}D_\alpha\right)\left(\bar{Z}^{\dot{\alpha}} - 4\bar{\partial}\bar{Y}^{\dot{\alpha}}\right) + 4i\bar{\partial}\bar{D}^2Y_\alpha. \tag{3.48}$$

Note that the real linear superfield Φ_T and and the chiral superfield $\mathcal{W}_T^\alpha$ are not independent. As shown in appendix B, they are related through

$$D^\alpha\mathcal{W}_{T\alpha} = -2\bar{\partial}\Phi_T,$$
$$\bar{D}^2D^\alpha\Phi_T = -4\partial\mathcal{W}_T^\alpha. \tag{3.49}$$

From these relations, we obtain

$$\left(\Box_4 + \partial\bar{\partial}\right)\Phi_T = 0, \qquad \left(\Box_4 + \partial\bar{\partial}\right)\mathcal{W}_T^\alpha = 0, \tag{3.50}$$

where we have used that $\mathcal{P}_\text{T}\Phi_T = \Phi_T$ and $\bar{D}^2D^2\mathcal{W}_T^\alpha = 16\Box_4\mathcal{W}_T^\alpha$. Namely, Φ_T and $\mathcal{W}_T^\alpha$ are on-shell. This stems from the fact that the 6D tensor multiplet contains a self-dual tensor field $B_{\mu\nu}^+$. In the 6D global SUSY theories, the tensor multiplet cannot be described as off-shell superfields,[9] and thus should be treated as external fields. As shown in refs. [14, 15], the off-shell description of the tensor multiplet becomes possible by combining the Weyl multiplet when the theory is promoted to SUGRA.

Therefore, from (2.18), the Lagrangian in the vector-tensor sector is

$$\mathcal{L}_\text{VT} = \oint_C \frac{d\zeta}{2\pi i\zeta}\int d^4\theta \, L_\text{VT}|$$
$$= \int d^4\theta\left\{-V_1(\mathcal{F}_1^{[2]})^* + V_0\mathcal{F}_0^{[2]} - \bar{V}_1\mathcal{F}_1^{[2]}\right\}\bigg|$$
$$= \int d^4\theta\left[-\frac{1}{2}V_1|\bar{D}^2\left\{\Phi_T\left(\bar{\partial}V - \bar{\Sigma}\right)\right\} + V_1|\mathcal{W}_T\mathcal{W}\right.$$
$$-V_0|\left\{\mathcal{W}_T^\alpha D_\alpha\left(\partial V - \Sigma\right) + D^\alpha\Phi_T\mathcal{W}_\alpha + \text{h.c.}\right\}$$
$$-2V_0|\Phi_T\left\{\left(\Box_4\mathcal{P}_\text{T} + \partial\bar{\partial}\right)V - \bar{\partial}\Sigma - \partial\bar{\Sigma}\right\}$$
$$\left.-\frac{1}{2}\bar{V}_1|D^2\left\{\Phi_T\left(\partial V - \Sigma\right)\right\} + \bar{V}_1|\bar{\mathcal{W}}_T\bar{\mathcal{W}}\right], \tag{3.51}$$

---

[9]This fact is explicitly shown in the harmonic superspace formulation in ref. [23].

where V_0 and V_1 are the coefficient superfields in the tropical superfield (2.9). Using $d^2\bar\theta = -\frac{1}{4}\bar{D}^2$ and performing the partial integrals, the above Lagrangian is rewritten as

$$\mathcal{L}_{\mathrm{VT}} = -\int d^2\theta\, \tilde\Sigma \mathcal{W}_T \mathcal{W} + \text{h.c.}$$
$$- \int d^4\theta \left\{ 2\bar{\tilde\Sigma} \Phi_T \left(\partial V - \Sigma \right) + \tilde{V} \mathcal{W}_T^\alpha D_\alpha \left(\partial V - \Sigma \right) + \tilde{V} D^\alpha \Phi_T \mathcal{W}_\alpha + \text{h.c.} \right\}$$
$$- \int d^4\theta\, 2\Phi_T \tilde{V} \left\{ \left(\Box_4 \mathcal{P}_{\mathrm{T}} + \partial\bar\partial \right) V - \bar\partial\Sigma - \partial\bar\Sigma \right\} \tag{3.52}$$

where

$$\tilde{V} \equiv V_0|, \qquad \tilde\Sigma \equiv \frac{1}{4}\bar{D}^2 V_1|. \tag{3.53}$$

The second line in (3.52) is further rewritten as

$$- \int d^4\theta \left\{ 2\bar{\tilde\Sigma} \Phi_T \left(\partial V - \Sigma \right) + \tilde{V} \mathcal{W}_T^\alpha D_\alpha \left(\partial V - \Sigma \right) + \tilde{V} D^\alpha \Phi_T \mathcal{W}_\alpha + \text{h.c.} \right\}$$
$$= - \int d^2\theta \left\{ \Sigma \tilde{\mathcal{W}} \mathcal{W}_T + \frac{1}{4}\bar{D}^2 \left(\Phi_T D^\alpha \tilde{V} \mathcal{W}_\alpha + \partial V D^\alpha \tilde{V} \mathcal{W}_{T\alpha} \right) \right\} + \text{h.c.}$$
$$+ \int d^4\theta\, \Phi_T \tilde{V} \left\{ 4\left(\Box_4 \mathcal{P}_{\mathrm{T}} + \partial\bar\partial \right) V - 2\bar\partial\Sigma - 2\partial\bar\Sigma \right\}$$
$$+ \int d^4\theta \left\{ 2\Phi_T (\bar\partial\tilde{V} - \bar{\tilde\Sigma}) \left(\partial V - \Sigma \right) + \text{h.c.} \right\}, \tag{3.54}$$

where we have used (3.49). Thus, $\mathcal{L}_{\mathrm{VT}}$ becomes

$$\mathcal{L}_{\mathrm{VT}} = -\int d^2\theta\, \left\{ \left(\tilde\Sigma \mathcal{W} + \Sigma \tilde{\mathcal{W}} \right) \mathcal{W}_T + \frac{1}{4}\bar{D}^2 \left(\Phi_T D^\alpha \tilde{V} \mathcal{W}_\alpha + \partial V D^\alpha \tilde{V} \mathcal{W}_{T\alpha} \right) \right\} + \text{h.c.}$$
$$+ \int d^4\theta\, 2\Phi_T \tilde{V} \left(\Box_4 \mathcal{P}_{\mathrm{T}} + \partial\bar\partial \right) V$$
$$+ \int d^4\theta \left\{ 2\Phi_T (\bar\partial\tilde{V} - \bar{\tilde\Sigma}) \left(\partial V - \Sigma \right) + \text{h.c.} \right\}, \tag{3.55}$$

where $\tilde{\mathcal{W}}_\alpha \equiv -\frac{1}{4}\bar{D}^2 D_\alpha \tilde{V}$. When the 6D vector multiplets (V, Σ) and $(\tilde{V}, \tilde\Sigma)$ are identical, (3.55) is simplified as

$$\mathcal{L}_{\mathrm{VT}} = -\int d^2\theta\, \left\{ 2\Sigma \mathcal{W} \mathcal{W}_T + \frac{1}{4}\bar{D}^2 \left(\Phi_T D^\alpha V \mathcal{W}_\alpha + \partial V D^\alpha V \mathcal{W}_{T\alpha} \right) \right\} + \text{h.c.}$$
$$+ \int d^4\theta\, 2\Phi_T \left\{ V \left(\Box_4 \mathcal{P}_{\mathrm{T}} + \partial\bar\partial \right) V + 2 \left(\bar\partial V - \bar\Sigma \right) \left(\partial V - \Sigma \right) \right\}. \tag{3.56}$$

This is our main result. This contains the result in ref. [1] as a special case: $\Phi_T = 1$ and $\mathcal{W}_T^\alpha = 0$, which corresponds to the case where the tensor multiplet is absent. In such a

case, (3.56) becomes

$$
\begin{aligned}
\mathcal{L}_{\mathrm{VT}} &= \int d^2\theta\, \mathcal{W}^2 + \text{h.c.} \\
&\quad + \int d^4\theta \left\{ V D^\alpha \mathcal{W}_\alpha + 2V\partial\bar{\partial}V + 4\left(\bar{\partial}V - \bar{\Sigma}\right)\left(\partial V - \Sigma\right) \right\} \\
&= \int d^2\theta\, \frac{1}{2}\mathcal{W}^2 + \text{h.c.} \\
&\quad + \int d^4\theta\, 2\left\{ 2\left(\bar{\partial}V - \bar{\Sigma}\right)\left(\partial V - \Sigma\right) - \bar{\partial}V\partial V \right\},
\end{aligned}
\tag{3.57}
$$

where we have used $d^2\bar{\theta} = -\frac{1}{4}\bar{D}^2$, and dropped total derivative terms. This agrees with (2.17) in ref. [1] after rescaling the superfields as $V \to \frac{1}{\sqrt{2}g}V$ and $\Sigma \to \frac{1}{2g}\phi$.

3.6 Dimensional reduction to 5D

Here we consider the dimensional reduction of (3.56) to five dimensions by neglecting the x^5-dependence of the $N = 1$ superfields. Then (3.49) becomes

$$
\begin{aligned}
D^\alpha \mathcal{W}_{T\alpha} &= -2\partial_4 \Phi_T, \\
\bar{D}^2 D^\alpha \Phi_T &= -4\partial_4 \mathcal{W}_T^\alpha.
\end{aligned}
\tag{3.58}
$$

Since the right-hand-side of the first equation is now real, $\mathcal{W}_T^\alpha$ satisfies the Bianchi identity $D^\alpha \mathcal{W}_{T\alpha} = \bar{D}_{\dot{\alpha}} \bar{\mathcal{W}}_T^{\dot{\alpha}}$. Hence it is a field-strength superfield.

$$
\mathcal{W}_T^\alpha = -\frac{1}{4}\bar{D}^2 D^\alpha V_T,
\tag{3.59}
$$

where V_T is a real superfield. Substituting this into the second constraint in (3.58), we obtain

$$
\bar{D}^2 D^\alpha \left(\Phi_T - \partial_4 V_T\right) = 0,
\tag{3.60}
$$

which indicates that

$$
\Phi_T = \partial_4 V_T - \Sigma_T - \bar{\Sigma}_T,
\tag{3.61}
$$

where[10]

$$
\Sigma_T \equiv \frac{\bar{D}^2 D^2}{16\square_4}\partial_4 V_T
\tag{3.62}
$$

is a chiral part of $\partial_4 V_T$. Then, the first constraint in (3.58) is rewritten as

$$
\left(\square_4 + \partial_4^2\right)\mathcal{P}_{\mathrm{T}}V_T = 0.
\tag{3.63}
$$

Namely, the 6D tensor multiplet becomes an (on-shell) 5D vector multiplet after the dimensional reduction.[11]

---

[10]Note that Φ_T is a real linear superfield, i.e., $\Phi_T = \mathcal{P}_{\mathrm{T}}\partial_4 V_T$.

[11]Although there exists a 5D tensor field among the fields obtained from the 6D tensor field B_{MN}^+ by the dimensional reduction, such a field is dual to a 5D vector field. The duality between the 5D tensor (gauge) multiplet and the 5D vector multiplet is explicitly shown in component fields in ref. [25].

As shown in appendix C, the Lagrangian (3.56) becomes the following expression after the dimensional reduction.

$$\mathcal{L}_{\mathrm{VT}}^{(5D)} = -\int d^2\theta \, C_{IJK}\Sigma^I \mathcal{W}^J \mathcal{W}^K + \mathrm{h.c.}$$
$$+ \int d^4\theta \, \frac{C_{IJK}}{3} \left\{ \left(\partial_4 V^I D^\alpha V^J - V^I \partial_4 D^\alpha V^J\right) \mathcal{W}_\alpha^K + \mathrm{h.c.} \right\}$$
$$+ \int d^4\theta \, \frac{2C_{IJK}}{3}\mathcal{V}^I \mathcal{V}^J \mathcal{V}^K, \tag{3.64}$$

where $(\Sigma^1, V^1, \Sigma^2, V^2) = (\Sigma, V, \Sigma_T, V_T)$, the symmetric constant tensor C_{IJK} is defined as $C_{112} = C_{121} = C_{211} = 1$ and the other components are zero, and

$$\mathcal{V}^I \equiv \partial_4 V^I - \Sigma^I - \bar{\Sigma}^I. \tag{3.65}$$

This agrees with the 5D supersymmetric Chern-Simons terms [1, 26].

3.7 Bilinear terms in tensor multiplets

In this subsection, we consider the Lagrangian terms that consist of only tensor multiplets. It is given by (2.16). The Lagrangian is expressed as

$$\mathcal{L}_{\mathrm{tensor}} = \oint_C \frac{d\zeta}{2\pi i \zeta} \int d^4\theta \, L_{\mathrm{tensor}} = \int d^4\theta \, \mathcal{T}_0^{[2]}|, \tag{3.66}$$

where

$$\mathcal{T}^{[2]} = \zeta \mathcal{T}_1^{[2]} + \mathcal{T}_0^{[2]} - \frac{1}{\zeta}(\mathcal{T}_1^{[2]})^*. \tag{3.67}$$

Using the expressions in section 3.3, $\mathcal{T}_0^{[2]}|$ is calculated as

$$\mathcal{T}_0^{[2]}| = \left\{ -i(D_\alpha \Phi)T^{2\alpha} + i(D'_\alpha \Phi)T^{1\alpha} + \mathrm{h.c.} \right\}| + \frac{1}{4}\Phi_T \tilde{\mathcal{A}}_0$$
$$= \left(-D^\alpha \Phi_T \bar{D}^2 \tilde{Y}_\alpha - \mathcal{W}_T^\alpha D_\alpha \tilde{X} + \mathrm{h.c.} \right) + \frac{1}{4}\Phi_T \tilde{\mathcal{A}}_0, \tag{3.68}$$

where

$$\tilde{\mathcal{A}}_0 = -8\partial\bar{\tilde{X}} - 4\bar{D}_{\dot{\alpha}} D^2 \bar{\tilde{Y}}^{\dot{\alpha}} + \tilde{\phi}_0. \tag{3.69}$$

Here we treat two tensor multiplets $(\Phi_T, \mathcal{W}_T^\alpha)$ originating from Φ and $(\tilde{X}, \tilde{Y}^\alpha)$ originating from $T^{i\alpha}$ as independent multiplets. The Lagrangian (3.66) is then expressed as

$$\mathcal{L}_{\mathrm{tensor}} = \int d^4\theta \left\{ \left(\Phi_T D^\alpha \bar{D}^2 \tilde{Y}_\alpha + D^\alpha \mathcal{W}_{T\alpha} \tilde{X} + \mathrm{h.c.} \right) + \frac{1}{4}\Phi_T \tilde{\mathcal{A}}_0 \right\}$$
$$= \int d^4\theta \left(\frac{1}{8}\Phi_T \tilde{\mathcal{A}}_0 + \Phi_T D^\alpha \bar{D}^2 \tilde{Y}_\alpha - 2\bar{\partial}\Phi_T \tilde{X} + \mathrm{h.c.} \right)$$
$$= \int d^4\theta \, \frac{1}{8}\Phi_T \left(\tilde{\mathcal{A}}_0 + 8D^\alpha \bar{D}^2 \tilde{Y}_\alpha + 16\bar{\partial}\tilde{X} + \mathrm{h.c.} \right)$$
$$= \int d^4\theta \, \Phi_T \left(\frac{1}{2}D^\alpha \bar{D}^2 \tilde{Y}_\alpha + \bar{\partial}\tilde{X} + \mathrm{h.c.} \right). \tag{3.70}$$

We have used (3.49), and dropped total derivative terms. At the last step, we have used that

$$\int d^4\theta \, \Phi_T \left(\tilde{\mathcal{A}}_0 + 4\bar{D}_{\dot{\alpha}} D^2 \bar{\tilde{Y}}^{\dot{\alpha}} + 8\partial\bar{\tilde{X}} \right) = \int d^4\theta \, \Phi_T \tilde{\phi}_0 = 0. \qquad (3.71)$$

This Lagrangian can be further rewritten as

$$\begin{aligned}
\mathcal{L}_{\text{tensor}} &= \int d^4\theta \, \left(-\frac{1}{2} \bar{D}^2 D^\alpha \Phi_T \tilde{Y}_\alpha - \bar{\partial}\Phi_T \tilde{X} + \text{h.c.} \right) \\
&= \int d^4\theta \, \left(-\frac{1}{2} \bar{D}^2 D^\alpha \Phi_T \tilde{Y}_\alpha - \frac{1}{2} \mathcal{W}_T^\alpha D_\alpha \tilde{X} + \text{h.c.} \right) \\
&= -\frac{1}{2} \int d^4\theta \, \left\{ \bar{D}^2 D^\alpha \Phi_T \tilde{Y}_\alpha + \mathcal{W}_T^\alpha \left(\tilde{Z}_\alpha - 4\partial\tilde{Y}_\alpha \right) + \text{h.c.} \right\} \\
&= -\frac{1}{2} \int d^4\theta \, \left\{ \left(\bar{D}^2 D^\alpha \Phi_T + 4\partial\mathcal{W}_T^\alpha \right) \tilde{Y}_\alpha + \text{h.c.} \right\}. \qquad (3.72)
\end{aligned}$$

At the last step, we have used the fact that $\mathcal{W}_T^\alpha$ and $\tilde{Z}^\alpha$ are a chiral and a linear superfields. This Lagrangian vanishes due to the second constraint in (3.49). However we can relax that constraint if we regard $\tilde{Y}_\alpha$ as the Lagrange multiplier. In that case, the constraint is obtained as the equation of motion for $\tilde{Y}_\alpha$. As shown in appendix B, that constraint is necessary in order for $\mathcal{F}^{[2]}$ defined in (2.19) to satisfy $\mathcal{D}_\alpha^{[1]}\mathcal{F}^{[2]} = 0$, which is relevant to the $N = 2$ SUSY invariance of the action. Thus, in such a case, the full $N = 2$ SUSY invariance of the vector-tensor coupling terms (3.56) is ensured only at the on-shell level.[12] Nevertheless, (3.72) is expected to play an important role when we promote the theory to SUGRA. It corresponds to (2.14) in ref. [23], which is described in the harmonic superspace.

3.8 Identification of component fields

Finally, we identify component fields of each $N = 1$ superfield. Here we focus on the bosonic fields.

A 6D vector field A_M is embedded into V and Σ as [1]

$$\begin{aligned}
V &= -(\theta\sigma^\mu\bar{\theta})A_\mu + \cdots, \\
\Sigma &= \frac{1}{2} (A_5 - iA_4) + \cdots, \qquad (3.73)
\end{aligned}$$

where the ellipses denote fermionic or auxiliary fields.

The 6D tensor multiplet contains a real scalar field σ and a self-dual tensor field B_{MN}^+, which satisfy [23]

$$\begin{aligned}
\left(\Box_4 + \partial\bar{\partial} \right) \sigma &= 0, \\
\partial_{[M} B_{NL]}^+ &= \frac{1}{6}\epsilon_{MNLPQR}\partial^P B^{+QR}, \qquad (3.74)
\end{aligned}$$

where ϵ_{MNLPQR} is the antisymmetric constant tensor. From (3.29) and (3.40), the gauge transformation for the tensor multiplet in (2.20) is expressed as

$$\begin{aligned}
\delta_G \left(Z_\alpha - 4\partial Y_\alpha \right) &= D_\alpha \left(\partial V_G - \Sigma_G \right), \\
\delta_G \bar{D}^2 Y_\alpha &= -\frac{1}{4} \bar{D}^2 D_\alpha V_G, \qquad (3.75)
\end{aligned}$$

---

[12]Half of the whole SUSY remains manifest at the off-shell level because the action is expressed in terms of $N = 1$ superfields.

where the vector multiplet (Σ_G, V_G) is the transformation parameter that satisfies the on-shell condition. Note that Φ_T and $\mathcal{W}_{T\alpha}$ are invariant under this transformation. In components, this gauge transformation is expressed as

$$\sigma \to \sigma, \quad B^+_{MN} \to B^+_{MN} + \partial_M \lambda_N - \partial_N \lambda_M, \tag{3.76}$$

where λ_M are the transformation parameters. From the conditions $D^2 \Phi_T = \bar{D}^2 \Phi_T = 0$ and (3.49), we find that σ and B^+_{MN} are embedded into Φ_T and $\mathcal{W}_{T\alpha}$ as

$$\Phi_T = \sigma - 2(\theta \sigma^\mu \bar{\theta}) \left\{ \partial_\mu B^+_{45} - \mathrm{Im}\,(\partial C_\mu) \right\} - \frac{1}{4}\theta^2 \bar{\theta}^2 \Box_4 \sigma + \cdots,$$

$$\mathcal{W}_{T\alpha} = \theta_\alpha \bar{\partial}\sigma + (\sigma^{\mu\nu}\theta)_\alpha \left\{ \bar{\partial} B^+_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu \right\} + \cdots, \tag{3.77}$$

where $C_\mu \equiv B^+_{\mu 4} + i B^+_{\mu 5}$, and $\mathcal{W}_{T\alpha}$ is expressed in the chiral basis $(x^\mu + i\theta\sigma^\mu\bar{\theta}, z, \bar{z}, \theta, \bar{\theta})$. Note that these expressions are invariant under (3.76).

4 Summary

We have derived the $N = 1$ superfield description of supersymmetric coupling terms among 6D tensor and vector multiplets from the projective superspace action provided in ref. [19]. This is necessary to describe 6D SUGRA in terms of $N = 1$ superfields. Our result contains the result in ref. [1] as a special case. It also reproduces the 5D supersymmetric Chern-Simons terms after the dimensional reduction.

The tensor multiplet is described by two complex spinor superfields Y_α and Z_α, where Z_α is constrained as $\bar{D}^2 Z_\alpha = 0$. They appear in the action in the forms of a real linear superfield Φ_T and a chiral spinor superfield $\mathcal{W}_{T\alpha}$ defined by (3.48). These superfields are constrained by (3.49), which leads to the on-shell conditions. Thus they should be treated as external fields. This stems from the fact that the 6D tensor multiplet contains a self-dual tensor field B^+_{MN}. As shown in ref. [14] in the component fields, the on-shell condition for the tensor multiplet can be relaxed when the theory couples to the gravity. Our result (3.56) provides a good starting point to obtain the $N = 1$ superfield description of 6D SUGRA. We will discuss this issue in the subsequent paper.

Acknowledgments

A Notations for spinors

A.1 Gamma matrices

The spacetime metric is

$$ds^2 = \eta_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu + (dx^4)^2 + (dx^5)^2, \tag{A.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

The 6D gamma matrices Γ^M $(M = 0, 1, \cdots, 5)$ are chosen as

$$\Gamma^M = \begin{pmatrix} & (\gamma^M)_{\underline{\alpha}\underline{\beta}} \\ (\tilde{\gamma}^M)^{\underline{\alpha}\underline{\beta}} & \end{pmatrix}, \tag{A.2}$$

where 4×4 matrices γ^M and $\tilde{\gamma}^M$ satisfy

$$\begin{aligned} \left(\gamma^M \tilde{\gamma}^N + \gamma^N \tilde{\gamma}^M\right)_{\underline{\alpha}}{}^{\underline{\beta}} &= -2\eta^{MN} \delta_{\underline{\alpha}}{}^{\underline{\beta}}, \\ \left(\tilde{\gamma}^M \gamma^N + \tilde{\gamma}^N \gamma^M\right)^{\underline{\alpha}}{}_{\underline{\beta}} &= -2\eta^{MN} \delta^{\underline{\alpha}}{}_{\underline{\beta}}, \end{aligned} \tag{A.3}$$

and are defined as

$$\begin{aligned} (\gamma^\mu)_{\underline{\alpha}\underline{\beta}} &= \begin{pmatrix} & -\sigma^\mu_{\alpha\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\gamma}\epsilon_{\gamma\beta} & \end{pmatrix}, \\ (\gamma^4)_{\underline{\alpha}\underline{\beta}} &= \begin{pmatrix} -i\epsilon_{\alpha\beta} & \\ & -i\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, & (\gamma^5)_{\underline{\alpha}\underline{\beta}} &= \begin{pmatrix} \epsilon_{\alpha\beta} & \\ & -\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \\ (\tilde{\gamma}^\mu)^{\underline{\alpha}\underline{\beta}} &= \begin{pmatrix} & \epsilon^{\alpha\gamma}\sigma^\mu_{\gamma\dot{\beta}} \\ -\epsilon_{\dot{\alpha}\dot{\gamma}}\bar{\sigma}^{\mu\dot{\gamma}\beta} & \end{pmatrix}, \\ (\tilde{\gamma}^4)^{\underline{\alpha}\underline{\beta}} &= \begin{pmatrix} -i\epsilon^{\alpha\beta} & \\ & -i\epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, & (\tilde{\gamma}^5)^{\underline{\alpha}\underline{\beta}} &= \begin{pmatrix} -\epsilon^{\alpha\beta} & \\ & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \end{aligned} \tag{A.4}$$

where the antisymmetric tensors $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ are chosen as $\epsilon^{12} = \epsilon_{21} = 1$. These matrices are anti-symmetric, i.e., $(\gamma^M)_{\underline{\alpha}\underline{\beta}} = -(\gamma^M)_{\underline{\beta}\underline{\alpha}}$ and $(\tilde{\gamma}^M)^{\underline{\alpha}\underline{\beta}} = -(\tilde{\gamma}^M)^{\underline{\beta}\underline{\alpha}}$. The 6D chirality matrix Γ_7 is defined by

$$\Gamma_7 \equiv \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 = \begin{pmatrix} \mathbf{1}_4 & \\ & -\mathbf{1}_4 \end{pmatrix}. \tag{A.5}$$

The antisymmetric tensors $\epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}$ and $\epsilon^{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}$ are given by

$$\epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} = \frac{1}{2}(\gamma^M)_{\underline{\alpha}\underline{\beta}}(\gamma_M)_{\underline{\gamma}\underline{\delta}}, \quad \epsilon^{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} = \frac{1}{2}(\tilde{\gamma}^M)^{\underline{\alpha}\underline{\beta}}(\tilde{\gamma}_M)^{\underline{\gamma}\underline{\delta}}. \tag{A.6}$$

Then it follows that

$$\frac{1}{2}\epsilon^{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}(\gamma^M)_{\underline{\gamma}\underline{\delta}} = (\tilde{\gamma}^M)^{\underline{\alpha}\underline{\beta}}, \quad \frac{1}{2}\epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}}(\tilde{\gamma}^M)^{\underline{\gamma}\underline{\delta}} = (\gamma^M)_{\underline{\alpha}\underline{\beta}}. \tag{A.7}$$

Since $\epsilon^{\underline{1234}} = 1 = -\epsilon^{12}\epsilon^{\dot{1}\dot{2}}$ and $\epsilon_{\underline{1234}} = 1 = -\epsilon_{12}\epsilon^{\dot{1}\dot{2}}$, these tensors are expressed in the 2-component notation as

$$\begin{aligned} \epsilon^{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} &= -\epsilon^{\alpha\beta}\epsilon_{\dot{\gamma}\dot{\delta}} - \epsilon^{\alpha\gamma}\epsilon_{\dot{\delta}\dot{\beta}} - \epsilon^{\alpha\delta}\epsilon_{\dot{\beta}\dot{\gamma}} - \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\gamma\delta} - \epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon^{\delta\beta} - \epsilon_{\dot{\alpha}\dot{\delta}}\epsilon^{\beta\gamma}, \\ \epsilon_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\delta}} &= -\epsilon_{\alpha\beta}\epsilon^{\dot{\gamma}\dot{\delta}} - \epsilon_{\alpha\gamma}\epsilon^{\dot{\delta}\dot{\beta}} - \epsilon_{\alpha\delta}\epsilon^{\dot{\beta}\dot{\gamma}} - \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\gamma\delta} - \epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon_{\delta\beta} - \epsilon^{\dot{\alpha}\dot{\delta}}\epsilon_{\beta\gamma}. \end{aligned} \tag{A.8}$$

A.2 Conjugation matrices

An 8-component Dirac spinor $\hat{\Psi}$ is decomposed into 4-component Weyl spinors as

$$\hat{\Psi} = \begin{pmatrix} \Psi_{\underline{\alpha}}^{(+)} \\ \Psi^{(-)\underline{\alpha}} \end{pmatrix}, \tag{A.9}$$

where the signs denote eigenvalues of Γ_7. The Dirac conjugate of $\hat{\Psi}$ is defined as

$$\bar{\hat{\Psi}} \equiv \hat{\Psi}^\dagger A, \tag{A.10}$$

where $\hat{A}$ satisfies

$$\hat{A}\Gamma^M\hat{A}^{-1} = (\Gamma^M)^\dagger. \tag{A.11}$$

The explicit form of $\hat{A}$ is given by

$$\hat{A} = \begin{pmatrix} & A \\ \tilde{A} & \end{pmatrix},$$

$$A^{\bar{\alpha}}{}_{\underline{\beta}} = \begin{pmatrix} & \epsilon^{\dot{\alpha}\dot{\beta}} \\ -\epsilon_{\alpha\beta} & \end{pmatrix}, \qquad \tilde{A}_{\bar{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} & \epsilon_{\dot{\alpha}\dot{\beta}} \\ -\epsilon^{\alpha\beta} & \end{pmatrix}, \tag{A.12}$$

where $\bar{\alpha}$ denotes a 4-component spinor index of the complex conjugate of the Weyl spinors.

Since $(\Gamma^M)^*$ form an equivalent representation of the Clifford algebra, there exists an invertible matrix $\hat{B}$ that satisfies

$$\hat{B}(\Gamma^M)^*\hat{B}^t = \Gamma^M. \tag{A.13}$$

An explicit form of $\hat{B}$ is given by

$$\hat{B} = \begin{pmatrix} B & \\ & B^\dagger \end{pmatrix}, \tag{A.14}$$

where

$$B_{\underline{\alpha}}{}^{\bar{\beta}} \equiv \begin{pmatrix} & \epsilon_{\alpha\beta} \\ -\epsilon^{\dot{\alpha}\dot{\beta}} & \end{pmatrix}, \qquad (B^*)_{\bar{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} & \epsilon_{\dot{\alpha}\dot{\beta}} \\ -\epsilon^{\alpha\beta} & \end{pmatrix}. \tag{A.15}$$

These matrices satisfy

$$BB^* = B^*B = -\mathbf{1}_4, \qquad B(\gamma^M)^*B^* = \gamma^M. \tag{A.16}$$

The charge conjugation matrix $\hat{C}$, which satisfies

$$\hat{C}\Gamma^M\hat{C}^{-1} = -(\Gamma^M)^t, \tag{A.17}$$

is constructed from $\hat{A}$ and $\hat{B}$ as

$$\hat{C} \equiv \hat{B}^\dagger\hat{A} = \begin{pmatrix} & C \\ \tilde{C} & \end{pmatrix}, \tag{A.18}$$

where

$$C^{\underline{\alpha}}_{\ \underline{\beta}} = \begin{pmatrix} -\delta^\alpha_{\ \beta} & \\ & -\delta_{\dot\alpha}^{\ \dot\beta} \end{pmatrix}, \quad \tilde{C}_{\underline{\alpha}}^{\ \underline{\beta}} = \begin{pmatrix} -\delta_\alpha^{\ \beta} & \\ & -\delta^{\dot\alpha}_{\ \dot\beta} \end{pmatrix}. \tag{A.19}$$

Thus the charge conjugation flips the 6D chirality.

The covariant conjugate of a spinor $\hat\Psi$ is defined as

$$\overline{\hat\Psi} \equiv \hat{B}\hat\Psi^*. \tag{A.20}$$

This operation is not a Z_2 transformation since

$$\overline{\overline{\hat\Psi}} = \overline{\hat{B}\hat\Psi^*} = \hat{B}(\hat{B}\hat\Psi^*)^* = \hat{B}\hat{B}^*\hat\Psi = -\hat\Psi. \tag{A.21}$$

For an SU(2)-doublet spinor $\hat\Psi^i$ ($i = 1,2$), a Z_2 transformation is obtained by combining the covariant conjugation with lowering the SU(2) index,

$$\hat\Psi^i \to \epsilon^{ij}\overline{\hat\Psi^j} = \epsilon^{ij}\hat{B}(\hat\Psi^j)^*. \tag{A.22}$$

Thus we can impose the SU(2)-Majorana condition,

$$\epsilon^{ij}\overline{\hat\Psi^j} = \hat\Psi^i \quad \Leftrightarrow \quad \overline{\hat\Psi^i} = \hat\Psi_i \equiv \epsilon_{ij}\hat\Psi^j. \tag{A.23}$$

Here the antisymmetric tensors ϵ^{ij} and ϵ_{ij} are chosen as $\epsilon^{12} = \epsilon_{21} = 1$. Since the covariant conjugation preserves the 6D chirality, we can impose this condition on 6D Weyl spinors. Namely, the SU(2)-Majorana-Weyl condition is expressed in the 4-component-spinor notation as

$$\left(\overline{\Psi^{(+)i}}\right)_{\underline{\alpha}} \equiv B_{\underline{\alpha}}^{\ \bar\beta}(\Psi^{(+)i*})_{\bar\beta} = \Psi^{(+)}_{i\underline{\alpha}} \equiv \epsilon_{ij}\Psi^{(+)j}_{\underline{\alpha}},$$
$$\left(\overline{\Psi^{(-)i}}\right)^{\underline{\alpha}} \equiv (B^\dagger)^{\underline{\alpha}}_{\ \bar\beta}(\Psi^{(-)i*})^{\bar\beta} = \Psi^{(-)\underline{\alpha}}_i \equiv \epsilon_{ij}\Psi^{(-)j\underline{\alpha}}. \tag{A.24}$$

In the two-component-spinor notation, the SU(2)-Majorana-Weyl spinors are expressed as

$$\Psi^{(+)1}_{\underline{\alpha}} = \begin{pmatrix} \chi^{(+)}_\alpha \\ \bar\lambda^{(+)\dot\alpha} \end{pmatrix}, \quad \Psi^{(+)2}_{\underline{\alpha}} = \begin{pmatrix} -\lambda^{(+)}_\alpha \\ \bar\chi^{(+)\dot\alpha} \end{pmatrix},$$
$$\Psi^{(-)1\underline{\alpha}} = \begin{pmatrix} \chi^{(-)\alpha} \\ \bar\lambda^{(-)}_{\dot\alpha} \end{pmatrix}, \quad \Psi^{(-)2\underline{\alpha}} = \begin{pmatrix} -\lambda^{(-)\alpha} \\ \bar\chi^{(-)}_{\dot\alpha} \end{pmatrix}. \tag{A.25}$$

A.3 Covariant spinor derivatives

We introduce the Grassmann coordinates $\Theta^{i\underline{\alpha}}$, which form an SU(2)-Majorana-Weyl spinor with the 6D chirality $-$. Then the covariant spinor derivatives are defined as

$$\mathcal{D}^i_{\underline{\alpha}} \equiv \epsilon^{ij}\frac{\partial}{\partial\Theta^{j\underline{\alpha}}} + i(\gamma^M)_{\underline{\alpha}\underline{\beta}}\Theta^{i\underline{\beta}}\partial_M = -\frac{\partial}{\partial\Theta^{\underline{\alpha}}_i} + i(\gamma^M)_{\underline{\alpha}\underline{\beta}}\Theta^{i\underline{\beta}}\partial_M, \tag{A.26}$$

which satisfies

$$\left\{\mathcal{D}^i_{\underline{\alpha}}, \mathcal{D}^j_{\underline{\beta}}\right\} = -2i\epsilon^{ij}(\gamma^M)_{\underline{\alpha}\underline{\beta}}\partial_M. \tag{A.27}$$

In the 2-component-spinor notation, $\Theta^{i\underline{\alpha}}$ are expressed as

$$\Theta^{1\underline{\alpha}} = \begin{pmatrix} \theta'^{\alpha} \\ -\bar{\theta}_{\dot{\alpha}} \end{pmatrix}, \quad \Theta^{2\underline{\alpha}} = \begin{pmatrix} \theta^{\alpha} \\ \bar{\theta}'_{\dot{\alpha}} \end{pmatrix}. \tag{A.28}$$

Then, the covariant spinor derivatives are expressed as

$$\mathcal{D}^{1}_{\underline{\alpha}} = \begin{pmatrix} D_{\alpha} \\ \bar{D}'^{\dot{\alpha}} \end{pmatrix}, \quad \mathcal{D}^{2}_{\underline{\alpha}} = \begin{pmatrix} -D'_{\alpha} \\ \bar{D}^{\dot{\alpha}} \end{pmatrix}, \tag{A.29}$$

where

$$\begin{aligned} D_{\alpha} &= \frac{\partial}{\partial \theta^{\alpha}} + i \left(\sigma^{\mu} \bar{\theta} \right)_{\alpha} \partial_{\mu} + \theta'_{\alpha} \bar{\partial}, \\ D'_{\alpha} &= \frac{\partial}{\partial \theta'^{\alpha}} + i \left(\sigma^{\mu} \bar{\theta}' \right)_{\alpha} \partial_{\mu} - \theta_{\alpha} \bar{\partial}, \\ \bar{D}^{\dot{\alpha}} &= \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} + i \left(\bar{\sigma}^{\mu} \theta \right)^{\dot{\alpha}} \partial_{\mu} + \bar{\theta}'^{\dot{\alpha}} \partial, \\ \bar{D}'^{\dot{\alpha}} &= \frac{\partial}{\partial \bar{\theta}'_{\dot{\alpha}}} + i \left(\bar{\sigma}^{\mu} \theta' \right)^{\dot{\alpha}} \partial_{\mu} - \bar{\theta}^{\dot{\alpha}} \partial, \end{aligned} \tag{A.30}$$

and $\partial \equiv \partial_4 - i \partial_5$. The algebra (A.27) is decomposed as

$$\begin{aligned} \left\{ D_{\alpha}, \bar{D}_{\dot{\alpha}} \right\} &= -2i\sigma^{\mu}_{\alpha\dot{\alpha}} \partial_{\mu}, & \left\{ D'_{\alpha}, \bar{D}'_{\dot{\alpha}} \right\} &= -2i\sigma^{\mu}_{\alpha\dot{\alpha}} \partial_{\mu}, \\ \left\{ D_{\alpha}, D'_{\beta} \right\} &= 2\epsilon_{\alpha\beta} \bar{\partial}, & \left\{ \bar{D}^{\dot{\alpha}}, \bar{D}'^{\dot{\beta}} \right\} &= 2\epsilon^{\dot{\alpha}\dot{\beta}} \partial, \\ \left\{ D_{\alpha}, D_{\beta} \right\} &= \left\{ D'_{\alpha}, D'_{\beta} \right\} = \left\{ D'_{\alpha}, \bar{D}_{\dot{\alpha}} \right\} = 0. \end{aligned} \tag{A.31}$$

We list some useful formulae following from this algebra.

$$\begin{aligned} \left[D_{\alpha}, \bar{D}^{2} \right] &= -4i\sigma^{\mu}_{\alpha\dot{\alpha}} \partial_{\mu} \bar{D}^{\dot{\alpha}}, & \left[\bar{D}_{\dot{\alpha}}, D^{2} \right] &= 4i\sigma^{\mu}_{\alpha\dot{\alpha}} \partial_{\mu} D^{\alpha}, \\ D'D &= DD' - 4\bar{\partial}, & \bar{D}'\bar{D} &= \bar{D}\bar{D}' - 4\partial, \\ D_{\alpha} D_{\beta} &= \frac{1}{2} \epsilon_{\alpha\beta} D^{2}, & \bar{D}_{\dot{\alpha}} \bar{D}_{\dot{\beta}} &= -\frac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{D}^{2}. \end{aligned} \tag{A.32}$$

B   Constraints on Φ_T and $\mathcal{W}_T^{\alpha}$

Here we derive the constraints in (3.49).

From the definition (3.48),

$$\begin{aligned} D^{\alpha} \mathcal{W}_{T\alpha} &= iD^{\alpha} \bar{D}^{2} \left(D_{\alpha} \bar{X} + 4\bar{\partial} Y_{\alpha} \right) \\ &= i\bar{D}_{\dot{\alpha}} D^{2} \bar{D}^{\dot{\alpha}} \bar{X} + 4i\bar{\partial} D^{\alpha} \bar{D}^{2} Y_{\alpha} \\ &= i\bar{D}_{\dot{\alpha}} D^{2} \left(\bar{Z}^{\dot{\alpha}} - 4\bar{\partial} \bar{Y}^{\dot{\alpha}} \right) + 4i\bar{\partial} D^{\alpha} \bar{D}^{2} Y_{\alpha} \\ &= -2\bar{\partial} \left(2i\bar{D}_{\dot{\alpha}} D^{2} \bar{Y}^{\dot{\alpha}} - 2iD^{\alpha} \bar{D}^{2} Y_{\alpha} \right) = -2\bar{\partial} \Phi_T. \end{aligned} \tag{B.1}$$

We have used that $\bar{D}^{\dot{\alpha}} \bar{X} = \bar{Z}^{\dot{\alpha}} - 4\bar{\partial} \bar{Y}^{\dot{\alpha}}$ and $D^{2} \bar{Z}^{\dot{\alpha}} = 0$.

The analyticity condition $\mathcal{D}_\alpha^{[1]}\mathcal{F}^{[2]} = 0$ is translated in the 2-component-spinor notation as

$$D_\alpha \mathcal{F}_1^{[2]} = 0, \qquad D_\alpha \mathcal{F}_0^{[2]} + D'_\alpha \mathcal{F}_1^{[2]} = 0,$$
$$\bar{D}'_{\dot\alpha} \mathcal{F}_1^{[2]} = 0, \qquad \bar{D}_{\dot\alpha} \mathcal{F}_1^{[2]} - \bar{D}'_{\dot\alpha} \mathcal{F}_0^{[2]} = 0. \tag{B.2}$$

Thus the $N = 1$ superfields $\mathcal{F}_0^{[2]}|$ and $\mathcal{F}_1^{[2]}|$ satisfy the following constraints.

$$D_\alpha \mathcal{F}_1^{[2]}| = 0,$$
$$D^2 \mathcal{F}_0^{[2]}| = -DD'\mathcal{F}_1^{[2]}| = -\left(D'D + 4\bar\partial\right)\mathcal{F}_1^{[2]}| = -4\bar\partial\mathcal{F}_1^{[2]}|. \tag{B.3}$$

From the explicit expressions in (3.47), we can see that the first constraint is satisfied. As for the second constraint, we can show that

$$D^2 \mathcal{F}_0^{[2]}| = -4\bar\partial\mathcal{F}_1^{[2]}| - \left(D^2\bar{D}_{\dot\alpha}\Phi_T + 4\bar\partial\bar{\mathcal{W}}_{T\dot\alpha}\right)\bar{\mathcal{W}}^{\dot\alpha}. \tag{B.4}$$

We have used the constraint (B.1). Comparing this with the second constraint in (B.3), we obtain

$$D^2 \bar{D}_{\dot\alpha}\Phi_T = -4\bar\partial\bar{\mathcal{W}}_{T\dot\alpha}. \tag{B.5}$$

C Derivation of 5D Lagrangian

We derive (3.64) from (3.56) after the dimensional reduction to 5D. By using (3.58) and (3.61), we can calculate

$$2\Phi_T \left\{ V\left(\Box_4 \mathcal{P}_{\mathrm{T}} + \partial_4^2\right)V + 2\left(\partial_4 V - \bar\Sigma\right)\left(\partial_4 V - \Sigma\right)\right\}$$

$$= \left(\partial_4 V_T - \Sigma_T - \bar\Sigma_T\right)VD^\alpha\mathcal{W}_\alpha - 2\left(\partial_4\Phi_T V + \Phi_T\partial_4 V\right)\partial_4 V$$

$$+4\Phi_T \left\{(\partial_4 V)^2 - \partial_4 V\left(\Sigma + \bar\Sigma\right) + \bar\Sigma\Sigma\right\}$$

$$= \left\{\frac{1}{2}\partial_4 V_T V D^\alpha\mathcal{W}_\alpha - \bar\Sigma_T V D^\alpha\mathcal{W}_\alpha + \frac{1}{2}D^\alpha\mathcal{W}_{T\alpha}V\partial_4 V + \text{h.c.}\right\}$$

$$+2\Phi_T \left\{(\partial_4 V)^2 - 2\partial_4 V\left(\Sigma + \bar\Sigma\right) + 2\bar\Sigma\Sigma\right\}$$

$$= \left\{-\frac{1}{2}D^\alpha\left(\partial_4 V_T V\right)\mathcal{W}_\alpha + \bar\Sigma_T D^\alpha V\mathcal{W}_\alpha - \frac{1}{2}D^\alpha\left(V\partial_4 V\right)\mathcal{W}_{T\alpha} + \text{h.c.}\right\}$$

$$+2\Phi_T\left(\partial_4 V - \Sigma - \bar\Sigma\right)^2 - 2\Phi_T\left(\Sigma^2 + \bar\Sigma^2\right). \tag{C.1}$$

We have also used $D^\alpha\mathcal{W}_\alpha = \bar{D}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}$, and dropped total derivative terms. Thus, after the dimensional reduction to 5D, (3.56) becomes

$$\mathcal{L}_{\mathrm{VT}}^{(5D)} = -\int d^2\theta \left\{2\Sigma\mathcal{W}\mathcal{W}_T + \frac{1}{4}\bar{D}^2\left(\Phi_T D^\alpha V\mathcal{W}_\alpha + \partial_4 V D^\alpha V\mathcal{W}_{T\alpha}\right)\right\} + \text{h.c.}$$

$$+\int d^4\theta \, 2\Phi_T \left\{V\left(\Box_4\mathcal{P}_{\mathrm{T}} + \partial_4^2\right)V + 2\left(\partial_4 V - \bar\Sigma\right)\left(\partial_4 V - \Sigma\right)\right\}$$

$$
= - \int d^2\theta \; 2\Sigma \mathcal{W} \mathcal{W}_T + \text{h.c.}
$$

$$
+ \int d^4\theta \; \left\{ \left(\partial_4 V_T - \Sigma_T - \bar{\Sigma}_T \right) D^\alpha V \mathcal{W}_\alpha + \partial_4 V D^\alpha V \mathcal{W}_{T\alpha} + \text{h.c.} \right\}
$$

$$
+ \int d^4\theta \; \left\{ \left(-\frac{1}{2} D^\alpha \left(\partial_4 V_T V \right) \mathcal{W}_\alpha + \bar{\Sigma}_T D^\alpha V \mathcal{W}_\alpha - \frac{1}{2} D^\alpha \left(V \partial_4 V \right) \mathcal{W}_{T\alpha} + \text{h.c.} \right) \right.
$$

$$
\left. + 2\Phi_T \left(\partial_4 V - \Sigma - \bar{\Sigma} \right)^2 - 2\Phi_T \left(\Sigma^2 + \bar{\Sigma}^2 \right) \right\}
$$

$$
= - \int d^2\theta \; \left(2\Sigma \mathcal{W} \mathcal{W}_T + \Sigma_T \mathcal{W}^2 \right) + \text{h.c.}
$$

$$
+ \int d^4\theta \left\{ \frac{1}{2} \left(\partial_4 V_T D^\alpha V - \partial_4 D^\alpha V_T V \right) \mathcal{W}_\alpha + \frac{1}{2} \left(\partial_4 V D^\alpha V - \partial_4 D^\alpha V V \right) \mathcal{W}_{T\alpha} + \text{h.c.} \right\}
$$

$$
+ \int d^4\theta \; 2 \left(\partial_4 V_T - \Sigma_T - \bar{\Sigma}_T \right) \left(\partial_4 V - \Sigma - \bar{\Sigma} \right)^2. \tag{C.2}
$$

We have dropped total derivative terms, and used that

$$
\int d^4\theta \; \Phi_T \Sigma^2 = -\frac{1}{4} \int d^2\theta \; \bar{D}^2 (\Phi_T \Sigma^2) = -\frac{1}{4} \int d^2\theta \; (\bar{D}^2 \Phi_T) \Sigma^2 = 0. \tag{C.3}
$$

Using (A.31) and (A.32), we can show that

$$
\left(\partial_4 V_T D^\alpha V - \partial_4 D^\alpha V_T V \right) \mathcal{W}_\alpha + \left(\partial_4 V D^\alpha V - \partial_4 D^\alpha V V \right) \mathcal{W}_{T\alpha} + \text{h.c.}
$$

$$
= 2 \left(\partial_4 V D^\alpha V_T - \partial_4 D^\alpha V V_T \right) \mathcal{W}_\alpha + \text{h.c.}, \tag{C.4}
$$

up to total derivatives. Thus, (C.2) is rewritten as

$$
\mathcal{L}_{\text{VT}}^{(5D)} = - \int d^2\theta \; \left(2\Sigma \mathcal{W} \mathcal{W}_T + \Sigma_T \mathcal{W}^2 \right) + \text{h.c.}
$$

$$
+ \int d^4\theta \left\{ \frac{1}{3} \left(\partial_4 V_T D^\alpha V - \partial_4 D^\alpha V_T V \right) \mathcal{W}_\alpha + \frac{1}{3} \left(\partial V D^\alpha V_T - \partial_4 D^\alpha V V_T \right) \mathcal{W}_\alpha \right.
$$

$$
\left. + \frac{1}{3} \left(\partial_4 V D^\alpha V - \partial_4 D^\alpha V V \right) \mathcal{W}_{T\alpha} + \text{h.c.} \right\}
$$

$$
+ \int d^4\theta \; 2 \left(\partial_4 V_T - \Sigma_T - \bar{\Sigma}_T \right) \left(\partial_4 V - \Sigma - \bar{\Sigma} \right)^2. \tag{C.5}
$$

If we relabel (Σ, V) and (Σ_T, V_T) as (Σ^1, V^1) and (Σ^2, V^2), this is expressed as (3.64).

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