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Exact RG flow equations and quantum gravity

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Abstract: We discuss the different forms of the functional RG equation and their relation to each other. In particular we suggest a generalized background field version that is close in spirit to the Polchinski equation as an alternative to the Wetterich equation to study Weinberg's asymptotic safety program for defining quantum gravity, and argue that the former is better suited for this purpose. Using the heat kernel expansion and proper time regularization we find evidence in support of this program in agreement with previous work.

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Contents

1 Introduction

Following the seminal work by Ken Wilson [1] many authors have discussed the formulation and consequences of continuum exact renormalization group (RG) equations for quantum field theory (QFT). Amongst these the most popular have been those of Polchinski [2] and Wetterich [3](see also [4–6]).[1] The former is a differential equation in RG "time" $\ln \Lambda$ for the Wilsonian effective action $I_\Lambda[\phi]$ obtained by integrating out the ultra-violet (UV) degrees of freedom down to some scale Λ. The latter is a differential equation for the so-called average effective action, obtained from the functional integral for the quantum effective action $\Gamma[\phi_c]$ by cutting off the integral over the eigenmodes of the kinetic operator of the QFT at some infra-red (IR) scale k. This produces a functional $\Gamma_k[\phi_c]$ such that its $k \to 0$ limit gives back $\Gamma[\phi_c]$. It is claimed that this equation defines the evolution all the way from an "initial" UV action all the way down to the deep IR $k \to 0$.

The standard model and Einstein's theory of gravity are usually regarded as effective field theories (EFT's). The UV completion of these EFT's is one of the main motivations for string theory. In the latter case it is expected that these EFT's are valid only up to the string scale, which is typically an order of magnitude or so below the four dimensional Planck scale.[2] It is commonly believed that above such a UV scale one needs to replace the EFT by string theory, with the parameters of the EFT being determined by the fundamental theory through matching conditions in the transitional region defined by the UV cutoff Λ.

[1]For reviews and references to recent work see for example [7–10]. For applications to the asymptotic safety program see [11–13] and references therein.

[2]If we have a large volume compactification the scale at which the EFT breaks down is the Kaluza-Klein scale.

An alternative to such a situation was proposed four decades ago by Weinberg [14, 15]. He argued that if the theory of gravity, or indeed gravity coupled to the standard model, possessed an ultra-violet fixed point with a finite number of relevant operators, then one would have a finite and predictive theory at all energy scales.

Such an eventuality would appear to eliminate the hope that a deeper understanding of the fundamental laws of nature would explain the values of the parameters of the standard model such as the Yukawa couplings, the CKM matrix elements, and the existence of three generations. On the other hand although it had been hoped that string theory would provide such an explanation, the discovery that there is an extremely large (if not infinite) landscape of (semi-realistic) solutions of the equations of string theory (with different numbers of generations etc.), means that in practice (at least with our current understanding of string theory), it is not possible to answer these questions. It is thus of great interest from a fundamental theoretical point of view to answer Weinberg's question.

It is of course well-known that Einstein's theory is not perturbatively renormalizable and will require an increasing number of counter-terms if one tries to calculate higher orders in the gravitational constant. The perturbation series is in effect an expansion in $g(E) = GE^2$ where G is the gravitational coupling constant ($\sqrt{8\pi G} = 1/M_P \sim 1/(10^{18}\text{GeV})$ and E is the energy scale which we are interested in. Thus it is valid only for energy scales below the Planck scale. Furthermore at a given energy scale to a given level of accuracy we need only a finite number of experimentally determined parameters to calculate any scattering process. However this procedure clearly breaks down as one approaches the Planck scale and certainly cannot address issues of post-Planckian physics. In contrast the "asymptotic safety" (AS) program initiated by Weinberg if indeed it can be realized in practice, should be able to calculate any gravitational process at arbitrarily high energies in terms of a finite number of parameters.

There is however a practical problem associated with trying to implement this program — the problem of truncation. The general Wilsonian effective action valid below some UV scale Λ will have an infinite number of local operators scaled by some (inverse) power of Λ. (see for instance equation (4.6) below.) In order to demonstrate the existence of an UV fixed point one needs to compute the beta functions, and clearly since one is not in the perturbative regime around the Gaussian fixed point, there is no small parameter that can give a controlled expansion to approximate the exact β-function. Consequently what has been resorted to in the literature has been a truncation procedure. This has been done in many works (see the reviews [11–13] and references therein). The procedure may be summarized as follows.

- Find a convenient truncation to a subset of the complete set of independent operators in the action and compute the β-functions for this subset. Then find a (non-trivial) fixed point (if one exists) and compute the scaling exponents there.

- Add new couplings and repeat the procedure.

- If the enlarged system does not admit a fixed point then clearly the fixed point of the original truncation was an artifact of the truncation. If it does then one can compare the fixed point coordinates of the original set in the two truncations. If these have been preserved up to small corrections in the enlarged truncation then this may be interpreted as evidence for a true fixed point of the complete theory.

This procedure has been checked against other methods of calculation in several condensed matter systems — for recent calculations showing good agreement see for example [16–20].

In the gravity case of course there is nothing to compare with and what has been done is the following. In pure gravity the fixed point derived from studying the truncation to Einstein gravity and a cosmological constant term has been shown to be stable under the addition of Riemann and Ricci squared terms as well as powers of R up to R^{35}. This has been taken as evidence that the fixed point found in the original truncation is indeed a true fixed point of the theory. Furthermore numerical investigations (see [21] and references therein), have shown the existence of an RG trajectory that connects such a fixed point with the Gaussian fixed point. This seems to point towards the existence of a QFT of gravity that accounts for the experimentally verified weak coupling calculations of Einstein's theory and is asymptotically complete.

On the other hand, the fact that explicit calculations of the UV fixed point values of the cosmological and gravitational coupling constants seem to depend only very weakly on the higher dimension operators, indicates that some deep principle that is yet to be understood is underlying this program. If this is uncovered then works such as that of [22–24] where standard model parameters are evaluated using asymptotic safety within some truncation ansatz, will indeed be true successes of the program. In the absence of such an understanding the experimental agreement of these calculations (for the Higgs and top quark masses) may still be interpreted as further evidence of the validity of the truncation in the sense that the higher dimension operators that in principle could have contributed to the calculation, are in fact negligible as in the pure gravity case.

Finally let us address the question of observables in quantum gravity. It is of course well-known that only quantities which are diffeomorphism invariant are observables in the theory. What the asymptotic safety program hopes to achieve then is to construct a Wilsonian effective action, which depends on only a finite number of relevant parameters[3] so that in principle gauge invariant observables such as the S-matrix can be calculated. The asymptotic safety program can also be of relevance for computing inflationary observables as has been explained by Weinberg in [25].

In this paper we will first review the two main versions of the Wilsonian exact RG equation and discuss the relation between them.[4] Then we will argue that the Wetterich version is ill-defined at the initial point of the evolution i.e. at the UV cutoff. Next we will derive a background field version of an exact RG equation which is close in spirit to the Polchinski equation, is well defined at the UV cutoff, and with proper time regularization, gives a simple expression for the β function.[5] Furthermore it avoids the problem of having

[3]This is clearly enunciated in [25].

[4]These equations are in effect RG improved one loop equations. As pointed out to the author by Joe Polchinski, such an equation was first derived by Weinberg in [14] (see section 8).

[5]The equation itself has been written down by other authors by conjecturing that a RG improved one-loop equation may be exact (see for example [8]). In this paper we establish the validity of this conjecture.

to deal with two sets of fields, a set of background fields as well as a set of expectation values of quantum fields (so there are two metrics for instance), that affects the background field formulation of the Wetterich equation. In practice much of the work that has been done towards establishing a fixed point for gravitational theories can actually be taken over for this RG equation for the Wilsonian action. We then discuss the beta function equations and the evidence for the existence of an ultra-violet fixed point with a finite number of relevant directions in a theory of gravity.

2 Review and comments

Our starting point is the (Euclidean) functional integral for connected correlation functions $W[J]$,

$$e^{-W[J]} = \int [d\phi] e^{-I[\phi] - \int \sqrt{g} J.\phi}. \tag{2.1}$$

Here I is the "classical" action and J is an external (classical) source. For simplicity we will work with scalar fields, but the expressions can be extended in the obvious ways to gauge fields, fermion fields, and metric fields, with appropriate tensor contractions and addition of gauge fixing terms DeWitt-Fadeev-Popov terms etc., and the replacement of traces and determinants by supertraces and superdeterminants. We will also employ a (commonly used) condensed notation so that for any fields ϕ, ψ, and a differential operator $K(x,y) \equiv -\nabla^2 \frac{1}{g^{1/4}(x)} \delta(x-y) \frac{1}{g^{1/4}(y)}$,

$$\phi.\psi \equiv \int \sqrt{g} d^4x \phi(x) \psi(x)$$
$$\phi.K.\psi \equiv \int \sqrt{g(x)} d^4x \int \sqrt{g(y)} d^4y \phi(x) K(x,y) \psi(y)$$

Let us now separate the kinetic and interaction parts and write

$$I[\phi] = I_0[\phi] + I_i[\phi],$$

with $I_0 \equiv \frac{1}{2}\phi.K.\phi]$ is the kinetic term and I_i is the interaction. By standard manipulations (2.1) may be rewritten as,

$$e^{-W[J]} = e^{-I_i\left[-\frac{\delta}{\delta J}\right]} e^{-\frac{1}{2}\mathrm{Tr}\ln K + \frac{1}{2} J.K^{-1}.J} \tag{2.2}$$

where K^{-1} is the Green's function associated with K. This expression is formal and (at least in a perturbative expansion in powers of the interaction) has divergences in any non-trivial field theory. Thus it needs to be regulated in the ultra-violet (UV). Assuming that K is a positive operator with eigenvalues p^2 we impose a cutoff Λ such that the modes $p^2 \gg \Lambda^2$ are suppressed. A simple hard cutoff would be to replace the Fourier transformed Green's function $\tilde{K}^{-1}(p^2)$ by $\tilde{K}_\Lambda^{-1}(p^2) = \tilde{K}^{-1}(p^2)\theta(\Lambda^2 - p^2)$. In flat space a smoothly cutoff propagator would be for instance of the form $\tilde{K}_\Lambda^{-1}(p^2) = \tilde{K}^{-1}(p^2)\exp(-p^2/\Lambda^2)$. Thus we have

$$e^{-W[J]} = e^{-I_{i\Lambda}\left[-\frac{\delta}{\delta J}\right]} e^{-\frac{1}{2}\mathrm{Tr}\ln K_\Lambda + \frac{1}{2} J.K_\Lambda^{-1}.J}. \tag{2.3}$$

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Demanding that the l.h.s. of this equation be independent of Λ gives us the following:

$$\begin{aligned}
0 &= e^{-I_{\mathrm{i}\Lambda}[-\frac{\delta}{\delta J}]}\left[-\frac{\partial I_{\mathrm{i}\Lambda}}{\partial\ln\Lambda}\left[-\frac{\delta}{\delta J}\right]-\frac{1}{2}\frac{\partial}{\partial\ln\Lambda}\mathrm{Tr}\ln\mathrm{K}_\Lambda+\frac{1}{2}J.\frac{d}{d\ln\Lambda}K_\Lambda^{-1}.J\right]e^{-\frac{1}{2}\mathrm{Tr}\ln K_\Lambda+\frac{1}{2}J.K_\Lambda^{-1}.J}\\
&= e^{-I_{\mathrm{i}\Lambda}[-\frac{\delta}{\delta J}]}\int[d\phi]e^{-I_{0\Lambda}[\phi]}\left[-\frac{\partial I_{\mathrm{i}\Lambda}}{\partial\ln\Lambda}[\phi]-\frac{1}{2}\frac{\partial}{\partial\ln\Lambda}\mathrm{Tr}\ln\mathrm{K}_\Lambda+\frac{1}{2}\frac{\delta}{\delta\phi}.\frac{d}{d\ln\Lambda}K_\Lambda^{-1}.\frac{\delta}{\delta\phi}\right]e^{-J.\phi}\\
&= \int[d\phi]e^{-I_{0\Lambda}[\phi]}\left[-\frac{\partial I_{\mathrm{i}\Lambda}}{\partial\ln\Lambda}[\phi]-\frac{1}{2}\frac{\partial}{\partial\ln\Lambda}\mathrm{Tr}\ln\mathrm{K}_\Lambda+\frac{1}{2}\frac{\delta}{\delta\phi}.\frac{d}{d\ln\Lambda}K_\Lambda^{-1}.\frac{\delta}{\delta\phi}\right]e^{-[I_{\mathrm{i}\Lambda}+J.\phi]} \quad (2.4)\\
&= \int[d\phi]\left\{-\frac{\partial I_{\mathrm{i}\Lambda}}{\partial\ln\Lambda}[\phi]-\frac{1}{2}\frac{\partial}{\partial\ln\Lambda}\mathrm{Tr}\ln\mathrm{K}_\Lambda+\frac{1}{2}\left(\frac{\delta I_{i\Lambda}}{\delta\phi}+J\right).\frac{d}{d\ln\Lambda}K_\Lambda^{-1}.\left(\frac{\delta I_{i\Lambda}}{\delta\phi}+J\right)\right.\\
&\quad\left.-\frac{1}{2}\frac{\delta}{\delta\phi}.\frac{d}{d\ln\Lambda}K_\Lambda^{-1}.\frac{\delta I_{i\Lambda}}{\delta\phi}\right\}e^{-[I_{0\Lambda}+I_{i\Lambda}+J.\phi]}. \quad (2.5)
\end{aligned}$$

The Polchinski equation results from using a external source current J which goes to zero at (momentum space) scales well below Λ. In this case the J dependent terms in the last equation vanish and we have the (slightly generalized) Polchinski RG equation,

$$\frac{\partial I_{\mathrm{i}\Lambda}}{\partial\ln\Lambda}[\phi]=-\frac{1}{2}\frac{\partial}{\partial\ln\Lambda}\mathrm{Tr}\ln\mathrm{K}_\Lambda+\frac{1}{2}\left(\frac{\delta I_{i\Lambda}}{\delta\phi}\right).\frac{d}{d\ln\Lambda}K_\Lambda^{-1}.\left(\frac{\delta I_{i\Lambda}}{\delta\phi}\right)-\frac{1}{2}\frac{\delta}{\delta\phi}.\frac{d}{d\ln\Lambda}K_\Lambda^{-1}.\frac{\delta I_{i\Lambda}}{\delta\phi}. \quad (2.6)$$

The slight generalization here is the occurrence of the first term, which in a general metric background needs to be kept[6] unlike in the original flat space case. Writing $I_\Lambda=I_{0\Lambda}+I_{i\Lambda}=I_{0\Lambda}+\sum_i g^i_\Lambda\Phi_i[\phi]$ and $\beta^i\equiv\frac{\partial g^i}{\partial\ln\Lambda}$ gives from (2.6) as is well known, a set of equations for the beta functions β_i.

In addition to this UV cutoff we will, in order to connect to the discussion of the so-called average effective action, introduce an infra-red (IR) cutoff k^2. So for example we could have in terms of a hard cutoff $\tilde{K}^{-1}_{\Lambda,k}(p^2)=\tilde{K}^{-1}(p^2)\theta(\Lambda^2-p^2)\theta(p^2-k^2)$, or in terms of a smooth cutoff $\tilde{K}^{-1}_{\Lambda,k}(p^2)=\tilde{K}^{-1}(p^2)[\exp(-p^2/\Lambda^2)-\exp(-p^2/k^2)]$. In this case (2.3) is replaced by

$$e^{-W_k[J]}=e^{-I_{i\Lambda}[-\frac{\delta}{\delta J}]}e^{-\frac{1}{2}\mathrm{Tr}\ln K_{\Lambda,k}+\frac{1}{2}J.K^{-1}_{\Lambda,k}.J}. \quad (2.7)$$

The dependence on the IR cutoff is indicated by the subscript k on W but it is still independent of the UV scale Λ. As discussed above this is equivalent to satisfying the Polchinski equation (2.6) since the replacement $K_\Lambda\to K_{\Lambda,k}$ has no effect on the derivation of that equation.

Let us now derive the so-called Wetterich equation in a slightly modified form, i.e. with both a UV cutoff as well as the original IR cutoff. But now we need to begin with the functional integral version of (2.7). So we have (putting $k\partial_k x\equiv\dot{x}$)

$$\begin{aligned}
\frac{\partial}{\partial\ln k}e^{-W_k[J]} &= -\int[d\phi]\frac{1}{2}\phi.\dot{K}_{k,\Lambda}.\phi e^{-\left[\frac{1}{2}\phi.K_{k,\Lambda}.\phi+I_{1\Lambda}[\phi]+J.\phi\right]}\\
&= -\frac{1}{2}\frac{\delta}{\delta J}.\dot{K}_{k,\Lambda}.\frac{\delta}{\delta J}e^{-W_k[J]}.
\end{aligned}$$

[6]This has already been observed in [26].

This leads to the RG equation

$$\dot{W}_k|_J = \frac{1}{2}\frac{\delta W_k}{\delta J}.\dot{K}_{k,\Lambda}.\frac{\delta W_k}{\delta J} - \frac{1}{2}\frac{\delta}{\delta J}.\dot{K}_{k,\Lambda}.\frac{\delta W_k}{\delta J}. \tag{2.8}$$

Defining the cutoff effective action by a Legendre transformation,

$$\Gamma_k[\phi_c] = W_k[J] - J.\phi_c, \;\; \phi_c = \frac{\delta W_k[J]}{\delta J} \Rightarrow \frac{\delta \Gamma_k[\phi_c]}{\delta \phi_c} = -J[\phi_c].$$

Also $\dot{\Gamma}_k|_{\phi_c} = \dot{W}_k|_J + \frac{\delta W_k[J]}{\delta J}.\dot{J} - \dot{J}.\phi_c = \dot{W}_k|_J$ and $\frac{\delta}{\delta J}.\dot{K}_{k,\Lambda}.\frac{\delta W_k}{\delta J} = -\mathrm{Tr}\dot{K}_{k,\Lambda}\left[\frac{\delta}{\delta\phi_c} \otimes \frac{\delta\Gamma_k[\phi_c]}{\delta\phi_c}\right]^{-1}$, giving from (2.8) a flow equation for the cutoff effective action,

$$\dot{\Gamma}_k[\phi_c] = \frac{1}{2}\mathrm{Tr}\dot{K}_{k,\Lambda}\left\{\phi_c \otimes \phi_c + \left[\frac{\delta}{\delta\phi_c} \otimes \frac{\delta\Gamma_k[\phi_c]}{\delta\phi_c}\right]^{-1}\right\}.$$

The effective average action defined by Wetterich is (except for the fact that here we are keeping Λ finite),

$$\Gamma_k^{\mathrm{w}}[\phi_c] \equiv \Gamma_k[\phi_c] - \frac{1}{2}\phi_c.R_{k,\Lambda}.\phi_c,$$

where $R_{k,\Lambda} \equiv K_{k,\Lambda} - K_{0,\Lambda}$ (so that $\dot{R}_{k,\Lambda} = \dot{K}_{k,\Lambda}$). The above equation now becomes

$$\dot{\Gamma}_k^{\mathrm{w}}[\phi_c] = \frac{1}{2}\mathrm{Tr}\dot{R}_{k,\Lambda}\left[\frac{\delta}{\delta\phi_c} \otimes \frac{\delta\Gamma_k^{\mathrm{w}}[\phi_c]}{\delta\phi_c} + R_{k,\Lambda}\right]^{-1}. \tag{2.9}$$

In the limit $k \to 0$, $\Gamma_k \to \Gamma$ and $R_{k,\Lambda} \to 0$, and we have the "final" condition

$$\Gamma_k^{\mathrm{w}} \to \Gamma, \text{ for } k \to 0.$$

To get the initial condition for Γ_k^W let us go back to the functional integral defining it.

$$\begin{aligned} e^{-\Gamma_k^{\mathrm{w}}[\phi_c]} &= \int [d\phi] e^{-\left[\frac{1}{2}\phi.K_{k,\Lambda}.\phi + I_{i\Lambda}[\phi] + J.(\phi-\phi_c) - \frac{1}{2}\phi_c.R_{k,\Lambda}.\phi_c\right]}\Big|_{J=-\delta\Gamma_k/\delta\phi_c} \\ &= \int [d\phi'] e^{-\left[\frac{1}{2}\phi_c.K_{k,\Lambda}.\phi_c + \frac{1}{2}\phi'.K_{k,\Lambda}.\phi' + I_{i\Lambda}[\phi_c+\phi'] + (J+\phi_c.K_{k,\Lambda}).\phi' - \frac{1}{2}\phi_c.R_{k,\Lambda}.\phi_c\right]}\Big|_{J=-\delta\Gamma_k/\delta\phi_c} \\ &= e^{-\left[\frac{1}{2}\phi_c.K_{0,\Lambda}.\phi_c + I_{i\Lambda}\left[\phi_c - \frac{\delta}{\delta J}\right] + \frac{1}{2}\mathrm{Tr}\ln K_{\Lambda,k}\right] + \frac{1}{2}\left(\bar{J}.K_{k,\Lambda}^{-1}.\bar{J}\right)}\Big|_{\bar{J}=-\delta\Gamma_k/\delta\phi_c+\phi_c.K_{k,\Lambda}.} \end{aligned} \tag{2.10}$$

When $k \to \Lambda$, $K_{k,\Lambda}^{-1} \to 0$ and up to a field independent infinite constant ($\frac{1}{2}\mathrm{Tr}\ln K_{\Lambda,\Lambda}$] which can be regularized to zero in heat kernel proper time regularization for instance), we have

$$\Gamma_k^{\mathrm{w}}[\phi_c] \to I_\Lambda[\phi_c] \text{ for } k \to \Lambda.$$

This establishes the fact that $\Gamma_k^{\mathrm{w}}[\phi_c]$ interpolates between the seed action (i.e. the Wilsonian action at the initial scale Λ) and the quantum effective action.

However the flow equation itself breaks down at the initial scale. This is because as $k \to \Lambda$, $K_{\Lambda,k}^{-1} \to 0$ and hence $R_{k,\Lambda} = K_{k,\Lambda} - K_{0,\Lambda} \to \infty$ and so in general does its log derivative. This means that the r.h.s. of (2.9) and hence the RG time derivative of Γ_k^W at the initial time is not well-defined. Thus it is not clear how the equation can be used to evolve the initial data, i.e. the Wilsonian action at scale Λ.

To be concrete let us take the regulated propagator to be

$$K_{k,\Lambda}^{-1}(x,y) =< x| \int_{1/\Lambda^2}^{1/k^2} ds e^{-\hat{K}s}|y> .$$

Then we have the operator relation (writing $K_{k,\Lambda}^{-1}(x,y) =< x|\hat{K}_{k,\Lambda}^{-1}|y>$ etc.)

$$\hat{R}_{k,\Lambda} = \hat{K}_{k,\Lambda} - \hat{K}_{0,\Lambda} = \hat{K}e^{\hat{K}/\Lambda^2}\left(e^{(1/k^2-1/\Lambda^2)\hat{K}} - 1\right)^{-1} \tag{2.11}$$

$$\dot{\hat{R}}_{k,\Lambda}R_{k,\Lambda}^{-1} = -2\frac{\hat{K}}{k^2}e^{(1/k^2-1/\Lambda^2)\hat{K}}\left(e^{(1/k^2-1/\Lambda^2)\hat{K}} - 1\right)^{-1} \tag{2.12}$$

Both these quantities diverge as $(1/k^2 - 1/\Lambda^2)^{-1}$ as $k \to \Lambda$. Thus the equation (2.9) is not well defined in this limit.

In the literature on the average effective action, in contrast to the above, there is no explicit UV cutoff. It is implicitly assumed that one can work with $\Lambda \to \infty$. In this case (2.12) becomes

$$\dot{\hat{R}}_{k,\infty}R_{k,\infty}^{-1} = -2\frac{\hat{K}}{k^2}e^{(1/k^2)\hat{K}}\left(e^{(1/k^2)\hat{K}} - 1\right)^{-1}$$

It is clear that $k \to \infty$ is now a well defined limit with $\dot{\hat{R}}_{k,\infty}R_{k,\infty}^{-1} \to -2$. However this assumes that the initial action I_Λ is well-defined in the $\Lambda \to \infty$ limit which is tantamount to assuming (at the very least) that the theory has a UV fixed point!

Actually the situation is worse than that since even if a fixed point existed as in QCD for example, the coupling $g_\Lambda \to 0$ as the cutoff $\Lambda \to \infty$ so that the action $-1/g_\Lambda^2 \int F^2$ does not exist.[7] One can of course get arbitrarily close to the fixed point with a well defined action but there is no finite action starting point as is assumed in the derivations of the Wetterich equation.[8] One needs to turn on the (marginally) relevant operator above at some large but finite cutoff Λ in order to flow away from the fixed point as one lowers the cutoff Λ. Thus even in this case the assumption that there is a meaningful $\Lambda \to \infty$ action is invalid.

The version of the Wetterich equation that is appropriate for gauge and gravity theories is formulated using the background field method. However this too has the above problem. Also it is clear from (2.10) that in the limit $k \to \Lambda$ there is now a background field dependent infinity which is absent if regularized for instance as in eqn (3.3). Furthermore this formulation requires the introduction of an effective action which depends on both a background field and the expectation value of the quantum field — thus it has two metrics for instance. In the next section we will suggest an alternative exact RG equation that is well defined at an arbitrary but finite UV scale Λ and is dependent only on the background field.

[7] One may redefine the gauge field to get a canonical kinetic term $A \to A_c = g_\Lambda A$, but in this case the gauge transformation $A_c \to \mathcal{G}^{-1}A_c\mathcal{G} + \frac{1}{g_\Lambda}\mathcal{G}^{-1}d\mathcal{G}$ is ill-defined in the limit $\Lambda \to \infty$.

[8] For alternative discussions of this so-called reconstruction problem for the effective average action see [27–29].

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3 An alternative RG equation

The quantum theory corresponding to a given classical action $I[\phi]$ is given by the quantum effective action $\Gamma(\phi_c)$ defined (implicitly and formally) by the formula

$$e^{-\Gamma(\phi_c)} = \int [d\phi] e^{-I[\phi]-J.(\phi-\phi_c)}\Big|_{J=-\partial\Gamma/\partial\phi_c}. \tag{3.1}$$

By translating the integration variable $\phi = \phi_c + \phi'$ we have the following expressions,

$$\begin{aligned} e^{-\Gamma(\phi_c)} &= \int [d\phi'] e^{-I[\phi_c+\phi']-J.\phi'}|_{J=-\partial\Gamma/\partial\phi_c} \\ &= \int [d\phi'] e^{-\left\{I[\phi_c]+\frac{1}{2}\phi'.\frac{\delta^2 I}{\delta\phi_c^2}.\phi'+I_{\rm i}[\phi_c,\phi']+\left(J+\frac{\delta I[\phi_c]}{\delta\phi_c}\right).\phi'\right\}}\Big|_{J=-\delta\Gamma/\delta\phi_c} \\ &= e^{-I[\phi_c]} e^{-\frac{1}{2}{\rm Tr}\ln K[\phi_c]} e^{-I_{\rm i}\left[\phi_c,-\frac{\delta}{\delta\bar{J}}\right]} e^{\frac{1}{2}\bar{J}.K[\phi_c]^{-1}.\bar{J}}\Big|_{\bar{J}=\delta I[\phi_c]/\partial\phi_c-\delta\Gamma/\delta\phi_c}. \end{aligned} \tag{3.2}$$

In the second line above $I_{\rm i}[\phi_c, \phi']$ contains all powers of ϕ' which are higher than quadratic in the expansion of $I[\phi_c+\phi']$, and the third line is the result of doing the Gaussian integral over ϕ'.

The above is a formal expression that needs to be regularized. A convenient way of doing this for our purposes is to introduce the Schwinger proper time regularization[9] as in the previous section, except that now the operators depend on the field ϕ_c.

$$K^{-1}_{k,\Lambda}(\phi_c; x, y) = <x\left|\int_{1/\Lambda^2}^{1/k^2} ds e^{-\hat{K}[\phi_c]s}\right|y>,\ \ln K_{k,\Lambda}[\phi_c; x, y] = -<x\left|\int_{1/\Lambda^2}^{1/k^2} \frac{ds}{s} e^{-\hat{K}[\phi_c]s}\right|y>. \tag{3.3}$$

So we replace (3.2) by

$$e^{-\Gamma_k(\phi_c)} = e^{-I_\Lambda[\phi_c]} e^{-\frac{1}{2}{\rm Tr}\ln K_{k,\Lambda}[\phi_c]} e^{-I_{\rm i}\left[\phi_c,-\frac{\delta}{\delta\bar{J}}\right]} e^{\frac{1}{2}\bar{J}.K_{k,\Lambda}[\phi_c]^{-1}.\bar{J}}\Big|_{\bar{J}=\delta I_\Lambda[\phi_c]/\partial\phi_c-\delta\Gamma_k/\delta\phi_c}. \tag{3.4}$$

Now we have the following limits:

$$k \to 0,\ \Gamma_k \to \Gamma;\ \ k \to \Lambda,\ \ K^{-1}_{k,\Lambda},\ \ln K_{k,\Lambda} \to 0,\ \Rightarrow \Gamma_k \to I_\Lambda. \tag{3.5}$$

It should be noted that the problematic limit $k \to \Lambda$ in the Wetterich case is taken care of by the use of the (regularized) Schwinger proper time representation for the one loop effective action — i.e. the second equation in (3.3). The necessity of the separate regularization of the logarithm of $\hat{K}$ is nothing but the well known phenomenon that any higher covariant derivative type regularization (which is essential for the discussion of a gauge theory) will not regularize the one-loop term. With these regularizations (3.3)[10] we have the well-defined expression (3.4).

[9]In general K will of course be a matrix over space-time indices as well as internal indices labelling the different fields as well as their components.

[10]One can of course use a whole class of such regularizations with a smooth cutoff function of the proper time s in the integrands. Here we just chose the simplest version.

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Differentiating w.r.t. $\ln k$ we have from (3.4) ($dt \equiv dk/k$)

$$
\begin{aligned}
-e^{-\Gamma_k(\phi_c)}\dot{\Gamma}_k[\phi_c] = \; & e^{-I_\Lambda[\phi_c]}e^{-\frac{1}{2}\mathrm{Tr}\ln K_{k,\Lambda}[\phi_c]}e^{-I_{\mathrm{i}\Lambda}\left(\phi_c,-\frac{\delta}{\delta \bar{J}}\right)} \times \\
& \left(-\frac{1}{2}\frac{d}{dt}\mathrm{Tr}\ln K_{k,\Lambda}[\phi_c] + \dot{\bar{J}}.K_{k,\Lambda}^{-1}.\bar{J} + \frac{1}{2}\bar{J}.\dot{K}_{k,\Lambda}^{-1}.\bar{J}\right) \times \\
& e^{\frac{1}{2}\bar{J}.K_{k,\Lambda}[\phi_c]^{-1}.\bar{J}}\Big|_{\bar{J}=\delta I_\Lambda[\phi_c]/\partial\phi_c - \delta\Gamma_k/\delta\phi_c}
\end{aligned} \tag{3.6}
$$

Now let us take the limit $k \to \Lambda$ of this equation. Using (3.5) we see that since $I_{\mathrm{i}}(\phi_c, -\frac{\delta}{\delta \bar{J}})$ is at least third order in $\delta/\delta J$, it will commute with the $\bar{J}$ terms up to terms which have at least one power of $\delta/\delta J$ acting on the last factor which is equal to 1 in this limit. Also in this limit $\bar{J} \to 0$. Thus we have the alternate RG equation

$$
\Lambda\frac{d}{d\Lambda}I_\Lambda[\phi_c] = \frac{1}{2}k\frac{d}{dk}\mathrm{Tr}\ln K_{k,\Lambda}[\phi_c]|_{k=\Lambda} = \mathrm{Tr}\exp\left\{-\frac{1}{\Lambda^2}\frac{\delta}{\delta\phi_c}\otimes\frac{\delta}{\delta\phi_c}I_\Lambda[\phi_c]\right\}, \tag{3.7}
$$

where in the last step we used (3.3). The same equation may be obtained by requiring the independence from Λ of Γ_k, as is required for consistency, in (3.4) and then taking the limit $k \to \Lambda$. Thus instead of (3.6) we have

$$
\begin{aligned}
0 = \; & e^{-I_\Lambda[\phi_c]}e^{-\frac{1}{2}\mathrm{Tr}\ln K_{k,\Lambda}[\phi_c]}e^{-I_{\mathrm{i}\Lambda}\left[\phi_c,-\frac{\delta}{\delta \bar{J}}\right]} \times \\
& \left\{-\Lambda\frac{d}{d\Lambda}I_\Lambda[\phi_c] - \Lambda\frac{d}{d\Lambda}\left(\frac{1}{2}\mathrm{Tr}\ln K_{k,\Lambda}[\phi_c] + I_{\mathrm{i}\Lambda}\left[\phi_c,-\frac{\delta}{\delta\bar{J}}\right]\right)\right. \\
& \left. + \left(\Lambda\frac{d}{d\Lambda}\bar{J}\right).K_{k,\Lambda}^{-1}.\bar{J} + \frac{1}{2}\bar{J}.\left(\Lambda\frac{d}{d\Lambda}K_{k,\Lambda}^{-1}\right).\bar{J}\right\} \times \\
& e^{\frac{1}{2}\bar{J}.K_{k,\Lambda}[\phi_c]^{-1}.\bar{J}}\Big|_{\bar{J}=\delta I_\Lambda[\phi_c]/\partial\phi_c - \delta\Gamma_k/\delta\phi_c}
\end{aligned} \tag{3.8}
$$

By the same argument as before in the limit $k \to \Lambda$ we get (after using the explicit regulator (3.3) in the last step)

$$
\begin{aligned}
\Lambda\frac{d}{d\Lambda}I_\Lambda[\phi_c] &= -\frac{1}{2}\Lambda\frac{d}{d\Lambda}\mathrm{Tr}\ln K_{k,\Lambda}[\phi_c]|_{k=\Lambda} = \mathrm{Tr}\lim_{k\to\Lambda}\int_{1/\Lambda^2}^{1/k^2}\frac{ds}{s}s\Lambda\frac{dK_\Lambda}{d\Lambda}e^{-sK_\Lambda[\phi_c]} + \mathrm{Tr}e^{-\frac{1}{\Lambda^2}K[\phi_c]} \\
&= \mathrm{Tr}\exp\left\{-\frac{1}{\Lambda^2}\frac{\delta}{\delta\phi_c}\otimes\frac{\delta}{\delta\phi_c}I_\Lambda[\phi_c]\right\},
\end{aligned}
$$

which is the same as (3.7).

The above discussion of course needs to be modified when the set of fields ϕ includes gauge (and graviton) fields. The quadratic "kinetic" term in ϕ' will have an additional "gauge fixing" term which will be also regularized in the same way. So we make the replacement (with α a gauge fixing parameter)

$$
K_{k,\Lambda}[\phi_c] \to K_{k,\Lambda}[\phi_c] + K_{k,\Lambda}^{\mathrm{GF}}[\phi_c,\alpha].
$$

In addition we have the ghost term which is just a determinant term and will give an additional factor

$$
e^{+\frac{1}{2}\mathrm{Tr}\ln K_{k,\Lambda}^{\mathrm{ghost}}[\phi_c]},
$$

on the r.h.s. of (3.4). Thus the RG equation (3.7) is replaced by

$$\begin{aligned}\Lambda\frac{d}{d\Lambda}I_\Lambda[\phi_c] &= \frac{1}{2}k\frac{d}{dk}\left\{\mathrm{Tr}\ln\left(K_{k,\Lambda}[\phi_c]+K^{\mathrm{GF}}_{k,\Lambda}[\phi_c,\alpha]\right)-\mathrm{Tr}\ln\mathrm{K}^{\mathrm{ghost}}_{\mathrm{k},\Lambda}\right\}|_{k=\Lambda},\\ &= \mathrm{Tr}\exp\left\{-\frac{1}{\Lambda^2}\left(I^{(2)}_\Lambda[\phi_c]+I^{(2)\mathrm{GF}}_\Lambda[\phi_c,\alpha]\right)\right\}-\mathrm{Tr}\exp\left\{-\frac{1}{\Lambda^2}I^{(2)\mathrm{ghost}}_\Lambda[\phi_c,\alpha]\right\}.\end{aligned} \tag{3.9}$$

In the last line we have again used our explicit proper time cutoff. Also $I^{(2)\mathrm{GF}}_\Lambda[\phi_c,\alpha]$, $I^{(2)\mathrm{ghost}}_\Lambda[\phi_c,\alpha]$ are the background dependent operators defining the gauge fixing and ghost terms (which are respectively quadratic in the quantum field ϕ' and the ghost fields $C,\bar{C}$). Also I_Λ is as before the Wilsonian action but now including also the gauge fixing and ghost terms with Λ dependent couplings.

We should point out at this stage that this exact RG equation is in fact simply an RG improvement of the proper time representation of the RG equation for the cutoff one-loop effective action. As a one loop equation it has been written down by many authors for example [30–32]. In a form that is the same as what we have above it is given in [8] (see equation (92)). The point of the above discussion was however to derive it as an exact equation for the Wilsonian effective action.

The action can be expanded in a complete set of local operators (after eliminating redundancies)

$$I_\Lambda = \sum \bar{g}^A(\Lambda)\Phi_A[\phi]. \tag{3.10}$$

The equation (3.7) is then in effect an RG improved one loop equation for the beta functions. It is (as is the case for the original form of the Polchinski equation) an exact equation for the evolution of the Wilson effective action. It is also the appropriate form to use for exploring the fixed points (if any) for gauge and gravitational field theories since it has manifest gauge invariance under the gauge transformation of ϕ_c.

In fact all one needs for the exploration of UV fixed points is the above equation. Indeed what has been done in the literature is completely equivalent to doing an operator truncation of this equation, since in practice the equation for the effective average action [3, 33] can be used only in a regime where the derivative expansion is valid. Thus in effect one is dealing with a derivative expansion in terms of local operators. In this sense Γ_k is not in any way different from the Wilsonian effective action. The validity of the derivative expansion requires that $\partial^2/k^2 \ll 1$ i.e. one cannot really take the $k \to 0$ limit. There is no sense in which one can get the non-local quantum effective action from the quasi-local Γ_k without being able to sum the relevant infinite series. However the former is not needed to get the RG equations or to establish the existence of a UV fixed point. In other words one does not need to have an action which interpolates from the initial Wilsonian action all the way to the quantum effective action $\Gamma[\phi_c]$ to explore the UV properties of the theory. Finally let us emphasize that this formulation avoids the problem of having two background fields such as having two metrics — which is the case for the Wetterich equation.

3.1 Gauge fixing dependence and observables in quantum gravity

DeWitt [34] has argued that the gauge fixing parameter is not renormalized. If this is the case we may assume that the gauge fixing parameters do not flow. Furthermore the

1PI effective action Γ at its extremum is in fact independent of the gauge fixing. On the other hand the Wilsonian action I_Λ and the average effective action Γ_k are gauge fixing dependent[11] and therefore so are the beta functions. Nevertheless the fixed points and the critical exponents should be gauge independent.

Currently what has been done is to check the gauge dependence by comparing the calculations in different gauges and to some extent this has been verified — see for example the discussion in section 7.3.3 of [13]. However clearly one needs a general argument establishing this.

One possibility would be to show that the different gauge choices are equivalent to reparametrizations in field space. An alternative approach to using gauge fixing is perhaps to use a cutoff version of the Vilkovisky-DeWitt gauge independent formulation of gauge theory [34]. We will leave further discussion of this problem to future work.

The question of gauge dependence also leads us to address what it is that a theory of quantum gravity hopes to calculate. As is well-known diffeomorphism invariance in quantum gravity implies that there are no local observables. In asymptotically flat or AdS backgrounds however it is possible to define an S-matrix that is well-defined as an observable. It may even be possible to do so in an asymptotically dS background [37]. Thus one would adopt the following procedure.

Consider the Wilsonian effective action for a theory of quantum gravity with a cutoff scale Λ. This would be an expansion in terms of (space-time integrals of) an infinite set of local operators which is valid for energy scales $E^2 \sim \partial\phi$ or $E^2 \sim R$ (where ϕ is any field and R is space-time curvature), such that $E < \Lambda$, so that the expansion in terms of local operators is valid. Thus for the pure gravity case we have,

$$I_\Lambda = \int d^4x\sqrt{g}\Big[\Lambda^4 g_0(\Lambda) + \Lambda^2 g_1(\Lambda)R + \big(g_{2a}(\Lambda)R_{\mu\nu}R^{\mu\nu} + g_{2b}(\Lambda)R^2 + g_{3b}R_{\ldots\ldots}R^{\cdots\cdot}\big) + \Lambda^{-2}\left(g_{3a}(\Lambda)RR_{\mu\nu}R^{\mu\nu} + \ldots\right) + \left(g_{3a}^{(1)}R\Box R + \ldots\right) + O\left(\Lambda^{-4}\right)\Big] + I_\Lambda^{\rm GF} + I_\Lambda^{\rm ghost}. \tag{3.11}$$

The asymptotic safety program hopes to establish that the dimensionless couplings have, in addition to the Gaussian fixed point $g_i = 0$ in the IR, also a non-trivial fixed point $g_i = g_i^*$, $\beta_i(g^*) = 0$ where not all g_i^* are zero. Furthermore only a finite number of these dimensionless couplings g_i are expected to be relevant. For instance in concrete calculations it appears that only g_0, g_1, g_2 are relevant, and so need to be determined by experiment. This then gives us a Wilsonian effective action which may be used to calculate the S-matrix for gravitons, once the background for the far past and the far future of the scattering process has been chosen to be (say) flat space, for arbitrarily high energies $E < \Lambda$ in terms of the three undetermined couplings.[12] Furthermore the BRST invariance of (3.11) ensures that the S-matrix is independent of the gauge fixing.[13]

[11]For a detailed discussion of this and a suggestion for a modified version of Γ_k see [35, 36].

[12]A concrete method of calculation (in the particular case of cosmology at least) has been discussed by Weinberg [25]. For a given energy E one needs to optimize the value of the cutoff Λ. In order to be able to ignore higher dimension operators one needs $E < \Lambda$. On the other hand in order to be able to ignore higher order radiative corrections the cutoff should not be much higher than E. This is of course very much in the spirit of perturbative QCD calculations as explained in the above reference.

[13]See for instance [34] and also [38, 39].

One issue that may affect this argument is the question of unitarity. Perturbatively it appears that any higher derivative theory has a propagating ghost — in particular in higher derivative gravity there appears to be a spin two ghost [40]. However this weak coupling argument may not be valid in the complete theory. Let us consider this in more detail. In the theory defined above (3.11) each dimensionless coupling will have an asymptotic expansion for large as well as small Λ. Thus defining the planck scale M_P by (the inverse of) the gravitational constant measured at long distances we have for instance with $g_i^{(j)}$ being pure numbers or (for $j > 0$) at most polynomials in $\ln \frac{\Lambda}{M_P}$,

$$\begin{aligned}
g_1(\Lambda) &= g_1^{(0)} + g_1^{(1)} \frac{M_P^2}{\Lambda^2} + \dots,\ \Lambda \gg M_P \\
&= \frac{M_P^2}{2\Lambda^2} + \tilde{g}_1^{(1)} + \tilde{g}_1^{(2)} \frac{\Lambda^2}{M_P^2} + \dots,\ \Lambda \ll M_P \\
g_2(\Lambda) &= g_2^{(0)} + g_2^{(1)} \frac{M_P^2}{\Lambda^2} + \dots,\ \Lambda \gg M_P \\
&= \tilde{g}_2^{(0)} + \tilde{g}_2^{(1)} \frac{\Lambda^2}{M_P^2} + \dots,\ \Lambda \ll M_P.
\end{aligned}$$

Now the existence of the perturbative ghost is inferred by looking at the propagator of the low energy theory (in a flat background i.e. writing $g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_P} h_{\mu\nu}$ truncated to four derivatives), which turns out to have two poles, one at zero mass corresponding to the graviton and another with squared mass $M_P^2/\tilde{g}_2^{(0)}$. But at this point the theory is strongly coupled and all higher derivative terms would also contribute and it is not at all clear that this putative ghost will survive in the full theory. Indeed it has been argued that such states decouple and that the S-matrix is unitary [38]. Furthermore it has been shown in toy models that the usual argument (even in weak coupling) for the existence of a ghost in quartic derivative theories is incorrect [41–43]. The Hamiltonian of the theory has to be interpreted not as a (Dirac) Hermitian one but as a PT symmetric one — in which case contrary to the naive expectation one has a unitary theory with a positive energy spectrum.

While the above arguments clearly do not imply that (3.11) defines a unitary quantum gravity, what they do show is that the naive argument for the existence of a perturbative ghost, does not mean that the correctly interpreted complete theory violates unitarity.

4 The beta function equations

4.1 General considerations

The formula (3.9) gives a straightforward way of evaluating the beta functions of any theory. In (3.10) let us introduce dimensionless couplings g^A by writing

$$\bar{g}^A = \Lambda^{4-n_A} g^A \tag{4.1}$$

where n_A is the physical (a.k.a. canonical or engineering) dimension of the operator Φ. Thus $n_A = 0$ for the unit operator (the cosmological constant term proportional to $\sqrt{g}$),

$n_A = 2$ for the Einstein-Hilbert term and for a scalar mass term, $n_A = 4$ for scalar, vector and fermionic kinetic terms and "renormalizable" interactions in the sense of perturbative QFT. Terms with physical dimensions $n_A > 4$ are the so-called "non-renormalizable" terms, amongst which one will have both higher derivative terms such as $\phi\Box^2\phi$, R^2, as well as higher powers of field operators such as $\phi^6, (\bar{\psi}\psi)^2$. Then the flow equation (3.7) (we ignore the complications of gauge fixing and ghosts for the moment), becomes an infinite set of coupled equations for the dimensionless couplings g_A:

$$\dot{g}^A + (4 - n_A)g^A = \Lambda^{n_A - 4}\mathrm{Tr}\,\exp\left\{-\frac{1}{\Lambda^2}\frac{\delta}{\delta\phi_c}\otimes\frac{\delta}{\delta\phi_c}I_\Lambda[\phi_c]\right\}\bigg|_{\Phi_A},\; A = 0, 1, 2, \ldots. \tag{4.2}$$

The instruction on the right is to isolate the coefficient of the operator Φ_A in the expansion of the trace. Also we've written $\dot{g} \equiv \frac{dg}{dt}$, $t = \ln\frac{\Lambda}{\Lambda_0}$ where Λ_0 is a fiducial scale which can be identified with the Planck scale (i.e. $\Lambda_0^2 = M_P^2 \equiv 1/8\pi G_N$ where G_N is Newton's constant measured at low energies).

The r.h.s. of (4.2) gives the contribution of quantum fluctuations to the various beta functions. It is succinctly given in this formula by the heat kernel trace whose expansion in powers of $1/\Lambda^2$ can be systematically worked out.[14]

These beta function equations take the general form

$$\dot{g}^A + (4 - n_A)g^A = \eta^A(\{g\}) \tag{4.3}$$

with $\eta^A = 0, \forall A$ when $g^A = 0, \forall A$ provided we replace $g^A \to \hat{g}^A \equiv (g^A)^{-1}$ for gravitational and gauge couplings. This means that there is always a Gaussian fixed point solution $\dot{g}^A = 0$, $g^A = 0$, $\forall A$. A non-trivial fixed point would exist if the infinite set of equations

$$(4 - n_A)g^A = \eta^A(\{g\}), \tag{4.4}$$

has real solutions $g^A = g_*^A$, with g_A^* finite for all A and $\neq 0$ for at least some couplings. We will argue below using the general structure of the heat kernel expansion, that this is indeed the case for gravity coupled to a scalar field theory.

The question of the nature of the fixed point and in particular its critical surface is determined by linearizing (4.3) around the fixed point so we have

$$\frac{d\delta g^A}{dt} = \sum_B\left(-(4 - n_A)\delta_B^A + \frac{\partial\eta^A(\{g\})}{\partial g^B}\bigg|_{g_*}\right)\delta g^B \equiv \sum_B D^A_B(g_*)\delta g^B. \tag{4.5}$$

The number of negative eigenvalues of the matrix $\mathbf{D}$ (i.e. the number of relevant directions) is then the dimensionality of the critical surface for an UV fixed point. A predictive (i.e. renormalizable) theory should have only a finite number of relevant directions and the corresponding couplings (at some fiducial scale) would need to be determined by experiment. The other (irrelevant) directions can then be set to their fixed point values.

To be more specific let us introduce the eigenvectors $\mathbf{u}$ of the matrix $\mathbf{D}$ with eigenvalues $\theta^{(J)}$ — i.e. $\mathbf{D}\mathbf{u}^{(J)} = \theta^{(J)}\mathbf{u}$. Also suppose that $\theta^{(0)}, \ldots \theta^{(R-1)} < 0$, while the rest are positive (or zero). In this case

$$\mathbf{u}^{(J)}(t) = e^{-|\theta^{(J)}|t}\mathbf{u}^{(J)}(0) \to 0,\; J = 0, \ldots R - 1,\; t \to \infty.$$

[14] In practice beyond the first few orders, it becomes extremely complicated though.

So the deviation from the fixed point value at the fiducial scale $\Lambda = \Lambda_0$ will need to be fixed by experiment. On the other hand for (the infinite set of) positive (or zero) eigenvalues we set $\mathbf{u}^{(J)}(0) = 0$, i.e. the corresponding couplings at the fiducial scale are equal to the fixed point values.

Now canonically in the standard model coupled to gravity there are only three operators with $n_A < 4$, namely the unit operator (cosmological constant term) with $n = 0$, the Higgs mass term with $n = 2$ and the Einstein-Hilbert term with $n = 2$. Then there are kinetic terms for all fields and the Yukawa couplings, with canonical dimension $n = 4$. All the other (infinite number of) operators have integral canonical dimensions with $n \geq 5$, i.e. $4 - n < -1$. Thus unless there are large anomalous dimensions (at the NGFP) one might expect the number of relevant operators to remain the same (or at least finite) as in the absence of quantum corrections. Obviously this is the case around the Gaussian fixed point. We will argue below that this is very likely to be the case around the UV fixed point as well.

4.2 The heat kernel expansion and the beta functions

In this section we will discuss the beta functions for a scalar theory coupled to gravity.[15] The action for the theory at some scale Λ is

$$\begin{aligned} I_\Lambda = & \int d^4x\sqrt{g}\Big[\Lambda^4 g_0(\Lambda) + \Lambda^2 g_1(\Lambda) R + \big(g_{2a}(\Lambda) R_{\mu\nu}R^{\mu\nu} + g_{2b}(\Lambda)R^2 + g_{3b}R_{....}R^{....}\big) \\ & +\Lambda^{-2}\left(g_{3a}(\Lambda) R R_{\mu\nu}R^{\mu\nu} + \ldots\right) + \left(g_{3a}^{(1)} R\Box R + \ldots\right) + O\big(\Lambda^{-4}\big)\Big] \\ & + \int d^4x\sqrt{g}\left[Z\left(\phi^2/\Lambda^2\right)\frac{1}{2}\phi(-\Box)\phi + V(\phi,\Lambda) + \xi(\phi,\Lambda)R + O\big(\partial^4\big)\right] \\ & + I_\Lambda^{(\mathrm{G.F.})} + I_\Lambda^{(\mathrm{ghost})}. \end{aligned} \tag{4.6}$$

Here

$$V(\phi,\Lambda) = \frac{1}{2}\lambda_1(\Lambda)\Lambda^2\phi^2 + \frac{1}{4!}\lambda_2(\Lambda)\phi^4 + \frac{1}{6!}\lambda_3(\Lambda)\Lambda^{-2}\phi^6 + \ldots, \tag{4.7}$$

$$Z\left(\frac{\phi^2}{\Lambda^2}\right) = Z_0 + \frac{1}{2}Z_1\frac{\phi^2}{\Lambda^2} + \ldots \tag{4.8}$$

$$\xi(\phi,\Lambda) = \frac{1}{2}\xi_1\phi^2 + \frac{1}{4!}\xi_2\phi^4 + \ldots \tag{4.9}$$

All coefficients g_i, λ_i, etc. are dimensionless. The field independent term in the potential has been included in the first line as an explicit cosmological constant term i.e. $\Lambda^4 g_0(\Lambda) = \Lambda_{CC}$. Also $\Lambda^2 g_1(\Lambda) = -1/16\pi G_N(\Lambda) \equiv -1/2\kappa^2(\Lambda)$ where $G_N(\Lambda)$ is Newton's constant at the scale Λ. To get the matrix $\frac{\delta}{\delta\phi_c}\otimes\frac{\delta}{\delta\phi_c}I_\Lambda[\phi_c] \equiv \mathbf{I}_\Lambda^{(2)}$ we expand around the background fields $g_{\mu\nu} \to g_{\mu\nu} + 2\kappa h_{\mu\nu}$, $\phi \to \phi + \hat{\phi}/\sqrt{Z_0}$ (dropping the subscript c), and identify the coefficients of $h\otimes h, h\otimes\hat{\phi}, \hat{\phi}\otimes\hat{\phi}$, to evaluate the second derivative matrix on the background fields.

[15] For previous treatments of this system based on the Wetterich equation see for example [44] and references therein.

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If one restricts the discussion to the two derivative action then this matrix operator takes the form

$$\mathbf{I}_\Lambda^{(2)} = -\nabla^2\mathbf{I} + \mathbf{E} \tag{4.10}$$

where $\mathbf{I}$ is the unit matrix on the space of symmetric transverse traceless tensors, vectors and scalars as well as space-time, and $\mathbf{E}$ is a matrix on the same space with matrix elements that are linear in the Riemann tensor, as well as ϕ dependent terms. The RG flow equation (3.9) will however generate the higher derivative terms in the action. Then in addition there will be terms ∇^4, etc in (4.10) and $\mathbf{E}$ will have higher dimension (i.e. greater than or equal to four) field dependent terms; higher powers of the Riemann tensor as well as its derivatives, in addition to higher dimension operators constructed out of ϕ and its derivatives and mixed terms such as $R\phi^2$ etc.

It is convenient to separate the constant part of the matrix $\mathbf{E}$ by writing

$$\mathbf{E} = \Lambda^2\mathbf{E}_0 + \hat{\mathbf{E}}. \tag{4.11}$$

Here the first constant term comes from the cosmological constant term (the first term of equation (4.6)) and the scalar mass term, so that (labelling the rows and columns of the matrix schematically with $h, \hat{\phi}$,

$$\mathbf{E}_0 = \begin{bmatrix} \frac{g_0}{g_1}\mathbf{I} & 0 \\ 0 & Z_0^{-1}\lambda_1(\Lambda) \end{bmatrix}.$$

The field dependent operator on the other hand has the structure

$$\hat{\mathbf{E}} = \begin{bmatrix} \hat{O}_2 & \hat{O}_1 \\ \hat{O}_1^{\mathrm{T}} & \hat{\hat{O}}_2 \end{bmatrix},$$

where the subscripts on the operators indicate the lowest (operator) dimension contained therein and we have suppressed the matrix indices. The off diagonal terms come from mixed derivatives such as $\delta^2/\delta g\delta\phi$ and are odd dimensional, starting with the dimension one operator which is linear in ϕ. Thus it would contain terms such as $g_1^{-1/2}Z_0^{-1/2}(\lambda_1(\Lambda)\Lambda\phi + \frac{1}{3!}\lambda_2(\Lambda)\Lambda^{-1}\phi^3 + \ldots)$ and $\Box\phi/(\sqrt{g_1 Z_0}\Lambda)$, etc. They will contribute to the traces of quadratic and higher powers of $\hat{\mathbf{E}}$ in the heat kernel expansion. The diagonal operators are even dimensional — starting with terms such as R and $g_1^{-1}\Lambda^{-2}V(\phi)$ in the case of the hh block $\hat{O}_2$, and with $\frac{1}{2}\lambda_2\phi^2 + \frac{1}{4!}\lambda_3\Lambda^{-2}\phi^4 + \ldots$and $2\xi(0)R + \ldots$, for the $\phi\phi$ block. In addition there is a ghost sector which we have suppressed for the moment.

To be explicit let us write out $\hat{\mathbf{E}}$ in the above theory of gravity coupled to a scalar field keeping only up to dimension two operators. Let us also label the rows and columns by $h_{\mu\nu}^{TF}$, $h \equiv ih_\lambda^\lambda$, C_μ, $\hat{\phi} \equiv \sqrt{Z_0}\delta\phi$ where the penultimate field is the diffeomorphism ghost, and the i in the trace comes from the rotation of the integration over the conformal mode to the imaginary axis to get a well-defined Euclidean functional integral. So including the gauge fixing term in Landau gauge (see [11, 33]) we have

$$\hat{O}_2 = 2\mathbf{U}[(\mathbf{I} - \mathbf{P}) - \mathbf{P}] + \kappa^2\hat{\mathbf{T}},$$

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where

$$U^{\mu\nu}_{\rho\sigma} = \frac{1}{2}R\delta^{\mu\nu}_{\rho\sigma} + \frac{1}{2}(g^{\mu\nu}R_{\rho\sigma} + g_{\rho\sigma}R^{\mu\nu}) - \delta^{(\mu}_{(\rho}R^{\nu)}_{\sigma)} - R^{(\mu\ \nu)}_{\ (\rho\ \sigma)}, \tag{4.12}$$

$$\hat{T}^{\lambda\sigma}_{\mu\nu} = \frac{1}{2}(g_{\mu\nu}T^{\lambda\sigma} + g^{\lambda\sigma}T_{\mu\nu}) + \left(\frac{\delta T_{\mu\nu}}{\delta g_{\lambda\sigma}} + \frac{\delta T^{\lambda\sigma}}{\delta g^{\mu\nu}}\right). \tag{4.13}$$

Here $T_{\mu\nu}$ is the stress-energy tensor of the matter sector and $\delta^{\mu\nu}_{\rho\sigma} = \delta^{(\mu}_{\rho}\delta^{\nu)}_{\sigma}$ ($X_{(\mu\nu)} \equiv \frac{1}{2}(X_{\mu\nu}+X_{\nu\mu})$) and $\mathbf{I}-\mathbf{P}$, $\mathbf{P} = [\frac{1}{4}g^{\mu\nu}g_{\lambda\sigma}]$ are projection operators onto the space of trace free and traced two index symmetric tensors. Thus $\hat{O}_2$ is a 10×10 matrix acting on the space of symmetric tensors which can be partially diagonalized into a 9×9 matrix on symmetric trace free tensors $h^{(TF)}_{\mu\nu}$ and a 1×1 acting on the trace part h^{μ}_{μ}. Including also the ghosts, the second diagonal block (labelled by $C^{\mu}, \hat{\phi}$) is (with $\mathbf{R} = [R_{\mu}^{\ \nu}]$)

$$\hat{\hat{O}}_2 = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \hat{V}''(\phi) + \hat{\xi}''(\phi)R \end{bmatrix},$$

where $\hat{V}''(\phi) = \frac{1}{2}\hat{\lambda}_2\phi^2 + \dots$ and $\hat{\xi}''(\phi) = \hat{\xi}_1 + \frac{1}{2}Z_0\hat{\xi}_2\phi^2 + \dots$with $\hat{\lambda}_i \equiv \lambda_i/(Z_0)^i$ and $\hat{\xi}_i \equiv \xi_i/(Z_0)^i$. The off-diagonal blocks take the form

$$\hat{O}_1 = \begin{bmatrix} \mathbf{0} & \frac{\kappa}{\sqrt{Z_0}}\mathbf{T}_{,\phi} \\ 0 & i\frac{\kappa}{\sqrt{Z_0}}T^{\mu}_{\mu}{}_{,\phi} \end{bmatrix}$$

where the row labels have been separated to correspond to the trace free and trace parts of $h_{\mu\nu}$. The i is a consequence of the rotation of the trace part (the conformal mode) of the graviton fluctuation to the imaginary axis.

The heat kernel coefficients for the first few terms of its expansion have been calculated (see for example [45]). As we said earlier it is useful to separate the constant piece so we have

$$\exp\left[-\Lambda^{-2}\left(-\nabla^2\mathbf{I} + \mathbf{E}\right)\right] = e^{-\mathbf{E}_0}\exp\left[-\Lambda^{-2}\left(-\nabla^2\mathbf{I} + \hat{\mathbf{E}}\right)\right]. \tag{4.14}$$

Our object is to calculate the trace of this operator over space-time and internal indices i.e. $\mathrm{Tr} = \int \mathrm{d^4x}\sqrt{\mathrm{g}}\mathrm{tr}$. Writing $\mathbf{K} \equiv -\nabla^2\mathbf{I} + \mathbf{E}$ we have the expansion,

$$\mathrm{Tr}e^{-\mathbf{K}/\Lambda^2} = \frac{1}{(4\pi)^2}\left[\Lambda^4 B_0(\mathbf{K}) + \Lambda^2 B_2(\mathbf{K}) + B_4(\mathbf{K}) + \Lambda^{-2}B_6(\mathbf{K}) + \dots\right], \tag{4.15}$$

$$B_n(\mathbf{K}) = \int d^4x\sqrt{g}\mathrm{tr}e^{-\mathbf{E}_0}\mathbf{b}_n. \tag{4.16}$$

Let us quote the first three b coefficients:

$$\mathbf{b}_0 = \mathbf{I}, \tag{4.17}$$

$$\mathbf{b}_2 = \frac{R}{6}\mathbf{I} - \hat{\mathbf{E}}, \tag{4.18}$$

$$\mathbf{b}_4 = \frac{1}{180}\left(R^{\cdots\cdot}R_{\cdots\cdot} - R^{\cdot\cdot}R_{\cdot\cdot} + \frac{5}{2}R^2 + 6\nabla^2 R\right)\mathbf{I} - \frac{1}{6}R\hat{\mathbf{E}} + \frac{1}{2}\hat{\mathbf{E}}^2 - \frac{1}{6}\nabla^2\hat{\mathbf{E}}. \tag{4.19}$$

These coefficients get rapidly more complicated. However for our purposes the exact numbers in front of the operators are not important. The only assumption we make is that they are $O(1)$ or less as is the case for the ones that have been calculated. In the general situation where one needs to keep higher derivative terms such as R^2 and $\phi\Box^2\phi$ which lead to fourth order derivative operators in (4.10), the coefficients of the fourth order operators (i.e. in $\mathbf{b}_{2r}$ for $r \geq 2$) will of course change, but will also affect the dimension 0 and 2 operators.

On the other hand for the scalar potential an exact RG equation is possible except for the constant (cosmological constant) term, which depends on all higher derivative couplings. The reason is that in computing the evolution of the potential one can treat the fields as constants so that the entire contibution will factor out as in (4.14) with $\mathbf{E}_0$ now containing all non-derivative terms (such as ϕ^{2n}), and $\hat{\mathbf{E}}$ now having at least two derivatives of the fields.[16]

We will leave the discussion of (untruncated) exact equations (and to what extent we can find a justification for the truncations that have been used) for future work. Here we will just follow what has been done with the Wetterich equation with our version of the Polchinski equation truncated to the lowest non-trivial operators for gravity coupled to a scalar field. Then the beta function equations for the dimension zero and two operators are given by (defining $\dot{x} = \Lambda\frac{d}{d\Lambda}x$, $\hat{\lambda}_i = \lambda_i/(Z_0)^i\,\gamma_i = \ln Z_i$),

$$\dot{g}_0 + 4g_0 = \frac{1}{(4\pi)^2}\left[10e^{-g_0/g_1} - 4 + e^{-\hat{\lambda}_1}\right], \tag{4.20}$$

$$\dot{g}_1 + 2g_1 = -\frac{1}{(4\pi)^2}\frac{1}{3}\left[13e^{-g_0/g_1} + 5 + \frac{1}{2}e^{-\hat{\lambda}_1}\left(1 - 6\hat{\xi}_1\right)\right], \tag{4.21}$$

$$\dot{\hat{\lambda}}_1 + \dot{\gamma}_0\hat{\lambda}_1 + 2\hat{\lambda}_1 = -\frac{e^{-\hat{\lambda}_1}}{(4\pi)^2}\left[\frac{\hat{\lambda}_2}{2} + \frac{1}{8}\frac{\hat{\lambda}_1^2}{g_1}\right] - \frac{5}{(4\pi)^2}e^{-g_0/g_1}\frac{\hat{\lambda}_1}{g_1}. \tag{4.22}$$

Note that in the last equation we've kept the contribution of the operator ϕ^4 since this is a measure of the error introduced by the truncation of the scalar field theory. It is useful also to consider these equations in terms of the following alternative variables

$$g_{{}_N}(\Lambda) = 2\kappa^2(\Lambda)\Lambda^2 = -\frac{1}{g_1(\Lambda)},\ 2\lambda_{CC} = \Lambda^2 2\kappa^2 g_0 = -\frac{g_0}{g_1}. \tag{4.23}$$

The beta function equations above then become[17]

$$\dot{\lambda}_{\rm CC} + 2\lambda_{\rm CC} = \frac{g_N}{(4\pi)^2}\left[\left(5 - \frac{13}{3}\lambda_{\rm CC}\right)e^{2\lambda_{\rm CC}} - \left(2 + \frac{5}{3}\lambda_{\rm CC}\right) + e^{-\hat{\lambda}_1}\left(\frac{1}{2} - \frac{1}{6}\lambda_{\rm CC}\left(1 - 6\hat{\xi}_1\right)\right)\right], \tag{4.24}$$

$$\dot{g}_N - 2g_N = -\frac{g_N^2}{(4\pi)^2}\frac{1}{3}\left[13e^{2\lambda_{\rm CC}} + 5 + \frac{1}{2}e^{-\hat{\lambda}_1}\left(1 - 6\hat{\xi}_1\right)\right], \tag{4.25}$$

$$\dot{\hat{\lambda}}_1 + \dot{\gamma}_0\hat{\lambda}_1 + 2\hat{\lambda}_1 = -\frac{e^{-\hat{\lambda}_1}}{(4\pi)^2}\left[\frac{\hat{\lambda}_2}{2} - \frac{1}{8}g_N\hat{\lambda}_1^2\right] + \frac{5}{(4\pi)^2}e^{2\lambda_{CC}}g_N\hat{\lambda}_1. \tag{4.26}$$

[16] It should be noted that in spite of the imaginary off-diagonal elements in $\mathbf{E}$, the final result for the beta functions which only involves traces of products (and derivatives) of these matrices is real.

[17] Note that in previous derivations of these equations there is a singularity $1/1 - 2\lambda_{CC})$. This is absent in our treatment and comes from the exact form given below i.e. $e^{2\lambda_{CC}} = 1/e^{-2\lambda_{CC}}$ which if expanded in the denominator and truncated to its leading two terms gives the above singularity which is clearly spurious.

In this form we see the Gaussian fixed point (GFP) at $g_N = \lambda_{CC} = \lambda_1 = \lambda_2 = \xi_1 = \xi_2 = 0$. Clearly there is also a non-Gaussian fixed point (NGFP) with for instance $g_N^* = 6(4\pi)^2[13e^{2\lambda_{CC}} + 5 + \frac{1}{2}e^{-\hat{\lambda}_1}(1-6\hat{\xi}_1)]^{-1}$. Note that this is clearly positive at least as long as ξ_1 is not $\gg O(1)$ as in Higgs inflation and that one may also get (effectively) a large negative contribution if there is a large number of scalar fields.

However to investigate the NGFP it is more transparent to consider the previous version i.e. (4.20), (4.21), (4.22). Putting the "time" derivatives to zero, we have a system of equations which determine the fixed point vales of all the couplings in terms of two undetermined parameters (both coming from scalar "mass" terms) which may be chosen to be (say) $\hat{\lambda}_1 = \hat{\lambda}_1^*$, $\hat{\xi}_1 = \hat{\xi}_1^*$. Then the first two fixed point equations (4.20)–(4.21) give one transcendental equation

$$2\lambda_{CC} = 3\frac{10e^{2\lambda_{CC}} - 4 + e^{-\hat{\lambda}_1^*}}{26e^{2\lambda_{CC}} + 10 + e^{-\hat{\lambda}_1^*}\left(1 - 6\hat{\xi}_1^*\right)}.$$

This determines λ_{CC}^* (as an $O(1)$ number) which when used in the second equation will determine g_1^* (which it should be noted must be negative giving a positive value for g_N). The third equation then determines the ϕ^4 coupling $\hat{\lambda}_2^*$. It also follows from the fact that the fixed point is at $O(1)$ values for g_0, g_1 (taking $\lambda_1^* > 0$ and $\xi_1^* = O(1)$), that the critical exponents are essentially given by the canonical values up to corrections $O(1/(4\pi)^2)$, so that this is indeed a UV fixed point with (at least) two relevant directions.

To get the beta function for λ_2 we need to consider the heat kernel expansion to the next order i.e. the $\mathbf{b}_4$ term. The coefficients of the beta function for the terms in the potential (to arbitrary order) are actually easily evaluated since they can be obtained by ignoring all derivative terms of ϕ as discussed before. In particular the beta function for ϕ^4 term is

$$\frac{1}{4!}\left(\dot{\lambda}_2 + \dot{\gamma}_0\hat{\lambda}_2\right) = \frac{e^{-\hat{\lambda}_1}}{(4\pi)^2}\left(\frac{1}{8}\hat{\lambda}_2^2 - \frac{1}{4!}\hat{\lambda}_3 + \frac{1}{3}\frac{1}{g_1}\hat{\lambda}_2\hat{\lambda}_1\right) + \frac{4g_N e^{2\lambda_{CC}}}{(4\pi)^2}\frac{1}{4!}\hat{\lambda}_2. \tag{4.27}$$

At a fixed point we already know $\hat{\lambda}_1, \hat{\lambda}_2, \lambda_{CC}, g_N$ (the last two and hence the last term on the r.h.s. above is of course known provided we ignore higher derivative terms), so this equation just determines $\hat{\lambda}_3$. This process obviously repeats itself ad infinitum with the beta function equation at a fixed point for $\hat{\lambda}_n$ determining $\hat{\lambda}_{n+1}$, since the set of fixed point equations for λ_m with $m < n$ have already determined all the other couplings which enter into this equation.[18]

Just focusing on the scalar sector in flat space it appears as if there is a one parameter family of fixed points since λ_1 does not appear to be determined by any of the fixed point equations. However it has been argued (see [7, 47] and references therein for flat space scalar field theory at least) the requirement of a scaling solution for all values of the dimensionless field $\tilde{\phi} \in \Re$ would restrict the allowed values of λ_1 to a discrete set for space time dimension $d = 3$ (in the sense that the potential blows up at some finite value of $\tilde{\phi}$

[18] This argument, for the case of flat space scalar field theory, seems to have been first given by Weinberg [14]. For comments on the usual statements about triviality of such theories see also [46] page 137.

otherwise) and has no solution in $d = 4$. However it is not clear that there is a physical requirement that the potential should be non-singular for all real values of $\tilde{\phi}$. In practice for finite Λ one works with $\tilde{\phi} \equiv \phi/\Lambda < \tilde{\phi}_s < 1$ so it is not obvious that a singularity (in the limit $\Lambda \to \infty$) at a finite value of $\tilde{\phi}_s$ is not physically reasonable. If indeed the scalar field sector is trivial, then for the asymptotic safety program to remain valid one needs to argue that the inclusion of gravitational corrections (such as the last term of (4.27)), somehow changes this conclusion.

The necessary inclusion of higher order terms such as R^2, implies that we have higher derivative terms in the kinetic operator; i.e. $\mathbf{K} = -\nabla^2 \mathbf{I} + \mathbf{E} + O(\nabla^4)$. These terms will not only affect $\mathbf{b}_4$ and the higher order coefficients but also the lower order ones and hence the beta-functions for the cosmological constant and Newton's constant as well. These calculations have been performed for operators whose highest derivative is quartic by generalizing standard heat kernel methods (see for example [48, 49]). Calculating these has been done for $f(R)$ theories and for a theory with $R_{\mu\nu}^2 + R^2$ (see for example [11] and references therein). The general conclusion is that the fixed point values obtained for the lowest order truncations (i.e. the solutions of (4.24)–(4.25) etc.) are not changed significantly.

However in general even with the inclusion of all the higher order contributions to $\mathbf{K}$, one expects that the coefficients of the generalized heat kernel expansion will all be of $O(1)$ or smaller as is the case for the ones that have actually been calculated, and that there is no singular behavior in this expansion. If true this implies that the anomalous dimension matrix has at most order one coefficients divided by $(4\pi)^2$. This means that the relevance or irrelevance of operators at the UV fixed point is determined by their canonical dimensions — except for the three R^2 operators and the ϕ^4 interaction (see eqn (4.5)) and perhaps a few more. Thus we see that the cosmological constant term and the Einstein Hilbert term are relevant[19] and all operators of dimension 6 and above are most probably irrelevant since to make them relevant, one would need large anomalous dimensions to cancel the canonical dimension term in (4.5) for $n_A - 4 > 2$. This is of course the general expectation that has been confirmed in some cases by detailed calculations in the literature. The relevance or irrelevance of the dimension four operators on the other hand can only be established by direct calculation. Truncating at this order, for instance in pure gravity, appears to give a non trivial fixed point value for these couplings (see for example [13] and references therein). The relevance (or irrelevance) of these classically marginal operators

[19] It is important to stress that these statements are manifest in terms of the natural couplings g_i defined in (4.6). If on the other hand one chose the inverses (such as $g_N = -g_1^{-1}$) then the fixed point values are at numerically large values ($O(4\pi)^2$). Thus (the equation (see (4.21) and (4.5)) for $\delta g_1 \equiv g_1 - g_1^*$ is

$$\dot{\delta g_1} + 2\delta g_1 = O\left(\frac{1}{(4\pi)^2}\right)$$

so that this is a relevant direction. On the other hand defining $\delta g_N = g_N - g_N^*$ the equation gives $\dot{\delta g}_N - 2\delta g_N \sim O(\frac{g_N}{(4\pi)^2}) g_N$ which of course can be used to establish that near the GFP this direction is relevant in the IR but near the NGFP which is strongly coupled the right hand side cannot be ignored, and restores the conclusion based on the analysis of the g_1 equation that direction is relevant in the UV. See for example [11, 12].

is however hard to establish since their scaling dimension could be significantly affected by higher derivative ("R^6" etc) terms. So it seems that the situation for these operators remains somewhat murky.

5 Comments and conclusions

Some comments on the general approach we have adopted here and its relation to the literature are in order.

1. The first equality of equation (3.7) (or (3.9)) resembles the Wetterich equation (2.9) if one uses $\frac{d}{dt}\ln K_{k,\Lambda} = \frac{\dot{K}_{k,\Lambda}}{K_{k,\Lambda}} = \frac{\dot{R}_{k,\Lambda}}{K_{k,\Lambda}}$. This would then reintroduce the problem mentioned at the end of section (2). The point is to use the Schwinger proper time expression (and regularization) of the one-loop determinant given in (3.3). Note that the difference (in the $k \to \Lambda$ limit) is an infinite term (which is not field independent in the background field version which is essential for the gauge theory discussion).

2. In practice, the average effective action is only used in the regime where the background field momentum modes are low compared to the cutoff k (which is usually identified as a IR cutoff) in order for the heat kernel expansion to be valid. So the entire discussion involving the so-called average effective action is completely equivalent to one involving the Wilsonian effective action. To put it another way as far as deriving the beta function equations go, one is effectively dealing with a floating UV cutoff rather than a IR cutoff. Although it is formally true that the definition (2.10) leads to the quantum effective action Γ in the limit $k \to 0$ it is difficult to recover in practice the non-localities of Γ starting from the quasi-local Γ_k since this would involve summing the infinite series of the derivative expansion.[20]

3. For the same reason as in the above, the RG equation (3.9) does not generate non-local terms — the expansion of the heat kernel is local as long as the mode numbers ("momenta") of the background fields are small compared to the cutoff. This is of course the well known statement that the Wilsonian effective action is (quasi) local. By the above arguments it follows that in actual calculations the average effective action is also being treated as being quasi-local.

4. The use of the equation (3.9) coupled with the heat kernel expansion and proper time regularization gives a simple expression for all the beta function equations of the theory. Of course as in other approaches one can find the fixed point only after truncating the number of operators.

5. To the extent that they have been computed, the heat kernel coefficients are order one or smaller. Given that there is no reason to expect anomalously large coefficients it is very plausible that there is only a finite number of relevant directions at the UV fixed point. In other words we expect that there are no large anomalous dimensions

[20] For partial summations recovering some aspects of non-locality see [50].

which can swamp the canonical dimensions of operators, which grow with the order of the terms in the derivative expansion. We found two relevant directions in agreement with previous work in pure gravity.

6. While in pure gravity the UV fixed point is completely determined, in gravity coupled to a scalar field theory (or to the standard model), there are two undetermined parameters corresponding to the scalar mass term and the $\phi^2 R$ term. However as in the case of flat space scalar field theory, it is possible that only a discrete set of values admit a scaling solution. All other couplings are then determined in terms of these.

Much work remains to be done to really establish the standard model coupled to gravity as a UV complete theory. While an iterative procedure for establishing the existence of a fixed point appears to be valid for the scalar potential the same is not the case for derivative interactions since higher derivative terms feed back into the flow equation for lower derivative terms. Nevertheless explicit calculations [11, 51] seem to show that the fixed point established (in pure gravity) is hardly affected by the inclusion of higher derivative terms. Also apart from giving a rigorous proof of the dimensionality of the critical surface, an important unresolved problem remains; that of establishing the unitarity of the theory, since in perturbation theory such higher derivative theories have ghosts. As discussed in subsection (3.1) at tree level these ghosts will appear at the (low energy) Planck mass. However at the scale of this putative ghost, in order to minimize radiative corrections in the Wilsonian action, the cutoff Λ needs to be chosen at around the same scale. But in this case all terms in the effective action will make similar contributions. On the other hand if the cutoff is chosen much larger the radiative corrections become important. Thus it is possible that (as discussed also in section VIII of [11] for instance) non-perturbative quantum effects change the spectrum at the putative ghost mass. It is also possible that higher derivative theories need to be reinterpreted as having PT invariant rather than Hermitian Hamiltonians as suggested in [41–43]. Clearly this is one of the most important issue that needs to be addressed in the asymptotic safety program.

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