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Non-supersymmetric Wilson loop in $\mathcal{N} = 4$ SYM and defect 1d CFT

Matteo Beccaria,[a] **Simone Giombi**[b] **and Arkady A. Tseytlin**[c,1]

[a] *Dipartimento di Matematica e Fisica Ennio De Giorgi, Università del Salento & INFN, Via Arnesano, 73100 Lecce, Italy*

[b] *Department of Physics, Princeton University, Princeton, NJ 08544, U.S.A.*

[c] *Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.*

E-mail: matteo.beccaria@le.infn.it, sgiombi@princeton.edu, tseytlin@imperial.ac.uk

Abstract: Following Polchinski and Sully (arXiv:1104.5077), we consider a generalized Wilson loop operator containing a constant parameter ζ in front of the scalar coupling term, so that $\zeta = 0$ corresponds to the standard Wilson loop, while $\zeta = 1$ to the locally supersymmetric one. We compute the expectation value of this operator for circular loop as a function of ζ to second order in the planar weak coupling expansion in $\mathcal{N} = 4$ SYM theory. We then explain the relation of the expansion near the two conformal points $\zeta = 0$ and $\zeta = 1$ to the correlators of scalar operators inserted on the loop. We also discuss the $\mathrm{AdS}_5 \times S^5$ string 1-loop correction to the strong-coupling expansion of the standard circular Wilson loop, as well as its generalization to the case of mixed boundary conditions on the five-sphere coordinates, corresponding to general ζ. From the point of view of the defect CFT_1 defined on the Wilson line, the ζ-dependent term can be seen as a perturbation driving a RG flow from the standard Wilson loop in the UV to the supersymmetric Wilson loop in the IR. Both at weak and strong coupling we find that the logarithm of the expectation value of the standard Wilson loop for the circular contour is larger than that of the supersymmetric one, which appears to be in agreement with the 1d analog of the F-theorem.

Keywords: AdS-CFT Correspondence, Wilson, 't Hooft and Polyakov loops, Supersymmetric Gauge Theory

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[1] Also at Lebedev Institute, Moscow.

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1 Introduction

The expectation value of the Wilson loop (WL) operator $\langle \mathrm{Tr}\, \mathcal{P} e^{i\int A}\rangle$ is an important observable in any gauge theory. In $\mathcal{N} = 4$ Super Yang-Mills (SYM), the Wilson-Maldacena loop (WML) [1, 2], which contains an extra scalar coupling making it locally-supersymmetric, was at the center of attention, but the study of the ordinary, "non-supersymmetric" WL is also of interest [3, 4] in the context of the AdS/CFT correspondence. Computing the large N expectation value of the standard WL for some simple contours (like circle or cusp) should produce new non-trivial functions of the 't Hooft coupling $\lambda = g^2 N$ which are no longer controlled by supersymmetry but may still be possible to determine using the underlying integrability of the theory. Another motivation comes from considering correlation functions of local operators inserted along the WL: this should produce a new example of AdS_2/CFT_1 duality, similar but different from the one recently discussed in the WML case [5, 6]. In the latter case, correlators of local operators on the 1/2-BPS Wilson line have a $OSp(4^*|4)$ 1d superconformal symmetry, while in the ordinary WL case one expects a non-supersymmetric "defect" CFT_1 with $SO(3)\times SO(6)$ "internal" symmetry.

On general grounds, for the standard WL defined for a smooth contour one should find that (i) all power divergences (that cancel in the WML case) exponentiate and factorize [7–12] and (ii) all logarithmic divergences cancel as the gauge coupling is not running in $\mathcal{N}=4$ SYM theory. Thus its large N expectation value should produce a nontrivial finite function of λ (after factorising power divergences, or directly, if computed in dimensional regularization).

It is useful to consider a 1-parameter family of Wilson loop operators with an arbitrary coefficient ζ in front of the scalar coupling which interpolates between the WL ($\zeta=0$) and the WML ($\zeta=1$) cases [4]

$$W^{(\zeta)}(C)=\frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\,\exp\oint_C d\tau\,\big[i\,A_\mu(x)\,\dot{x}^\mu+\zeta\Phi_m(x)\,\theta^m\,|\dot{x}|\,\big],\qquad \theta_m^2=1\,. \tag{1.1}$$

We may choose the direction θ_m of the scalar coupling in (1.1) to be along 6-th direction, i.e. $\Phi_m\theta^m=\Phi_6$. Below we shall sometimes omit the expectation value brackets using the notation

$$\mathrm{WL}:\quad \langle W^{(0)}\rangle\equiv W^{(0)}\,,\qquad\qquad \mathrm{WML}:\quad \langle W^{(1)}\rangle\equiv W^{(1)}\,. \tag{1.2}$$

Ignoring power divergences, for generic ζ the expectation value $\langle W^{(\zeta)}\rangle$ for a smooth contour may have additional logarithmic divergences but it should be possible to absorb them into a renormalization of the coupling ζ, i.e.[1]

$$\langle W^{(\zeta)}\rangle\equiv W\big(\lambda;\zeta(\mu),\mu\big)\,,\qquad\qquad \mu\frac{\partial}{\partial\mu}W+\beta_\zeta\frac{\partial}{\partial\zeta}W=0\,, \tag{1.3}$$

where μ is a renormalization scale and the beta-function is, to leading order at weak coupling [4]

$$\beta_\zeta=\mu\frac{d\zeta}{d\mu}=-\frac{\lambda}{8\pi^2}\zeta(1-\zeta^2)+\mathcal{O}(\lambda^2)\,. \tag{1.4}$$

The WL and WML cases in (1.2) are the two conformal fixed points $\zeta=0$ and $\zeta=1$ where the logarithmic divergences cancel out automatically.[2] Given that the SYM action is invariant under the change of sign of Φ_m the fixed point points $\zeta=\pm1$ are equivalent (we may resstrict ζ to be non-negative in (1.1)).

Our aim below will be to compute the leading weak and strong coupling terms in the WL expectation value for a circular contour in the planar limit. As is well known, the

[1] Here there is an analogy with a partition function of a renormalizable QFT: if g_b is bare coupling depending on cutoff Λ one has $Z_\mathrm{b}(g_\mathrm{b}(\Lambda),\Lambda)=Z(g(\mu),\mu)$, $\mu\frac{dZ}{d\mu}=\mu\frac{\partial Z}{\partial\mu}+\beta(g)\frac{\partial Z}{\partial g}=0$, $\beta=\mu\frac{dg}{d\mu}$. In the present case the expectation value depends on μ via μR where R is the radius of the circle (which we often set to 1). A natural choice of renormalization point is then $\mu=R^{-1}$.

[2] As the expectation value of the standard WL has no logarithmic divergences, combined with the fact that the straight line (or circle) preserves a subgroup of 4d conformal group this implies that one should have a 1d conformal $SL(2,R)$ invariance for the corresponding CFT on the line for all λ.

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circular WML expectation value can be found exactly due to underlying supersymmetry; in the planar limit [13–15] (see also [16])

$$W^{(1)}(\text{circle}) = \frac{2}{\sqrt{\lambda}}\, I_1(\sqrt{\lambda}) = \begin{cases} 1+\frac{\lambda}{8}+\frac{\lambda^2}{192}+\cdots\,, & \lambda \ll 1\,, \\ \sqrt{\frac{2}{\pi}}\,\frac{1}{(\sqrt{\lambda})^{3/2}}\,e^{\sqrt{\lambda}}\left(1-\frac{3}{8\sqrt{\lambda}}+\cdots\right), & \lambda \gg 1\,. \end{cases} \tag{1.5}$$

For a straight line the expectation value of the WML is 1, and then for the circle its non-trivial value can be understood as a consequence of an anomaly in the conformal transformation relating the line to the circle [14]. As this anomaly is due to an IR behaviour of the vector field propagator [14], one may wonder if the same anomaly argument may apply to the WL as well. Indeed, in this case ($\zeta = 0$) there are no additional logarithmic divergences and then after all power divergences are factorized or regularized away one gets $W^{(0)}(\text{line}) = 1$; then the finite part of $W^{(0)}(\text{circle})$ may happen to be the same as in the WML case (1.5).[3]

Some indication in favour of this is that the leading strong and weak coupling terms in the circular WL happen to be the same as in the WML case. The leading strong-coupling term is determined by the volume of the same minimal surface (AdS_2 with circle as a boundary) given by $2\pi(\frac{1}{a}-1)$ and (after subtracting the linear divergence) thus has the universal form $\langle W^{(\zeta)}\rangle \sim e^{\sqrt{\lambda}}$. At weak coupling, the circular WL and WML also have the same leading-order expectation value (again after subtracting linear divergence) $\langle W^{(\zeta)}\rangle = 1 + \frac{1}{8}\lambda + O(\lambda^2)$.

However, as we shall see below, the subleading terms in WL in both weak and strong coupling expansion start to differ from the WML values, i.e. $\langle W^{(\zeta)}(\text{circle})\rangle$ develops dependence on ζ. This implies, in particular, that the conformal anomaly argument of [14] does not apply for $\zeta = 0$.[4]

Explicitly, we shall find that at weak coupling (in dimensional regularization)

$$\langle W^{(\zeta)}\rangle = 1 + \frac{1}{8}\lambda + \left[\frac{1}{192} + \frac{1}{128\,\pi^2}(1-\zeta^2)^2\right]\lambda^2 + \mathcal{O}(\lambda^3)\,. \tag{1.6}$$

This interpolates between the WML value in (1.5) and the WL value ($\zeta = 0$)

$$W^{(0)} = 1 + \frac{1}{8}\lambda + \left(\frac{1}{192} + \frac{1}{128\,\pi^2}\right)\lambda^2 + \mathcal{O}(\lambda^3)\,. \tag{1.7}$$

Note that the 2-loop correction in (1.7) to the WML value in (1.5) has a different transcendentality; it would be very interesting to find the all-order generalization of (1.7), i.e. the counterpart of the exact Bessel function expression in (1.5) in the standard WL case. It is tempting to conjecture that the highest transcendentality part of $\langle W\rangle$ at each order in the

[3]The conjecture that the circular WL may have the same value as the locally-supersymmetric WML one runs, of course, against the derivation of the expectation value of the latter based on localization [15] as there is no reason why the localisation argument should apply in the standard WL case.

[4]This may be attributed to the presence of extra (power) divergences that do not cancel automatically in the standard WL case. For generic ζ there are also additional logarithmic divergences that break conformal invariance.

perturbative expansion is the same for supersymmetric and non-supersymmetric Wilson loops and hence given by (1.5).

The expression (1.6) passes several consistency checks. The UV finiteness of the two-loop λ^2 term is in agreement with ζ-independence of the one-loop term (cf. (1.3), (1.4) implying that UV logs should appear first at the next λ^3 order). The derivative of (log of) (1.6) over ζ is proportional to the beta-function (1.4)

$$\frac{\partial}{\partial\zeta}\log\langle W^{(\zeta)}\rangle = \mathcal{C}\,\beta_\zeta\,, \qquad\qquad \mathcal{C}=\frac{\lambda}{4}+\mathcal{O}(\lambda^2)\,, \tag{1.8}$$

where $\mathcal{C}=\mathcal{C}(\lambda,\zeta)$ should not have zeroes. This implies that the conformal points $\zeta=1$ and $\zeta=0$ are extrema (minimum and maximum) of $\langle W^{(\zeta)}\rangle$. This is consistent with the interpretation of $\langle W^{(\zeta)}\rangle$ as a 1d partition function on S^1 that may be computed in conformal perturbation theory near $\zeta=1$ or $\zeta=0$ conformal points. Indeed, eq.(1.8) may be viewed as a special $d=1$ case of the relation $\frac{\partial F}{\partial g_i}=\mathcal{C}^{ij}\beta_j$ for free energy F on a sphere S^d computed by perturbing a CFT$_d$ by a linear combination of operators g_iO^i (see, e.g., [17, 18]).

In the present case, the flow [4] is driven by the scalar operator $\Phi_m\theta^m=\Phi_6$ in (1.1) restricted to the line, and the condition $\frac{\partial}{\partial\zeta}\langle W^{(\zeta)}\rangle\big|_{\zeta=0,1}=0$ means that its one-point function vanishes at the conformal points, as required by the 1d conformal invariance. The parameter ζ may be viewed as a "weakly relevant" (nearly marginal up to $\mathcal{O}(\lambda)$ term, cf. (1.4)) coupling constant running from $\zeta=0$ in the UV (the ordinary Wilson loop) to $\zeta=1$ in the IR (the supersymmetric Wilson loop). Note that our result (1.6) implies that

$$\log\langle W^{(0)}\rangle \;>\; \log\langle W^{(1)}\rangle\;. \tag{1.9}$$

Hence, viewing $\langle W^{(\zeta)}\rangle=Z_{S^1}$ as a partition function of a 1d QFT on the circle, this is precisely consistent with the F-theorem [17–21], which in $d=1$ (where it is analogous to the g-theorem [22, 23] applying to the boundary of a 2d theory) implies

$$\widetilde{F}_{\rm UV}>\widetilde{F}_{\rm IR}\,, \qquad\qquad \widetilde{F}\equiv\sin\tfrac{\pi d}{2}\,\log Z_{S^d}\Big|_{d=1}=\log Z_{S^1}=-F\,. \tag{1.10}$$

Moreover, we see that $\langle W^{(\zeta)}\rangle$ decreases monotonically as a function of ζ from the non-supersymmetric to the supersymmetric fixed point.

The second derivative of $\langle W^{(\zeta)}\rangle$ which from (1.8) is thus proportional to the derivative of the beta-function (1.4)

$$\frac{\partial^2}{\partial\zeta^2}\log\langle W^{(\zeta)}\rangle\bigg|_{\zeta=0,1}=\mathcal{C}\,\frac{\partial\beta_\zeta}{\partial\zeta}\bigg|_{\zeta=0,1}\,, \tag{1.11}$$

should, on the other hand, be given by the integrated 2-point function of Φ_6 restricted to the line and should thus be determined by the corresponding anomalous dimensions. Indeed, $\frac{\partial\beta_\zeta}{\partial\zeta}\big|_{\zeta=0,1}$ reproduces [4] the anomalous dimensions [3] of Φ_6 at the $\zeta=1$ and $\zeta=0$ conformal points

$$\begin{aligned}\Delta(\zeta)-1=\frac{\partial\beta_\zeta}{\partial\zeta}&=\frac{\lambda}{8\pi^2}(3\zeta^2-1)+\mathcal{O}(\lambda^2)\,,\\ \Delta(1)=1+\frac{\lambda}{4\pi^2}+\ldots\,, &\qquad\qquad \Delta(0)=1-\frac{\lambda}{8\pi^2}+\ldots\,.\end{aligned} \tag{1.12}$$

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Again, this is a special case of a general relation between the second derivative of free energy on S^d at conformal point and anomalous dimensions found in conformal perturbation theory. We shall explicitly verify this relation between $\frac{\partial^2}{\partial\zeta^2}\langle W^{(\zeta)}\rangle\big|_{\zeta=0,1}$ and the integrated 2-point function of Φ_6 inserted into the circular Wilson loop in section 3 below.

The interpretation of $\langle W^{(\zeta)}\rangle$ as a partition function of an effective 1d QFT is strongly supported by its strong-coupling representation as the AdS$_5 \times S^5$ string theory partition function on a disc with mixed boundary conditions [4] for S^5 coordinates (in particular, Dirichlet for $\zeta = 1$ and Neumann for $\zeta = 0$ [3]). As we will find in section 4, in contrast to the large λ asymptotics of the WML $\langle W^{(1)}\rangle \sim (\sqrt{\lambda})^{-3/2} e^{\sqrt{\lambda}} + \dots$ in (1.5), in the standard WL one gets

$$\langle W^{(0)}\rangle \sim \sqrt{\lambda}\, e^{\sqrt{\lambda}} + \dots \,, \tag{1.13}$$

so that the F-theorem inequality (1.9), (1.10) is satisfied also at strong coupling. At strong coupling, the counterpart of the Φ_6 perturbation near the $\zeta = 0$ conformal point is an extra boundary term (which to leading order is quadratic in S^5 coordinates) added to the string action with Neumann boundary condition to induce the boundary RG flow to the other conformal point.[5] The counterpart of ζ in (1.1) is a (relevant) coupling $\varkappa = \mathrm{f}(\zeta;\lambda)$ (which is 0 for $\zeta = 0$ and ∞ for $\zeta = 1$) has the beta function (see 4.2) $\beta_\varkappa = (-1 + \frac{5}{\sqrt{\lambda}})\varkappa + \dots$. This implies that strong-coupling dimensions of Φ_6 near the two conformal points should be (in agreement with [3, 6])

$$\Delta - 1 = \pm\Big(-1 + \frac{5}{\sqrt{\lambda}} + \dots\Big), \quad \text{i.e.}\ \ \Delta(0) = \frac{5}{\sqrt{\lambda}} + \dots\,, \qquad \Delta(1) = 2 - \frac{5}{\sqrt{\lambda}} + \dots\,. \tag{1.14}$$

This paper is organized as follows. In section 2 we shall compute the two leading terms in the planar weak-coupling expansion of the circular WL. The structure of the computation will be similar to the one in the WML case in [13] (see also [24]) but now the integrands (and thus evaluating the resulting path-ordered integrals) will be substantially more complicated. We shall then generalize to any value of ζ in (1.1) obtaining the expression in (1.6).

In section 3 we shall elaborate on the relation between the expansion of the generalized WL (1.6) near the conformal points and the correlators of scalar operators inserted on the loop. In section 4 we shall consider the strong-coupling (string theory) computation of the circular WL to 1-loop order in AdS$_5 \times S^5$ superstring theory generalizing the previous discussions in the WML case. We shall also discuss the general ζ case in section 4.2.

Some concluding remarks will be made in section 5. In appendix A we shall comment on cutoff regularization. In appendix B we shall explain different methods of computing path-ordered integrals on a circle appearing in the 2-loop ladder diagram contribution to the generalized WL.

[5] In particular, the boundary term is independent of the fermionic fields. When restricted to the AdS$_2$ minimal surface dual to the Wilson loop, they will be assumed to have the usual unitary $\Delta = 3/2$ boundary behaviour along the whole RG flow. Instead, for the S^5 scalars the boundary deformation induces unitary mixed boundary conditions and only in Dirichlet case we have unbroken supersymmetry [4].

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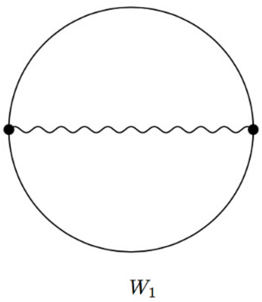

Figure 1. Gauge field exchange diagram contributing the standard Wilson loop at the leading order. In the Wilson-Maldacena loop case there is an additional scalar exchange contribution.

2 Weak coupling expansion

Let us now consider the weak-coupling ($\lambda = g^2 N \ll 1$) expansion in planar $\mathcal{N} = 4$ SYM theory and compute the first two leading terms in the expectation value for the generalized circular Wilson loop (1.1)

$$\langle W^{(\zeta)}\rangle = 1 + \lambda\, W_1^{(\zeta)} + \lambda^2\, W_2^{(\zeta)} + \cdots . \tag{2.1}$$

We shall first discuss explicitly the standard Wilson loop $W^{(0)}$ in (1.2) comparing it to the Wilson-Maldacena loop $W^{(1)}$ case in [13] and then generalize to an arbitrary value of the parameter ζ.

2.1 One-loop order

The perturbative computation of the WML was discussed in [13] (see also [24]) that we shall follow and generalize. The order λ contribution is[6]

$$W_1^{(1)}(C) = \frac{1}{(4\,\pi)^2} \oint_C d\tau_1 d\tau_2\, \frac{|\dot{x}(\tau_1)|\,|\dot{x}(\tau_2)| - \dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{|x(\tau_1) - x(\tau_2)|^2} . \tag{2.2}$$

Here the term $\dot{x}(\tau_1)\cdot\dot{x}(\tau_2)$ comes from the vector exchange (see figure 1) and the term $|\dot{x}(\tau_1)|\,|\dot{x}(\tau_2)|$ from the scalar exchange. This integral is finite for a smooth loop. In particular, for the straight line $x^\mu(\tau) = (\tau, 0, 0, 0)$, the numerator in $W_1^{(1)}$ is zero and thus

$$W_1^{(1)}(\text{line}) = 0. \tag{2.3}$$

For the circular loop, $x^\mu(\tau) = (\cos\tau, \sin\tau, 0, 0)$, the integrand in (2.2) is constant

$$\frac{|\dot{x}(\tau_1)|\,|\dot{x}(\tau_2)| - \dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{|x(\tau_1) - x(\tau_2)|^2} = \frac{1}{2} \tag{2.4}$$

and thus, in agreement with (1.5), (2.1)

$$W_1^{(1)}(\text{circle}) = \frac{1}{(4\pi)^2}\,(2\pi)^2\,\frac{1}{2} = \frac{1}{8} . \tag{2.5}$$

[6]There is a misprint in the overall coefficient in [13] corrected in [24].

The analog of (2.2) in the case of the standard WL is found by omitting the scalar exchange $|\dot{x}(\tau_1)|\,|\dot{x}(\tau_2)|$ term in the integrand. The resulting integral will have linear divergence (see appendix A) that can be factorized or automatically ignored using dimension regularization for the vector propagator with parameter $\omega = 2-\varepsilon \to 2$. If we replace the dimension 4 by $d = 2\omega \equiv 4-2\varepsilon$ the standard Euclidean 4d propagator becomes

$$\Delta(x) = (-\partial^2)^{-1} = \frac{\Gamma(\omega-1)}{4\pi^\omega}\frac{1}{|x|^{2\omega-2}} \ . \tag{2.6}$$

Then

$$W_1^{(0)} = \frac{1}{(4\,\pi)^2}\oint d\tau_1 d\tau_2\,\frac{-\dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{|x(\tau_1)-x(\tau_2)|^2} \ \to\ \frac{\Gamma(\omega-1)}{16\pi^\omega}\oint d\tau_1 d\tau_2\,\frac{-\dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{|x(\tau_1)-x(\tau_2)|^{2\omega-2}} \, . \tag{2.7}$$

In the infinite line case we get $(L\to\infty)$[7]

$$\int_0^L d\tau_1 \int_0^L d\tau_2\,\frac{1}{|\tau_1-\tau_2|^{2\omega-2}} = 2\int_0^L d\tau\,\frac{L-\tau}{\tau^{2\,(\omega-1)}} = \frac{L^{4-2\omega}}{2-\omega}\,\frac{1}{3-2\omega} \to 0 \ . \tag{2.8}$$

The formal integral here is linearly divergent. If we use dimensional regularization to regulate both UV and IR divergences (analytically continuing from $\omega>2$ region) we get as in (2.3)

$$W_1^{(0)}(\text{line}) = 0. \tag{2.9}$$

In the case of a circle, we may use (2.2), (2.5) to write $(\omega \equiv 2-\varepsilon \to 2)$

$$\begin{aligned} W_1^{(0)}(\text{circle}) &= W_1^{(1)}(\text{circle}) - \frac{\Gamma(\omega-1)}{16\pi^\omega}\oint d\tau_1 d\tau_2\,\frac{1}{|x(\tau_1)-x(\tau_2)|^{2\omega-2}} \\ &= \frac{1}{8} - \frac{\Gamma(\omega-1)}{2^{2\omega+2}\pi^\omega}\oint d\tau_1 d\tau_2\left[\sin^2\frac{\tau_{12}}{2}\right]^{1-\omega} . \end{aligned} \tag{2.10}$$

The integral here may be computed, e.g., by using the master-integral in eq. (G.1) of [25][8]

$$\begin{aligned} \mathcal{M}(a,b,c) &\equiv \oint d\tau_1 d\tau_2 d\tau_3\left[\sin^2\frac{\tau_{12}}{2}\right]^a\left[\sin^2\frac{\tau_{23}}{2}\right]^b\left[\sin^2\frac{\tau_{13}}{2}\right]^c \\ &= 8\,\pi^{3/2}\,\frac{\Gamma\left(\frac{1}{2}+a\right)\Gamma\left(\frac{1}{2}+b\right)\Gamma\left(\frac{1}{2}+c\right)\Gamma(1+a+b+c)}{\Gamma(1+a+c)\Gamma(1+b+c)\Gamma(1+a+b)} \, , \end{aligned} \tag{2.11}$$

i.e.

$$\oint d\tau_1 d\tau_2\left[\sin^2\frac{\tau_{12}}{2}\right]^{1-\omega} = \frac{1}{2\pi}\,\mathcal{M}(1-\omega,0,0) = \frac{4\,\pi^{3/2}\,\Gamma\left(-\frac{1}{2}+\varepsilon\right)}{\Gamma(\varepsilon)} = -8\pi^2\varepsilon + \mathcal{O}(\varepsilon^2). \tag{2.12}$$

Plugging this into (2.10), we get the same result as in (2.5):

$$W_1^{(0)}(\text{circle}) = \frac{1}{8} \ . \tag{2.13}$$

Thus the leading-order expectation values for the WML and WL are the same for both the straight line and the circle.

[7] We use that $\int_0^L d\tau_1 \int_0^L d\tau_2\, f(|\tau_1-\tau_2|) = 2\int_0^L d\tau\ (L-\tau)\, f(\tau)$.

[8] Alternative direct methods of computing similar integrals are discussed in appendix B. We also note that such 2-point and 3-point integrals can be viewed as a special $d=1$ case of the conformal integrals on S^d used in [17, 26].

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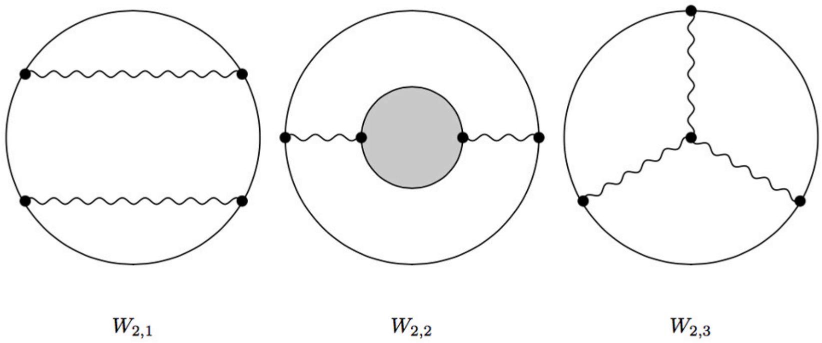

Figure 2. Order λ^2 contributions to the standard Wilson loop. The middle diagram contains the full self-energy 1-loop correction in SYM theory (with vector, ghost, scalar and fermion fields in the loop). For the Wilson-Maldacena loop there are additional diagrams with scalar propagators instead of some of the vector ones.

2.2 Two-loop order

At order λ^2 there are three types of planar contributions to the Wilson loop in (2.14) shown in figure 2 that we shall denote as

$$W_2^{(\zeta)} = W_{2,1}^{(\zeta)} + W_{2,2}^{(\zeta)} + W_{2,3}^{(\zeta)} \ . \tag{2.14}$$

In the WML case it was found in [13] that the ladder diagram contribution $W_{2,1}^{(1)}$ is finite. While the self-energy part $W_{2,2}^{(1)}$ and the internal-vertex part $W_{2,3}^{(1)}$ are separately logarithmically divergent (all power divergences cancel out in WML case), their sum is finite; moreover, the finite part also vanishes in 4 dimensions (in Feynman gauge)

$$W_{2,2}^{(1)} + W_{2,3}^{(1)} = 0 \ . \tag{2.15}$$

In the WL case, using dimensional regularization to discard power divergences, we find that the ladder diagram $W_{2,1}^{(0)}$ in figure 2 has a logarithmic singularity (i.e. a pole in $\varepsilon = 2 - \omega$). The same is true for both the self-energy diagram $W_{2,2}^{(0)}$ and the internal-vertex diagram $W_{2,3}^{(0)}$. However, their sum in (2.14) turns out to be finite (in agreement with the general expectation for a conformal WL operator in a theory where the gauge coupling is not running).[9]

Let us now discuss each of these contributions in turn.

2.2.1 Ladder contribution

The planar ladder diagram $W_{2,1}$ in figure 2 arises from the quartic term in the expansion of the Wilson loop operator (1.1). It is convenient to split the integration region into 4! ordered domains, i.e. $\tau_1 > \tau_2 > \tau_3 > \tau_4$ and similar ones. Before the Wick contractions, all

[9] If one uses power UV cutoff $a \to 0$ the remaining power divergences universally factorize as an exponential factor $\exp(-k\,\frac{L}{a})$ where L is the loop length. This can be interpreted as a mass renormalization of a test particle moving along the loop.

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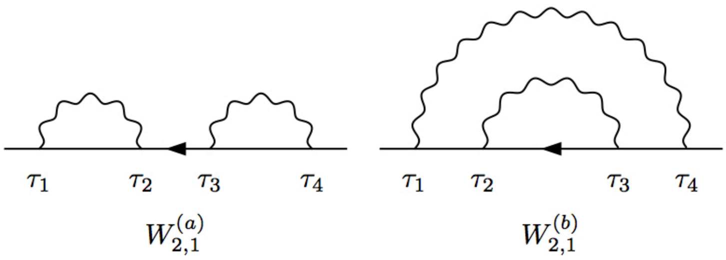

Figure 3. Two of planar diagrams of ladder type $W_{2,1} = W^{(a)}_{2,1} + W^{(b)}_{2,1}$ with path-ordered four points $\tau_1, \ldots, \tau_4$ in the WL ($\zeta = 0$) case. For general ζ one needs also to add similar diagrams with scalar propagators.

these are equivalent and cancel the 4! factor from the expansion of the exponential. There are two different planar Wick contractions shown in figure 3.

In the WML case the expression for the first one is [13][10]

$$W^{(1)}_{2,1a} = \frac{\left[\Gamma(\omega-1)\right]^2}{64\,\pi^{2\omega}} \oint_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\, \frac{(|\dot{x}^{(1)}|\,|\dot{x}^{(2)}| - \dot{x}^{(1)}\cdot\dot{x}^{(2)})(|\dot{x}^{(3)}|\,|\dot{x}^{(4)}| - \dot{x}^{(3)}\cdot\dot{x}^{(4)})}{(|x^{(1)}-x^{(2)}|^2\,|x^{(3)}-x^{(4)}|^2)^{\omega-1}}. \tag{2.16}$$

The second diagram has a similar expression with $(1,2,3,4) \to (1,4,2,3)$. In the WML case these two contributions are equal and finite. Setting $\omega = 2$ we find that the integrand in (2.16) in the circle case is constant as in (2.4). As a result,

$$W^{(1)}_{2,1} = W^{(1)}_{2,1a} + W^{(1)}_{2,1b} = 2 \times \frac{1}{64\,\pi^4}\,\frac{(2\pi)^4}{4!}\left(\frac{1}{2}\right)^2 = \frac{1}{192}\,. \tag{2.17}$$

This already reproduces the coefficient of the λ^2 term in (1.5) (consistently with the vanishing (2.15) of the rest of the contributions [13]).

The corresponding expression in the WL case is found by dropping the scalar field exchanges, i.e. the $|\dot{x}|$ terms in the numerator of (2.16). Then for the circle we get

$$\begin{aligned} W^{(0)}_{2,1a} &= \frac{[\Gamma(\omega-1)]^2}{64\,\pi^{2\omega}} \int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\, \frac{\cos\tau_{12}\,\cos\tau_{34}}{(4\,\sin^2\frac{\tau_{12}}{2}\;4\,\sin^2\frac{\tau_{34}}{2})^{\omega-1}}\,, \\ W^{(0)}_{2,1b} &= \frac{[\Gamma(\omega-1)]^2}{64\,\pi^{2\omega}} \int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\, \frac{\cos\tau_{14}\,\cos\tau_{23}}{(4\,\sin^2\frac{\tau_{14}}{2}\;4\,\sin^2\frac{\tau_{23}}{2})^{\omega-1}}\,. \end{aligned} \tag{2.18}$$

The computation of these integrals is discussed in appendix B. Setting $\omega = 2 - \varepsilon$ we get

$$\begin{aligned} W^{(0)}_{2,1a} &= \tfrac{[\Gamma(1-\varepsilon)]^2}{64\,\pi^{2\,(2-\varepsilon)}} \left[\tfrac{\pi^2}{\varepsilon} + 3\,\pi^2 + \tfrac{\pi^4}{6} + \mathcal{O}(\varepsilon)\right] = \tfrac{1}{64\,\pi^2\,\varepsilon} + \tfrac{1}{384} + \tfrac{3}{64\,\pi^2} + \tfrac{\gamma_{\rm E}+\log\pi}{32\,\pi^2} + \mathcal{O}(\varepsilon), \\ W^{(0)}_{2,1b} &= \tfrac{[\Gamma(1-\varepsilon)]^2}{64\,\pi^{2\,(2-\varepsilon)}} \left[\tfrac{\pi^2}{2} + \tfrac{\pi^4}{6} + \mathcal{O}(\varepsilon)\right] = \tfrac{1}{384} + \tfrac{1}{128\,\pi^2} + \mathcal{O}(\varepsilon). \end{aligned} \tag{2.19}$$

The total ladder contribution in the WL case is thus

$$W^{(0)}_{2,1} = W^{(0)}_{2,1a} + W^{(0)}_{2,1b} = \frac{1}{64\,\pi^2\,\varepsilon} + \frac{1}{192} + \frac{7}{128\,\pi^2} + \frac{\gamma_{\rm E} + \log\pi}{32\,\pi^2} + \mathcal{O}(\varepsilon). \tag{2.20}$$

[10] Here $x^{(i)} = x(\tau_i)$ and $d^4\boldsymbol{\tau} \equiv d\tau_1 d\tau_2 d\tau_3 d\tau_4$.

2.2.2 Self-energy contribution

It is convenient to represent the contribution $W_{2,2}$ of the self-energy diagram in figure 2 as

$$W^{(\zeta)}_{2,2} = -\frac{[\Gamma(\omega-1)]^2}{8\,\pi^{\omega}(2-\omega)(2\omega-3)}\,\widetilde{W}^{(\zeta)}_1\,, \tag{2.21}$$

where, in the WML case, one has [13]

$$\widetilde{W}^{(1)}_1 = \frac{1}{16\pi^{\omega}}\oint d\tau_1 d\tau_2\,\frac{|\dot{x}(\tau_1)|\,|\dot{x}(\tau_2)| - \dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{\left[|x(\tau_1)-x(\tau_2)|^2\right]^{2\omega-3}}\,. \tag{2.22}$$

Again, the expression in the WL case is obtained by simply dropping the scalar exchange $|\dot{x}(\tau_1)|\,|\dot{x}(\tau_2)|$ term in the numerator of (2.22):

$$\widetilde{W}^{(0)}_1 = \frac{1}{16\pi^{\omega}}\oint d\tau_1 d\tau_2\,\frac{-\dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{\left[|x(\tau_1)-x(\tau_2)|^2\right]^{2\omega-3}}\,. \tag{2.23}$$

Altough (2.23) is very similar to $W^{(0)}_1$ in (2.7), for $\omega\neq 2$ there is a difference in the power in the denominator. Specializing to the circle case we find (using the integral (2.11))

$$\begin{aligned}
\widetilde{W}^{(1)}_1 &= 2^{3-4\omega}\pi^{-\omega}\oint d\tau_1 d\tau_2\left[\sin^2\frac{\tau_{12}}{2}\right]^{4-2\omega} = \frac{2^{3-4\omega}\pi^{-\omega}}{2\pi}\,\mathcal{M}(4-2\omega,0,0)\\
&= \frac{1}{8}+\frac{1}{8}\log\pi\,\varepsilon+\mathcal{O}(\varepsilon^2)\,, \qquad\qquad (2.24)\\
\widetilde{W}^{(0)}_1 &= -4^{1-2\omega}\pi^{-\omega}\oint d\tau_1 d\tau_2\left[\sin^2\frac{\tau_{12}}{2}\right]^{3-2\omega} + 2^{3-4\omega}\pi^{-\omega}\oint d\tau_1 d\tau_2\left[\sin^2\frac{\tau_{12}}{2}\right]^{4-2\omega}\\
&= \frac{1}{8}+\frac{1}{8}(2+\log\pi)\,\varepsilon+\mathcal{O}(\varepsilon^2). \qquad\qquad (2.25)
\end{aligned}$$

Then from (2.21) we get

$$W^{(1)}_{2,2} = -\frac{1}{64\,\pi^2\,\varepsilon}-\frac{1}{32\,\pi^2}-\frac{\gamma_{\rm E}}{32\,\pi^2}-\frac{\log\pi}{32\,\pi^2}+\mathcal{O}(\varepsilon), \tag{2.26}$$

$$W^{(0)}_{2,2} = -\frac{1}{64\,\pi^2\,\varepsilon}-\frac{1}{16\,\pi^2}-\frac{\gamma_{\rm E}}{32\,\pi^2}-\frac{\log\pi}{32\,\pi^2}+\mathcal{O}(\varepsilon)\,. \tag{2.27}$$

Note that the difference between the WL and WML self-energy contributions is finite

$$W^{(0)}_{2,2} = W^{(1)}_{2,2} - \frac{1}{32\,\pi^2}\,. \tag{2.28}$$

2.2.3 Internal-vertex contribution

In the WML case, the internal-vertex diagram contribution in figure 2 has the following expression [13]

$$\begin{aligned}
W^{(1)}_{2,3} = -\frac{1}{4}\oint d^3\boldsymbol{\tau}\,\varepsilon(\tau_1,\tau_2,\tau_3)\,&\left[|\dot{x}^{(1)}|\,|\dot{x}^{(3)}| - \dot{x}^{(1)}\cdot\dot{x}^{(3)}\right]\\
&\times\dot{x}^{(2)}\cdot\frac{\partial}{\partial x^{(1)}}\int d^{2\omega}y\,\Delta(x^{(1)}-y)\,\Delta(x^{(2)}-y)\,\Delta(x^{(3)}-y), \qquad (2.29)
\end{aligned}$$

where $\Delta(x)$ is the propagator (2.6), $d^3\boldsymbol{\tau} \equiv d\tau_1 d\tau_2 d\tau_3$ and $\varepsilon(\tau_1,\tau_2,\tau_3)$ is the totally anti-symmetric path ordering symbol equal to 1 if $\tau_1 > \tau_2 > \tau_3$. Using the Feynman parameter representation for the propagators and specializing to the circle case (2.29) becomes

$$\begin{aligned} W^{(1)}_{2,3} &= \frac{\Gamma(2\omega-2)}{2^{2\omega+5}\,\pi^{2\omega}} \int_0^1 [d^3\boldsymbol{\alpha}] \oint d^3\boldsymbol{\tau}\,\epsilon(\tau_1,\tau_2,\tau_3) \\ &\quad \times \left(1-\cos\tau_{13}\right) \frac{\alpha\,(1-\alpha)\,\sin\tau_{12} + \alpha\,\gamma\,\sin\tau_{23}}{Q^{2\,\omega-2}}, \end{aligned} \tag{2.30}$$

$$[d^3\boldsymbol{\alpha}] \equiv d\alpha\, d\beta\, d\gamma\, (\alpha\beta\gamma)^{\omega-2}\,\delta(1-\alpha-\beta-\gamma)\,, \tag{2.31}$$

$$Q \equiv \alpha\,\beta\,(1-\cos\tau_{12}) + \beta\,\gamma\,(1-\cos\tau_{23}) + \gamma\,\alpha\,(1-\cos\tau_{13})\;. \tag{2.32}$$

The corresponding WL expression is found by omitting the scalar coupling term $|\dot{x}^{(1)}|\,|\dot{x}^{(3)}|$, i.e. by replacing the factor $(1-\cos\tau_{13})$ by $(-\cos\tau_{13})$. We can then represent the WL contribution as

$$W^{(0)}_{2,3} = W^{(1)}_{2,3} - \frac{\Gamma(2\omega-2)}{2^{2\omega+5}\pi^{2\omega}}\, J(\omega), \tag{2.33}$$

$$J(\omega) \equiv \int_0^1 [d^3\boldsymbol{\alpha}] \oint d^3\boldsymbol{\tau}\,\epsilon(\tau_1,\tau_2,\tau_3)\; \frac{\alpha\,(1-\alpha)\,\sin\tau_{12} + \alpha\,\gamma\,\sin\tau_{23}}{Q^{2\,\omega-2}}\;. \tag{2.34}$$

In the WML case one finds that (2.30) is related to $W^{(1)}_{2,2}$ [13]

$$W^{(1)}_{2,3} = -W^{(1)}_{2,2} + \mathcal{O}(\varepsilon)\,, \tag{2.35}$$

where $W^{(1)}_{2,2}$ was given in (2.26). Thus to compute $W^{(0)}_{2,3}$ it remains to determine $J(\omega)$. Let us first use that

$$\begin{aligned} \oint d^3\boldsymbol{\tau}\,\varepsilon(\tau_1,\tau_2,\tau_3)\; F(\tau_1,\tau_2,\tau_3) &= \oint_{\tau_1>\tau_2>\tau_3} d^3\boldsymbol{\tau}\, \Big[F(\tau_1,\tau_2,\tau_3) - F(\tau_1,\tau_3,\tau_2) \\ &\quad + F(\tau_2,\tau_3,\tau_1) - F(\tau_2,\tau_1,\tau_3) + F(\tau_3,\tau_1,\tau_2) - F(\tau_3,\tau_2,\tau_1)\Big], \end{aligned} \tag{2.36}$$

and relabel the Feynman parameters in each term. Then $J(\omega)$ takes a more symmetric form

$$J(\omega) = 8\int_0^1 [d^3\boldsymbol{\alpha}] \oint_{\tau_1>\tau_2>\tau_3} d^3\boldsymbol{\tau}\; \frac{(\alpha\,\beta+\beta\,\gamma+\gamma\,\alpha)\,\sin\frac{\tau_{12}}{2}\,\sin\frac{\tau_{13}}{2}\,\sin\frac{\tau_{23}}{2}}{Q^{2\,\omega-2}}\;. \tag{2.37}$$

Using the double Mellin-Barnes representation (see, for instance, [27])

$$\frac{1}{(A+B+C)^\sigma} = \frac{1}{(2\,\pi\,i)^2}\frac{1}{\Gamma(\sigma)}\int_{-i\,\infty}^{+i\,\infty} du\,dv\, \frac{B^u\,C^v}{A^{\sigma+u+v}}\,\Gamma(\sigma+u+v)\,\Gamma(-u)\,\Gamma(-v), \tag{2.38}$$

we can further rewrite (2.37) as

$$\begin{aligned} J(\omega) &= \frac{8}{(2\pi i)^2\, 2^{2\omega-2}\Gamma(2\omega-2)} \oint_{\tau_1>\tau_2>\tau_3} d^3\boldsymbol{\tau} \int du dv \int_0^1 d\alpha\, d\beta\, d\gamma\, (\alpha\beta\gamma)^{\omega-2}(\alpha\beta+\beta\gamma+\gamma\alpha) \\ &\quad \times \Gamma(2\omega-2+u+v)\Gamma(-u)\Gamma(-v)\, \frac{\left(\beta\gamma\sin^2\frac{\tau_{23}}{2}\right)^u \left(\alpha\beta\sin^2\frac{\tau_{12}}{2}\right)^v}{\left(\gamma\alpha\sin^2\frac{\tau_{13}}{2}\right)^{2\omega-2+u+v}}\, \sin\frac{\tau_{12}}{2}\,\sin\frac{\tau_{13}}{2}\,\sin\frac{\tau_{23}}{2}\,. \end{aligned} \tag{2.39}$$

Integrating over α,β,γ using the relation

$$\int_0^1 \prod_{i=1}^N d\alpha_i\, \alpha_i^{\nu_i-1}\, \delta\Big(1-\sum_i \alpha_i\Big) = \frac{\Gamma(\nu_1)\cdots\Gamma(\nu_N)}{\Gamma(\nu_1+\cdots+\nu_N)}\,, \tag{2.40}$$

gives the following representation for J

$$J(\omega) = -\frac{1}{\pi^2\, 2^{2\omega-3}}\frac{1}{\Gamma(2\omega-2)\Gamma(3-\omega)}\int_{-i\infty}^{+i\infty} du \int_{-i\infty}^{+i\infty} dv\;\; X(u,v)\; T(u,v)\;, \tag{2.41}$$

$$X(u,v) \equiv \left(\frac{1}{u+v+\omega-1} - \frac{1}{u+\omega-1} - \frac{1}{v+\omega-1}\right) \tag{2.42}$$
$$\times\,\Gamma(2\omega-2+u+v)\Gamma(-u)\Gamma(-v)\,\Gamma(2-u-\omega)\,\Gamma(2-v-\omega)\,\Gamma(u+v+\omega)\,,$$

$$T(u,v) \equiv \oint_{\tau_1>\tau_2>\tau_3} d^3\boldsymbol{\tau}\; \frac{\left(\sin^2\frac{\tau_{23}}{2}\right)^{u+1/2}\,\left(\sin^2\frac{\tau_{12}}{2}\right)^{v+1/2}}{\left(\sin^2\frac{\tau_{13}}{2}\right)^{2\omega-2+u+v-1/2}}\;. \tag{2.43}$$

A remarkable feature of (2.41), familiar in computations of similar integrals, is that the integrand is symmetric in the three τ_i variables as one can show using a suitable linear change of the Mellin-Barnes integration parameters u, v.[11] As a result, we may effectively replace $T(u,v)$ by $\frac{1}{3!}$ of the integrals along the full circle:

$$T(u,v) \to \frac{1}{3!}\oint_0^{2\pi} d^3\boldsymbol{\tau}\; \frac{\left(\sin^2\frac{\tau_{23}}{2}\right)^{u+1/2}\,\left(\sin^2\frac{\tau_{12}}{2}\right)^{v+1/2}}{\left(\sin^2\frac{\tau_{13}}{2}\right)^{2\omega-2+u+v-1/2}}. \tag{2.44}$$

Using again the master integral (2.11), we find the following expression for $J(\omega)$ as a double integral

$$J(\omega) = -\frac{8\,\pi^{3/2}}{3!\,\pi^2\,2^{2\omega-3}}\frac{1}{\Gamma(2\omega-2)\Gamma(3-\omega)}\int_{-i\infty}^{+i\infty} du \int_{-i\infty}^{+i\infty} dv\, X(u,v)$$
$$\times\,\frac{\Gamma(u+1)\,\Gamma(v+1)\,\Gamma\left(\frac{9}{2}-2\omega\right)\,\Gamma(-u-v-2\omega+3)}{\Gamma(u+v+2)\,\Gamma(-u-2\omega+4)\,\Gamma(-v-2\omega+4)}. \tag{2.45}$$

Writing all factors in $X(u,v)$ in (2.42) in terms of Γ-functions we end up with

$$J(\omega) = \frac{\pi^{3/2}}{3\times 2^{2\omega-7}}\frac{1}{\Gamma(2\omega-2)\Gamma(3-\omega)}\int_{-i\infty}^{+i\infty}\frac{du}{2\,\pi\, i}\int_{-i\infty}^{+i\infty}\frac{dv}{2\,\pi\, i}\;R(u,v)\,, \tag{2.46}$$

$$R(u,v) = \Gamma(2\omega-2+u+v)\Gamma(-u)\Gamma(-v)\,\Big[\Gamma(1-u-\omega)\,\Gamma(2-v-\omega)\,\Gamma(u+v+\omega)$$
$$+\,\Gamma(2-u-\omega)\,\Gamma(1-v-\omega)\,\Gamma(u+v+\omega) + \Gamma(2-u-\omega)\,\Gamma(2-v-\omega)\,\Gamma(u+v+\omega-1)\Big]$$
$$\times\,\frac{\Gamma(u+1)\Gamma(v+1)\Gamma\left(\frac{9}{2}-2\omega\right)\Gamma(-u-v-2\omega+3)}{\Gamma(u+v+2)\Gamma(-u-2\omega+4)\Gamma(-v-2\omega+4)}. \tag{2.47}$$

[11] For instance, the exchange of τ_1 and τ_3 is compensated by redefining $(u,v)\to(u',v')$ with $u+\frac{1}{2} = -(2\omega-2+u'+v'-1/2)$, $\;-(2\omega-2+u+v-1/2) = u'+1/2$, that is $u = 2-u'-v'-2\omega,\; v = v'$. This change of variables leaves invariant the other part $T(u,v)$ of the integrand: it takes the same form when written in terms of u', v'.

This integral can be computed using the algorithms described in [28] and by repeated application of Barnes first and second lemmas [29]. The result expanded in $\varepsilon = 2-\omega \to 0$ is

$$J(2-\varepsilon) = \frac{8\,\pi^2}{\varepsilon} - 8\,\pi^2\,(2\,\log 2 - 3) + \mathcal{O}(\varepsilon). \tag{2.48}$$

Using this in (2.33) gives

$$W_{2,3}^{(0)} = W_{2,3}^{(1)} - \frac{1}{64\,\pi^2\,\varepsilon} - \frac{1}{64\,\pi^2} - \frac{\gamma_{\rm E} + \log\pi}{32\pi^2} + \mathcal{O}(\varepsilon). \tag{2.49}$$

2.2.4 Total contribution to standard Wilson loop

From (2.28) and (2.49) we get

$$W_{2,2}^{(0)} + W_{2,3}^{(0)} = -\frac{1}{64\,\pi^2\,\varepsilon} - \frac{3}{64\,\pi^2} - \frac{\gamma_{\rm E} + \log\pi}{32\pi^2} + \mathcal{O}(\varepsilon).\,, \tag{2.50}$$

i.e. in contrast to the WML case (2.15), (2.35) the sum of the self-energy and internal vertex diagrams is no longer zero and is logarithmically divergent. The divergence is cancelled once we add the ladder contribution in (2.20). Thus the total contribution to the WL expectation value at order λ^2 found from (2.20), (2.50) is finite

$$W_2^{(0)} = W_{2,1}^{(0)} + W_{2,2}^{(0)} + W_{2,3}^{(0)} = \frac{1}{192} + \frac{1}{128\,\pi^2}\,, \qquad\qquad W_2^{(0)} = W_2^{(1)} + \frac{1}{128\,\pi^2}\,. \tag{2.51}$$

Thus, using (2.1), (2.13), we get the final result for the expectation value of the ordinary Wilson loop

$$W^{(0)} = 1 + \frac{1}{8}\lambda + \left(\frac{1}{192} + \frac{1}{128\,\pi^2}\right)\lambda^2 + \mathcal{O}(\lambda^3)\ . \tag{2.52}$$

We conclude that the weak-coupling expectation values for the circular WML and WL start to differ from order λ^2.

2.3 Generalization to any ζ

Let us now generalize the above results for the leading and subleading term in the weak-coupling expansion (2.1) of the circular Wilson loop to the case of the generalized WL, i.e. to any value of the parameter ζ in (1.1). The computation follows the same lines as above.

At leading order in λ we find the same result as in the circular WML (2.5) and WL (2.13) cases, i.e., after subtracting the linear divergence, the quantity W_1 in (2.1) has the universal (independent on ζ) value

$$W_1^{(\zeta)} = \frac{1}{8}\ . \tag{2.53}$$

Explicitly, using again dimensional regularization, we find as in (2.2), (2.10), (2.12)

$$\begin{aligned} W_1^{(\zeta)} &= \frac{\Gamma(\omega-1)}{16\pi^\omega}\oint d\tau_1 d\tau_2\,\frac{\zeta^2 - \dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{|x(\tau_1)-x(\tau_2)|^{2\omega-2}} \\ &= \frac{1}{8} - \frac{(1-\zeta^2)\,\Gamma(\omega-1)}{16\pi^\omega}\oint \frac{d\tau_1 d\tau_2}{(4\,\sin^2\frac{\tau_{12}}{2})^{\omega-1}} = \frac{1}{8} + \frac{1}{8}(1-\zeta^2)\varepsilon + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{2.54}$$

where we set $\omega = 2 - \varepsilon$ and retained a term of order ε as this will contribute to the final result at order λ^2 in our dimensional regularization scheme upon replacing the bare with renormalized coupling. To order λ, however, one can safely remove this term yielding (2.53).

Turning to λ^2 order, the ladder diagram contributions in figure 2 generalizing the $\zeta = 0$ expressions (2.18) are

$$
\begin{aligned}
W^{(\zeta)}_{2,1a} &= \frac{[\Gamma(\omega-1)]^2}{64\,\pi^{2\omega}} \int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\, \frac{(\zeta^2 - \cos\tau_{12})\,(\zeta^2 - \cos\tau_{34})}{\left(4\,\sin^2\frac{\tau_{12}}{2}\ 4\,\sin^2\frac{\tau_{34}}{2}\right)^{\omega-1}}, \\
W^{(\zeta)}_{2,1b} &= \frac{[\Gamma(\omega-1)]^2}{64\,\pi^{2\omega}} \int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\, \frac{(\zeta^2 - \cos\tau_{14})\,(\zeta^2 - \cos\tau_{23})}{\left(4\,\sin^2\frac{\tau_{14}}{2}\ 4\,\sin^2\frac{\tau_{23}}{2}\right)^{\omega-1}}.
\end{aligned}
\tag{2.55}
$$

The result of their rather involved computation generalizing (2.19) is (see appendix B)

$$
\begin{aligned}
W^{(\zeta)}_{2,1a} &= \frac{[\Gamma(1-\varepsilon)]^2}{64\,\pi^{2\,(2-\varepsilon)}} \left[\frac{\pi^2\,(1-\zeta^2)}{\varepsilon} + \pi^2\,(1-\zeta^2)(3-\zeta^2) + \frac{\pi^4}{6} + \mathcal{O}(\varepsilon)\right], \\
W^{(\zeta)}_{2,1b} &= \frac{[\Gamma(1-\varepsilon)]^2}{64\,\pi^{2\,(2-\varepsilon)}} \left[\frac{\pi^2}{2}\,(1-\zeta^2)^2 + \frac{\pi^4}{6} + \mathcal{O}(\varepsilon)\right],
\end{aligned}
\tag{2.56}
$$

with the sum being

$$
W^{(\zeta)}_{2,1} = W^{(\zeta)}_{2,1a} + W^{(\zeta)}_{2,1b} = \frac{1}{192} + (1-\zeta^2)\left[\frac{1}{64\,\pi^2\,\varepsilon} + \frac{1}{128\,\pi^2}\,(7-3\,\zeta^2) + \frac{\log\pi + \gamma_{\rm E}}{32\,\pi^2}\right] + \mathcal{O}(\varepsilon).
\tag{2.57}
$$

For the self-energy contribution in figure 2 we find the expression (2.21) where now

$$
\begin{aligned}
\widetilde{W}^{(\zeta)}_1 &= \frac{1}{16\pi^\omega} \oint d\tau_1 d\tau_2\, \frac{\zeta^2\,|\dot{x}(\tau_1)|\,|\dot{x}(\tau_2)| - \dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{\left[|x(\tau_1) - x(\tau_2)|^2\right]^{2\omega-3}} \\
&= \zeta^2\,\widetilde{W}^{(1)}_1 + (1-\zeta^2)\,\widetilde{W}^{(0)}_1 = \frac{1}{8} + \frac{1}{8}\left[2\,(1-\zeta^2) + \log\pi\right] + \mathcal{O}(\varepsilon),
\end{aligned}
\tag{2.58}
$$

with $\widetilde{W}^{(1)}_1$ and $\widetilde{W}^{(0)}_1$ given by (2.22), (2.24) and (2.23), (2.25). Substituting this into (2.21), we get

$$
W^{(\zeta)}_{2,2} = \zeta^2\, W^{(1)}_{2,2} + (1-\zeta^2)\left[-\frac{1}{64\,\pi^2\,\varepsilon} - \frac{1}{16\,\pi^2} - \frac{\gamma_E + \log\pi}{32\,\pi^2}\right] + \mathcal{O}(\varepsilon)\,,
\tag{2.59}
$$

where $W^{(1)}_{2,2}$ is given by (2.26).

The internal-vertex diagram contribution in figure 2 generalizing (2.33) is

$$
W^{(\zeta)}_{2,3} = W^{(1)}_{2,3} - (1-\zeta^2)\,\frac{\Gamma(2\omega-2)}{2^{2\omega+5}\pi^{2\omega}}\,J(\omega)\,,
\tag{2.60}
$$

where J is given by (2.34), (2.48) and $W^{(1)}_{2,3}$ is given by (2.35), (2.26), i.e.

$$
W^{(\zeta)}_{2,3} = -W^{(1)}_{2,2} + (1-\zeta^2)\left[-\frac{1}{64\,\pi^2\,\varepsilon} - \frac{1}{64\,\pi^2} - \frac{\gamma_{\rm E} + \log\pi}{32\pi^2}\right] + \mathcal{O}(\varepsilon)\ .
\tag{2.61}
$$

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Summing up the separate contributions given in (2.57), (2.59) and (2.61) we find that the $\frac{1}{\varepsilon} \sim \log a$ logarithmic divergences cancel out, and we get the finite expression

$$W_2^{(\zeta)} = W_{2,1}^{(\zeta)} + W_{2,2}^{(\zeta)} + W_{2,3}^{(\zeta)} = \frac{1}{192} + \frac{1}{128\pi^2}\,(1-\zeta^2)\,(1-3\zeta^2)\,. \tag{2.62}$$

The final result for the Wilson loop expectation value to order λ^2 that follows from (2.54) and (2.62) is then

$$\langle W^{(\zeta)}\rangle = 1 + \lambda\left(\frac{1}{8} - \frac{1}{8}\zeta^2\varepsilon\right) + \lambda^2\left[\frac{1}{192} + \frac{1}{128\pi^2}\,(1-\zeta^2)\,(1-3\zeta^2)\right] + \mathcal{O}(\lambda^3)\,. \tag{2.63}$$

Here it is important to retain the order $\zeta^2\varepsilon$ part in the 1-loop term in (2.54): despite the cancellation of all $\frac{1}{\varepsilon}$ terms to this order, ζ in the order λ term is a bare coupling that contains poles that may effectively contribute at higher orders.

Despite λ not running in $d = 4$ the presence of the linear in ζ term in the beta-function (1.4) implies that the present case is best treated as a 2-coupling $g_i = (\lambda, \zeta)$ theory. In general, if $d = 4 - 2\varepsilon$ and we have a set of near-marginal couplings g_i with mass dimensions $u_i\varepsilon$ the bare couplings may be expressed in terms of the dimensionless renormalized couplings g_i as

$$g_{i\mathrm{b}} = \mu^{u_i\varepsilon}\left[g_i + \frac{1}{\varepsilon}K_i(g) + \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)\right], \qquad \mu\frac{dg_{i\mathrm{b}}}{d\mu} = 0, \tag{2.64}$$

$$\beta_i(g) = \mu\frac{dg_i}{d\mu} = -\varepsilon u_i g_i - u_i K_i + \sum_j u_j g_j \frac{\partial}{\partial g_j}K_i\,. \tag{2.65}$$

In the present case we may choose dimensions so that the gauge field and scalars Φ_m in the bare SYM action $\frac{N}{\lambda_{\mathrm{b}}}\int d^dx(F^2 + D\Phi D\Phi + \ldots)$ have dimension 1 so that λ_b has dimension 2ε, i.e. $\lambda_{\mathrm{b}} = \mu^{2\varepsilon}\lambda$, or $u_\lambda = 2$ (and of course $K_\lambda = 0$). As the Wilson line integrand in (1.1) should have dimension 1, that means ζ_b should have dimension zero, i.e. $u_\zeta = 0$.[12] Then from (2.64), (2.65) we learn that (using (1.4))

$$\zeta_{\mathrm{b}} = \zeta + \frac{1}{\varepsilon}K_\zeta + \mathcal{O}\left(\frac{1}{\varepsilon^2}\right), \qquad \beta_\zeta = u_\lambda\lambda\frac{\partial}{\partial\lambda}K_\zeta\,, \qquad K_\zeta = \frac{1}{2}\beta_\zeta = \frac{\lambda}{16\pi^2}\zeta(\zeta^2-1)\,. \tag{2.66}$$

The coupling ζ in (2.63) should actually be the bare coupling; replacing it with the renormalized coupling according to (2.66) and then sending $\varepsilon \to 0$ we find the expression in (1.6), i.e.

$$\langle W^{(\zeta)}\rangle = 1 + \frac{1}{8}\lambda + \left[\frac{1}{192} + \frac{1}{128\pi^2}\,(1-\zeta^2)^2\right]\lambda^2 + \mathcal{O}(\lambda^3)\,. \tag{2.67}$$

As we shall discuss in appendix B.3, there is an alternative regularization procedure in which the full 2-loop expression in (2.67) comes just from the type (b) ladder diagram contribution in (2.56) and thus the use of the evanescent 1-loop term in (2.63) is not required.

[12] This is natural as the dimension of the Wilson line integral is not changed. Note that the same is true if one redefines the SYM fields by a power of gauge coupling g: then dimension of Φ is canonical $\frac{d-2}{2} = 1 - \varepsilon$ but $g\,\Phi$ that then enters the Wilson loop (1.1) still has dimension 1.

3 Relation to correlators of scalar operators on the Wilson loop

The ζ-dependence of the generalized WL (1.1) can be viewed as being to due to multiple insertions of the scalar operators on the loop. It is of interest to relate the expression (2.67) to what is known about 2-point functions of (scalar) operators on the line or circle (see [3, 5, 6, 30–34]). Let us choose the scalar coupling in (1.1) to be along 6-th direction, i.e. $\Phi_m\theta^m = \Phi_6$ and denote the remaining 5 scalars not coupled directly to the loop as Φ_a $(a = 1, \dots, 5)$. Let us also choose the contour to be straight line $x^\mu = (\tau, 0, 0, 0)$ along the Euclidean time direction $x^0 = t$ so that the exponent in (1.1) is simply $\int dt(iA_t + \zeta\Phi_6)$. For $\zeta = 1$ or $\zeta = 0$ when the loop preserves the conformal symmetry the 2- (and higher) point functions of conformal operators inserted along the line can be interpreted as correlators in an effective (defect) 1d CFT. For example, for $\zeta = 1$

$$\langle\!\langle O(t_1)O(t_2)\rangle\!\rangle_{\rm line} \equiv \langle \mathrm{Tr}\,\mathcal{P}\big[O(x_1)O(x_2)\; e^{\int dt(iA_t+\Phi_6)}\big]\rangle = \frac{C}{|t_{12}|^{2\Delta}} \quad . \tag{3.1}$$

Here in $\langle \mathrm{Tr} \dots\rangle$ the operator $O(x)$ is a gauge-theory operator in the adjoint representation restricted to the line (with exponential factors appearing between and after $O(x_n(t_n))$ according to path ordering to preserve gauge invariance). We also use that in the WML case for a straight line the normalization factor is trivial, i.e. $\langle\!\langle 1\rangle\!\rangle = 1$. Similar relation can be written for a circular loop using the map $t \to \tan\frac{\tau}{2}$

$$\langle\!\langle O(\tau_1)O(\tau_2)\rangle\!\rangle_{\rm circle} = \frac{C}{|2\sin\frac{\tau_{12}}{2}|^{2\Delta}}\,. \tag{3.2}$$

Here the gauge-theory expectation value is to be normalized with the non-trivial circle WML factor (1.5) so that once again $\langle\!\langle 1\rangle\!\rangle = 1$. In the $\zeta = 0$ case one is to use (2.52) as the corresponding normalization factor. In what follows $\langle\!\langle \dots\rangle\!\rangle$ will refer to the expectation value in the effective CFT on the circle.

The simplest example is the insertion of the "orthogonal" scalars Φ_a into the WML (3.1) in which case the dimension is protected, $\Delta = 1$, while the norm is related to the Bremsstrahlung function $B(\lambda)$ [34][13]

$$\langle\!\langle \Phi_a(\tau_1)\Phi_b(\tau_2)\rangle\!\rangle = \delta_{ab}\frac{C_0(\lambda)}{\big|2\sin\frac{\tau_{12}}{2}\big|^2}\ , \qquad\qquad C_0 = 2B(\lambda)\,, \tag{3.3}$$

$$B(\lambda) \equiv \frac{1}{2\pi^2}\frac{d}{d\log\lambda}\langle W^{(1)}\rangle = \frac{\sqrt{\lambda}\,I_2(\sqrt{\lambda})}{4\pi^2\,I_1(\sqrt{\lambda})}\,, \tag{3.4}$$

$$C_0(\lambda \ll 1) = \frac{\lambda}{8\pi^2} - \frac{\lambda^2}{192\pi^2} + \mathcal{O}(\lambda^3)\,, \quad C_0(\lambda \gg 1) = \frac{\sqrt{\lambda}}{2\pi^2} - \frac{3}{4\pi^2} + \mathcal{O}\Big(\frac{1}{\sqrt{\lambda}}\Big)\,. \tag{3.5}$$

[13] Let us recall that the leading tree level value of the 2-point coefficient $C = \frac{\lambda}{8\pi^2} + \dots$ (with $\lambda \equiv g^2N$) is found by taking into the account that the adjoint scalar field is $\Phi = \Phi^r t^r$ with propagator $\langle\Phi^r(x)\Phi^{r'}(0)\rangle = \frac{g^2\,\delta^{rr'}}{4\pi^2\,x^2}$ $(r = 1, \dots, N^2-1$ is the SU(N) algebra index) where the generators satisfy $\mathrm{Tr}(t_r t_{r'}) = \frac{1}{2}\,\delta_{rr'}$, $t_r t_r = \frac{1}{2}N\,\mathrm{I}$. The trace $\delta^{rr'}\delta_{rr'} = N^2-1$ produces the factor of N^2 in the planar limit.

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The operator Φ_6 which couples to the loop in this $\zeta = 1$ case, on the other hand, gets renormalized and its scaling dimension is a non-trivial function of λ. At small λ one gets[14]

$$\langle\!\langle \Phi_6(\tau_1)\Phi_6(\tau_2)\rangle\!\rangle = \frac{C(\lambda)}{\left|2\sin\frac{\tau_{12}}{2}\right|^{2\Delta}}\,, \qquad C = \frac{\lambda}{8\pi^2} + \mathcal{O}(\lambda^2)\,, \qquad \Delta = 1 + \frac{\lambda}{4\pi^2} + \mathcal{O}(\lambda^2)\;. \tag{3.6}$$

Here the anomalous dimension can be obtained by direct computation [3] or by taking the derivative of the beta-function (1.4) at the $\zeta = 1$ conformal point [4] as in (1.12). The leading term in C is the same as in (3.3), (3.5) as it comes just from the free-theory correlator. At strong coupling the "transverse" scalars Φ_a should correspond to massless string coordinates y_a in S^5 directions (with $\Delta = \Delta_+ = 1$, cf. (4.7)) [6] while Φ_6 should correspond [35] to the 2-particle world-sheet state $y_a y_a$ (see section 4.2), with dimension $\Delta = 2\Delta_+ + \mathcal{O}(\frac{1}{\sqrt{\lambda}}) = 2 + \mathcal{O}(\frac{1}{\sqrt{\lambda}})$ [4]. The subleading term in

$$\Delta = 2 - \frac{5}{\sqrt{\lambda}} + \mathcal{O}\Big(\frac{1}{(\sqrt{\lambda})^2}\Big), \tag{3.7}$$

computed in [6] has a negative sign consistent with a possibility of a smooth interpolation to the weak-coupling expansion in (3.6) (see also section 4.2).

In the case of the standard WL with no scalar coupling ($\zeta = 0$) the defect CFT_1 has unbroken SO(6) symmetry and thus all 6 scalars have the same correlators:

$$\langle\!\langle \Phi_m \rangle\!\rangle = 0\,, \qquad \langle\!\langle \Phi_m(\tau_1)\Phi_n(\tau_2)\rangle\!\rangle = \delta_{mn}\frac{C(\lambda)}{\left|2\sin\frac{\tau_{12}}{2}\right|^{2\Delta}}\,, \tag{3.8}$$

$$C = \frac{\lambda}{8\pi^2} + \mathcal{O}(\lambda^2)\,, \qquad \Delta = 1 - \frac{\lambda}{8\pi^2} + \mathcal{O}(\lambda^2)\;. \tag{3.9}$$

Here the leading free-theory term in C is the same as in (3.5), (3.6) as it comes just from the free-theory correlator. The anomalous dimension in (3.9) found by direct computation in [3] is again the same as the derivative of the beta-function (1.4) at the $\zeta = 0$ conformal point [4] (see (1.12)). At strong coupling, i.e. in the string theory description where the S^5 coordinates are to be subject to the Neumann boundary conditions restoring the $O(6)$ symmetry, one expects to find [3]

$$\Delta = \frac{5}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right), \tag{3.10}$$

which is consistent with the negative sign of the anomalous dimension at weak coupling in (3.9), suggesting that it decreases to zero at strong coupling.[15]

[14]Definition of a good conformal operator may require subtraction of a non-zero constant one-point function on the circle, which may depend on the regularization scheme.

[15]It is interesting to notice that the data (3.7), (3.10) about strong-coupling dimensions of Φ_6 near $\zeta = 0$ and near $\zeta = 1$ is consistent with the relation [4] $2\Delta_+ + 2\Delta_- = 2$, i.e. $[\frac{5}{\sqrt{\lambda}} + O(\frac{1}{(\sqrt{\lambda})^2})] + [2 - \frac{5}{\sqrt{\lambda}} + O(\frac{1}{(\sqrt{\lambda})^2})] = 2 + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^2})$. Here $2\Delta_\pm$ are dimensions of perturbations near the two ends of the flow between the Dirichlet and Neumann b.c. which may be interpreted as being driven by the "double-trace"-like operator constructed out of a massless 2d scalar with strong-coupling dimensions $\Delta_+ = 1$ and $\Delta_- = 0$ (see section 4.2).

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As a test of our perturbative calculation of the expectation value (2.67) of the generalized WL (1.1), let us now relate its expansion near the conformal points $\zeta = 0, 1$ to the above expressions for the 2-point functions of the Φ_6 operator. The expectation value of $W^{(\zeta)}$ for the circular contour ($|\dot{x}| = 1$) expanded near $\zeta = 0$ may be written as

$$\langle W^{(\zeta)} \rangle = W^{(0)} \left[1 + \zeta \left\langle\!\!\left\langle \oint d\tau\, \Phi_6(x(\tau)) \right\rangle\!\!\right\rangle + \frac{\zeta^2}{2} \left\langle\!\!\left\langle \oint d\tau\, \Phi_6(x(\tau)) \oint d\tau'\, \Phi_6(x(\tau')) \right\rangle\!\!\right\rangle + \mathcal{O}(\zeta^3) \right], \tag{3.11}$$

where $\langle\!\langle \ldots \rangle\!\rangle$ is defined as in (3.1) but now for $\zeta = 0$, i.e. with only the gauge field coupling $i \int d\tau \dot{x}^\mu A_\mu$ in the exponent and the normalization factor $W^{(0)} \equiv \langle W^{(0)} \rangle$ has weak-coupling expansion given in (2.52). The order ζ (tadpole) term here vanishes automatically as in (3.8) due to the SO(6) symmetry, consistently with the conformal invariance. We may compute the ζ^2 term here

$$\langle W^{(\zeta)} \rangle_{\zeta^2} = \frac{\zeta^2}{2} W^{(0)} \int_0^{2\pi} d\tau \int_0^{2\pi} d\tau'\, \langle\!\langle \Phi_6(\tau)\, \Phi_6(\tau') \rangle\!\rangle \,, \tag{3.12}$$

directly using the conformal 2-point function (3.8) with generic $C(\lambda)$ and $\Delta(\lambda) \equiv 1 + \gamma(\lambda)$. Doing the integral over τ as in (2.12) and then expanding in small λ using (3.9) we obtain[16]

$$\begin{aligned} \langle W^{(\zeta)} \rangle_{\zeta^2} &= \zeta^2 W^{(0)}\, C(\lambda)\, \frac{\pi^{3/2}\, \Gamma\left(-\frac{1}{2} - \gamma(\lambda)\right)}{2^{1+2\gamma(\lambda)}\, \Gamma(-\gamma(\lambda))} \\ &= \zeta^2 W^{(0)}\, C(\lambda)\, \pi^2 \gamma(\lambda) \left[1 + \mathcal{O}(\gamma^2)\right] \;=\; -\zeta^2 \frac{\lambda^2}{64\pi^2} + \mathcal{O}(\lambda^3) \,. \end{aligned} \tag{3.13}$$

This precisely matches the term of order $\lambda^2\zeta^2$ in (2.67). Comparing to the general relation (1.11), the higher order terms in the anomalous dimension $\gamma(\lambda)$ can be absorbed into the relation between $\mathcal{C}$ in (1.8) and C in (3.8).

Next, let us consider the expansion of the WL (1.1), (2.67) near the supersymmetric conformal point $\zeta = 1$. The term of order $\zeta - 1$ in this expansion is expected to vanish by conformal symmetry (provided a possible tadpole contribution is suitably subtracted),[17] and the term of order $(\zeta - 1)^2$ is to be related to the integrated two-point function on the supersymmetric WL

$$\langle W^{(\zeta)} \rangle_{(\zeta-1)^2} = \frac{1}{2} (\zeta - 1)^2\, W^{(1)} \int_0^{2\pi} d\tau \int_0^{2\pi} d\tau'\, \langle\!\langle \Phi_6(\tau)\, \Phi_6(\tau') \rangle\!\rangle_{\zeta=1} \,. \tag{3.14}$$

Inserting here the conformal 2-point function (3.6) and we get the same integral as in (3.12), (3.13). Plugging in the values for $C = \frac{\lambda}{8\pi^2} + O(\lambda^2)$ and $\gamma = \frac{\lambda}{4\pi^2} + \mathcal{O}(\lambda^2)$ from (3.6) we get

$$\langle W^{(\zeta)} \rangle_{(\zeta-1)^2} = \frac{\lambda^2}{32\pi^2} (\zeta - 1)^2 + \mathcal{O}(\lambda^3) \,, \tag{3.15}$$

[16] This integral is similar to the one in (2.24) and thus can be found by an analytic continuation in γ. Alternatively, we may use a cutoff regularization, see appendix A.

[17] As we have seen above, the dimensional regularization scheme that leads to (1.8) and thus implies the vanishing of the tadpole at the conformal point effectively preserves the conformal invariance.

which is indeed in precise agreement with the term of order $(1-\zeta)^2$ in the expansion of (2.67) near $\zeta = 1$

$$\langle W^{(\zeta)}\rangle = \langle W^{(1)}\rangle\left\{1+\frac{\lambda^2}{32\pi^2}\left[(\zeta-1)^2+(\zeta-1)^3+\frac{1}{4}(\zeta-1)^4\right]+\mathcal{O}(\lambda^3)\right\}. \tag{3.16}$$

We may also compare the higher order terms in the small ζ or small $(1-\zeta)$ expansion to integrated higher-point conformal correlators of the $\zeta = 0$ and $\zeta = 1$ CFT's. The absence of the ζ^3 term (and other ζ^{2n+1} terms) in the expansion near the $\zeta = 0$ is in agreement with the vanishing of the odd-point scalar correlators that follows from the $\Phi_m \to -\Phi_m$ symmetry of the SYM action. At the same time, the 3-point scalar Φ_6 correlator at the $\zeta = 1$ point is non-trivial (cf. also [36, 37]). In general, on the 1/2-BPS circular WL we should have

$$\langle\!\langle \Phi_6(\tau_1)\,\Phi_6(\tau_2)\Phi_6(\tau_3)\rangle\!\rangle_{\zeta=1} = \frac{C_3(\lambda)}{\left|2\sin\frac{\tau_{12}}{2}\right|^{\Delta}\left|2\sin\frac{\tau_{23}}{2}\right|^{\Delta}\,\left|2\sin\frac{\tau_{31}}{2}\right|^{\Delta}}\,, \tag{3.17}$$

where at weak coupling $\Delta = 1+\gamma(\lambda)$ is the same as in (3.6), i.e. $\gamma = \frac{\lambda}{4\pi^2}+\mathcal{O}(\lambda^2)$, and we should have $C_3 = c_3\lambda^2+\mathcal{O}(\lambda^3)$. Integrating (3.17) using (2.11) and then expanding in small λ we get as in (3.13), (3.14)

$$\begin{aligned}\langle W^{(\zeta)}\rangle_{(\zeta-1)^3} &= \frac{1}{3!}(\zeta-1)^3\,\langle W^{(1)}\rangle \oint d\tau_1\, d\tau_2\, d\tau_3\, \langle\!\langle \Phi_6(\tau_1)\Phi_6(\tau_2)\Phi_6(\tau_3)\rangle\!\rangle_{\zeta=1} \\ &= (\zeta-1)^3\langle W^{(1)}\rangle\, C_3\,\frac{\pi^{3/2}\,\Gamma\left(-\frac{\gamma}{2}\right)\Gamma\left(-\frac{1}{2}-\frac{3}{2}\gamma\right)}{3\cdot 2^{1+3\gamma}\,[\Gamma(-\gamma)]^3} = -\frac{8}{3}\pi^2(\zeta-1)^3 C_3\big[1+\mathcal{O}(\lambda)\big].\end{aligned} \tag{3.18}$$

Comparing (3.18) to (3.16) we conclude that

$$C_3 = -\frac{3\lambda^2}{256\pi^4}+\mathcal{O}(\lambda^3)\ . \tag{3.19}$$

The ζ^4 term in the expansion of (2.67) should be related to the integrated value of the 4-point correlator of Φ_6. To λ^2 order it is given just by the product of the two 2-point contributions (corresponding to the two ladder graphs; the third ordering is subleading in the planar limit)

$$\begin{aligned}\langle\!\langle \Phi_6(\tau_1)\Phi_6(\tau_2)\Phi_6(\tau_3)\Phi_6(\tau_4)\rangle\!\rangle =& \big[G_0(\tau_1,\tau_2)\,G_0(\tau_3,\tau_4)+G_0(\tau_1,\tau_4)\,G_0(\tau_2,\tau_3) \\ &+\mathcal{O}(\lambda^3)\big]\theta(1,2,3,4)+\text{permutations}\,,\end{aligned} \tag{3.20}$$

where $G_0(\tau_1,\tau_2) = \frac{\lambda}{8\pi^2}\frac{1}{|2\sin\frac{\tau_{12}}{2}|^2}$ is the leading term in the 2-point correlator (3.8) of Φ_6 at $\zeta = 0$ and $\theta(1,2,3,4) = \theta(\tau_1-\tau_2)\theta(\tau_2-\tau_3)\theta(\tau_3-\tau_4)$.

To understand the precise relation between the integrated 4-point correlator and the ζ^4 term in $\langle W^{(\zeta)}\rangle$ in (2.67) one should follow the logic of conformal perturbation theory by a nearly-marginal operator O with dimension $\Delta = d-\epsilon$ (see, e.g., [18]). In the present case of $d = 1$ near the $\zeta = 0$ point we have $O = \Phi_6$ with dimension $\Delta = 1-\epsilon$, $\epsilon \equiv -\gamma = \frac{\lambda}{8\pi^2}+\ldots \ll 1$ (see (3.9)). Then the dimension 1 perturbation $\zeta_{\rm b}O$ where the bare coupling $\zeta_{\rm b}$ is related to the dimensionless renormalized one by $\zeta_{\rm b} = \mu^\epsilon(\zeta+\frac{\lambda}{16\pi^2\epsilon}\zeta^3+\ldots)$ corresponding to the beta-function (1.4), i.e. $\beta_\zeta = -\epsilon\,\zeta + \frac{\lambda}{8\pi^2}\zeta^3 + \ldots$. Computing $\langle W^{(\zeta)}\rangle$ in an expansion in powers of ζ_b we get for the λ^2 term: $\langle W^{(\zeta)}\rangle = \langle W^{(0)}\rangle\big[1 + \lambda^2(k_2\zeta_{\rm b}^2 + k_4\zeta_{\rm b}^4) + \mathcal{O}(\zeta_{\rm b}^6)\big]$ where $k_2 = -\frac{1}{64\pi^2}$ is the contribution of the integrated 2-point function given by (3.13) and $k_4 = \frac{1}{64\pi^2}(\pi^2 + \frac{1}{2}\pi^2)$ is the contribution of the integral of (3.20), i.e. the sum of the ζ^4 terms in the two ladder diagrams in (2.56). Similarly to what happened in the dimensional regularization case in (2.63), here the quadratic term contributes to the quartic one once expressed in terms of the renormalized coupling. Using $\zeta_{\rm b} = \zeta + \frac{1}{2}\zeta^3 + \ldots$ we get $k_2\zeta_{\rm b}^2 + k_4\zeta_{\rm b}^4 = k_2\zeta^2 + k_4'\zeta^4 + \ldots$, where $k_4' = k_4 + k_2 = \frac{1}{128\pi^2}$ which is in agreement with the ζ^4 coefficient in (2.67). Similar considerations should apply to the $(\zeta-1)^4\lambda^2$ term in the expansion (3.16) near $\zeta = 1$.

4 Strong coupling expansion

As discussed in [3, 4], the $\mathrm{AdS}_5 \times S^5$ string description of the standard Wilson loop should be given by the path integral with Dirichlet boundary condition along the boundary of AdS_5 and Neumann (instead of Dirichlet for the Wilson-Maldacena loop) condition for the S^5 coordinates. The case of the generalized WL (1.1) may then correspond to mixed boundary conditions [4]. Below we shall first discuss the subleading strong-coupling correction to the standard WL ($\zeta = 0$) comparing it to the more familiar WML ($\zeta = 1$) case and then consider the general ζ case.

The strong coupling expansion of the straight-line or circular WL will be represented by the string partition function with the same AdS_2 world-sheet geometry as in the WML case [38]. As the AdS_2 is a homogeneous-space, the log of the string partition function should be proportional to the volume of AdS_2 [39, 40]. In the straight-line case the volume of AdS_2 with infinite ($L \to \infty$) line as a boundary is linearly divergent as $\frac{L}{a}$. Thus the straight-line WL should be given just by an exponent of this linear 2d IR divergence. Linear UV divergences in WL for a smooth contour are known to factorize in general at weak coupling [8]. After the separation of this linear divergence the straight-line WL should be thus equal to 1 as in the case of the locally-supersymmetric WML. The same should be true for the generalized WL (1.1).

Similar arguments apply in the case of the circular WL where the minimal surface is again the AdS_2 but now with a circle as its boundary. In this case the volume is (we fix the radius to be 1)

$$V_{\mathrm{AdS}_2} = 2\pi\left(\frac{1}{a} - 1\right), \tag{4.1}$$

i.e. has a finite part and thus the expectation value may be a non-trivial function of string tension $\frac{\sqrt{\lambda}}{2\pi}$. After factorizing the linearly divergent factor, the leading strong-coupling term will then have a universal $\sqrt{\lambda}$ form

$$\langle W^{(\zeta)}\rangle \equiv e^{-F^{(\zeta)}(\lambda)}\,, \qquad F^{(\zeta)} = -\sqrt{\lambda} + F_1^{(\zeta)} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{4.2}$$

The subleading terms $F_1^{(\zeta)} + \ldots$ will, however, differ due to the different boundary conditions in the S^5 directions.

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4.1 Standard Wilson loop

Let us consider the 1-loop string correction in the standard WL case following the same approach as used in the WML case in [39–41]. As the fluctuation determinants for all the 2d fields (3 AdS_5 bosons with m^2=2, 8 fermions with $m^2 = 1$ and ghosts) except the S^5 massless scalars are the same, the ratio of the WML and WL expectation values (1.2) should be proportional the ratio of the 1-loop string partition functions with the Dirichlet and Neumann boundary conditions in the five S^5 directions:

$$\frac{\langle W^{(1)}\rangle}{\langle W^{(0)}\rangle} = \frac{e^{-F^{(1)}}}{e^{-F^{(0)}}} = \mathcal{N}_0^{-1} \left[\frac{\det(-\nabla^2)_{\rm D}}{\det'(-\nabla^2)_{\rm N}}\right]^{-5/2} \left[1+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right] . \tag{4.3}$$

Here $-\nabla^2$ is the massless scalar wave operator in AdS_2 and $\mathcal{N}_0$ is the normalization factor of the S^5 zero modes present in the Neumann case

$$\mathcal{N}_0 = c_0\, (\sqrt{\lambda})^{5/2} , \tag{4.4}$$

with c_0 being a numerical constant (representing contributions of renormalized volume of AdS_2 and volume of S^5).[18] The 1-loop corrections to $F^{(\zeta)}$ are thus related by

$$F_1^{(1)} - F_1^{(0)} = 5 \left[\frac{1}{2}\log\det(-\nabla^2)_{\rm D} - \frac{1}{2}\log\det'(-\nabla^2)_{\rm N}\right] + \log\mathcal{N}_0 \ . \tag{4.5}$$

To compute this correction we may use the general result for the difference of effective actions with standard (D or +) and alternate (N or -) boundary conditions for a scalar with mass m in AdS_{d+1} [43, 44]

$$\begin{aligned}\delta\Gamma = \Gamma_+ - \Gamma_- &= \frac{1}{2}\log\det(-\nabla^2+m^2)_{\rm D} - \frac{1}{2}\log\det(-\nabla^2+m^2)_{\rm N} \\ &= \frac{1}{2}\sum_{\ell=0}^{\infty} c_{d,\ell}\,\log\frac{\Gamma\left(\ell+\frac{d}{2}-\nu\right)}{\Gamma\left(\ell+\frac{d}{2}+\nu\right)}, \qquad c_{d,\ell} = (2\ell+d-1)\,\frac{(\ell+d-2)!}{\ell!(d-1)!} ,\end{aligned} \tag{4.6}$$

where ν is defined by

$$m^2 = \Delta(\Delta-d) , \qquad \Delta_\pm = \frac{d}{2}\pm\nu, \qquad \nu\equiv\sqrt{\frac{d^2}{4}+m^2} \ . \tag{4.7}$$

In the present case of $d=1$ and $m=0$ the $\ell=0$ term with $\Gamma\left(\ell+\frac{d}{2}-\nu\right)$ is singular and should be dropped: this corresponds to projecting out the constant 0-mode present in the Neumann case. Then in the limit $d\to 1$ and $\nu\to\frac{1}{2}$ in (4.6) we get (projecting out 0-mode)

$$\delta\Gamma' = -\sum_{\ell=1}^{\infty}\log\ell = \lim_{s\to 0}\frac{d}{ds}\sum_{\ell=1}^{\infty}\ell^{-s} = \zeta_{\rm R}'(0) = -\frac{1}{2}\log(2\pi) \ . \tag{4.8}$$

One may also give an alternative derivation of (4.8) using the relation between the AdS_{d+1} bulk field and S^d boundary conformal field partition functions: $Z_-/Z_+ = Z_{\rm conf}$ (see [44–46]). For a massive scalar in AdS_{d+1} associated to an operator with dimension

[18] In general, one is to separate the 0-mode integral and treat it exactly (cf. [42]).

Δ_+, the boundary conformal (source) field has canonical dimension $\Delta_- = d - \Delta_+$ and thus the kinetic term $\int d^d x\, \varphi(-\partial^2)^\nu \varphi$, with $\nu = \Delta_+ - \frac{d}{2}$. In the present case of the massless scalar in AdS_2 we have $d = 1$, $\Delta_+ = 1$, $\Delta_- = 0$ and $\nu = \frac{1}{2}$. The induced boundary CFT has thus the kinetic operator $\partial \equiv (-\partial^2)^{1/2}$ defined on S^1 and thus we find again (4.8)

$$\delta\Gamma' = -\log \frac{Z_+}{Z_-} = -\frac{1}{4} \log \det{}'(-\partial^2) = -\sum_{\ell=1}^{\infty} \log \ell \ , \tag{4.9}$$

where we fixed the normalization constant in the S^1 eigen-value to be 1.

It is interesting to note that the zero-mode contribution in (4.5) may be included automatically by "regularizing" the $m \to 0$ or $\nu \to \frac{1}{2}$ limit in (4.6), (4.7). One may expect that for the Neumann boundary conditions which are non-supersymmetric in the world-sheet theory [4] the massless S^5 scalars y^a may get 1-loop correction to their mass $m^2 = -\frac{k}{\sqrt{\lambda}} + \mathcal{O}\big(\frac{1}{(\sqrt{\lambda})^2}\big) \to 0$.[19] Then $\nu = \frac{1}{2} - \frac{k}{\sqrt{\lambda}} + \dots$ and $\Delta_- = \frac{k}{\sqrt{\lambda}} + \dots$; for the agreement with (3.10) we need to fix $k = 5$. We then get an extra $-\frac{1}{2}\log|m^2| = -\frac{1}{2}\log\frac{k}{\sqrt{\lambda}}$ term from the $\ell = 0$ term in (4.6), i.e.

$$\delta\Gamma = \delta\Gamma' - \frac{1}{2}\log|m^2| = -\frac{1}{2}\log(2\pi) + \frac{1}{2}\log\sqrt{\lambda} - \frac{1}{2}\log k \ . \tag{4.10}$$

This agrees with (4.5), (4.8) if we set $c_0 = k^{-5/2}$. Finally, from (4.5), (4.6) we find

$$F_1^{(0)} = F_1^{(1)} - 5\delta\Gamma = F_1^{(1)} - 5\,\delta\Gamma' - \log\mathcal{N}_0 = F_1^{(1)} + \frac{5}{2}\log(2\pi) - \left(\frac{5}{2}\log\sqrt{\lambda} + \log c_0\right) . \tag{4.11}$$

Let us now recall that the direct computation of the determinants in the string 1-loop partition function for the circular WML gives (after using (4.1) and separating out the linear divergence) [39–41] (see also [49–51])

$$F_1^{(1)} = \frac{1}{2}\log(2\pi) \ . \tag{4.12}$$

At the same time, the exact gauge-theory result (1.5) for the WML implies that the total correction to the leading strong-coupling term should, in fact, be

$$F_{1\,\rm tot}^{(1)} = \frac{1}{2}\log(2\pi) - \log 2 + \frac{3}{2}\log\sqrt{\lambda} \ . \tag{4.13}$$

The $\frac{3}{2}\log\sqrt{\lambda}$ term may be attributed to the normalization of the three Möbius symmetry zero modes on the disc [14], but the remaining $\log 2$ difference still remains to be understood.

It is then natural to conjecture that for the standard WL expanded at strong coupling the total value of the subleading term at strong coupling should be given by (4.11) where the first term is replaced by (4.13), i.e.

$$F_{1\,\rm tot}^{(0)} = F_{1\,\rm tot}^{(1)} + \frac{5}{2}\log(2\pi) + \log\mathcal{N}_0 = \ 3\log(2\pi) - \log(2c_0) - \log\sqrt{\lambda} \ . \tag{4.14}$$

[19] This correction may be found by computing the 1-loop contribution to the propagator of y^a in AdS_2 background. Similar correction to scalar propagator with alternate b.c. should appear in higher spin theories in the context of vectorial AdS/CFT (there the effective coupling is $1/N$ instead of $1/\sqrt{\lambda}$), see e.g. [47]. Note that having a correction to the mass of a world-sheet excitation here does not run against the usual 2d conformal invariance constraint as we are expanding near a non-trivial background and are effectively in a physical gauge where the conformal freedom is fixed (cf. [48]).

We then conclude that while the leading $\lambda \gg 1$ prediction for the log of the expectation value $\tilde{F}^{(\zeta)} \equiv \log\langle W^{(\zeta)}\rangle = -F^{(\zeta)}_{\rm tot}$ for the circular WML and WL is the same $\sqrt{\lambda}$ in (4.2), the subleading term in $\tilde{F}^{(0)}$ is larger than that in $\tilde{F}^{(1)}$ by $\log \mathcal{N}_0 = \frac{5}{2}\log\sqrt{\lambda} + \dots$. This appears to be in agreement with a similar behavior (1.9) observed at weak coupling and thus with the 1d analog of the F-theorem (1.10).

While the strong-coupling behaviour of WML $\langle W^{(1)}\rangle \sim (\sqrt{\lambda})^{-3/2} e^{\sqrt{\lambda}} + \dots$ follows from the exact Bessel function expression in (1.5), one may wonder which special function may give the above strong-coupling asymptotics $\langle W^{(0)}\rangle \sim \sqrt{\lambda}\, e^{\sqrt{\lambda}} + \dots$ of the standard WL.

4.2 General case

Turning to the case of generic $0 < \zeta < 1$, one may imagine computing $\langle W^{(\zeta)}(\lambda)\rangle$ exactly to all orders in the weak-coupling expansion and expressing it in terms of the renormalized coupling ζ (in some particular scheme). One may then re-expand the resulting function at strong coupling (as in (1.5)) expecting to match $F^{(\zeta)}_1$ in (4.2) with (4.13) and (4.14) at the two conformal points.

A way to set up the strong-coupling (string-theory) computation for an arbitrary value of ζ may not be a priori clear as non-conformal WL operators need not have a simple string-theory description. Below we shall develop a heuristic but rather compelling suggestion of [4]. Starting with the $AdS_5 \times S^5$ string action and considering a minimal surface ending, e.g., on a line at the boundary of AdS_5 we may choose a static string gauge where $x^0 = \tau,\ z = \sigma$ so that the induced metric is the AdS_2 one: $ds^2 = \frac{1}{\sigma^2}(d\sigma^2 + d\tau^2)$; in what follows we identify z and σ.[20] Let the 5 independent S^5 coordinates be y^a (with the embedding coordinates being, e.g., $Y_a = \frac{y_a}{1+\frac{1}{4}y^2}$, $Y_6 = \frac{1-\frac{1}{4}y^2}{1+\frac{1}{4}y^2}$, $ds^2_{S^5} = \frac{dy_a dy_a}{(1+\frac{1}{4}y^2)^2}$). In the WL case they are subject to the Neumann condition $\partial_z y^a\big|_{z\to 0} = 0$. One may then start with this Neumann (i.e. standard WL) case and perturb the corresponding string action $I^{(0)}$ by a boundary term that should induced the flow towards the other (Dirichlet or WML) fixed point

$$I(\varkappa) = I^{(0)} + \delta I\,, \qquad I^{(0)} = T\int d\tau dz \Big(\frac{1}{2}\sqrt{h}h^{pq}\partial_p y^a \partial_q y^a + \dots\Big), \quad T = \frac{\sqrt{\lambda}}{2\pi}\,, \tag{4.15}$$

$$\delta I = -\varkappa T \int d\tau\, Y_6\,, \quad Y_6 = \sqrt{1 - Y_a Y_a} = 1 - \frac{1}{2}y_a y_a + \dots\ . \tag{4.16}$$

In $I^{(0)}$ we give only the part depending quadratically on S^5 coordinates and h_{mn} is the induced AdS_2 metric.

Here $\varkappa$ is a new coupling constant which should be a strong-coupling counterpart of ζ: $\varkappa = 0$ should correspond to $\zeta = 0$ and $\varkappa = \infty$ to $\zeta = 1$. Y_6 is then the counterpart of the operator Φ_6 in (1.1) perturbing the $\zeta = 0$ conformal point at weak coupling.

Note that for the AdS_2 metric $ds^2 = z^{-2}(dz^2 + d\tau^2)$ with the boundary at $z = a \to 0$ the boundary metric is $ds = a^{-1}d\tau$ and thus it may be more natural to write δI in (4.16) as $\delta I = -\kappa T \int ds\, Y_6$ so that $\varkappa = a^{-1}\kappa$. Then κ will always appear together with the AdS_2 IR cutoff factor a^{-1} which, on the other hand, can be also interpreted — from the world-sheet theory point of view — as playing the same role as a UV cutoff Λ.

[20] In the case of the circular boundary $ds^2 = \frac{1}{\sinh^2\sigma}(d\sigma^2 + d\tau^2)$.

The variation of the action $I(\varkappa)$ implies that to linear order in y^a it should satisfy the massless wave equation in AdS_2 (so that near the AdS_2 world-sheet boundary $y^a = z^{\Delta_+}u^a + z^{\Delta_-}v^a + \mathcal{O}(z^2) = z\,u^a + v^a + \mathcal{O}(z^2)$) subject to the mixed (Robin) boundary condition[21]

$$(-\partial_z + \varkappa)y^a\big|_{z\to 0} = 0\,, \qquad \text{i.e.} \qquad -u^a + \varkappa\, v^a = 0\,. \tag{4.17}$$

The parameter $0 \le \varkappa \le \infty$ thus interpolates between the Neumann and Dirichlet boundary conditions conditions. Note that in general one may add, instead of Y_6, in (4.16) any linear combination $\theta^m Y_m$ with $\theta_m^2 = 1$ (cf. (1.1)) and the S^5 part of (4.15) as $\partial^p Y^m \partial_p Y_m$, with $Y^m Y_m = 1$. Then the boundary condition becomes $\big[-\partial_z Y_m + \varkappa(\theta_m - \theta^k Y_k Y_m)\big]\big|_{z\to 0} = 0$. For θ_m along 6-th axis this reduces to (4.17) to linear order in y_a.

Like ζ at weak coupling (1.4), the new boundary coupling $\varkappa$ will need to be renormalized, i.e. it will be running with 2d UV scale.[22] In (4.16) $\varkappa$ is a renormalized coupling of effective mass dimension 1. In general, in the bare action one should have $\delta I_{\rm b} = -\Lambda\varkappa_{\rm b} T\int d\tau Y_6$, where $\Lambda\varkappa_{\rm b} = \mu\varkappa\big[1 + K(\frac{1}{\sqrt{\lambda}})\log\frac{\Lambda}{\mu}\big] + \ldots$, with $\Lambda \to \infty$ being a UV cutoff and $\varkappa_b$ and $\varkappa$ being dimensionless. We may choose the renormalization scale μ to be fixed as $\mu = R^{-1}$ in terms of the radius R and set $R = 1$, i.e. measuring scales in units of R; then we may effectively treat $\varkappa$ as dimensionless.[23]

Dimensionless renormalized $\varkappa$ should be a non-trivial (scheme-dependent) function of the renormalized dimensionless parameter ζ and the string tension or 't Hooft coupling λ

$$\varkappa = {\rm f}(\zeta;\lambda)\,, \qquad {\rm f}(0;\lambda) = 0\,, \quad {\rm f}(1;\lambda \gg 1) = \infty\,. \tag{4.18}$$

Lack of information about this function prevents one from direct comparison of weak-coupling and strong-coupling pictures. Just as an illustration, one may assume that at large λ one has $\varkappa = \frac{\zeta}{1-\zeta}$, ensuring the right limits (cf. (4.17)).

The boundary $\varkappa$-term in (4.15) may be viewed as a special case of an "open-string tachyon" coupling depending on S^5 coordinates:

$$\delta I_{\rm b} = \Lambda\int d\tau\, \mathcal{T}_{\rm b}(y)\,, \qquad \Lambda\mathcal{T}_{\rm b} = \mu\Big[\mathcal{T} - \log\frac{\Lambda}{\mu}(\alpha' D^2 + \ldots)\mathcal{T} + \ldots\Big]\,, \tag{4.19}$$

$$\beta_{\mathcal{T}} = \mu\frac{d\mathcal{T}}{d\mu} = -\mathcal{T} - \alpha' D^2\mathcal{T} + \ldots\,, \qquad \alpha' = \frac{R^2}{\sqrt{\lambda}}\,. \tag{4.20}$$

Here D^2 is the Laplacian on S^5 (of radius R that we set to 1) and $\beta_{\mathcal{T}}$ is the corresponding renormalization group function [52, 53].[24] The $\mathcal{T} = \varkappa Y_6$ term in $I(\varkappa)$ in (4.15) is the

[21]The tangent vector to the boundary is $t^p = (0, z)$ and the outward normal to the boundary is $n^p = (-z, 0)$, so that $h^{pq} = n^p n^q + t^p t^q$.

[22]As already mentioned above, in the present case of the boundary of the AdS_2 world sheet being at $z \to 0$ it is natural to add to the boundary term a factor of $z^{-1} = a^{-1} \to \infty$ that may then be interpreted as playing the same role as the world-sheet UV cutoff Λ; then this running may be interpreted as a flow with AdS_2 cutoff.

[23]In the case of the circular boundary the dependence on the radius R that drops out at the conformal points remains for generic value of $\varkappa$ or ζ. One may fix, for example, $\mu R = 1$ as a renormalization condition, or rescale $\varkappa$ by R to make it dimensionless.

[24]Similar expression for the closed-string tachyon beta-function has familiar extra factors of 2 and $\frac{1}{2}$: $\beta_{\rm T} = -2{\rm T} - \frac{1}{2}\alpha'\nabla^2{\rm T} + \ldots$.

eigen-function of the Laplacian with eigenvalue 5 (e.g. for small y_a one has $D^2 Y_6 = (\partial_y^2 + \ldots)(-\frac{1}{2} y_a y_a + \ldots) = -5 + \ldots$).[25] As a result, we should expect to find that $\varkappa$ should be renormalized according to

$$\Lambda \varkappa_{\rm b} = \mu\varkappa\Big(1 + \frac{5}{\sqrt{\lambda}} \log\frac{\Lambda}{\mu} + \ldots\Big)\,, \qquad \beta_\varkappa = \mu\frac{d\varkappa}{d\mu} = \Big(-1 + \frac{5}{\sqrt{\lambda}} + \ldots\Big)\varkappa + \ldots\,. \quad (4.21)$$

This beta-function then gives another derivation of the strong-coupling dimension (3.10) of the perturbing operator near the WL ($\zeta = 0$) or $\varkappa = 0$ fixed point: the coefficient of the linear term in the beta-function should be the anomalous dimension or $\Delta - 1$.[26] This operator identified as Φ_6 from the weak-coupling point of view is thus naturally associated with the quadratic $y_a y_a$ perturbation in (4.16) [6, 35].

Note that in the opposite WML ($\zeta = 1$) or $\varkappa \to \infty$ limit we may expect to find the same linear beta-function but with the opposite coefficient, as seen by rewriting the RG equation in (4.21) as $\mu\frac{d\varkappa^{-1}}{d\mu} = -\big(-1 + \frac{5}{\sqrt{\lambda}} + \ldots\big)\varkappa^{-1} + \ldots$, with now $\varkappa^{-1} \to 0$ (an alternative is to reverse the UV and IR limits, i.e. $\log\mu \to -\log\mu$). Then the strong-coupling dimension of Φ_6 should be given by $\Delta - 1 = 1 - \frac{5}{\sqrt{\lambda}} + \ldots$ in agreement with (3.7).

Another way to derive (4.21) is to use the general expression for the divergence of the determinant of a 2d scalar Laplacian in curved background subject to the Robin boundary condition $(\partial_n + \kappa)\phi\big|_\partial = 0$ as in (4.17) [54, 55] (see also appendix B in [56] for a review)

$$\Gamma_\infty = \frac{1}{2}\log\det(-\nabla^2 + X)\Big|_\infty = -\frac{1}{2}A_0\Lambda^2 - A_1\Lambda - A_2\log\Lambda\,, \quad (4.22)$$
$$A_0 = \frac{1}{4\pi}\int d^2x\sqrt{g}\,, \quad A_1 = \frac{1}{8\sqrt{\pi}}\int_\partial ds\,, \qquad A_2 = \frac{1}{6}\chi - \frac{1}{4\pi}\int d^2x\sqrt{g}X - \frac{1}{2\pi}\int_\partial ds\,\kappa\,.$$

Here χ is the Euler number and $L = \int_\partial ds$ is the length of the boundary. In the present massless case $X = 0$ and for the Euclidean AdS$_2$ we have $\chi = 1$. For the circular boundary at $z = a \to 0$ we have (for $R = 1$) $L = 2\pi a^{-1}$. To compare this to (4.17) we note that for an outward normal to the boundary of AdS$_2$ we have $(\partial_n + \kappa)\phi\big|_\partial = (-z\partial_z + \kappa)\phi\big|_{z=a}$ so that we need to identify $a^{-1}\kappa$ with $\varkappa$ in (4.17). Taking into account the factor of 5 for massless scalars y_a we thus find the same $\varkappa\log\Lambda$ divergence as in (4.21).[27]

[25] In general, the eigenfunctions of Laplacian on S^5 are $C_{m_1\ldots m_J}Y^{m_1}\ldots Y^{m_J}$ (where $C_{m_1\ldots m_J}$ is totally symmetric and traceless) with eigenvalue $J(J+4)$. For example, one may consider $(Y_1 + iY_2)^J$. In $J = 1$ case we may choose any linear combination $C_m Y^m$ or any of six Y_m which will have the eigenvalue 5.

[26] To recall, the argument for the strong-coupling dimension $\Delta(0) = \frac{5}{\sqrt{\lambda}} + \ldots$ of the scalar operator on the WL in [3] was based on considering AdS$_2$ in global coordinates as conformal to a strip $ds^2 = \frac{1}{\sin^2\sigma}(dt^2 + d\sigma^2)$ where $0 \le \sigma < \pi$. Then the Hamiltonian with respect to global time is the dilatation operator and the mode constant in σ should be the primary, and its energy is the conformal dimension. The Hamiltonian of quantized massless particle moving on S^5 is then proportional to the Laplacian on S^5 with the eigenvalue $\frac{\alpha'}{R^2}J(J+4)$ with the present case being that of $J = 1$ (in the $\zeta = 0$ case the dimension of all 6 scalars is the same due to unbroken $O(6)$ symmetry).

[27] Note that (4.22) directly applies only for a finite non-zero $\varkappa$ (including $\varkappa = 0$ of the Neumann condition). In the Dirichlet case ($\kappa \to \infty$) the sign of A_1 is reversed and the boundary contribution to the logarithmic divergence (the last term in A_2) is absent. Thus the D-limit or $\varkappa \to \infty$ can not be taken directly in (4.22) (see also [57]). The logarithmic χ divergence and the quadratic divergence are universal, so they cancel in the difference of effective actions with different boundary conditions. Linear divergence has the opposite sign for the Dirichlet and Neumann or Robin b.c.; that means it cancels in the difference of effective actions for the Robin and the Neumann conditions (4.25).

Explicitly, in the case of 5 massless scalars in AdS_{d+1} with spherical boundary and mixed boundary conditions (4.17) the analog of (4.6) gives [43, 44] (see eqs. (3.2), (5.2) in [44])

$$F_1(\varkappa) - F_1(0) = \frac{5}{2}\sum_{\ell=0}^{\infty} c_{d,\ell}\, \log\left(1+\varkappa\, q_\ell\right), \qquad q_\ell = \frac{2^{2\nu}\,\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{\Gamma\left(\ell+\frac{d}{2}-\nu\right)}{\Gamma\left(\ell+\frac{d}{2}+\nu\right)}R^{2\nu}, \quad (4.23)$$

where $c_{d,\ell}$ is the same as in (4.6) and in the present $d=1$ case $c_{d,0}=1$, $c_{d,\ell>0}=2$. Since $\varkappa$ and ζ are related by (4.18) the connection to previous notation in (4.5), (4.11), (4.12) is

$$F(\varkappa) \equiv F^{(\zeta)}, \qquad F(\infty) \equiv F^{(1)}, \qquad F(0) \equiv F^{(0)}\,. \quad (4.24)$$

Then from (4.23)

$$F_1(\varkappa) - F_1^{(0)} = 5\sum_{\ell=1}^{\infty}\log\left(1+\varkappa\,\ell^{-1}\right) + \tfrac{5}{2}\log\left(1+\varkappa\,|m^{-2}|\right)\,. \quad (4.25)$$

Here we effectively set the radius R to 1 absorbing it into $\varkappa$ (which will then be dimensionless) and isolated the contribution of the $\ell=0$ mode (using that for $m^2\to 0$ we have $\nu = \frac{1}{2}+m^2+\ldots$). The limit $\varkappa\to 0$ of (4.25) is smooth provided it is taken before $m^2\to 0$ one. The limit $\varkappa\to\infty$ in (4.25) may be formally taken before the summation and then (using $\sum_{\ell=1}^{\infty} 1+\frac{1}{2} = \zeta_{\rm R}(0)+\frac{1}{2}=0$) we recover the previous $\zeta=1$ result in (4.8), (4.10), (4.11).

Using (4.8), (4.10), (4.11) we may instead consider the difference between $F_1(\varkappa)$ and $F_1(\infty)\equiv F_1^{(1)}$, i.e.

$$F_1(\varkappa) - F_1(\infty) = 5\sum_{\ell=1}^{\infty}\log\left(\ell+\varkappa\right) + \tfrac{5}{2}\log\varkappa\,, \qquad \varkappa>0\,. \quad (4.26)$$

where we assumed $\varkappa>0$ to drop 1 in the log in the second term in (4.25) and observed that the constant S^5 zero mode contribution $\sim\log|m^2|$ which is present only in the N-case ($\varkappa=0$) then cancels out. An alternative is to rewrite (4.26) in the form that has regular expansion near $\varkappa=\infty$

$$F_1(\varkappa) - F_1(\infty) = 5\sum_{\ell=1}^{\infty}\log(1+\varkappa^{-1}\ell)\,, \quad (4.27)$$

where we used again the zeta-function regularization ($\zeta_{\rm R}(0)+\frac{1}{2}=0$). Note that this expression comes out of the general expression in [44] or (4.23) if we interchange the roles of Δ_+ and Δ_- (i.e. set $\nu=-\frac{1}{2}$) and replace $\varkappa\to\varkappa^{-1}$. The infinite sum in (4.26) or (4.27) contains the expected logarithmic UV divergence as in (4.21), (4.22) ($\epsilon=\Lambda^{-1}\to 0$) as can be seen using an explicit cutoff, $\sum_{\ell=1}^{\infty} e^{-\epsilon\ell}\log\left(\ell+\varkappa\right) \to \varkappa\sum_{\ell=1}^{\infty} e^{-\epsilon\ell}\ell^{-1}+\ldots = -\varkappa\log\epsilon+\ldots$ (we ignore power divergence as in (4.8)). In general, the term linear in $\varkappa$ in the finite part is thus scheme-dependent. The finite part of (4.27) can be found using derivative of the Hurwitz zeta-function or simply expanding the log in powers of $\varkappa^{-1}\ell$ and then using the zeta-function to define the sum over ℓ. As a result,

$$F_{1\,\mathrm{fin}}(\varkappa) - F_1(\infty) = 5\varkappa(\log\varkappa - 1) - 5\log\big[\Gamma(1+\varkappa)\big] + \tfrac{5}{2}\log(2\pi\varkappa) \qquad (4.28)$$
$$= 5\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,\zeta_{\mathrm{R}}(-n)\,\varkappa^{-n} = -\frac{5}{12\varkappa} + \frac{1}{72\varkappa^3} - \frac{1}{252\varkappa^5} + \mathcal{O}\left(\frac{1}{\varkappa^7}\right).$$

Taken with the opposite sign, i.e. $\tilde{F}_{1\,\mathrm{fin}}(\varkappa) - \tilde{F}_1(\infty)$, this expression is a positive monotonically decreasing function which is consistent with the F-theorem (1.9), (1.10).

5 Concluding remarks

In this paper we computed the λ^2 term in the expectation value of the generalized circular Wilson loop (1.6) depending on the parameter ζ. The computation is considerably more involved than in the Wilson-Maldacena loop case [13]. In particular, in dimensional regularization, to obtain the finite λ^2 part one needs to take into account the "evanescent" dependence of the 1-loop term on the bare value of ζ. It would be useful to extend the perturbative computation of $\langle W^{(\zeta)}\rangle$ to λ^3 order to see if the ladder diagrams may still be giving the most relevant contributions, with a hope to sum them up to all orders (at least in the standard WL case).

The circular loop expectation value $\langle W^{(\zeta)}\rangle$ admits a natural interpretation as a special $d = 1$ case of a partition function on d-sphere and thus satisfies a $d = 1$ analog of F-theorem: we demonstrated the inequality (1.9) at first subleading orders at both weak and strong coupling.

The 2-loop term (1.6) in $\langle W^{(\zeta)}\rangle$ determined in this paper effectively encodes several previously known results about the defect CFT_1 defined on the Wilson line: the 1-loop beta-function for ζ [4] and the related anomalous dimensions of the scalar operator Φ_6 near the two conformal points $\zeta = 1$ and $\zeta = 0$ [3]. It would be interesting to further study the spectrum and correlation functions of operator insertions on the non-supersymmetric ($\zeta = 0$) Wilson line. A particularly interesting insertion is the displacement operator $D_i \sim F_{ti}$, which has protected dimension $\Delta = 2$ as a consequence of conformal symmetry (see e.g. [58]). The normalization of its two-point correlation function is an important observable of the CFT, which should be a non-trivial function of the 't Hooft coupling. This observable is also expected to appear in the small angle expansion of the cusp anomalous dimension, or in the expectation value of the WL at second order of small deformations of the loop around the circular shape. In the case of the supersymmetric Wilson-Maldacena loop, the analogous observable, known as "Bremsstrahlung function", can be determined exactly by localization [34] as well as integrability [59, 60]. It would be very interesting to find the corresponding quantity in the non-supersymmetric Wilson loop case.

Motivated by the 2-loop expression (1.6) one may make a bold conjecture[28] that to all orders in λ the renormalized expression for the circular loop will depend on ζ only through

[28] We thank R. Roiban for a discussion of the possible exact structure of $\langle W^{(\zeta)}\rangle$ and this suggestion.

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the combination $(1-\zeta^2)\lambda$, i.e. will have the form

$$\langle W^{(\zeta)}\rangle = W^{(1)}(\lambda)\Big[1+\mathcal{Z}\big((1-\zeta^2)\lambda\big)\Big]\,, \qquad\qquad \mathcal{Z}(x)=\sum_{n=2}^{\infty}c_n x^n\,, \tag{5.1}$$

where $W^{(1)}(\lambda)$ is the exact expression for the WML given in (1.5). If (in some particular renormalization scheme) all $c_n>0$ then for $0\le\zeta\le1$ this function will have the minimum at $\zeta=1$ and the maximim at $\zeta=0$, in agreement with the expected structure of the β-function in (1.8) and the F-theorem (1.9). The standard WL expectation value will be given by $W^{(0)}(\lambda)=W^{(1)}(\lambda)\big[1+\mathcal{Z}(\lambda)\big]$. One may also try to determine the coefficients c_n by using that at each λ^n order the term ζ^{2n} with the highest power of ζ should come from the ladder graphs. The large λ behavior of the WL in (1.13), (4.14) suggests that one should have $\mathcal{Z}(\lambda\gg1)\sim\lambda^{5/4}$.

While localization does not apply to the non-supersymmetric circular Wilson loop case, it would be very interesting to see if $\langle W^{(\zeta)}\rangle$, and, more generally, the spectrum of local operator insertions on the loop, may be determined exactly in the planar limit using the underlying integrability of the large N theory.

Another important direction is to understand better the strong-coupling side, in particular, shed light on the precise correspondence between the "strong-coupling" and "weak-coupling" parameters $\varkappa$ and ζ in (4.18). A related question is about the detailed comparison of the expansion of the Wilson loop expectation value near the conformal points to correlation functions of scalar operator insertions at strong coupling.

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A Cut-off regularization

We can compute the leading order λ contribution to the Wilson loop in (2.7) using the explicit UV cutoff $a\to0$ by replacing x^2 in the vector field propagator by x^2+a^2. Then in the line case we get (cf. (2.8))

$$\begin{aligned} W_1^{(0)} &= -\frac{1}{(4\pi)^2}\int_0^L d\tau_1\int_0^L d\tau_2\,\frac{1}{(\tau_1-\tau_2)^2+a^2} = -\frac{2}{(4\pi)^2}\int_0^L d\tau\,\frac{L-\tau}{\tau^2+a^2}\\ &= -\frac{L}{16\,\pi\,a}+\frac{1+\log{(L/a)}}{8\,\pi^2}+\mathcal{O}(a). \end{aligned}\tag{A.1}$$

Here for $L \to \infty$ only the first term is relevant and this linear divergence is to be subtracted out. For the circle, we have as in (2.10) $W_1^{(0)} = \frac{1}{8} + \delta W_1^{(0)}$ where

$$\delta W_1^{(0)} = -\frac{1}{4\pi}\int_0^\pi \frac{d\tau}{4\sin^2\frac{\tau}{2} + a^2} = -\frac{1}{4\,a\,\sqrt{a^2+4}} = -\frac{1}{8\,a} + \frac{a}{64} + \mathcal{O}(a^2). \tag{A.2}$$

The linear divergence here is the same as in (A.1) after the identification of L with the circle length 2π. Subtracting this linear divergence we get the same result as in (2.10), (2.12).

The computation of the integral in (3.12), (3.13) can be done using similar cutoff a

$$\begin{aligned} I(a,\gamma) &= \int_0^{2\pi} \frac{d\tau}{(4\sin^2\frac{\tau}{2} + a^2)^{1+\gamma}} = 2\pi\, a^{-2-2\gamma}\ {}_2F_1\left(\frac{1}{2}, 1+\gamma, 1, -\frac{4}{a^2}\right) \\ &= a^{-2-2\gamma}\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+\gamma\right)\,a}{\Gamma(1+\gamma)} + \mathcal{O}(a^3)\right] + \left[\frac{\sqrt{\pi}\,\Gamma\left(-\frac{1}{2}-\gamma\right)}{2^{1+2\gamma}\,\Gamma(-\gamma)} + \mathcal{O}(a)\right]. \end{aligned} \tag{A.3}$$

Expanding in $\gamma \to 0$, the first term gives just a power divergence with no finite $\mathcal{O}(a^0)$ part. The leading finite part in the second bracket is the same as given in (3.13) found by directly computing the integral in (3.12) using an analytic continuation. Subtracting power divergence we get $I(a \to 0, \gamma) = \pi\,\gamma + \mathcal{O}(\gamma^2)$ in agreement with (3.13). One can check that the expansions $a \to 0$ and $\gamma \to 0$ here commute.

B Computing 2-loop circle integrals

B.1 Expansion method

The circle integrals that appear in the expectation value of the circular WL can be computed by using the commonly used expansion method (see [61], appendix B of [62] and appendix G of [25]). Let us first illustrate it on the example of the the 1-loop integral in (2.10) or (A.3). Expanding power of sine-function as a series of exponents and setting $\alpha \equiv \omega - 1 = 1 - \varepsilon$ we get[29]

$$\begin{aligned} W_1^{(0)}(\alpha) &= -\frac{1}{16\,\pi^{\alpha+1}}\int_0^{2\pi} d\tau_1 d\tau_2\ \frac{\cos\tau_{12}}{(4\,\sin^2\frac{\tau_{12}}{2})^\alpha} \\ &= -\frac{1}{16\,\pi^{\alpha+1}}\,e^{-i\,\pi\,\alpha}\sum_{n=0}^{\infty}\binom{-2\alpha}{n}(-1)^n\int_0^{2\pi} d\tau_1 d\tau_2\ \cos(\tau_1-\tau_2)\,e^{i\,(n+\alpha)(\tau_1-\tau_2)} \\ &= -\frac{2^{-2(\alpha+1)}\pi^{\frac{3}{2}-\alpha}\alpha\cos(\pi\alpha)}{\Gamma(2-\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)} = \frac{1}{8} + \frac{1}{8}(1+\log\pi)\,\varepsilon + \mathcal{O}(\varepsilon^2)\,, \end{aligned} \tag{B.1}$$

which is in agreement with (2.10), (2.12).

At two loops, we need the integrals in (2.55); stripping off irrelevant factors these are

$$\mathcal{W}_{2(a)}^{(\zeta)}(\alpha) = \int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\ \frac{(\zeta^2-\cos\tau_{12})\,(\zeta^2-\cos\tau_{34})}{(4\,\sin^2\frac{\tau_{12}}{2}\ 4\,\sin^2\frac{\tau_{34}}{2})^\alpha}, \tag{B.2}$$

$$\mathcal{W}_{2(b)}^{(\zeta)}(\alpha) = \int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\ \frac{(\zeta^2-\cos\tau_{14})\,(\zeta^2-\cos\tau_{23})}{(4\,\sin^2\frac{\tau_{14}}{2}\ 4\,\sin^2\frac{\tau_{23}}{2})^\alpha}. \tag{B.3}$$

[29] Here we omit the overall factor $\Gamma(2-\omega)$ that does not contribute to the final result.

Applying the expansion procedure as in (B.1), we finally obtain for $\mathcal{W}^{(\zeta)}_{2^{(a)}}(\alpha)$

$$\begin{aligned}\mathcal{W}^{(\zeta)}_{2^{(a)}}(\alpha) = \frac{1}{\pi(\alpha-1)}\Bigg\{&2^{1-4\alpha}\zeta^2\Gamma\left(\frac{1}{2}-\alpha\right)^2\Gamma(\alpha-1)^2\Big[\pi^2(\alpha-1)^2[(\alpha-1)\zeta^2+4\alpha]\\&+[(\alpha-1)\zeta^2+2]\sin^2(\pi\alpha)+(\alpha-1)^2\sin^2(\pi\alpha)\Big((\alpha-1)\zeta^2\psi^{(1)}(1-\alpha)\\&+[\zeta^2-\alpha(\zeta^2+4)]\psi^{(1)}(\alpha-1)\Big)\Big]\Bigg\}\\&+\frac{4\pi^2(\alpha-1)\alpha\Gamma(1-2\alpha)^2\big[\pi^2(\alpha-1)\alpha\csc^2(\pi\alpha)-(\alpha-1)\alpha\psi^{(1)}(\alpha-1)-1\big]}{\Gamma(2-\alpha)^4},\end{aligned}\tag{B.4}$$

where $\psi^{(1)}(z)$ is the derivative of the digamma function. Its expansion around $\alpha=1$ gives the expression in (2.56)

$$\mathcal{W}^{(\zeta)}_{2^{(a)}}(1-\varepsilon)=\frac{\pi^2(1-\zeta^2)}{\varepsilon}+\pi^2(3-\zeta^2)(1-\zeta^2)+\frac{\pi^4}{6}+\mathcal{O}(\varepsilon)\,.\tag{B.5}$$

For $\mathcal{W}^{(\zeta)}_{2^{(b)}}(\alpha)$ a similar calculation gives

$$\begin{aligned}\mathcal{W}^{(\zeta)}_{2^{(b)}}(\alpha)=&\frac{2\pi^6\big[(\alpha-1)\zeta^2+\alpha\big]^2\csc^2(2\pi\alpha)}{3\Gamma(1-\alpha)^2\Gamma(2-\alpha)^2\Gamma(2\alpha)^2}\\&+\frac{4\pi^4(\alpha-4)^2(\alpha-3)^2(\zeta^2-1)\big(2\alpha[(\alpha-2)\zeta^2+\alpha-1]+\zeta^2+1\big)\csc^2(2\pi\alpha)}{\Gamma(1-\alpha)^2\Gamma(5-\alpha)^2\Gamma(2\alpha)^2}\\&\times{}_3F_2(1,\alpha,\alpha;3-\alpha,3-\alpha;1)\\&-\frac{12\pi^4(\zeta^2-1)^2\csc^2(2\pi\alpha)}{\Gamma(4-\alpha)^2\Gamma(-\alpha)^2\Gamma(2\alpha)^2}\,{}_3F_2(2,\alpha+1,\alpha+1;4-\alpha,4-\alpha;1)\\&-\frac{32\pi^4(\zeta^2-1)^2\csc^2(2\pi\alpha)}{(\alpha-4)^2(\alpha-3)^2(\alpha-2)^2(\alpha-1)^2(\alpha+1)^2\Gamma(-\alpha-1)^4\Gamma(2\alpha+1)^2}\\&\times{}_3F_2(3,\alpha+2,\alpha+2;5-\alpha,5-\alpha;1)-\frac{4\pi^4(\alpha-1)^2(\alpha\zeta^2+\alpha-\zeta^2)^2\csc^2(2\pi\alpha)}{\Gamma(1-\alpha)^2\Gamma(3-\alpha)^2\Gamma(2\alpha)^2}\\&\times{}_5F_4(1,1,1,\alpha,\alpha;2,2,3-\alpha,3-\alpha;1).\end{aligned}\tag{B.6}$$

Expanding this around $\alpha=1$, we obtain the finite expression given in (2.56)

$$\mathcal{W}^{(\zeta)}_{2^{(b)}}(1-\varepsilon)=\frac{1}{2}\pi^2(1-\zeta^2)^2+\frac{\pi^4}{6}+\mathcal{O}(\varepsilon)\,.\tag{B.7}$$

B.2 Method based on Fourier representation

In the expansion method, the intermediate calculations are not manifestly real and cancellation of imaginary parts is often due to non-trivial relations between infinite sums. A simpler approach closer to the analysis in momentum space is based on the Fourier representation of the real even function $(4\sin^2\frac{x}{2})^{-\alpha}$

$$\frac{1}{\left(4\sin^2\frac{x}{2}\right)^{\alpha}}=\frac{1}{2}\mathrm{a}_0(\alpha)+\sum_{n=1}^{\infty}\mathrm{a}_n(\alpha)\cos(nx)\,,\tag{B.8}$$

$$\mathrm{a}_n(\alpha)=\frac{1}{\pi}\int_0^{2\pi}dx\frac{\cos(nx)}{\left(4\sin^2\frac{x}{2}\right)^{\alpha}}=\frac{\sec(\pi\alpha)\,\Gamma(n+\alpha)}{\Gamma(2\,\alpha)\,\Gamma(n-\alpha+1)}\tag{B.9}$$

Note that near $\alpha=1$ we have $\mathrm{a}_0=\frac{\sec(\pi\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)\Gamma(2\alpha)}=\alpha-1+\mathcal{O}(\alpha-1)^2$.

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Let us define the integrals

$$\mathcal{I}^{(a)}_{\alpha,\beta} = \int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\, \frac{1}{\left(4\sin^2\frac{\tau_{12}}{2}\right)^\alpha\ \left(4\sin^2\frac{\tau_{34}}{2}\right)^\beta}\,,$$
$$\mathcal{I}^{(b)}_{\alpha,\beta} = \int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\, \frac{1}{\left(4\sin^2\frac{\tau_{14}}{2}\right)^\alpha\ \left(4\sin^2\frac{\tau_{23}}{2}\right)^\beta} \tag{B.10}$$

Using the identity

$$(\zeta^2-\cos\tau_{12})\,(\zeta^2-\cos\tau_{34}) = 4\sin^2\frac{\tau_{12}}{2}\sin^2\frac{\tau_{34}}{2} + 2\left(\sin^2\frac{\tau_{12}}{2}+\sin^2\frac{\tau_{34}}{2}\right)(\zeta^2-1)+(\zeta^2-1)^2, \tag{B.11}$$

we find that the ladder integrals in (B.3) may be written as

$$\mathcal{W}^{(\zeta)}_{2^{(a)}}(\alpha) = \frac{1}{4}\mathcal{I}^{(a)}_{\alpha-1,\alpha-1} + \frac{1}{2}\left(\mathcal{I}^{(a)}_{\alpha-1,\alpha}+\mathcal{I}^{(a)}_{\alpha,\alpha-1}\right)(\zeta^2-1)+\mathcal{I}^{(a)}_{\alpha,\alpha}(\zeta^2-1)^2, \tag{B.12}$$
$$\mathcal{W}^{(\zeta)}_{2^{(b)}}(\alpha) = \frac{1}{4}\mathcal{I}^{(b)}_{\alpha-1,\alpha-1} + \frac{1}{2}\left(\mathcal{I}^{(b)}_{\alpha-1,\alpha}+\mathcal{I}^{(b)}_{\alpha,\alpha-1}\right)(\zeta^2-1)+\mathcal{I}^{(b)}_{\alpha,\alpha}(\zeta^2-1)^2. \tag{B.13}$$

To compute the integrals (B.10), we use the representation (B.8) and the integrals

$$\int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\,\cos(n\tau_{12})\cos(m\tau_{34}) = \begin{cases} 0, & m,n>0,\\ \frac{2\pi^2}{n^2}, & m=0,n>0,\\ \frac{2\pi^2}{m^2}, & n=0,m>0,\\ \frac{2\pi^4}{3}, & n=m=0, \end{cases} \tag{B.14}$$

$$\int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\,\cos(n\tau_{14})\cos(m\tau_{23}) = \begin{cases} 0, & m\neq n>0,\\ -\frac{\pi^2}{n^2}\,, & m=n>0,\\ \frac{2\pi^2}{m^2}\,, & n=0,m>0,\\ -\frac{2\pi^2}{n^2}, & m=0,n>0,\\ \frac{2\pi^4}{3}, & n=m=0. \end{cases} \tag{B.15}$$

As a result,

$$\mathcal{I}^{(a)}_{\alpha,\beta} = \frac{\pi^4}{6}\mathrm{a}_0(\alpha)\mathrm{a}_0(\beta) + \sum_{n=1}^{\infty}\frac{\pi^2}{n^2}\left[\mathrm{a}_0(\alpha)\mathrm{a}_n(\beta)+\mathrm{a}_0(\beta)\mathrm{a}_n(\alpha)\right]\,, \tag{B.16}$$
$$\mathcal{I}^{(b)}_{\alpha,\beta} = \frac{\pi^4}{6}\mathrm{a}_0(\alpha)\mathrm{a}_0(\beta) + \sum_{n=1}^{\infty}\frac{\pi^2}{n^2}\left[\mathrm{a}_0(\alpha)\mathrm{a}_n(\beta)-\mathrm{a}_0(\beta)\mathrm{a}_n(\alpha)-\mathrm{a}_n(\alpha)\mathrm{a}_n(\beta)\right]\,. \tag{B.17}$$

Plugging this into (B.12) and using the explicit expression of the Fourier coefficients (B.9), this yields

$$\mathcal{W}^{(\zeta)}_{2^{(a)}}(\alpha) = \frac{2\pi^6\left[(\alpha-1)\zeta^2+\alpha\right]^2\csc^2(2\pi\alpha)}{3\Gamma(1-\alpha)^2\Gamma(2-\alpha)^2\Gamma(2\alpha)^2} + \sum_{n=1}^{\infty}\frac{2\pi^3\,\left[(\alpha-1)\zeta^2+\alpha\right]\csc(\pi\alpha)\sec^2(\pi\alpha)}{n^2\Gamma(1-\alpha)\Gamma(2-\alpha)\Gamma(2\alpha)^2\Gamma(n-\alpha+2)}$$
$$\times\left[\alpha^2-\alpha+n^2+\zeta^2(\alpha-n-1)(\alpha+n-1)\right]\Gamma(n+\alpha-1). \tag{B.18}$$

Evaluating the sum gives

$$\mathcal{W}^{(\zeta)}_{2^{(a)}}(\alpha) = \frac{2\pi^6\left[(\alpha-1)\zeta^2+\alpha\right]^2\csc^2(2\pi\alpha)}{3\Gamma(1-\alpha)^2\Gamma(2-\alpha)^2\Gamma(2\alpha)^2} + \frac{4^{1-2\alpha}\pi\left[(\alpha-1)\zeta^2+\alpha\right]\Gamma(\frac{1}{2}-\alpha)}{\Gamma(1-\alpha)^2}$$
$$\times\left[\frac{(\zeta^2-1)\Gamma(\frac{1}{2}-\alpha)}{(\alpha-1)^3} - \frac{\pi^{3/2}2^{2\alpha+1}\left[(\alpha-1)\zeta^2+\alpha\right]{}_4F_3(1,1,1,\alpha;2,2,3-\alpha;1)}{\sin(2\pi\alpha)\Gamma(3-\alpha)\Gamma(2\alpha)}\right]. \tag{B.19}$$

One can check that (B.19) is equal to (B.4) by using the identity

$${}_4F_3(1,1,1,\alpha;2,2,3-\alpha;1) = \frac{(\alpha-2)\left[\pi^2\left(1-6\csc^2(\pi\alpha)\right)+6\psi^{(1)}(\alpha-1)\right]}{12(\alpha-1)}. \tag{B.20}$$

Using (B.17) in (B.13), we find in a similar way that

$$\mathcal{W}^{(\zeta)}_{2^{(b)}}(\alpha) = \frac{2\pi^6\left[(\alpha-1)\zeta^2+\alpha\right]^2\csc^2(2\pi\alpha)}{3\Gamma(1-\alpha)^2\Gamma(2-\alpha)^2\Gamma(2\alpha)^2}$$
$$-\sum_{n=1}^{\infty}\frac{\pi^2\left[\alpha^2-\alpha+n^2+\zeta^2(\alpha-n-1)(\alpha+n-1)\right]^2\,\Gamma(n+\alpha-1)^2}{n^2\cos^2(\pi\alpha)\Gamma(2\alpha)^2\,\Gamma(n-\alpha+2)^2}. \tag{B.21}$$

Evaluating the infinite sum gives the expression which is in agreement with (B.6). In particular, for $\alpha\to 1$, one finds again (B.7).

Let us note that it is easy to extract the $\alpha\to 1$ expansion of expressions like (B.21) without computing the unwieldy closed form (B.6): one is to separate the leading contribution at large n in the sum. For instance,

$$-\sum_{n=1}^{\infty}\frac{\pi^2\left[\alpha^2-\alpha+n^2+\zeta^2(\alpha-n-1)(\alpha+n-1)\right]^2\Gamma(n+\alpha-1)^2}{n^2\cos^2(\pi\alpha)\Gamma(2\alpha)^2\Gamma(n-\alpha+2)^2}$$
$$= -\sum_{n=1}^{\infty}\left[\frac{\pi^2(\zeta^2-1)^2\sec^2(\pi\alpha)n^{4\alpha-4}}{\Gamma(2\alpha)^2} + \mathcal{O}\big((\alpha-1)\,n^{4\alpha-6}\big)\right]$$
$$= -\frac{\pi^2(\zeta^2-1)^2\sec^2(\pi\alpha)}{\Gamma(2\alpha)^2}\,\zeta_{\rm R}(4-4\alpha)+\mathcal{O}(\alpha-1) \overset{\alpha\to 1}{=} \frac{\pi^2}{2}\,(1-\zeta^2)^2\,, \tag{B.22}$$

where $\zeta_{\rm R}$ is the Riemann zeta-function. Adding the $\alpha\to 1$ limit of the first line of (B.21), i.e. $\frac{\pi^4}{6}$, we reproduce the expression in (B.7).

B.3 Alternative approach: expansion and summation directly in $d = 4$

The ladder integrals in (2.55) were computed using dimensional regularization with the analytic continuation parameter $\alpha = \omega - 1 = \frac{d}{2} - 1 = 1 - \varepsilon \to 1$. The expansion method and its improved Fourier representation version used to compute these integrals involve infinite summations that produce meromorphic functions of α that are then evaluated near the physical value $\alpha = 1$. Instead of using analytic continuation in α one may use a simple alternative approach: first set $\alpha = 1$, use expansion procedure, do the τ-integrals and then regularize the resulting infinite sums.

For example, starting with $\mathcal{W}^{(\zeta)\,(a)}(\alpha)$ in (B.2), setting $\alpha = 1$ and using (B.8), i.e. $\frac{1}{4\sin^2\frac{x}{2}} = -\sum_{n=1}^{\infty} n\,\cos(nx)$, and (B.14) we get (the same expression is found of course by setting $\alpha = 1$ in (B.18))

$$\mathcal{W}^{(\zeta)\,(a)}(1) = 2\,\pi^2\,(1-\zeta^2)\sum_{n=1}^{\infty}\frac{1}{n} + \frac{\pi^4}{6}. \tag{B.23}$$

Comparing to (B.5), we see that the pole $\frac{1}{\varepsilon}$ there corresponds to the logarithmically divergent sum $2\sum_{n\geq 1}\frac{1}{n}$ in (B.23). The finite parts of (B.5) and (B.23) (which are, in general, scheme-dependent) do not match. The reason for this disagreement can be understood as follows. The finite term of order ζ^4 in (B.2) comes from the integral

$$\int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\boldsymbol{\tau}\,\frac{1}{\left(4\sin^2\frac{\tau_{12}}{2}4\sin^2\frac{\tau_{34}}{2}\right)^{\alpha}} = \mathrm{a}_0(\alpha)\left[\frac{\pi^4}{6}\mathrm{a}_0(\alpha) + \sum_{n=1}^{\infty}\frac{2\pi^2}{n^2}\mathrm{a}_n(\alpha)\right] = \pi^2 + \mathcal{O}(\alpha-1) \tag{B.24}$$

where we used that

$$\sum_{n=1}^{\infty}\frac{2\pi^2}{n^2}\mathrm{a}_n(\alpha) = \frac{\pi^2}{\alpha-1} + \mathcal{O}(1)\,, \qquad \mathrm{a}_0(\alpha) = \alpha - 1 + \mathcal{O}\big((\alpha-1)^2\big). \tag{B.25}$$

The direct way of setting $\alpha = 1$ before summation misses extra finite $\frac{0}{0}$ term as $\mathrm{a}_0(1)$ is set to zero from the start.

This subtlety does not appear in the case of $\mathcal{W}^{(\zeta)\,(b)}(\alpha)$ which does not have a pole near $\alpha = 1$ (see (B.7)). Indeed, direct evaluation at $\alpha = 1$ (or setting $\alpha = 1$ in (B.21) before summation) gives, in agreement with (B.7) or (2.56),

$$\mathcal{W}^{(\zeta)}_{2(b)}(1) = \frac{\pi^4}{6} - \sum_{n=1}^{\infty}\pi^2(1-\zeta^2)^2 = \frac{\pi^4}{6} - \pi^2(1-\zeta^2)^2\,\zeta_{\mathrm{R}}(0) = \frac{\pi^4}{6} + \frac{\pi^2}{2}\,(1-\zeta^2)^2, \tag{B.26}$$

where we used ζ-function regularization for the linearly divergent sum.

This direct procedure thus gives a vanishing ζ^4 contribution from the type (a) ladder diagram integral, i.e. the full ζ^4 term in the final result (2.67) comes just from the type (b) integral, avoiding the use of the evanescent bare coupling terms in (2.63), (2.66) required in dimensional regularization.

A weak point of this regularization method is that it is difficult to apply it to the self-energy and internal-vertex diagrams in figure 2 where the Mellin double integral representation is quite useful when combined with dimensional regularization. Nevertheless, it is remarkable that in this prescription the only finite contribution to the 2-loop term in (2.67) should come just from the ladder type (b) diagram, i.e. the logarithmically divergent (and scheme-dependent finite) parts from other diagrams should cancel against each other.

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