

RECEIVED: November 22, 2016

REVISED: December 13, 2016

ACCEPTED: December 20, 2016

PUBLISHED: January 9, 2017

A_∞/L_∞ structure and alternative action for WZW-like superstring field theory

Keiyu Goto^{a,1} and Hiroaki Matsunaga^{b,c}

^a*Institute of Physics, University of Tokyo,
Komaba, Meguro-ku, Tokyo 153-8902, Japan*

^b*Institute of Physics, Academy of Sciences of the Czech Republic,
Na Slovance 2, Prague 8, Czech Republic*

^c*Yukawa Institute for Theoretical Physics, Kyoto University,
Kyoto 606-8502, Japan*

E-mail: keiyu.goto.9@gmail.com, matsunaga@fzu.cz

ABSTRACT: We propose new gauge invariant actions for open NS, heterotic NS, and closed NS-NS superstring field theories. They are based on the large Hilbert space, and have Wess-Zumino-Witten-like expressions which are the \mathbb{Z}_2 -reversed versions of the conventional WZW-like actions. On the basis of the procedure proposed in [arXiv:1505.01659](https://arxiv.org/abs/1505.01659), we show that our new WZW-like actions are completely equivalent to A_∞/L_∞ actions proposed in [arXiv:1403.0940](https://arxiv.org/abs/1403.0940) respectively.

KEYWORDS: String Field Theory, Superstrings and Heterotic Strings

ARXIV EPRINT: [1512.03379](https://arxiv.org/abs/1512.03379)

¹Present address: Komazawa, Setagaya-ku, Tokyo 154-0012, Japan.

Contents

1	Introduction and summary	1
2	WZW-like structure from the L_∞ formulation	8
2.1	Construction of L_∞ -product and L_∞ action	9
2.2	Alternative WZW-like form of L_∞ action	12
2.3	Variation of the action S_η	15
2.4	Gauge transformations	16
3	Alternative parameterisation for closed NS string field theory	17
3.1	Large space parameterisation	18
3.2	Equivalence of the actions in the different parameterizations	20
4	Conclusions and discussions	21
A	Basic facts of A_∞ and L_∞	23
B	Derivation of (2.71)	28
C	Embedding to the large Hilbert space of NS closed string	29
D	Open NS superstrings with stubs	31
D.1	Alternative WZW-like relations and action $S_\eta[\varphi]$	33
D.2	Small-space parametrization: $\varphi = \Psi$	34
D.3	Large-space parametrization: $\varphi = \Phi$	35
E	NS-NS sector	37
E.1	Alternative Wess-Zumino-Witten-like relations	38
E.2	Alternative WZW-like action $S_{\eta\tilde{\eta}}[\varphi]$	38
E.3	Small-space parameterisation: $\varphi = \Phi$	40
E.4	Large-space parametrization: $\varphi = \Psi$	41

1 Introduction and summary

Recently, the formulation of superstring field theory has progressed gradually: actions for superstring field theories were constructed and their properties have been clarified [1–4]. In addition to these, 1PI effective field theory approach has provided a good insight into self-dual gauge theory and supergravity [5]. One trigger of these developments was given by the establishment of the A_∞/L_∞ formulation: the gauge invariant actions for open NS string [6], heterotic NS string, and closed NS-NS string [7] were constructed by giving

the prescription to get the string products satisfying A_∞/L_∞ relations. This prescription was extended to the cases including the Ramond sector and the equations of motion were provided in the work of [8]. At least at the tree level, these theories reproduce S-matrices of perturbative superstring theory with insertions of picture-changing operators at external lines [9]. One of our aims in this paper is to develop the understanding of the relation between these A_∞/L_∞ theories and other theories such as WZW-like theories.¹

While these A_∞/L_∞ theories work well, we know that WZW-like theories [16–21] would also give the same results from [22, 23].² Actually, we have clear understandings for open superstrings: for the NS sector, the relation between the A_∞ action given in [6] and the Berkovits WZW-like action [16] is clarified by the works of [29–31], and for the NS and R sectors, the equivalence of the complete action from which the equations of motion given in [8] are derived and the complete action proposed in [2] is provided by the work of [4].³ It would be important to extend these understandings of open strings to the case of closed strings and to construct complete actions for heterotic and type II theories. However, it seems to be difficult to discuss them on the basis of the same procedure⁴ as [29] and known WZW-like actions [18–21]: it necessitates other insights because of *the skew* between A_∞/L_∞ actions and these conventional WZW-like actions, which we explain. It would be helpful to consider their dual versions.

Recall that the Berkovits theory is formulated on the large Hilbert space, which is the state space whose superconformal ghost sector is spanned by $\xi(z)$, $\eta(z)$, and $\phi(z)$ [33]. In this paper, we write η for the zero mode of $\eta(z)$ for brevity. An NS string field Φ of the Berkovits theory is a Grassmann even, ghost number 0, and picture number 0 state in the large Hilbert space: $\eta\Phi \neq 0$, and all string products are defined by Witten’s star product [34]. Using a real parameter $t \in [0, 1]$ and a path $\Phi(t)$ connecting $\Phi(0) = 0$ and $\Phi(1) = \Phi$, the action of the Berkovits theory is given by *the Wess-Zumino-Witten-type form*

$$S_{\text{Berkovits}}[\Phi] = \langle e^{-\Phi}\eta e^\Phi, e^{-\Phi}Qe^\Phi \rangle + \int_0^1 dt \langle e^{-\Phi(t)}\partial_t e^{\Phi(t)}, \llbracket e^{-\Phi(t)}\eta e^{\Phi(t)}, e^{-\Phi(t)}Qe^{\Phi(t)} \rrbracket^* \rangle,$$

where Q is the BRST operator, $\langle A, B \rangle$ is the BPZ inner product of A and B , and $\llbracket A, B \rrbracket^*$ is the graded commutator for the star products of A and B . One can find that this action has nonlinear gauge invariances given by

$$\delta e^\Phi = e^\Phi(\eta\Omega_B) + (Q\Lambda_B)e^\Phi,$$

where Λ_B and Ω_B are gauge parameters. Like wise $\llbracket A, B \rrbracket^*$, in this paper, we write $\llbracket d_1, d_2 \rrbracket$ for the graded commutator of operators d_1 and d_2 ,

$$\llbracket d_1, d_2 \rrbracket = d_1 d_2 - (-)^{d_1 d_2} d_2 d_1.$$

The upper index of $(-)^d$ denotes the grading of the operator d , namely, its ghost number.

¹For other approaches, see also [10–15].

²For the R sector, see [24–28].

³See also a new result given by T. Erler, Y. Okawa and T. Takezaki, *JHEP* **08** (2016) 012 [[arXiv:1602.02582](https://arxiv.org/abs/1602.02582)].

⁴By taking another approach, namely by constructing an L_∞ -morphism connecting their equations of motion, one can discuss the *on-shell equivalence* of L_∞ actions [7] and conventional WZW-like actions [19, 21]. The existence of the L_∞ -morphism provides the equivalence of the solution spaces of two equations of motion up to Q -exact terms, which implies the on-shell states match each other. See [32] for more details.

Conventional WZW-like form. It was shown in [19] that Berkovits action can be written as a *WZW-like form*

$$S_{\text{Berkovits}}[\Phi] = \int_0^1 \langle \tilde{A}_t[\Phi(t)], \eta \tilde{A}_Q[\Phi(t)] \rangle$$

with *WZW-like functionals* $\tilde{A}_Q = \tilde{A}_Q[\Phi]$ of the dynamical string field Φ defined by

$$\tilde{A}_Q[\Phi(t)] \equiv e^{-\Phi(t)}(Qe^{\Phi(t)}), \quad \tilde{A}_t[\Phi(t)] \equiv e^{-\Phi(t)}(\partial_t e^{\Phi(t)}).$$

The equation of motion is given by the t -independent form

$$\eta \tilde{A}_Q[\Phi] = 0,$$

and we can represent gauge transformations using nilpotent operators $Q_{\tilde{A}_Q[\Phi]}$ and η ,

$$\tilde{A}_\delta[\Phi] = Q_{\tilde{A}_Q[\Phi]} \tilde{\Lambda} + \eta \tilde{\Omega},$$

where $\tilde{\Lambda} \equiv e^\Phi \Lambda_B e^{-\Phi}$ and $\tilde{\Omega} \equiv \Omega_B$ are redefined gauge parameters, and

$$\tilde{A}_\delta[\Phi] \equiv e^{-\Phi}(\delta e^\Phi), \quad Q_{\tilde{A}_Q[\Phi]} \tilde{\Lambda} \equiv Q \tilde{\Lambda} + m_2(\tilde{A}_Q, \tilde{\Lambda}) + m_2(\tilde{\Lambda}, \tilde{A}_Q).$$

We write $m_2(A, B)$ for the star product of A and B . A significant feature of this WZW-like form of the action is that one can obtain all properties of the action by using not explicit forms of functionals $\tilde{A}_Q[\Phi]$, $\tilde{A}_t[\Phi]$ but only specific algebraical relations of these functionals, which we call *WZW-like relations*:

$$Q \tilde{A}_Q[\Phi] + m_2(\tilde{A}_Q[\Phi], \tilde{A}_Q[\Phi]) = 0,$$

$$(-)^d d \tilde{A}_Q[\Phi] = Q_{\tilde{A}_Q[\Phi]} \tilde{A}_d[\Phi],$$

where $d = \partial_t, \delta, \eta$ and the upper index of $(-)^A$ denotes the Grassmann parity of A . While the first relation provides *the constraint* for $\tilde{A}_Q[\Phi]$, the second relation specifies its derivatives and the properties of the equation of motion. We call $\tilde{A}_d[\Phi]$ *an associated field*.

The first relation implies that the WZW-like functional $\tilde{A}_Q[\Phi]$, which we call *a pure-gauge-like field*,⁵ gives a solution of the Maurer-Cartan equation for the bosonic open string products

$$M^B = Q + m_2,$$

which consists of the BRST operator Q and the star product m_2 . Recall that M^B satisfies A_∞ -relations: $Q^2 = 0$, $Q m_2(A, B) - m_2(QA, B) - (-)^A m_2(A, QB) = 0$, and $m_2(m_2(A, B), C) - m_2(A, m_2(B, C)) = 0$. The derivation properties for $d = \partial_t, \delta, \eta$

⁵The Maurer-Cartan equation for M^B and the explicit form of $\tilde{A}_Q[\Phi]$ have the same forms as the equation of motion and pure gauge field in bosonic open string field theory respectively. Actually, one can construct $\tilde{A}_Q[\Phi] \equiv \tilde{A}_Q[1; \Phi]$ as the $\tau = 1$ value solution $\tilde{A}_Q[\tau; \Phi]$ of the differential equation $\partial_\tau \tilde{A}_Q[\tau; \Phi] = Q_{\tilde{A}_Q[\tau; \Phi]} \tilde{A}_Q[\tau; \Phi]$ with the initial condition $\tilde{A}_Q[0; \Phi] = 0$, which is just the defining equation of the pure gauge in bosonic theory.

also hold: $[[d, Q]] = 0$ and $d m_2(A, B) - m_2(dA, B) - (-)^{dA} m_2(A, dB) = 0$. In this paper, we symbolically⁶ write

$$[[\mathbf{M}^B, \mathbf{M}^B]] = 0, \quad [[\mathbf{d}, \mathbf{M}^B]] = 0$$

for the A_∞ -relations of \mathbf{M}^B and the d -derivation properties of \mathbf{M}^B . These properties yield the WZW-like relations, and then, the constraint equation and the equation of motion are given by

$$\text{Constraint (C-WZW)} : \quad Q \tilde{A}_Q[\Phi] + m_2(\tilde{A}_Q[\Phi], \tilde{A}_Q[\Phi]) = 0,$$

$$\text{E.O.M. (C-WZW)} : \quad \eta \tilde{A}_Q[\Phi] = 0.$$

This construction is extended to more generic case: open NS superstrings with stubs, heterotic NS strings [19], and closed NS-NS strings [21]. As an example, we consider the Berkovits theory with stubs. The starting point is a set of generic bosonic open string products [35]

$$\mathbf{M}_{\text{stub}}^B = Q + m_2^{\text{st}} + m_3^{\text{st}} + m_4^{\text{st}} + \dots$$

which are nonassociative but satisfy A_∞ -relations and derivation properties for $d = \partial_t, \delta, \eta$:

$$[[\mathbf{M}_{\text{stub}}^B, \mathbf{M}_{\text{stub}}^B]] = 0, \quad [[\mathbf{d}, \mathbf{M}_{\text{stub}}^B]] = 0.$$

Using these $\mathbf{M}_{\text{stub}}^B$, d , and the NS open string field φ , one can construct a pure-gauge-like field $\tilde{A}_Q^{\text{st}}[\varphi]$ and an associated field $\tilde{A}_d^{\text{st}}[\varphi]$ via the same type of the defining differential equations as those of the theory without stubs. The resultant theory is given by the following action

$$S_{\text{stub}}[\varphi] = \int_0^1 dt \langle \tilde{A}_t^{\text{st}}[\varphi], \eta \tilde{A}_Q^{\text{st}}[\varphi] \rangle,$$

and characterized by the pair of equations:

$$\text{Constraint (C-WZW)} : \quad Q \tilde{A}_Q^{\text{st}}[\varphi] + m_2^{\text{st}}(\tilde{A}_Q^{\text{st}}[\varphi], \tilde{A}_Q^{\text{st}}[\varphi]) + \sum_{n=3}^{\infty} m_n^{\text{st}}(\overbrace{\tilde{A}_Q^{\text{st}}[\varphi], \dots, \tilde{A}_Q^{\text{st}}[\varphi]}^n) = 0,$$

$$\text{E.O.M. (C-WZW)} : \quad \eta \tilde{A}_Q^{\text{st}}[\varphi] = 0.$$

However, when we compare this conventional WZW-like action and the A_∞ action given in [7], we notice that there exists a skew: A_∞ theory is characterised by the pair of equations

$$\text{Constraint } (A_\infty) : \quad \eta \Psi = 0,$$

$$\text{E.O.M. } (A_\infty) : \quad Q \Psi + M_2(\Psi, \Psi) + \sum_{n=3}^{\infty} M_n(\overbrace{\Psi, \dots, \Psi}^n) = 0, \tag{1.1}$$

where Ψ is the NS open string field of A_∞ theory and $\{M_n\}_{n=2}^{\infty}$ are NS superstring products. While the BRST operator Q and the string products $\{m_n^{\text{st}}\}_{n=2}^{\infty}$ determine the constraint

⁶We use bold fonts for coalgebraic operations or notions. See appendix A.

equation for the ingredients of the conventional WZW-like theory, that of A_∞ theory is supplied by η . This skew is just the obstacle to obtain the off-shell equivalence of two theory by the naive way.

Alternative WZW-like form. It is known that Berkovits action can be also written as *alternative WZW-like form*

$$S_{\text{Berkovits}}[\Phi] = - \int_0^1 dt \langle A_t[\Phi(t)], Q A_\eta[\Phi(t)] \rangle \tag{1.2}$$

with *alternative WZW-like functionals* $A_\eta = A_\eta[\Phi]$ of the dynamical string field Φ defined by

$$A_\eta[\Phi] \equiv (\eta e^{\Phi(t)}) e^{-\Phi(t)}, \quad A_t[\Phi] \equiv (\partial_t e^{\Phi(t)}) e^{-\Phi(t)}. \tag{1.3}$$

Then, the equation of motion is given by the t -independent form

$$Q A_\eta[\Phi] = 0,$$

and we can represent gauge transformations using nilpotent operators D_η^* and Q ,

$$A_\delta[\Phi] = D_\eta^* \Omega + Q \Lambda,$$

where $\Lambda \equiv \Lambda_B$ and $\Omega \equiv e^{-\Phi} \Omega_B e^\Phi$ are redefined gauge parameters, and

$$A_\delta[\Phi] \equiv (\delta e^\Phi) e^{-\Phi}, \quad D_\eta^* \Omega \equiv \eta \Omega - m_2(A_\eta[\Phi], \Omega) - (-)^\Omega m_2(\Omega, A_\eta[\Phi]).$$

A significant feature of this alternative WZW-like form is that, as the conventional case, one can obtain all properties of the action by using only specific algebraical relations of alternative WZW-like functionals, which we also call *WZW-like relations*:

$$\begin{aligned} \eta A_\eta[\Phi] - m_2(A_\eta[\Phi], A_\eta[\Phi]) &= 0, \\ (-)^d d A_\eta[\Phi] &= D_\eta^* A_d[\Phi], \end{aligned} \tag{1.4}$$

where $d = \partial_t, \delta, Q$. Therefore, we can say that this type of WZW-like theory also belongs to the category of so-called the WZW-like formulation. While the first relation provides the constraint for $A_\eta[\Phi]$, the second relation specifies the properties of the equation of motion.

Note that this type of WZW-like theory, a dual version of the conventional WZW-like theory, is characterized by the pair of equations

$$\text{Constraint (A-WZW)} : \quad \eta A_\eta[\Phi] - m_2(A_\eta[\Phi], A_\eta[\Phi]) = 0,$$

$$\text{E.O.M. (A-WZW)} : \quad Q A_\eta[\Phi] = 0.$$

One can find that there is no skew between this type of WZW-like theory and A_∞ theory. Actually, as demonstrated in [29], by decomposing the NS superstring product $\mathbf{M} = \mathbf{Q} + \mathbf{M}_2 + \mathbf{M}_3 + \dots$ given in [6] as $\mathbf{M} = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}$ and by using a redefined string field

$A_\eta[\Psi] \equiv \pi_1 \widehat{\mathbf{G}}(1 - \Psi)^{-1}$, we can transform the pair of equations (1.1) characterizing the A_∞ action into

$$\text{Constraint } (A_\infty) : \quad \eta A_\eta[\Psi] - m_2(A_\eta[\Psi], A_\eta[\Psi]) = 0,$$

$$\text{E.O.M. } (A_\infty) : \quad Q A_\eta[\Psi] = 0.$$

Therefore, we expect that one can construct WZW-like actions which are off-shell equivalent to A_∞ and L_∞ actions proposed in [7] as this type of WZW-like theories.

Main results of this paper. What is the starting point of this type of WZW-like theory? We can read it from the constraint for $A_\eta[\Phi]$ or $A_\eta[\Psi]$. For the Berkovits theory, it is given by the following A_∞ -product:

$$D_*^\eta \equiv \eta - m_2,$$

which we call *the dual products* of \mathbf{M} of [6]. We write D_2^η for $-m_2$. Then, one can check that $\eta + D_2^\eta$ satisfies the A_∞ -relations of η : $\eta^2 = 0$, $\eta D_2^\eta(A, B) - D_2^\eta(\eta A, B) - (-)^A D_2^\eta(A, \eta B) = 0$, and $D_2^\eta(D_2^\eta(A, B), C) - D_2^\eta(A, D_2^\eta(B, C)) = 0$. The derivation properties for $d = \partial_t, \delta, Q$ also hold: $[[d, \eta]] = 0$ and $d D_2^\eta(A, B) - D_2^\eta(dA, B) - (-)^A D_2^\eta(A, dB) = 0$. We write them as follows:

$$[[D_*^\eta, D_*^\eta]] = 0, \quad [[d, D_*^\eta]] = 0,$$

which give the starting point of our alternative WZW-like theory. Actually, one can construct the pure-gauge-like field $A_\eta[\Phi] = (\eta e^\Phi) e^{-\Phi}$ of (1.3) and the action (1.2) using this $D_*^\eta = \eta - m_2$: the pure-gauge-like field $A_\eta[\Phi]$ in (1.3) is given by the $\tau = 1$ value $A_\eta[\Phi] \equiv A_\eta[\tau = 1; \Phi]$ of the solution $A_\eta[\tau; \Phi]$ of the differential equation

$$\frac{\partial}{\partial \tau} A_\eta[\tau; \Phi] = \eta \Phi - m_2(A_\eta[\tau; \Phi], \Phi) + m_2(\Phi, A_\eta[\tau; \Phi])$$

with the initial condition $A_\eta[\tau = 0; \Phi] = 0$, where $\tau \in [0, 1]$ is a real parameter.

In this paper, we show that a nonassociative extended version of this construction gives our new WZW-like theory, which provides the equivalence of A_∞/L_∞ formulation and WZW-like formulation. The starting point is the following A_∞ products

$$D^\eta = \eta + D_2^\eta + D_3^\eta + D_4^\eta + \dots$$

satisfying the A_∞ -relations of η and derivation properties for $d = \partial_t, \delta, Q$:

$$[[D^\eta, D^\eta]] = 0, \quad [[d, D^\eta]] = 0.$$

One can construct these products by the dual products of $\mathbf{M} = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}$ given in [7], namely $D^\eta \equiv \widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1}$. Then, using these dual products D^η , we propose alternative WZW-like actions which are equivalent to A_∞ actions proposed in [7] as this type of WZW-like theories

$$S_\eta[\varphi] = - \int_0^1 dt \langle A_t[\varphi(t)], Q A_\eta[\varphi(t)] \rangle, \tag{1.5}$$

where φ is a dynamical NS string field, $t \in [0, 1]$ is a real parameter. The WZW-like functionals $A_\eta[\varphi]$ and $A_t[\varphi]$ satisfy the nonassociative extended versions of (1.4):

$$\eta A_\eta[\varphi] - m_2(A_\eta[\varphi], A_\eta[\varphi]) + \sum_{n=3}^{\infty} D_n^\eta(\overbrace{A_\eta[\varphi], \dots, A_\eta[\varphi]}^n) = 0, \quad (1.6)$$

$$(-)^d d A_\eta[\varphi] = D_\eta A_d[\varphi],$$

where now the A_η -shifted dual product D_η is given by

$$D_\eta \Omega \equiv \eta \Omega - m_2(A_\eta[\varphi], \Omega) - (-)^\Omega m_2(\Omega, A_\eta[\varphi]) + \sum_{\text{cyclic } n=2}^{\infty} \sum (-)^{\text{sign}} D_{n+1}^\eta(\overbrace{A_\eta[\varphi], \dots, A_\eta[\varphi]}^n, \Omega).$$

Note that this WZW-like action for generic open NS strings just reduces to the alternative WZW-like form (1.2) of the Berkovits theory when we take the star product. We would like to emphasize that we do not need a specific form of $A_\eta[\varphi]$ or $A_t[\varphi]$ as a functional of given dynamical string field φ but only their properties (1.6) to show the properties of the action: its variation, equations of motion, gauge invariance, and so on. In appendix D, we will give explicit forms of two realizations of these functionals A_η and A_d using two different dynamical string fields: Ψ in the small Hilbert space and Φ in the large Hilbert space. Namely, the equivalence of A_∞ and WZW-like actions for open superstring field theory with stubs.

In the above, we take open NS theory as an example to grab a feature of our alternative WZW-like approach and its necessity. The details of the above open NS theory are discussed in appendix D. In the following sections, our main topic is “heterotic NS theory”: we consider the L_∞ action and explain how its WZW-like properties arise. On the basis of the WZW-like structure naturally arising from L_∞ actions, we propose new gauge invariant actions for heterotic NS (and NS-NS strings in appendix E), as well as that for open NS strings with stubs which we introduced in (1.5). These actions are \mathbb{Z}_2 -reversed versions of the conventional ones [19, 21], and we show that they are completely equivalent to the A_∞/L_∞ actions proposed in [7].

We expect that our new WZW-like actions would provide a first step to construct complete actions for heterotic and type II string field theories.⁷ Actually, an action for open superstring field theory including the R sector was constructed [2] by starting with this type of WZW-like action: the R string field couples to the Berkovits theory for the NS sector gauge-invariantly on the basis of (not the conventional but) this type of WZW-like gauge structure. We would like to mention that although we expect that our new WZW-like actions are also equivalent to the conventional WZW-like actions, the all order equivalence of these has not been proven: we will show lower-order equivalence to the conventional WZW-like actions only.

This paper is organized as follows. In section 2, after a brief review of the L_∞ formulation, we clarify a WZW-like structure naturally arising from it. We show that the L_∞ action can be written in our (alternative) WZW-like form: the functionals appearing in the

⁷See a new result given by K. Goto and H. Kunitomo, [arXiv:1606.07194](https://arxiv.org/abs/1606.07194).

action satisfy alternative WZW-like relations, the \mathbb{Z}_2 -reversed versions of the conventional WZW-like relations given in [19], which guarantees the gauge invariance of the action. The on-shell condition and the gauge transformation of the (alternative) WZW-like action are derived by using only the (alternative) WZW-like relations. We also see how the gauge parameters appearing in the WZW-like form are parameterised by that in the L_∞ form. Then, we conclude the L_∞ action gives one realization of the alternative WZW-like action $S_\eta[\Phi]$ parametrized by the dynamical string field Φ in the small Hilbert space. In section 3, we provide another realisation of the alternative WZW-like action using the string field V in the large Hilbert space. The functionals satisfying our WZW-like relations can be defined by the differential equations which are the \mathbb{Z}_2 -reversed versions of those given in [19]. Utilizing them, we construct a new gauge invariant action $S_\eta[V]$ for heterotic NS string field theory. We derive the condition for the equivalence of the new action $S_\eta[V]$ and the L_∞ action on the basis of the procedure demonstrated in [29]: these are different parameterizations of the same WZW-like structure and action. Then we also derive the relation between two dynamical string fields Φ and V from the equivalence condition. We end with some conclusions and discussions.

Basic facts and definitions of the coalgebraic notation of A_∞/L_∞ algebras are summarized in appendix A. In appendix B, we derive a formula which is used in section 2. In appendix C, we consider the trivially embedding of the string field of the L_∞ action belonging to the small Hilbert space into the string field in the large Hilbert space, and show the embedded action can also be written in the WZW-like form. Appendices D and E are devoted to the open NS and the closed NS-NS theories, respectively.

2 WZW-like structure from the L_∞ formulation

In this section, we clarify a WZW-like structure naturally arising from the L_∞ formulation for NS heterotic string field theory. After a brief review of the L_∞ formulation [7], we show that the L_∞ action can be written in our (alternative) WZW-like form. We see that the functionals appearing in the action satisfy alternative WZW-like relations, the \mathbb{Z}_2 -reversed version of the conventional WZW-like relations in [19], which guarantees the gauge invariance of the action.

Preliminaries. The product of n closed strings is described by a multilinear map $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$, where \wedge is the *symmetrized tensor product* satisfying $\Phi_1 \wedge \Phi_2 = (-)^{\deg(\Phi_1)\deg(\Phi_2)} \Phi_2 \wedge \Phi_1$. A map $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$ with degree 1 naturally induces a map from the *symmetrized tensor algebra* $\mathcal{S}(\mathcal{H}) = \mathcal{H}^{\wedge 0} \oplus \mathcal{H}^{\wedge 1} \oplus \mathcal{H}^{\wedge 2} \oplus \dots$ to $\mathcal{S}(\mathcal{H})$ itself, called a *coderivation*. A map $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}'$ with degree 0 also naturally induces a map from $\mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\mathcal{H}')$, called a *cohomomorphism*. Since it is convenient to write the functionals of the string field and the action in terms of them, we briefly introduce the rules of their actions here.

The coderivation $\mathbf{d}_n : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$ is naturally derived from a map $d_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$ with degree one. It act on $\Phi_1 \wedge \cdots \wedge \Phi_N \in \mathcal{H}^{\wedge N \geq n} \subset \mathcal{S}(\mathcal{H})$ as

$$\begin{aligned} \mathbf{d}_n(\Phi_1 \wedge \cdots \wedge \Phi_N) &= (d_n \wedge \mathbb{I}_{N-n})(\Phi_1 \wedge \cdots \wedge \Phi_N) \\ &= \sum_{\sigma} \frac{(-)^\sigma}{n!(N-n)!} d_n(\Phi_{\sigma(1)}, \dots, \Phi_{\sigma(n)}) \wedge \Phi_{\sigma(n+1)} \wedge \cdots \wedge \Phi_{\sigma(N)}, \end{aligned} \quad (2.1)$$

and vanishes when acting on $\mathcal{H}^{\wedge N \leq n}$. The graded commutator of two coderivations \mathbf{b}_n and \mathbf{c}_m , $[[\mathbf{b}_n, \mathbf{c}_m]]$, is a coderivation derived from the map $[[b_n, c_m]] : \mathcal{H}^{\wedge n+m-1} \rightarrow \mathcal{H}$ which is defined by

$$[[b_n, c_m]] = b_n(c_m \wedge \mathbb{I}_{n-1}) - (-)^{\deg(b_n)\deg(c_m)} c_m(b_n \wedge \mathbb{I}_{m-1}). \quad (2.2)$$

A set of degree zero multilinear maps $\{f_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}'\}_{n=0}^{\infty}$ naturally induces a cohomorphism $\widehat{f} : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}')$, which acts on $\Phi_1 \wedge \cdots \wedge \Phi_n \in \mathcal{H}^{\wedge n} \subset \mathcal{S}(\mathcal{H})$ as

$$\begin{aligned} \widehat{f}(\Phi_1 \wedge \cdots \wedge \Phi_n) &= \sum_{i \leq n} \sum_{k_1 < \cdots < k_i} e^{\wedge f_0} \wedge f_{k_1}(\Phi_1, \dots, \Phi_{k_1}) \wedge f_{k_2-k_1}(\Phi_{k_1+1}, \dots, \Phi_{k_2}) \wedge \\ &\quad \cdots \wedge f_{k_i-k_{i-1}}(\Phi_{k_{i-1}+1}, \dots, \Phi_n). \end{aligned} \quad (2.3)$$

For definitions and their more details, see appendix A.

2.1 Construction of L_∞ -product and L_∞ action

In this subsection, we briefly review the construction of the NS superstring product \mathbf{L} and the action $S_{\text{EKS}}[\Phi]$ for heterotic string field theory in the L_∞ formulation originally given in [7].

NS superstring product \mathbf{L} . Let us review the construction of the NS string products $\mathbf{L}[\tau] = \sum_{p=0}^{\infty} \tau^p \mathbf{L}_{p+1}$ satisfying L_∞ -relation $[[\mathbf{L}[\tau], \mathbf{L}[\tau]]] = 0$, the cyclicity $\mathbf{L}^\dagger = -\mathbf{L}$, and the η -derivation $[[\eta, \mathbf{L}[\tau]]] = 0$. The $(p+1)$ -product \mathbf{L}_{p+1} carries the ghost number $1-2p$ and the picture number p . The products $\mathbf{L}[\tau]$ consist of η , ξ , and Zwiebach's bosonic string products $\mathbf{L}^{\text{B}} = \mathbf{Q} + \mathbf{L}_2^{\text{B}} + \mathbf{L}_3^{\text{B}} + \cdots$ satisfying $[[\mathbf{L}^{\text{B}}, \mathbf{L}^{\text{B}}]] = 0$ [36]. First, we focus on the condition for $\mathbf{L}[\tau]$ to satisfy the L_∞ relations,

$$[[\mathbf{L}[\tau], \mathbf{L}[\tau]]] = 0. \quad (2.4)$$

The L_∞ relations holds if we define $\mathbf{L}[\tau]$ as a solution of the differential equations

$$\partial_\tau \mathbf{L}[\tau] = [[\mathbf{L}[\tau], \lambda^{[0]}[\tau]]], \quad (2.5)$$

with the initial condition $\mathbf{L}[\tau=0] = Q$. Here $\lambda^{[0]}[\tau] = \sum_{p=0}^{\infty} \tau^p \lambda_{p+2}^{[0]}$ are called gauge products. We can take any $\lambda^{[0]}$ as long as they carry correct quantum numbers: the $(p+2)$ -product $\lambda_{p+2}^{[0]}$ carries ghost number $-2(p+1)$ and picture number $p+1$. The solution for the differential equations is given by the similarity transformation of Q ,

$$\mathbf{L}[\tau] = \widehat{\mathbf{G}}^{-1}[\tau] \mathbf{Q} \widehat{\mathbf{G}}[\tau], \quad (2.6)$$

where $\widehat{\mathbf{G}}$ is an invertible cohomomorphism defined by the path-ordered exponential of the gauge products $\lambda^{[0]}_{[\tau]}$ as follows

$$\widehat{\mathbf{G}}_{[\tau]} = \overleftarrow{\mathcal{P}} \exp \left(\int_0^\tau d\tau' \lambda^{[0]}_{[\tau']} \right), \quad \widehat{\mathbf{G}}^{-1}_{[\tau]} = \overrightarrow{\mathcal{P}} \exp \left(- \int_0^\tau d\tau' \lambda^{[0]}_{[\tau']} \right). \quad (2.7)$$

Here $\leftarrow(\rightarrow)$ on \mathcal{P} denotes that the operator at later time acts from the right (left). The differential equations hold since the path-ordered exponentials satisfy

$$\partial_\tau \widehat{\mathbf{G}}_{[\tau]} = \widehat{\mathbf{G}}_{[\tau]} \lambda^{[0]}_{[\tau]}, \quad \partial_\tau \widehat{\mathbf{G}}^{-1}_{[\tau]} = -\lambda^{[0]}_{[\tau]} \widehat{\mathbf{G}}^{-1}_{[\tau]}. \quad (2.8)$$

We may check directly the L_∞ relations:

$$\mathbf{L}^2 = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}} \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}} = \widehat{\mathbf{G}}^{-1} \widehat{\mathbf{Q}} \mathbf{Q} \widehat{\mathbf{G}} = 0. \quad (2.9)$$

The cyclicity and the η -derivation properties follow from the suitable choice of $\lambda^{[0]}$.

Second, let us consider the cyclicity. For \mathbf{L} to be cyclic, it is sufficient to choose $\lambda^{[0]}$ to be BPZ-odd, so that $\widehat{\mathbf{G}}^{-1} = \widehat{\mathbf{G}}^\dagger$:

$$\mathbf{L}^\dagger = (\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}})^\dagger = -\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}} = -\mathbf{L}. \quad (2.10)$$

Third, we check η acts as a derivation on \mathbf{L} , namely the η -derivation properties

$$\llbracket \eta, \mathbf{L}_{[\tau]} \rrbracket = 0. \quad (2.11)$$

A construction of the suitable gauge product $\lambda^{[0]}$ is given in [7]. It is helpful to consider a series of generating functions

$$\mathbf{L}(s, \tau) = \sum_{d=0}^{\infty} s^d \mathbf{L}^{[d]}_{[\tau]} = \sum_{d=0}^{\infty} \sum_{p=0}^{\infty} s^d \tau^p \mathbf{L}^{[d]}_{p+d+1}, \quad (2.12)$$

including $\mathbf{L}_{[\tau]} = \mathbf{L}^{[0]}_{[\tau]}$ and $\mathbf{L}_N^{\mathbf{B}} = \mathbf{L}_N^{[N-1]}$, where the superscript $[d]$ denotes the picture deficit relative to what is needed for the NS products. They satisfy the L_∞ -relations and η -derivation properties,

$$\llbracket \mathbf{L}(s, \tau), \mathbf{L}(s, \tau) \rrbracket = 0, \quad (2.13)$$

$$\llbracket \eta, \mathbf{L}(s, \tau) \rrbracket = 0, \quad (2.14)$$

if we define $\mathbf{L}(s, \tau)$ as a solution for the following differential equations,

$$\partial_\tau \mathbf{L}(s, \tau) = \llbracket \mathbf{L}(s, \tau), \lambda(s, \tau) \rrbracket, \quad (2.15)$$

$$\partial_s \mathbf{L}(s, \tau) = \llbracket \eta, \lambda(s, \tau) \rrbracket. \quad (2.16)$$

Here $\lambda(s, \tau)$ is a series of generating functions for the gauge products

$$\lambda(s, \tau) = \sum_{d=0}^{\infty} s^d \lambda^{[d]}_{[\tau]} = \sum_{d=0}^{\infty} \sum_{p=0}^{\infty} s^d \tau^p \lambda^{[d]}_{d+p+2}, \quad (2.17)$$

including $\lambda_{[\tau]} = \lambda^{[0]}_{[\tau]}$. The expansion of the differential equations (2.15) and (2.16) in powers of s and τ provides the recursive definition of $\mathbf{L}_N^{[d]}$ and $\lambda_N^{[d]}$. Note that the explicit forms of $\lambda_N^{[d]}$ is not determined uniquely even if we require all the desired properties, which follows from the arbitrariness of the inverse of η in (2.16). Then we obtain the suitable $\lambda^{[0]}_{[\tau]}$ which provides η -derivation properties of $\mathbf{L}_{[\tau]} = \mathbf{L}^{[0]}_{[\tau]}$, the s^0 part of (2.14). Hereafter we take $\tau = 1$ and omit the argument:

$$\mathbf{L} := \mathbf{L}_{[\tau=1]}. \quad (2.18)$$

Action for heterotic string field theory in the L_∞ formulation. A gauge-invariant action in the L_∞ formulation is constructed by [7], using the above products $\mathbf{L} = \{L_k\}_{k \geq 1}$. The dynamical string field Φ in the L_∞ formulation carries ghost number 2 and picture number -1 and belongs to the small Hilbert space $\mathcal{H}_{\text{small}}$: $\eta\Phi = 0$. The action is written as follows:

$$S_{\text{EKS}}[\Phi] = \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \langle \xi\Phi, L_{n+1}(\overbrace{\Phi, \Phi, \dots, \Phi}^{n+1}) \rangle, \quad (2.19)$$

where $L_1 = Q$ and $\langle A, B \rangle$ is the BPZ inner product with the $c_0^- = \frac{1}{2}(c_0 - \tilde{c}_0)$ -insertion.

Let us introduce the t -parametrized string field $\Phi(t)$ with $t \in [0, 1]$ satisfying $\Phi(0) = 0$ and $\Phi(1) = \Phi$, which is a path connecting 0 and the string field Φ in the space of string fields. Utilizing this $\Phi(t)$, the action can be represented in the following form:

$$\begin{aligned} S_{\text{EKS}}[\Phi] &= \int_0^1 dt \frac{\partial}{\partial t} \left(\sum_{n=0}^{\infty} \frac{1}{(n+2)!} \langle \xi\Phi(t), L_{n+1}(\overbrace{\Phi(t), \dots, \Phi(t)}^{n+1}) \rangle \right) \\ &= \int_0^1 dt \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \langle \partial_t \xi\Phi(t), L_{n+1}(\overbrace{\Phi(t), \dots, \Phi(t)}^{n+1}) \rangle. \end{aligned} \quad (2.20)$$

Here we use $\Phi(t) = \eta\xi\Phi(t)$, η -derivation property, and the cyclicity of \mathbf{L} in order to obtain the form in which both ∂_t and ξ act on the first slot of the inner product. Note that this t -dependence is topological and it does not appear in the variation of the action, as we will see later. We can represent the action in the coalgebraic notation as follows:

$$S_{\text{EKS}}[\Phi] = \int_0^1 dt \langle \pi_1(\xi_t e^{\wedge\Phi(t)}), \pi_1(\mathbf{L}(e^{\wedge\Phi(t)})) \rangle, \quad (2.21)$$

where $e^{\wedge\Phi}$ is the *group-like element* defined by

$$e^{\wedge\Phi} = \mathbf{1} + \Phi + \frac{1}{2}\Phi \wedge \Phi + \frac{1}{3!}\Phi \wedge \Phi \wedge \Phi + \dots, \quad (2.22)$$

π_1 is a projector from the symmetrized tensor algebra to the single-state space

$$\pi_1(\Phi_0 + \Phi_1 \wedge \Phi_2 + \Phi_3 \wedge \Phi_4 \wedge \Phi_5 + \dots) = \Phi_0, \quad (2.23)$$

and ξ_t is a coderivation⁸ derived from the linear map $\xi\partial_t : \Phi \mapsto \xi\partial_t\Phi$. (See appendix A.)

⁸We will define ξ_d for the more general class of d later. For ∂_t , definitions are equivalent.

The variation of the action can be taken as

$$\delta S_{\text{EKS}}[\Phi] = \langle \xi \delta \Phi, \pi_1(\mathbf{L}(e^{\wedge \Phi})) \rangle, \quad (2.24)$$

and then the equation of motion is given by

$$\pi_1 \mathbf{L}(e^{\wedge \Phi}) = 0. \quad (2.25)$$

Since $\mathbf{L}^2 = 0$, the action is invariant under the gauge transformation

$$\delta \Phi = \pi_1 \mathbf{L}(\lambda \wedge e^{\wedge \Phi}), \quad (2.26)$$

where the gauge parameter λ is in the small Hilbert space and carries ghost number 1 and picture number -1 .

2.2 Alternative WZW-like form of L_∞ action

The action in the L_∞ formulation can be transformed as

$$\begin{aligned} S_{\text{EKS}}[\Phi] &= \int_0^1 dt \langle \pi_1(\xi_t e^{\wedge \Phi(t)}), \pi_1(\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})) \rangle \\ &= \int_0^1 dt \langle \pi_1(\widehat{\mathbf{G}}(\xi_t e^{\wedge \Phi(t)})), \pi_1 \mathbf{Q}(\widehat{\mathbf{G}}(e^{\wedge \Phi(t)})) \rangle. \end{aligned} \quad (2.27)$$

See [32] for heterotic strings, and see also [29] for open strings. We find that the functionals $\Psi_\eta = \Psi_\eta[\Phi]$ of the dynamical string field Φ defined by

$$\Psi_\eta[\Phi(t)] \equiv \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}), \quad (2.28)$$

$$\Psi_d[\Phi(t)] \equiv \pi_1 \widehat{\mathbf{G}}(\xi_d e^{\wedge \Phi(t)}), \quad (2.29)$$

appear in the action, and we will find that these functionals play important roles. One can show that, by introducing a certain set of products satisfying L_∞ -relations,

$$\eta, \quad [\cdot, \cdot]^\eta, \quad [\cdot, \cdot, \cdot]^\eta, \quad \dots, \quad (2.30)$$

the functionals $\Psi_\eta[\Phi(t)]$ and $\Psi_d[\Phi(t)]$ satisfy the (alternative) WZW-like relations:

$$0 = \eta \Psi_\eta + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \overbrace{[\Psi_\eta, \dots, \Psi_\eta]}^{n+1}{}^\eta, \quad (2.31)$$

$$(-)^d d \Psi_\eta = \eta \Psi_d + \sum_{k=1}^{\infty} \frac{1}{k!} \overbrace{[\Psi_\eta, \dots, \Psi_\eta, \Psi_d]}^k{}^\eta, \quad (2.32)$$

which are \mathbb{Z}_2 -reversed versions of conventional WZW-like relations in [19]. In this subsection, after defining these L_∞ products (2.30), which we call the dual L_∞ products for EKS's L_∞ products, we confirm a pair of fields (2.28) and (2.29) satisfy the WZW-like relations (2.31) and (2.32). We write \mathbf{L}^η for this set of dual L_∞ products of (2.30).

Construction of the dual L_∞ product \mathbf{L}^η . The dual L_∞ products \mathbf{L}^η can be constructed using the cohomomorphism $\widehat{\mathbf{G}}$ which provides the NS heterotic string products $\mathbf{L} = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}$.⁹ The product \mathbf{L}^η is defined as the similarity transformation of η :

$$\mathbf{L}^\eta = \widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1} = \sum_{n=1}^{\infty} \mathbf{L}_n^\eta. \quad (2.33)$$

These \mathbf{L}^η carry odd degree, and the n -product \mathbf{L}_n^η carries ghost number $3 - 2n$ and picture number $n - 2$. \mathbf{L}^η satisfy the L_∞ -relations, which follow from its definition:

$$(\mathbf{L}^\eta)^2 = \widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1} \widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1} = \widehat{\mathbf{G}} \eta \eta \widehat{\mathbf{G}}^{-1} = 0. \quad (2.34)$$

The Q -derivation properties of \mathbf{L}^η follow from η -derivatioies property of \mathbf{L} :

$$\mathbf{Q} \mathbf{L}^\eta = \widehat{\mathbf{G}} (\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}) \eta \widehat{\mathbf{G}}^{-1} = -\widehat{\mathbf{G}} \eta (\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}) \widehat{\mathbf{G}}^{-1} = -\mathbf{L}^\eta \mathbf{Q}. \quad (2.35)$$

The cyclicity of L^η follows from that the gauge products are BPZ-odd:

$$(\mathbf{L}^\eta)^\dagger = (\widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1})^\dagger = (\widehat{\mathbf{G}}^{-1})^\dagger \eta^\dagger \widehat{\mathbf{G}}^\dagger = -\widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1}. \quad (2.36)$$

The dual L_∞ products \mathbf{L}^η is the generating function for (2.30). Namely, we define

$$[B_1, \dots, B_n]^\eta := \pi_1 \mathbf{L}_n^\eta (B_1 \wedge \dots \wedge B_n) \quad (2.37)$$

for any states $B_1, \dots, B_n \in \mathcal{H}$. In terms of $[B_1, \dots, B_n]^\eta$, the dual products satisfy the following L_∞ relations, Q -derivation properties, and cyclicity:

$$\sum_{\sigma} \sum_{k=1}^n \frac{1}{k!(n-k)!} (-)^{|\sigma|} [[B_{i_{\sigma(1)}}, \dots, B_{i_{\sigma(k)}}]^\eta, B_{i_{\sigma(k+1)}}, \dots, B_{i_{\sigma(n)}}]^\eta = 0, \quad (2.38)$$

$$Q[B_1, \dots, B_n]^\eta + \sum_{i=1}^n (-)^{B_1 + \dots + B_{k-1}} [B_1, \dots, QB_k, \dots, B_n]^\eta = 0, \quad (2.39)$$

$$\langle B_1, [B_2, \dots, B_{n+1}]^\eta \rangle = (-)^{B_1 + B_2 + \dots + B_n} \langle [B_1, \dots, B_n]^\eta, B_{n+1} \rangle, \quad (2.40)$$

where $(-)^{\sigma}$ is the sign factor of the permutation $\{B_{\sigma(1)}, \dots, B_{\sigma(n)}\}$.

For any state $A \in \mathcal{H}$, the A -shifted products of \mathbf{L}^η are defined by

$$[B_1, B_2, \dots, B_n]_A^\eta = \sum_{m=0}^{\infty} \frac{1}{m!} [\overbrace{A, A, \dots, A}^m, B_1, B_2, \dots, B_n]^\eta. \quad (2.41)$$

They also satisfy L_∞ relations if A is a solution for the Maurer-Cartan equation of \mathbf{L}^η :

$$0 = \pi_1 \mathbf{L}^\eta (e^{\wedge A}) = \eta A + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \overbrace{[A, \dots, A]^\eta}^{n+1}. \quad (2.42)$$

⁹See also appendix D or section 3.2 of [4] for generic properties of these types of products.

Let Ψ_η be a solution for this Maurer-Cartan equation of \mathbf{L}^η . In particular we write D_η for the Ψ_η -shifted 1-product as

$$D_\eta B := [B]_{\Psi_\eta}^\eta = \sum_{m=0}^{\infty} \frac{1}{m!} [\overbrace{\Psi_\eta, \Psi_\eta, \dots, \Psi_\eta}^m, B]^\eta. \quad (2.43)$$

From the L_∞ -relation of the Ψ_η -shifted \mathbf{L}^η , we find that D_η is nilpotent,

$$(D_\eta)^2 B = -[\pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta}), B]_{\Psi_\eta}^\eta = 0, \quad (2.44)$$

and that D_η acts on the Ψ_η -shifted 2-product $[B_1, B_2]_{\Psi_\eta}^\eta$ as a derivation,

$$D_\eta [B_1, B_2]_{\Psi_\eta}^\eta + [D_\eta B_1, B_2]_{\Psi_\eta}^\eta + (-)^{B_1} [B_1, D_\eta B_2]_{\Psi_\eta}^\eta = -[\pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta}), B_1, B_2]_{\Psi_\eta}^\eta = 0. \quad (2.45)$$

Note that the shifted products are BPS odd, which follows from that of \mathbf{L}^η ,

$$\langle B_1, [B_2, \dots, B_{n+1}]_{\Psi_\eta}^\eta \rangle = (-)^{B_1+B_2+\dots+B_n} \langle [B_1, \dots, B_n]_{\Psi_\eta}^\eta, B_{n+1} \rangle. \quad (2.46)$$

WZW-like relations. Let us confirm a pair of fields (2.28) and (2.29) satisfy the WZW-like relations, which can be represented as follows:

$$\pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta}) = 0, \quad (2.47)$$

$$(-)^d d \Psi_\eta = D_\eta \Psi_d. \quad (2.48)$$

The first relation (2.47) directly follows from the fact that Φ belongs to small space:

$$\mathbf{L}^\eta(e^{\wedge \Psi_\eta[\Phi(t)]}) = \mathbf{L}^\eta(e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})}) = (\widehat{\mathbf{G}} \boldsymbol{\eta} \widehat{\mathbf{G}}^{-1}) \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) = \widehat{\mathbf{G}} \boldsymbol{\eta}(e^{\wedge \Phi(t)}) = 0. \quad (2.49)$$

We call $\Psi_\eta[\Phi(t)]$ satisfying (2.47) the *pure-gauge-like field*.

The second relation (2.48) can be confirmed similarly. The operator \mathbf{d} which we focus on is the derivation on \mathbf{L}^η . For example, we can take $d = Q$, ∂_t , or δ . Their derivation property on \mathbf{L}^η leads to $\llbracket \widehat{\mathbf{G}}^{-1} \mathbf{d} \widehat{\mathbf{G}}, \boldsymbol{\eta} \rrbracket = 0$, and we can define the coderivation $\boldsymbol{\xi}_d$ such that

$$\widehat{\mathbf{G}}^{-1} \mathbf{d} \widehat{\mathbf{G}} = (-)^d \llbracket \boldsymbol{\eta}, \boldsymbol{\xi}_d \rrbracket. \quad (2.50)$$

Note that for the operator \mathbf{d} which commutes with $\widehat{\mathbf{G}}$, such as ∂_t and δ , $\boldsymbol{\xi}_d$ is a coderivation derived form $\mathbf{d}\boldsymbol{\xi}$. Then, utilizing this $\boldsymbol{\xi}_d$, the following relation holds:

$$\begin{aligned} (-)^d \mathbf{d} \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) &= (-)^d \widehat{\mathbf{G}}(\widehat{\mathbf{G}}^{-1} \mathbf{d} \widehat{\mathbf{G}})(e^{\wedge \Phi(t)}) \\ &= \widehat{\mathbf{G}} \boldsymbol{\eta} \boldsymbol{\xi}_d(e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \widehat{\mathbf{G}} \boldsymbol{\xi}_d(e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \left(\pi_1 \widehat{\mathbf{G}} \boldsymbol{\xi}_d(e^{\wedge \Phi(t)}) \wedge e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})} \right). \end{aligned} \quad (2.51)$$

Since $D_\eta = \pi_1 \mathbf{L}^\eta(\mathbb{I} \wedge e^{\wedge \Psi_\eta})$, we can see $\Psi_\eta[\Phi(t)]$ and $\Psi_d[\Phi(t)]$ satisfy the WZW-like relation (2.48) using (2.28), (2.29), (2.50), and (2.51). We call $\Psi_d[\Phi(t)]$ satisfying (2.48) the *associated field*.

Then the action in the L_∞ formulation can be written in the alternative WZW-like form:

$$S_{\text{EKS}}[\Phi] = \int_0^1 dt \langle \Psi_t[\Phi(t)], Q\Psi_\eta[\Phi(t)] \rangle. \quad (2.52)$$

The variation of the action can be taken easily using the WZW-like relations (2.47) and (2.48), and the gauge invariances also follows from them, which can be seen in a similar (but \mathbb{Z}_2 -reversed) manner to those in [19]. We will see them in the next subsection.

2.3 Variation of the action S_η

Let us take the variation of the action. Note that in the computation here we use only the WZW-like relations (2.47) and (2.48), and we do not necessitate the explicit forms of the functionals Ψ_η and Ψ_d . Therefore, even if the functionals Ψ_η and Ψ_d have different forms, the variation of the action can be taken in the same manner as long as they satisfy the WZW-like relations. In this subsection, we write $\Psi_\eta(t)$ for $\Psi_\eta[\Phi(t)]$ and so on for brevity.

First, consider the variation of the integrand of (2.52),

$$\delta \langle \Psi_t(t), Q\Psi_\eta(t) \rangle = \langle \delta\Psi_t(t), Q\Psi_\eta(t) \rangle + \langle \Psi_t(t), Q\delta\Psi_\eta(t) \rangle. \quad (2.53)$$

Utilizing the following relation following from (2.48),

$$0 = \llbracket d_1, d_2 \rrbracket \Psi_\eta = (-)^{d_1+d_2} D_\eta (d_1 \Psi_{d_2} - (-)^{d_1 d_2} d_2 \Psi_{d_1} + (-)^{d_2 d_1 + d_2} [\Psi_{d_2}, \Psi_{d_1}]_{\Psi_\eta}^\eta), \quad (2.54)$$

the first term can be transformed into

$$\langle \delta\Psi_t(t), Q\Psi_\eta(t) \rangle = \langle \partial_t \Psi_\delta(t) + [\Psi_\delta(t), \Psi_t(t)]_{\Psi_\eta(t)}^\eta, Q\Psi_\eta(t) \rangle. \quad (2.55)$$

Utilizing $\llbracket D_\eta, d \rrbracket B = -[d\Psi_\eta, B]_{\Psi_\eta}^\eta$, the second term can be transformed into

$$\begin{aligned} \langle \Psi_t(t), Q\delta\Psi_\eta(t) \rangle &= \langle \Psi_t(t), QD_\eta \Psi_\delta(t) \rangle \\ &= \langle D_\eta Q\Psi_t(t), \Psi_\delta(t) \rangle \\ &= \langle -QD_\eta \Psi_t(t) - [Q\Psi_\eta(t), \Psi_t(t)]_{\Psi_\eta(t)}^\eta, \Psi_\delta(t) \rangle. \end{aligned} \quad (2.56)$$

The second terms of (2.55) and (2.56) are canceled because of the cyclicity of the Ψ_η -shifted \mathbf{L}^η . Then we find that the variation of the integrand of (2.52) becomes a total derivative of t :

$$\begin{aligned} \delta \langle \Psi_t(t), Q\Psi_\eta(t) \rangle &= \langle \partial_t \Psi_\delta(t), Q\Psi_\eta(t) \rangle - \langle QD_\eta \Psi_t(t), \Psi_\delta(t) \rangle \\ &= \langle \partial_t \Psi_\delta(t), Q\Psi_\eta(t) \rangle + \langle \Psi_\delta(t), Q\partial_t \Psi_\eta(t) \rangle \\ &= \partial_t \langle \Psi_\delta(t), Q\Psi_\eta(t) \rangle. \end{aligned} \quad (2.57)$$

Integrating over t , the variation of the action is given by

$$\int_0^1 dt \delta \langle \Psi_t(t), Q\Psi_\eta(t) \rangle = \int_0^1 dt \partial_t \langle \Psi_\delta(t), Q\Psi_\eta(t) \rangle = \langle \Psi_\delta(1), Q\Psi_\eta(1) \rangle, \quad (2.58)$$

where the pure-gauge-like field $\Psi_\eta(t)$ and the associated field $\Psi_d(t)$ vanish at $t = 0$. Then the variation of the action becomes

$$\delta S_{\text{EKS}} = \langle \Psi_\delta[\Phi], Q\Psi_\eta[\Phi] \rangle. \quad (2.59)$$

We find that the variation of the action does not depend on t , and therefore t -dependence is topological.

Equations of motion. One can derive the on-shell condition from the variation of the action in the WZW-like form (2.59),

$$Q\Psi_\eta[\Phi] = 0. \quad (2.60)$$

For completion, let us discuss the equivalence of (2.60) and (2.25).

The latter (2.25) can be transformed into the following form:

$$\pi_1 \mathbf{L}(e^{\wedge\Phi}) = \pi_1 \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}(e^{\wedge\Phi}) = \pi_1 \widehat{\mathbf{G}}^{-1} \mathbf{Q}(e^{\wedge\Psi_\eta[\Phi]}) = \pi_1 \widehat{\mathbf{G}}^{-1}((Q\Psi_\eta[\Phi]) \wedge e^{\wedge\Psi_\eta[\Phi]}). \quad (2.61)$$

Since $\pi_1 \widehat{\mathbf{G}}^{-1}(\cdot \wedge e^{\wedge\Psi_\eta})$ is invertible, (2.60) and (2.25) are equivalent:

$$\pi_1 \mathbf{L}(e^{\wedge\Phi}) = 0 \iff Q\Psi_\eta[\Phi] = 0. \quad (2.62)$$

Note that the overall factor $\pi_1 \widehat{\mathbf{G}}^{-1}(\cdot \wedge e^{\wedge\Psi_\eta})$ comes from the difference of the $\xi\delta\Phi$ and Ψ_δ .

2.4 Gauge transformations

It follows from the nilpotency of Q and D_η that the WZW-like action is invariant under the following form of the gauge transformations,

$$\Psi_\delta = D_\eta\Omega + Q\Lambda, \quad (2.63)$$

where Ω and Λ are gauge parameters belonging to the large Hilbert space, which carry ghost numbers 0 and 0, and picture numbers 1 and 0, respectively. In this subsection, we see how the gauge transformation (2.26) can be represented in the WZW-like form (2.63).

Let us consider the associated field $\Psi_\delta[\Phi] = \pi_1 \widehat{\mathbf{G}}(\xi_\delta e^{\wedge\Phi}) = \pi_1 \widehat{\mathbf{G}}((\xi\delta\Phi) \wedge e^{\wedge\Phi})$, with $\delta\Phi$ being the gauge transformation (2.26) in the L_∞ -formulation:

$$\delta\Phi = \pi_1 \mathbf{L}(\lambda \wedge e^{\wedge\Phi}) = \pi_1 \mathbf{L}\eta(\xi\lambda \wedge e^{\wedge\Phi}) = -\eta\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge\Phi}). \quad (2.64)$$

We find $\Psi_\delta[\Phi]$ can be transformed as follows:

$$\begin{aligned} \Psi_\delta[\Phi] &= \pi_1 \widehat{\mathbf{G}}\left(\left(-\xi\eta\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge\Phi})\right) \wedge e^{\wedge\Phi}\right) \\ &= \pi_1 \widehat{\mathbf{G}}\left(\left(\eta\xi\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge\Phi})\right) \wedge e^{\wedge\Phi}\right) - \pi_1 \widehat{\mathbf{G}}\left(\left(\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge\Phi})\right) \wedge e^{\wedge\Phi}\right) \\ &= \pi_1 \widehat{\mathbf{G}}\eta\left(\left(\xi\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge\Phi})\right) \wedge e^{\wedge\Phi}\right) - \pi_1 \widehat{\mathbf{G}}\left(\mathbf{L}(\xi\lambda \wedge e^{\wedge\Phi}) + \xi\lambda \wedge \pi_1 \mathbf{L}(e^{\wedge\Phi}) \wedge e^{\wedge\Phi}\right) \\ &= \pi_1 \mathbf{L}^\eta \widehat{\mathbf{G}}\left(\left(\xi\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge\Phi})\right) \wedge e^{\wedge\Phi}\right) - \pi_1 \mathbf{Q}\widehat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge\Phi}) - \pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge \pi_1 \mathbf{L}(e^{\wedge\Phi}) \wedge e^{\wedge\Phi}) \\ &= \pi_1 \mathbf{L}^\eta\left(\pi_1 \widehat{\mathbf{G}}\left(\left(\xi\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge\Phi})\right) \wedge e^{\wedge\Phi}\right) \wedge e^{\wedge\pi_1 \widehat{\mathbf{G}}(e^{\wedge\Phi})}\right) - Q\pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge\Phi}) - \Delta_T[\lambda, \Phi] \\ &= D_\eta\Omega[\lambda, \Phi] + Q\Lambda[\lambda, \Phi] - \Delta_T[\lambda, \Phi]. \end{aligned} \quad (2.65)$$

While the first two terms of (2.65) correspond to (2.63) with the gauge parameters $\Omega = \Omega[\lambda, \Phi]$ and $\Lambda = \Lambda[\lambda, \Phi]$ parameterized by λ ,

$$\Omega[\lambda, \Phi] = \pi_1 \widehat{\mathbf{G}}\left(\left(\xi\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge\Phi})\right) \wedge e^{\wedge\Phi}\right), \quad (2.66)$$

$$\Lambda[\lambda, \Phi] = -\pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge\Phi}), \quad (2.67)$$

the third term of (2.65),

$$\Delta_T[\lambda, \Phi] = \pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge \pi_1 \mathbf{L}(e^{\wedge\Phi}) \wedge e^{\wedge\Phi}), \quad (2.68)$$

corresponds to the trivial gauge transformation of the WZW-like action. Thus, the gauge transformation in the L_∞ -formulation (2.26) can be written in the WZW-like form (2.63).

Trivial gauge transformation. Trivial gauge transformation is a transformation proportional to the equations of motion. Schematically, it is of the following form,

$$\delta_\mu \varphi^i = (\text{EOM})_j \mu^{ji}, \quad \mu^{ji} = -(-)^{ij} \mu^{ij}, \quad (2.69)$$

and its gauge invariance follows from the symmetric property of μ^{ji} ,

$$\delta_\mu S = (\text{EOM})_i \delta_\mu \varphi^i = (\text{EOM})_i (\text{EOM})_j \mu^{ji} = 0. \quad (2.70)$$

They are no physical significance, but in general they may appear in the algebra of the nontrivial gauge transformations, and in the context of the Batalin-Vilkovisky quantization [37, 38] it is convenient to consider them.¹⁰

In our case, by the almost same computation with [31], when $A + B + C = \text{even}$, cyclicity of the cohomomorphism \widehat{H} is written as follows:

$$\langle \pi_1 \widehat{H}(A \wedge B \wedge e^{\wedge\Phi}), \pi_1 \widehat{H}(C \wedge e^{\wedge\Phi}) \rangle = -(-)^B \langle \pi_1 \widehat{H}(A \wedge e^{\wedge\Phi}), \pi_1 \widehat{H}(B \wedge C \wedge e^{\wedge\Phi}) \rangle. \quad (2.71)$$

The derivation is in appendix B. We take $\widehat{H} = \widehat{\mathbf{G}}$, $A = \xi\lambda$, which is even, and $B = C = \pi_1 \mathbf{L}e^{\wedge\Phi}$, which are odd. The invariance under the trivial gauge transformation (2.68),¹¹

$$\begin{aligned} \langle -\Delta_T[\lambda, \Phi], Q\Psi_\eta[\Phi] \rangle &= \langle -\pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge \pi_1 \mathbf{L}(e^{\wedge\Phi}) \wedge e^{\wedge\Phi}), \pi_1 \widehat{\mathbf{G}}(\pi_1 \mathbf{L}(e^{\wedge\Phi}) \wedge e^{\wedge\Phi}) \rangle \\ &= -\langle \pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge\Phi}), \pi_1 \widehat{\mathbf{G}}(\pi_1 \mathbf{L}(e^{\wedge\Phi}) \wedge \pi_1 \mathbf{L}(e^{\wedge\Phi}) \wedge e^{\wedge\Phi}) \rangle \\ &= 0, \end{aligned} \quad (2.72)$$

follows from the symmetric property:

$$\pi_1 \mathbf{L}e^{\wedge\Phi} \wedge \pi_1 \mathbf{L}e^{\wedge\Phi} = 0. \quad (2.73)$$

3 Alternative parameterisation for closed NS string field theory

Let φ be a some dynamical string field and $t \in [0, 1]$ be a real parameter. We write $\varphi(t)$ for a path satisfying $\varphi(t=0) = 0$ and $\varphi(1) = \varphi$. In section 2.3, we saw that the gauge invariance of the action is provided by only the algebraic relations (2.47) and (2.48). In other words, once the functionals $\Psi_\eta = \Psi_\eta[\varphi]$ and $\Psi_d = \Psi_d[\varphi]$ satisfying the WZW-like relations,

$$0 = \pi_1 \mathbf{L}^\eta(e^{\wedge\Psi_\eta[\varphi]}), \quad (-)^d d\Psi_\eta[\varphi] = D_\eta \Psi_d[\varphi], \quad (3.1)$$

¹⁰For more detail, see [39].

¹¹Note that, although the expression does not contain explicit WZW-like on-shell condition $Q\Psi_\eta$, the on-shell equivalence (2.62) guarantees it is trivial gauge transformation.

are obtained, the gauge invariant action can be constructed as

$$S_\eta[\varphi] = \int_0^1 dt \langle \Psi_t[\varphi(t)], Q\Psi_\eta[\varphi(t)] \rangle. \tag{3.2}$$

This form of the action is the \mathbb{Z}_2 -reversed version of that in [19], which we call the alternative WZW-like form. We found that by taking $\varphi = \Phi$, (2.28), and (2.29), the L_∞ action $S_{\text{EKS}}[\Phi]$ gives one realisation of this WZW-like action,

$$S_\eta[\Phi] = S_{\text{EKS}}[\Phi]. \tag{3.3}$$

In this section we provides another realisation of these functionals, which is parameterized by the string field V in the large Hilbert space. We first see the pure-gauge-like and the associated functional fields can be defined by the differential equations, which are the \mathbb{Z}_2 -reversed versions of the construction in [19]. Then, the WZW-like action parameterized by the string field V ,

$$S_\eta = S_\eta[V], \tag{3.4}$$

is given in terms of them. The equivalence of these actions in the different parameterizations is shown by the almost same procedure performed in [29].

3.1 Large space parameterisation

Let V be a dynamical string field which belongs to the large Hilbert space and carries ghost number 1 and picture number 0. In this subsection, we provides the another parameterization of the pure-gauge-like field $\Psi_\eta = \Psi_\eta[V]$ and the associated fields $\Psi_d = \Psi_d[V]$. A set of differential equations which are the \mathbb{Z}_2 -reversed version of those in [19] give these parameterizations so that WZW-like relations (2.47) and (2.48) hold. Utilizing these functionals, a new gauge invariant action for the string field V is constructed in the (alternative) WZW-like form.

Pure-gauge-like field $\Psi_\eta = \Psi_\eta[V]$. A pure-gauge-like (functional) field $\Psi_\eta[V]$ satisfying $\pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta[V]}) = 0$ is the solution of the the Maurer-Cartan equation for \mathbf{L}^η . Therefore, $\Psi_\eta[V]$ is obtained by mimicking the pure gauge construction of [19]. First, we introduce a real parameter $\tau \in [0, 1]$. Second, we solve the differential equation

$$\begin{aligned} \frac{\partial}{\partial \tau} \Psi_\eta[\tau; V] &= D_\eta(\tau)V \\ &= \eta V + \sum_{k=1}^{\infty} \frac{1}{k!} \overbrace{[\Psi_\eta[\tau; V], \dots, \Psi_\eta[\tau; V], V]^k}^k, \end{aligned} \tag{3.5}$$

with the initial condition

$$\Psi_\eta[\tau = 0; V] = 0. \tag{3.6}$$

Finally, we set $\tau = 1$ and obtain $\Psi_\eta[V]$ as the $\tau = 1$ value solution,

$$\Psi_\eta[V] := \Psi_\eta[\tau = 1; V]. \tag{3.7}$$

One can check that the differential equation actually provides a solution for the Maurer-Cartan equation

$$\mathcal{F}(\Psi_\eta[\tau; V]) = \mathbf{L}^\eta e^{\wedge \Psi_\eta[\tau; V]} = 0, \quad (3.8)$$

by differentiating it by τ :

$$\partial_\tau \mathcal{F}(\Psi_\eta) = \pi_1 \mathbf{L}^\eta (\partial_\tau \Psi_\eta \wedge e^{\wedge \Psi_\eta[\tau]}) = D_\eta (\partial_\tau \Psi_\eta) = D_\eta D_\eta V = -[\mathcal{F}(\Psi_\eta), V]_{\Psi_\eta}^\eta. \quad (3.9)$$

With the initial condition $\mathcal{F}(\Psi_\eta[\tau = 0; V]) = 0$, this differential equation ensures $\mathcal{F}(\Psi_\eta[\tau; V]) = 0$ for arbitrary τ , and then (3.8) holds.

Associated fields $\Psi_d = \Psi_d[V]$. The associated fields $\Psi_d[V]$ are the functionals satisfying

$$d\Psi_\eta[V] = (-)^d D_\eta \Psi_d[V]. \quad (3.10)$$

To derive the differential equation which defines $\Psi_d[V]$, let us introduce

$$\mathcal{I}(\tau) \equiv D_\eta(\tau) \Psi_d[\tau; V] - (-)^d d\Psi_\eta[\tau; V] \quad (3.11)$$

and consider its differentiation by τ :

$$\partial_\tau \mathcal{I}(\tau) = [V, \mathcal{I}(\tau)]_{\Psi_\eta}^\eta + D_\eta \left(\partial_\tau \Psi_d - dV - [V, \Psi_d]_{\Psi_\eta}^\eta \right). \quad (3.12)$$

We define the functional field $\Psi_d[\tau; V]$ by the differential equation

$$\partial_\tau \Psi_d[\tau; V] = dV + [V, \Psi_d[\tau; V]]_{\Psi_\eta[\tau; V]}^\eta \quad (3.13)$$

with the initial condition $\Psi_d[\tau = 0; V] = 0$. Then, the equation (3.12) becomes $\partial_\tau \mathcal{I}(\tau) = [V, \mathcal{I}(\tau)]_{\Psi_\eta}^\eta$, and leads to the vanishing of $\mathcal{I}(\tau)$ at arbitrary τ since $\mathcal{I}(\tau = 0) = 0$, which means that $\Psi_d[\tau; V]$ defined by (3.13) actually satisfies the WZW-like relation (3.10). We set $\tau = 1$ and obtain the associated (functional) field $\Psi_d[V]$ as the $\tau = 1$ value solution

$$\Psi_d[V] := \Psi_d[\tau = 1; V]. \quad (3.14)$$

Action $S_\eta = S_\eta[V]$. We write $V(t)$ for a path satisfying $V(0) = 0$ and $V(1) = V$ with a real parameter $t \in [0, 1]$. Utilizing the pure gauge string field $\Psi_\eta[V]$ and associated fields $\Psi_d[V]$ which are defined by (3.5) and (3.13) to satisfy the WZW-like relations (3.8) and (3.10), one can construct a new gauge invariant action as follows:

$$S_\eta[V] = \int_0^1 dt \langle \Psi_t[V(t)], Q \Psi_\eta[V(t)] \rangle. \quad (3.15)$$

The variation of the action can be taken in the same manner as that in section 2.3,

$$\delta S_\eta[V] = \langle \Psi_\delta[V], Q \Psi_\eta[V] \rangle, \quad (3.16)$$

and the equation of motion can be read off from it,

$$Q \Psi_\eta[V] = 0. \quad (3.17)$$

Since D_η and Q are nilpotent, the action is invariant under the gauge transformations,

$$\Psi_\delta[V] = D_\eta \Omega + Q \Lambda, \quad (3.18)$$

where Ω and Λ are gauge parameters belonging to the large Hilbert space, which carry ghost numbers 0 and 0 and picture numbers 1 and 0, respectively.

3.2 Equivalence of the actions in the different parameterizations

In this subsection, we show the equivalence of our new action $S_\eta[V]$ and L_∞ action $S_{\text{EKS}}[\Phi]$, by identifying the pure-gauge-like functional fields $\Psi_\eta[V]$ and $\Psi_\eta[\Phi]$. We also derive the relation of two dynamical string fields V and Φ from this identification. Let us consider the identification of the pure-gauge-like fields in the both parameterisations:

$$\pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) = \Psi_\eta[\Phi(t)] \equiv \Psi_\eta[V(t)] = \int_0^1 d\tau D_\eta(\tau)V(t). \quad (3.19)$$

Apparently it provides the equivalence of the equations of motions $Q\Psi_\eta[V] = Q\Psi_\eta[\Phi]$. In addition, one can see that it also provides the equivalence of the two actions $S_\eta[V]$ and $S_\eta[\Phi] = S_{\text{EKS}}[\Phi]$, which are two different parametrisations of the same (alternative) WZW-like theory.

Approach 1. Under the identification $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$, the associated fields Ψ_d in two parameterizations are equivalent up to D_η -exact terms $\Psi_d[\Phi] = \Psi_d[V] + (D_\eta\text{-exact terms})$, which is guaranteed by the WZW-like relation $(-)^d d\Psi_\eta = D_\eta\Psi_d$. We thus find

$$D_\eta(\Psi_d[\Phi(t)] - \Psi_d[V(t)]) = (-)^d d(\Psi_\eta[\Phi(t)] - \Psi_\eta[V(t)]) = 0. \quad (3.20)$$

The D_η -exact terms in the associated fields do not affect to the WZW-like relation $(-)^d d\Psi_\eta = D_\eta\Psi_d$. Recall that there exists the arbitrariness to add D_η -exact terms in the associated fields. Besides, since $Q\Psi_\eta$ is D_η -exact, the difference between $\Psi_t[\Phi]$ and $\Psi_t[V]$ does not contribute to the action. Then, two actions are shown to be equivalent:

$$S_{\text{EKS}}[\Phi] = \int_0^1 dt \langle \Psi_t[\Phi(t)], Q\Psi_\eta[\Phi(t)] \rangle = \int_0^1 dt \langle \Psi_t[V(t)], Q\Psi_\eta[V(t)] \rangle = S_\eta[V]. \quad (3.21)$$

The correspondence of the gauge parameters is given by (2.66) and (2.67), with the mixing of the trivial transformation (2.68). Note that we only use the WZW-like relations here, and therefore this identification provides the equivalence of the WZW-like actions in the arbitrary parameterizations as long as the WZW-like relations hold.

Approach 2. We write ξ for an operator satisfying $[[\eta, \xi]] = 1$. Let us consider a linear operator f defined by

$$f \equiv \sum_{n=0}^{\infty} (\xi(D_\eta - \eta))^n. \quad (3.22)$$

One can quickly check this f satisfies $[[f\xi, D_\eta]] = 0$. Since $Q\Psi_\eta = -D_\eta\Psi_Q$, $(D_\eta)^2 = 0$, and

$$\begin{aligned} \Psi_t &= (f\xi D_\eta + D_\eta f\xi)\Psi_t = f\xi \partial_t \Psi_\eta + D_\eta f\xi \Psi_t \\ &= \sum_{n=0}^{\infty} (\xi(D_\eta - \eta))^n \xi \partial_t \Psi_\eta + D_\eta f\xi \Psi_t, \end{aligned} \quad (3.23)$$

we can rewrite the action as the following single functional form:

$$S_\eta = \int_0^1 dt \langle \Psi_t(t), Q\Psi_\eta(t) \rangle = \sum_{n=0}^{\infty} \int_0^1 dt \langle (\xi(D_\eta(t) - \eta))^n \xi \partial_t \Psi_\eta(t), Q\Psi_\eta(t) \rangle. \quad (3.24)$$

This form of the WZW-like action consists of the functional field Ψ_η and operators Q , ξ , and ∂_t . Hence the identification (3.19) automatically provides the equivalence of two actions.

Relation of the fields Φ and V , and partial-gauge-fixing. The identification $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$ can be solved by Φ . Exponentiating both hand side, $e^{\wedge\Psi_\eta[\Phi]} = e^{\wedge\Psi_\eta[V]}$, and using the property of the cohomomorphism and group-like element, the identification becomes

$$e^{\wedge\Psi_\eta[\Phi]} = e^{\wedge\pi_1\widehat{\mathbf{G}}(e^{\wedge\Phi})} = \widehat{\mathbf{G}}e^{\wedge\Phi} = e^{\wedge\Psi_\eta[V]}. \quad (3.25)$$

Since $\widehat{\mathbf{G}}$ is invertible, by acting $\widehat{\mathbf{G}}^{-1}$ and by projecting by π_1 , the condition of the field corresponding relation which provides $S_\eta[V]$ from the L_∞ -action $S_{\text{EKS}}[\Phi]$ is obtained:

$$\Phi[V] = \pi_1\widehat{\mathbf{G}}^{-1}(e^{\wedge\Psi_\eta[V]}). \quad (3.26)$$

Expanding it in powers of V , it reads

$$\begin{aligned} \Phi[V] = \eta V - \frac{1}{2}\eta\lambda_2^{[0]}(V, \eta V) + \frac{1}{12}\eta\left(-\lambda_3^{[0]}(V, \eta V, \eta V) + 2\lambda_2^{[0]}(\lambda_2^{[0]}(V, \eta V), \eta V) \right. \\ \left. - \lambda_2^{[0]}(V, \lambda_2^{[0]}(\eta V, \eta V)) + 2\lambda_2^{[0]}(V, \eta\lambda_2^{[0]}(V, \eta V))\right) + O(V^4), \end{aligned} \quad (3.27)$$

where $\lambda_2^{[0]}$ and $\lambda_3^{[0]}$ are gauge products. (See section 2 or [7] for their explicit forms of $\lambda^{[0]}$.)

The identification $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$ can be solved also by V when the η -symmetry is fixed. By expanding $V = V[\Phi] = V_1(\Phi) + V_2(\Phi, \Phi) + V_3(\Phi, \Phi, \Phi) + \dots$ in powers of Φ , and by acting ξ on the both hand sides of $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$, one can determine V_n perturbatively. The simplest choice of the partial-gauge-fixing condition is $\xi V = 0$, which provides $\xi\eta V = V$. Then the explicit form of the partially-gauge-fixed string field $V(\Phi)$ which provides the L_∞ -action $S_{\text{EKS}}[\Phi]$ from $S_\eta[V]$ is obtained as follows:

$$\begin{aligned} V[\Phi] = \xi\Phi + \frac{1}{2}\xi\eta\lambda_2^{[0]}(\xi\Phi, \Phi) + \frac{1}{12}\xi\eta\left(\lambda_3^{[0]}(\xi\Phi, \Phi, \Phi) - 2\lambda_2^{[0]}(\lambda_2^{[0]}(\xi\Phi, \Phi), \Phi) + \lambda_2^{[0]}(\xi\Phi, \lambda_2^{[0]}(\Phi, \Phi)) \right. \\ \left. + \lambda_2^{[0]}(\xi\Phi, \eta\lambda_2^{[0]}(\xi\Phi, \Phi)) + 3\lambda_2^{[0]}(\xi\eta\lambda_2^{[0]}(\xi\Phi, \Phi), \Phi)\right) + O(V^4). \end{aligned} \quad (3.28)$$

If we choose the form of \mathbf{G} as given in [7], it reads

$$\begin{aligned} V[\Phi] = \xi\Phi + \frac{1}{3!}\xi[\xi\Phi, \Phi] + \frac{1}{4!}\left(\frac{1}{4}\xi[X\xi\Phi, \Phi, \Phi] + \frac{1}{4}\xi X[\xi\Phi, \Phi, \Phi] + \frac{1}{2}\xi[X\Phi, \xi\Phi, \Phi] \right. \\ \left. + \frac{1}{3}\xi[\xi\Phi, \xi[\Phi, \Phi]] - \frac{2}{3}\xi[\xi\Phi, [\xi\Phi, \Phi]] + \frac{4}{3}\xi[\Phi, \xi[\xi\Phi, \Phi]]\right) + O(V^4). \end{aligned} \quad (3.29)$$

4 Conclusions and discussions

In this paper, we clarified a WZW-like structure naturally arising from the L_∞ formulation: the pure-gauge-like field $\Psi_\eta[\Phi] \equiv \pi_1\widehat{\mathbf{G}}(e^{\wedge\Phi})$ and the associated fields $\Psi_d[\Phi] \equiv \pi_1\widehat{\mathbf{G}}(\xi_d e^{\wedge\Phi})$ satisfy the alternative WZW-like relations, which are \mathbb{Z}_2 -reversed versions of the conventional WZW-like relations given in [19]. We found that once WZW-like functionals $\Psi_\eta[\varphi]$ and $\Psi_d[\varphi]$ of some dynamical string field φ satisfying these (alternative) WZW-like relations are given, one can construct a gauge invariant WZW-like action $S_\eta[\varphi]$ in terms of them. The L_∞ action S_{EKS} just gives one realisation of this (alternative) WZW-like action $S_\eta[\Phi]$ parameterised by Φ . On the basis of this procedure, we constructed a new WZW-like

action(s) as another parameterisation of the WZW-like action $S_\eta[V]$ by using the string field V living in the large Hilbert space. (For open NS or NS-NS theory, see appendix D or E, respectively.) The pure-gauge-like field $\Psi_\eta[V]$ and the associated fields $\Psi_d[V]$ can be defined by the differential equations which are the \mathbb{Z}_2 -reversed versions of those in [19]. We showed the equivalence of our new action $S_\eta[V]$ and the L_∞ action $S_\eta[\Phi] \equiv S_{\text{EKS}}[\Phi]$ on the basis of the procedure demonstrated in [29]. The direct relation between two dynamical string fields Φ and V , a field redefinition, can be derived from this equivalence with partially gauge-fixing of V or trivial uplift of Φ .

Towards the equivalence to the conventional WZW-like action. Although we expect the equivalence between our (alternative) WZW-like action $S_\eta[V]$ and Berkovits-Okawa-Zwiebach's (conventional) WZW-like action $S_{\text{BOZ}}[V_c]$, their relation remains to be understood. We write V for the dynamical NS heterotic string field of the alternative WZW-like theory, and write V_c for that of the conventional WZW-like theory as follows:

$$S_\eta[V] = \frac{1}{2} \langle V, Q\eta V \rangle + \frac{\kappa}{3!} \langle V, Q[V, \eta V]^\eta \rangle + \frac{\kappa^2}{4!} \langle V, Q([V, \eta V, \eta V]^\eta + [V, [V, \eta V]^\eta]^\eta) \rangle + \dots,$$

$$S_{\text{BOZ}}[V_c] = \frac{1}{2} \langle \eta V_c, QV_c \rangle + \frac{\kappa}{3!} \langle \eta V_c, [V_c, QV_c] \rangle + \frac{\kappa^2}{4!} \langle \eta V_c, ([V_c, QV_c, QV_c] + [V_c, [V_c, QV_c]]) \rangle + \dots.$$

Both string fields V and V_c belong to the large Hilbert space. At least, perturbatively, one can check their equivalence: we set $V_c = V + O(\kappa^2)$ because of $\mathbf{L}_2^\eta = -\mathbf{L}_2^{\text{BOS}}$, which implies that the first nontrivial order is κ^2 , namely the quartic interaction. Let us check how the equivalence can be shown in this order. It would be crucial that the 3-products \mathbf{L}_3^η and \mathbf{L}_3^{B} are made from the same gauge product $\lambda_3^{[1]}$:

$$\mathbf{L}_3^\eta = -\frac{1}{2} \llbracket \mathbf{Q}, \lambda_3^{[1]} \rrbracket, \quad \mathbf{L}_3^{\text{B}} = \frac{1}{2} \llbracket \boldsymbol{\eta}, \lambda_3^{[1]} \rrbracket.$$

Utilizing them, the quartic interactions in both actions can be written in term of $\lambda_3^{[1]}$ as

$$\langle V, Q[V, \eta V, \eta V]^\eta \rangle = -\frac{1}{2} \langle QV, \lambda_3^{[1]}(QV, \eta V, \eta V) \rangle + \langle QV, \lambda_3^{[1]}(V, Q\eta V, \eta V) \rangle, \quad (4.1)$$

$$\langle \eta V_c, [V_c, QV_c, QV_c] \rangle = -\frac{1}{2} \langle \eta V_c, \lambda_3^{[1]}(\eta V_c, QV_c, QV_c) \rangle - \langle \eta V_c, \lambda_3^{[1]}(V_c, Q\eta V_c, QV_c) \rangle. \quad (4.2)$$

The difference between the quartic interactions comes from the second terms of (4.1) and (4.2):

$$S_{\eta,4}[V] - S_{\text{wzw},4}[V_c] = \frac{\kappa^2}{4!} \langle Q\eta V, 2\lambda_3^{[1]}(V, \eta V, QV) \rangle + O(\kappa^3), \quad (4.3)$$

which can be compensated by the following off-shell field redefinition,

$$V_c = V + \frac{2\kappa^2}{4!} \lambda_3^{[1]}(V, \eta V, QV) + O(\kappa^3). \quad (4.4)$$

Thus, under the identification of the string fields V_c and V by (4.4), two actions $S_\eta[V]$ and $S_{\text{wzw}}[V_c]$ are equivalent at κ^2 .

To prove their all-order equivalence, it would be helpful to characterize the conventional WZW-like action in terms of the gauge products $\lambda_N^{[d]}$. In fact, Q_G , the pure-gauge shifted

BRST operator which plays a key role in conventional WZW-like formulation, can be written as $Q_G = \mathcal{E}_V^{-1} Q \mathcal{E}_V$ using a linear map \mathcal{E}_V [32]. It may provide a solution of this problem to understand the relation between two cohomomorphisms \mathcal{E}_V and $\widehat{\mathbf{G}}$.

Ramond sector of closed superstring fields. The concept of the WZW-like structure based on the dual product \mathbf{L}^η will play crucial roles in construction of the complete action of heterotic string field theory including both NS and R sectors. As [2], our WZW-like action $S_\eta[V]$ will provide a good starting point of the construction of complete actions.¹² It is expected that as demonstrated in [4] for open superstrings, the WZW-like structure including the Ramond sector would be written by the L_∞ products $\mathbf{L}^\eta = \widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1}$ which are dual to the L_∞ products of [8]: $\widetilde{\mathbf{L}} = \widehat{\mathbf{G}}^{-1}(\mathbf{Q} + \dots) \widehat{\mathbf{G}}$ satisfying $[[\eta, \widetilde{\mathbf{L}}]] = 0$.

It is also expected that the action for closed NS-R and R-NS strings can be constructed in the same manner as that for heterotic string. For closed R-R strings, it is not clear whether or how the kinetic term can be constructed with no constraint yet. If the kinetic term can be constructed, it may be possible to construct the complete action of type II string on the basis of the concept of the WZW-like structure arising from the dual product \mathbf{L}^η .

Acknowledgments

The authors would like to thank Hiroshi Kunitomo and Yuji Okawa for discussions and comments. H.M. would like to express his gratitude to his doctor for medical treatments. The work of H.M. was supported in part by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

A Basic facts of A_∞ and L_∞

In this section, we give a short review of coalgebraic description for A_∞/L_∞ algebras. For more details, see [40–42] or some mathematical manuscripts.

Coalgebras: tensor algebra $\mathcal{T}(\mathcal{H})$ and its symmetrization $\mathcal{S}(\mathcal{H})$. Let \mathcal{C} be a set. Suppose that a coproduct $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes' \mathcal{C}$ is defined on \mathcal{C} and it is coassociative

$$(\Delta \otimes' \mathbb{1})\Delta = (\mathbb{1} \otimes' \Delta)\Delta. \tag{A.1}$$

Then, the pair (\mathcal{C}, Δ) is called a *coalgebra*. We write $\mathcal{T}(\mathcal{H})$ for a *tensor algebra* of a graded vector space \mathcal{H} :

$$\mathcal{T}(\mathcal{H}) = \mathcal{H}^{\otimes 0} \oplus \mathcal{H}^{\otimes 1} \oplus \mathcal{H}^{\otimes 2} \oplus \dots. \tag{A.2}$$

One can define a coassociative coproduct $\Delta : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}) \otimes' \mathcal{T}(\mathcal{H})$ by

$$\Delta(\Phi_1 \otimes \dots \otimes \Phi_n) \equiv \sum_{k=0}^n (\Phi_1 \otimes \dots \otimes \Phi_k) \otimes' (\Phi_{k+1} \otimes \dots \otimes \Phi_n) \tag{A.3}$$

¹²See a new result given by K. Goto and H. Kunitomo, [arXiv:1606.07194](https://arxiv.org/abs/1606.07194).

with $\otimes' \equiv \otimes$, where $\Phi_1 \otimes \dots \otimes \Phi_n \in \mathcal{H}^{\otimes n} \subset \mathcal{T}(\mathcal{H})$. Then, the pair $(\mathcal{T}(\mathcal{H}), \Delta)$ gives a coalgebra. As well as $\mathcal{T}(\mathcal{H})$, its symmetrization $\mathcal{S}(\mathcal{H})$ also gives a coalgebra. Recall that the *symmetrized tensor product* \wedge for $\Phi_1, \Phi_2 \in \mathcal{H}$ is defined by

$$\Phi_1 \wedge \Phi_2 \equiv \Phi_1 \otimes \Phi_2 + (-)^{\deg(\Phi_1)\deg(\Phi_2)} \Phi_2 \otimes \Phi_1, \quad (\text{A.4})$$

and it satisfies the following properties for $\Phi_1, \dots, \Phi_n \in \mathcal{H}$:

$$\Phi_1 \wedge \Phi_2 = (-)^{\deg(\Phi_1)\deg(\Phi_2)} \Phi_2 \wedge \Phi_1, \quad (\text{A.5})$$

$$(\Phi_1 \wedge \Phi_2) \wedge \Phi_3 = \Phi_1 \wedge (\Phi_2 \wedge \Phi_3), \quad (\text{A.6})$$

$$\Phi_1 \wedge \Phi_2 \wedge \dots \wedge \Phi_n = \sum_{\sigma} (-)^{\sigma} \Phi_{\sigma(1)} \otimes \Phi_{\sigma(2)} \otimes \dots \otimes \Phi_{\sigma(n)}. \quad (\text{A.7})$$

We write $\mathcal{H}^{\wedge n}$ for the vector space spanned by n -fold symmetrized tensor $\Phi_1 \wedge \dots \wedge \Phi_n$, and $\mathcal{S}(\mathcal{H})$ for the symmetrization of $\mathcal{T}(\mathcal{H})$, which is called the *symmetrized tensor algebra* $\mathcal{S}(\mathcal{H})$:

$$\mathcal{S}(\mathcal{H}) = \mathcal{H}^{\wedge 0} \oplus \mathcal{H}^{\wedge 1} \oplus \mathcal{H}^{\wedge 2} \oplus \dots. \quad (\text{A.8})$$

A coassociative coproduct $\Delta : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}) \otimes' \mathcal{S}(\mathcal{H})$ can be defined by

$$\Delta(\Phi_1 \wedge \dots \wedge \Phi_n) \equiv \sum_{k=0}^n \sum_{\sigma} (-)^{\sigma} (\Phi_{\sigma(1)} \wedge \dots \wedge \Phi_{\sigma(k)}) \otimes' (\Phi_{\sigma(k+1)} \wedge \dots \wedge \Phi_{\sigma(n)}) \quad (\text{A.9})$$

with $\otimes' \equiv \otimes$, where $\Phi_1 \wedge \dots \wedge \Phi_n \in \mathcal{H}^{\wedge n}$ and σ runs over $(k, n-k)$ -unshuffle. Then, the pair $(\mathcal{S}(\mathcal{H}), \Delta)$ gives coalgebra.

In the case of open superstring field theory, \mathcal{H} is the state space of open superstrings, which is a \mathbb{Z}_2 -graded vector space, and its grading which we call degree is given by the Grassmann parity minus one mod 2. Then, the pair $(\mathcal{T}(\mathcal{H}), \Delta)$ is the Fock space of open superstrings. On the other hand, in the case of closed superstring field theory, \mathcal{H} is the state space of closed superstrings, which is a \mathbb{Z}_2 -graded vector space, and its grading which we call degree is given by the Grassmann parity mod 2. Then, the pair $(\mathcal{S}(\mathcal{H}), \Delta)$ is the Fock space of closed superstrings.

Multi-linear maps as a coderivation. A linear operator $m : \mathcal{C} \rightarrow \mathcal{C}$ which raise the degree one is called *coderivation* if it satisfies

$$\Delta m = (m \otimes \mathbb{1})\Delta + (\mathbb{1} \otimes m)\Delta. \quad (\text{A.10})$$

Multilinear maps with degree 1 and 0 naturally induce the maps from $\mathcal{T}(\mathcal{H})$ to $\mathcal{T}(\mathcal{H})$ or from $\mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\mathcal{H})$. They are called a coderivation and a cohomomorphism respectively.

Recall that from a n -fold multilinear product b_n of $\Phi_1, \dots, \Phi_n \in \mathcal{H}$, one can define a linear map on $\mathcal{H}^{\otimes n}$, which we write $b_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}$, by

$$b_n(\Phi_1 \otimes \Phi_2 \otimes \dots \otimes \Phi_n) \equiv b_n(\Phi_1, \Phi_2, \dots, \Phi_n), \quad (\text{A.11})$$

where $\Phi_1 \otimes \dots \otimes \Phi_n \in \mathcal{H}^{\otimes n}$. Let $A : \mathcal{H}^{\otimes k} \rightarrow \mathcal{H}^{\otimes l}$ and $B : \mathcal{H}^{\otimes m} \rightarrow \mathcal{H}^{\otimes n}$ be multilinear maps. One can naturally define a product of these maps $A \otimes B : \mathcal{H}^{\otimes k+m} \rightarrow \mathcal{H}^{\otimes l+n}$ by

$$A \otimes B(\Phi_1 \otimes \dots \otimes \Phi_{k+m}) = \sum_k (-)^{B(\Phi_1 + \dots + \Phi_k)} A(\Phi_1 \otimes \dots \otimes \Phi_k) \otimes B(\Phi_{k+1} \otimes \dots \otimes \Phi_{k+m}). \quad (\text{A.12})$$

Then, the identity operator on $\mathcal{H}^{\otimes n}$ and $\mathcal{H}^{\wedge n}$ is defined by

$$\mathbb{I}_n = \overbrace{\mathbb{I} \otimes \mathbb{I} \otimes \dots \otimes \mathbb{I}}^n = \frac{1}{n!} \overbrace{\mathbb{I} \wedge \mathbb{I} \wedge \dots \wedge \mathbb{I}}^n. \quad (\text{A.13})$$

Using a degree one map $b_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}$, one can naturally construct a coderivation $\mathbf{b}_n : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ by

$$\mathbf{b}_n \Phi = \sum_{k=0}^{N-n} (\mathbb{I}_k \otimes b_n \otimes \mathbb{I}_{N-n-k}) \Phi, \quad \Phi \in \mathcal{H}^{\otimes N \geq n} \subset \mathcal{T}(\mathcal{H}), \quad (\text{A.14})$$

and \mathbf{b}_n vanishes when acting on $\mathcal{H}^{\otimes N \leq n}$.

Similarly, a symmetric multilinear product b_n of $\Phi_1, \dots, \Phi_n \in \mathcal{H}$ naturally defines a map $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$ by

$$b_n(\Phi_1 \wedge \Phi_2 \wedge \dots \wedge \Phi_n) = b_n(\Phi_1, \Phi_2, \dots, \Phi_n). \quad (\text{A.15})$$

The symmetric tensor product of two multilinear maps $A : \mathcal{H}^{\wedge k} \rightarrow \mathcal{H}^{\wedge l}$ and $B : \mathcal{H}^{\wedge m} \rightarrow \mathcal{H}^{\wedge n}$, $A \wedge B : \mathcal{H}^{\wedge k+m} \rightarrow \mathcal{H}^{\wedge l+n}$, can also be defined naturally by

$$A \wedge B(\Phi_1 \wedge \dots \wedge \Phi_{k+m}) = \sum_{\sigma} (-)^{\sigma} A(\Phi_{\sigma(1)} \wedge \dots \wedge \Phi_{\sigma(k)}) \wedge B(\Phi_{\sigma(k+1)} \wedge \dots \wedge \Phi_{\sigma(k+m)}). \quad (\text{A.16})$$

A degree map $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$ naturally gives a coderivation $\mathbf{b}_n : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$ defined by

$$\mathbf{b}_n \Phi = (b_n \wedge \mathbb{I}_{N-n}) \Phi, \quad \Phi \in \mathcal{H}^{\wedge N \geq n} \subset \mathcal{S}(\mathcal{H}), \quad (\text{A.17})$$

and \mathbf{b}_n vanishes when acting on $\mathcal{H}^{\wedge N \leq n}$. For example, we find that \mathbf{b}_1 acts as follows:

$$\begin{aligned} \mathbf{b}_1 : \quad 1 &\rightarrow 0 \\ \Phi_1 &\rightarrow b_1(\Phi_1) \\ \Phi_1 \wedge \Phi_2 &\rightarrow b_1(\Phi_1) \wedge \Phi_2 + (-)^{\deg(\Phi_1)\deg(b_1)} \Phi_1 \wedge b_1(\Phi_2). \end{aligned} \quad (\text{A.18})$$

In particular, the coderivation $\mathbf{b}_0 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$ derived from a map $b_0 : \mathcal{H}^0 \rightarrow \mathcal{H}$ is given by

$$\mathbf{b}_0 \Phi = (b_0 \wedge \mathbb{I}_N) \Phi = b_0 \wedge \Phi, \quad \Phi \in \mathcal{H}^{\wedge N} \subset \mathcal{S}(\mathcal{H}) \quad (\text{A.19})$$

and acts as

$$\begin{aligned} \mathbf{b}_0 : \quad 1 &\rightarrow b_0 \\ \Phi_1 &\rightarrow b_0 \wedge \Phi_1 \\ \Phi_1 \wedge \Phi_2 &\rightarrow b_0 \wedge \Phi_1 \wedge \Phi_2. \end{aligned} \quad (\text{A.20})$$

Given two coderivations \mathbf{b}_n and \mathbf{c}_m which are derived from $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$ and $c_m : \mathcal{H}^{\wedge m} \rightarrow \mathcal{H}$ respectively, the graded commutator $[[\mathbf{b}_n, \mathbf{c}_m]]$ becomes the coderivation derived from the map $[[b_n, c_m]] : \mathcal{H}^{\wedge n+m-1} \rightarrow \mathcal{H}$ which is defined by

$$[[b_n, c_m]] = b_n(c_m \wedge \mathbb{I}_{n-1}) - (-)^{\deg(b_n)\deg(c_m)} c_m(b_n \wedge \mathbb{I}_{m-1}). \quad (\text{A.21})$$

Multilinear maps as a cohomomorphism. Given two coalgebras C, C' , a *cohomomorphism* $\widehat{f} : C \rightarrow C'$ is a map of degree zero satisfying

$$\Delta \widehat{f} = (\widehat{f} \otimes' \widehat{f}) \Delta. \quad (\text{A.22})$$

A set of degree zero multilinear maps $\{f_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}'\}_{n=0}^{\infty}$ naturally induces a cohomomorphism $\widehat{f} : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}')$, which we denote as $\widehat{f} = \{f_n\}_{n=0}^{\infty}$. Its action on $\Phi_1 \wedge \cdots \wedge \Phi_n \in \mathcal{H}^{\wedge n} \subset \mathcal{S}(\mathcal{H})$ is defined by

$$\begin{aligned} \widehat{f}(\Phi_1 \wedge \cdots \wedge \Phi_n) &= \sum_{i \leq n} \sum_{k_1 < \cdots < k_i} e^{\wedge f_0} \wedge f_{k_1}(\Phi_1, \dots, \Phi_{k_1}) \wedge f_{k_2 - k_1}(\Phi_{k_1+1}, \dots, \Phi_{k_2}) \wedge \\ &\quad \cdots \wedge f_{k_i - k_{i-1}}(\Phi_{k_{i-1}+1}, \dots, \Phi_n). \end{aligned} \quad (\text{A.23})$$

Its explicit actions are given as follows:

$$\begin{aligned} \widehat{f}: \quad 1 &\rightarrow e^{\wedge f_0} \\ \Phi &\rightarrow e^{\wedge f_0} \wedge f_1(\Phi) \\ \Phi_1 \wedge \Phi_2 &\rightarrow e^{\wedge f_0} \wedge f_1(\Phi_1) \wedge f_1(\Phi_2) + e^{\wedge f_0} \wedge f_2(\Phi_1 \wedge \Phi_2). \end{aligned} \quad (\text{A.24})$$

Cyclic A_{∞} and cyclic L_{∞} . Let \mathcal{H} be a graded vector space and $\mathcal{T}(\mathcal{H})$ be its tensor algebra. A *weak A_{∞} -algebra* $(\mathcal{H}, \mathbf{M})$ is a coalgebra $\mathcal{T}(\mathcal{H})$ with a coderivation $\mathbf{M} = \mathbf{M}_0 + \mathbf{M}_1 + \mathbf{M}_2 + \dots$ satisfying

$$(\mathbf{M})^2 = 0. \quad (\text{A.25})$$

We denote the collection of the multilinear maps $\{M_k\}_{k \geq 0}$ also by \mathbf{M} . In particular, if $\mathbf{M}_0 = 0$, $(\mathcal{H}, \mathbf{M})$ is called an *A_{∞} -algebra*. Note that for fixed n , the equation (A.25) gives

$$\mathbf{M}_n \cdot \mathbf{M}_1 + \mathbf{M}_{n-1} \cdot \mathbf{M}_2 + \cdots + \mathbf{M}_2 \cdot \mathbf{M}_{n-1} + \mathbf{M}_1 \cdot \mathbf{M}_n = 0. \quad (\text{A.26})$$

We can act it on $B_1 \otimes B_2 \otimes \cdots \otimes B_n \in \mathcal{H}^{\otimes n}$ to get the A_{∞} relations for the maps $\{M_k\}$:

$$\sum_{k=0}^{n-i} (-)^{B_1 + \cdots + B_k} M_{n-i+1}(B_1, \dots, B_k, M_i(B_{k+1}, \dots, B_{k+i}), B_{k+i+1}, \dots, B_n) = 0. \quad (\text{A.27})$$

Let $\langle \cdot, \cdot \rangle : \mathcal{H}^{\otimes 2} \rightarrow \mathbb{C}$ be the BPZ inner product, which gives the graded symplectic form $\langle \omega | : \mathcal{H}^{\otimes 2} \rightarrow \mathbb{C}$:

$$\langle A, B \rangle = (-)^A \langle \omega | A \otimes B. \quad (\text{A.28})$$

Given the operator \mathcal{O}_n , we can define its BPZ-conjugation \mathcal{O}_n^{\dagger} as follows:

$$\langle \omega | \mathbb{I} \otimes \mathcal{O}_n = \langle \omega | \mathcal{O}_n^{\dagger} \otimes \mathbb{I}. \quad (\text{A.29})$$

A pair $(\mathcal{T}(\mathcal{H}), \mathbf{M}, \omega)$ is called *cyclic A_{∞} -algebra* if each M_n is BPZ-odd,

$$M_n^{\dagger} = -M_n. \quad (\text{A.30})$$

Let \mathcal{H} be a graded vector space and $\mathcal{S}(\mathcal{H})$ be its symmetrized tensor algebra. A *weak L_{∞} -algebra* $(\mathcal{H}, \mathbf{L})$ is a coalgebra $\mathcal{S}(\mathcal{H})$ with a coderivation $\mathbf{L} = \mathbf{L}_0 + \mathbf{L}_1 + \mathbf{L}_2 + \dots$ satisfying

$$(\mathbf{L})^2 = 0. \quad (\text{A.31})$$

We denote the collection of the multilinear maps $\{L_k\}_{k \geq 0}$ also by \mathbf{L} . In particular, if $\mathbf{L}_0 = 0$, $(\mathcal{H}, \mathbf{L})$ is called an L_∞ -algebra.

In the case of an L_∞ -algebra, the part of (A.31) that correspond to an n -fold multilinear map $\mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$ is given by

$$\mathbf{L}_n \cdot \mathbf{L}_1 + \mathbf{L}_{n-1} \cdot \mathbf{L}_2 + \cdots + \mathbf{L}_2 \cdot \mathbf{L}_{n-1} + \mathbf{L}_1 \cdot \mathbf{L}_n = 0. \quad (\text{A.32})$$

We can act it on $B_1 \wedge B_2 \wedge \cdots \wedge B_n \in \mathcal{H}^{\wedge n}$ to get the L_∞ relations for the multilinear maps $\{L_k\}$:

$$0 = \sum_{i+j=n+1} \sum_{\sigma}' (-)^\sigma L_j(L_i(B_{\sigma(1)}, \dots, B_{\sigma(i)}), B_{\sigma(i+1)}, \dots, B_{\sigma(n)}). \quad (\text{A.33})$$

A set $(\mathcal{S}(\mathcal{H}), \mathbf{L}, \omega)$ is called *cyclic L_∞ -algebra* if each L_n is BPZ-odd,

$$L_n^\dagger = -L_n. \quad (\text{A.34})$$

Projector and group-like element. We can naturally define a *projector* $\pi_1 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{H}$ whose action on $\Phi \in \mathcal{S}(\mathcal{H})$ is given by

$$\pi_1 \Phi = \Phi_1, \Phi = \sum_{n=1}^{\infty} \Phi_1 \wedge \cdots \wedge \Phi_n \in \mathcal{S}(\mathcal{H}). \quad (\text{A.35})$$

Note that π_1 acts trivially on \mathcal{H} and commutes with one-coderivations.

Let \mathcal{H}_0 be the degree zero part of \mathcal{H} . The following exponential map of $\Phi \in \mathcal{H}_0$,

$$e^{\wedge \Phi} = \mathbf{1} + \Phi + \frac{1}{2} \Phi \wedge \Phi + \frac{1}{3!} \Phi \wedge \Phi \wedge \Phi + \cdots, \quad (\text{A.36})$$

is called a *group-like element*. It satisfies

$$\Delta e^{\wedge \Phi} = e^{\wedge \Phi} \wedge e^{\wedge \Phi}. \quad (\text{A.37})$$

The action of a coderivation on a group-like element is given by

$$\mathbf{b}_n(e^{\wedge \Phi}) = \frac{1}{n!} b_n(\Phi^{\wedge n}) \wedge e^{\wedge \Phi}, \quad (\text{A.38})$$

where we promise $0! = 1$. Note that we can not distinguish a one-coderivation \mathbf{b}_1 derived from a linear map $b_1 : \mathcal{H} \rightarrow \mathcal{H}; \Phi \mapsto b_1(\Phi)$ and a zero-coderivation \mathbf{b}_0 derived from $b_0 : \mathcal{H}^{\wedge 0} \rightarrow \mathcal{H}; 1 \mapsto b_0 = b_1(\Phi)$ when acting on group-like element,

$$\mathbf{b}_0(e^{\wedge \Phi}) = b_0 \wedge (e^{\wedge \Phi}) = b_1(\Phi) \wedge (e^{\wedge \Phi}) = \mathbf{b}_1(e^{\wedge \Phi}). \quad (\text{A.39})$$

One of the important property of a cohomomorphism is its action on the group-like element:

$$\widehat{\Delta} \widehat{f}(e^{\wedge \Phi}) = (\widehat{f} \otimes \widehat{f}) \Delta e^{\wedge \Phi} = (\widehat{f} \otimes \widehat{f}) e^{\wedge \Phi} \wedge e^{\wedge \Phi} = \widehat{f}(e^{\wedge \Phi}) \wedge \widehat{f}(e^{\wedge \Phi}). \quad (\text{A.40})$$

We can see that the cohomomorphism preserves the group-like element: $\widehat{f}(e^{\wedge \Phi}) = e^{\wedge \Phi'}$.

Utilizing the projector and the group-like element, the *Maurer-Cartan element* for an L_∞ -algebra $(\mathcal{H}, \mathbf{L})$ is given by

$$\mathcal{F}_\Phi := \pi_1 \mathbf{L}(e^{\wedge \Phi}) = \mathbf{L}_1(\Phi) + \frac{1}{2} \mathbf{L}_2(\Phi \wedge \Phi) + \frac{1}{3!} \mathbf{L}_3(\Phi \wedge \Phi \wedge \Phi) + \cdots. \quad (\text{A.41})$$

The *Maurer-Cartan equation* for an L_∞ -algebra $(\mathcal{H}, \mathbf{L})$ is given by $\mathcal{F}_\Phi = 0$, which correspond to the on-shell condition in string field theory based on the L_∞ -algebra $(\mathcal{H}, \mathbf{L})$.

B Derivation of (2.71)

For cyclic cohomomorphism \widehat{H} , the following relation holds:

$$\langle \omega | \pi_2 = \langle \omega | \pi_2 \widehat{H} = \langle \omega | \nabla(\pi_1 \otimes' \pi_1) \Delta \widehat{H} = \langle \omega | \nabla(\pi_1 \widehat{H} \otimes' \pi_1 \widehat{H}) \Delta, \quad (\text{B.1})$$

where π_2 is the projector $\mathcal{S}(\mathcal{H}) \rightarrow \mathcal{H} \wedge \mathcal{H}$, and ∇ is the product of the tensor algebra $(\mathcal{T}(\mathcal{H}), \nabla)$.

Let us consider its action

$$\begin{aligned} & \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \\ & + (-)^{AB} \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \\ & + (-)^{BC} \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi}. \end{aligned} \quad (\text{B.2})$$

The left hand side vanishes since π_2 is acting on $\mathcal{H}^{\wedge n \geq 3}$. For the right hand side, the first term of (B.2) becomes

$$\begin{aligned} & \langle \omega | \nabla(\pi_1 \widehat{H} \otimes' \pi_1 \widehat{H}) \Delta \left(\frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \right) \\ & = \langle \omega | \nabla(\pi_1 \widehat{H} \otimes' \pi_1 \widehat{H}) \left(\frac{1}{1-\Phi} \otimes' \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \right. \\ & \quad + \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes' \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \\ & \quad + \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes' \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \\ & \quad \left. + \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \otimes' \frac{1}{1-\Phi} \right). \end{aligned} \quad (\text{B.3})$$

Hereafter we assume $A + B + C = \text{even}$. The inner product in the large Hilbert space is

$$\langle a, b \rangle = (-)^a \langle \omega | a \otimes b, \quad (\text{B.4})$$

and the 1st term and 4th term of (B.3) cancel since $\langle a, b \rangle = (-)^{(a+1)(b+1)} \langle b, a \rangle$. Now, gathering the contribution from all terms of (B.2), and utilizing the identities

$$a \wedge e^{\wedge \Phi} = \frac{1}{1-\Phi} \otimes a \otimes \frac{1}{1-\Phi}, \quad (\text{B.5})$$

$$a \wedge b \wedge e^{\wedge \Phi} = \frac{1}{1-\Phi} \otimes a \otimes \frac{1}{1-\Phi} \otimes b \otimes \frac{1}{1-\Phi} + (-)^{ab} \frac{1}{1-\Phi} \otimes b \otimes \frac{1}{1-\Phi} \otimes a \otimes \frac{1}{1-\Phi}, \quad (\text{B.6})$$

one can obtain the BPZ property for \widehat{H} :

$$0 = (-)^B \langle \pi_1 \widehat{H}(A \wedge B \wedge e^{\wedge \Phi}), \pi_1 \widehat{H}(C \wedge e^{\wedge \Phi}) \rangle + \langle \pi_1 \widehat{H}(A \wedge e^{\wedge \Phi}), \pi_1 \widehat{H}(B \wedge C \wedge e^{\wedge \Phi}) \rangle. \quad (\text{B.7})$$

Note that in the computation some terms cancel by the symmetric property of the inner product.

C Embedding to the large Hilbert space of NS closed string

In this appendix, we consider the trivial embedding of the NS closed string field Φ in the L_∞ -formulation into the string field \tilde{V} in the large Hilbert space, and see that the trivially embedded action is also WZW-like form. In other words, we provide a parameterization of the pure-gauge-like field and the associated fields by another large space string field \tilde{V} . In particular, we discuss the relation between the gauge transformations in two representations.

We define the trivial embedding by replacing $\xi\Phi$ to \tilde{V} which is a string field belonging to the large Hilbert space, so that under the partial-gauge-fixing condition $\xi\tilde{V} = 0$, the trivial solution $\tilde{V} = \xi\Phi$ reproduce the L_∞ -formulation parameterized by Ψ , the string field in the small Hilbert space. The action for \tilde{V} is given by

$$\begin{aligned} S_{\text{EKS}}[\tilde{V}] &= \int_0^1 dt \langle \partial_t \tilde{V}(t), \pi_1(\mathbf{L}(e^{\wedge\eta\tilde{V}(t)})) \rangle \\ &= \int_0^1 dt \langle \pi_1 \hat{\mathbf{G}}((\partial_t \tilde{V}(t)) \wedge e^{\wedge\eta\tilde{V}(t)}), Q \pi_1 \hat{\mathbf{G}}(e^{\wedge\eta\tilde{V}(t)}) \rangle. \end{aligned} \quad (\text{C.1})$$

One can confirm the functionals appearing in the action

$$\Psi_\eta[\tilde{V}] = \pi_1 \hat{\mathbf{G}}(e^{\wedge\eta\tilde{V}}), \quad (\text{C.2})$$

$$\Psi_d[\tilde{V}] = \pi_1 \hat{\mathbf{G}}(d\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) \quad (\text{for } d = \partial_t, \delta) \quad (\text{C.3})$$

satisfy the WZW-like relations:

$$0 = \pi_1 \mathbf{L}^\eta(e^{\wedge\Psi_\eta[\tilde{V}]}) \quad (-)^d d \Psi_\eta[\tilde{V}] = D_\eta \Psi_d[\tilde{V}]. \quad (\text{C.4})$$

The first relation follows from the projection invariance $e^{\wedge\eta\tilde{V}} = \eta\xi e^{\wedge\eta\tilde{V}}$ as (2.49). To check the second relation, consider

$$\Psi_d[\tilde{V}] = \pi_1 \hat{\mathbf{G}}(d(\eta\xi + \xi\eta)\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) = \pi_1 \hat{\mathbf{G}}(d\xi\eta\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) + (-)^d D_\eta \pi_1 \hat{\mathbf{G}}(d\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}), \quad (\text{C.5})$$

where we used

$$\begin{aligned} (-)^d \pi_1 \hat{\mathbf{G}}(d\eta\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) &= \pi_1 \hat{\mathbf{G}}\eta(d\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) = \pi_1 \mathbf{L}^\eta \hat{\mathbf{G}}(d\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) \\ &= \pi_1 \mathbf{L}^\eta(\pi_1 \hat{\mathbf{G}}(d\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) \wedge \hat{\mathbf{G}}e^{\wedge\eta\tilde{V}}) = D_\eta \pi_1 \hat{\mathbf{G}}(d\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}). \end{aligned} \quad (\text{C.6})$$

Since the second term vanishes when acted by D_η , the same computation as (2.51) can be applied to show $(-)^d d \Psi_\eta[\tilde{V}] = D_\eta \Psi_d[\tilde{V}]$.

In fact, the embedding procedure $\xi\Phi \rightarrow \tilde{V}$ has an ambiguity: one can consider the replacing $\xi\Phi = \xi\eta\xi\Phi \rightarrow \xi\eta\tilde{V}$. However, the difference of Ψ_d produced by the difference of the procedure is only D_η -exact term as seen in (C.6), which does not affect to the WZW-like relation $(-)^d d \Psi_\eta = D_\eta \Psi_d$. Besides, since $Q\Psi_\eta$ is D_η -exact, the D_η -exact part of Ψ_t does not affect the action, that is, the action does not depend on this ambiguity. Here we fix this ambiguity by requiring the invertibility of $\Psi_{d=\delta, \partial_t}$ as functions of $d\tilde{V}$ [31].

For derivation d which does not commute with $\widehat{\mathbf{G}}$, such as Q , we define $\Psi_d[\widetilde{V}]$ by

$$\Psi_d[\widetilde{V}] = \pi_1 \widehat{\mathbf{G}} (\pi_1 \widehat{\mathbf{G}}^{-1} d \widehat{\mathbf{G}} (\widetilde{V} \wedge \int e^{\wedge \eta \widetilde{V}}) \wedge e^{\wedge \eta \widetilde{V}}), \quad (\text{C.7})$$

where

$$\int e^{\wedge \eta \widetilde{V}} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\eta \widetilde{V})^{\wedge n}. \quad (\text{C.8})$$

For $d = Q$, it reads

$$\Psi_Q[\widetilde{V}] = \pi_1 \widehat{\mathbf{G}} (\pi_1 \mathbf{L} (\widetilde{V} \wedge \int e^{\wedge \eta \widetilde{V}}) \wedge e^{\wedge \eta \widetilde{V}}). \quad (\text{C.9})$$

To check the WZW-like relations, utilizing $\widehat{\mathbf{G}}^{-1} d \widehat{\mathbf{G}} = (-)^d \llbracket \boldsymbol{\eta}, \boldsymbol{\xi}_d \rrbracket$, we write $\Psi_d[\widetilde{V}]$ as

$$\begin{aligned} \Psi_d[\widetilde{V}] &= \pi_1 \widehat{\mathbf{G}} (\pi_1 \widehat{\mathbf{G}}^{-1} d \widehat{\mathbf{G}} (\widetilde{V} \wedge \int e^{\wedge \eta \widetilde{V}}) \wedge e^{\wedge \eta \widetilde{V}}) \\ &= \pi_1 \widehat{\mathbf{G}} (\pi_1 (-)^d \llbracket \boldsymbol{\eta}, \boldsymbol{\xi}_d \rrbracket (\widetilde{V} \wedge \int e^{\wedge \eta \widetilde{V}}) \wedge e^{\wedge \eta \widetilde{V}}) \\ &= \pi_1 \widehat{\mathbf{G}} (\pi_1 \boldsymbol{\xi}_d e^{\wedge \eta \widetilde{V}} \wedge e^{\wedge \eta \widetilde{V}}) + (-)^d D_\eta \pi_1 \widehat{\mathbf{G}} (\pi_1 \boldsymbol{\xi}_d (\widetilde{V} \wedge \int e^{\wedge \eta \widetilde{V}}) \wedge e^{\wedge \eta \widetilde{V}}). \end{aligned} \quad (\text{C.10})$$

Since the second term vanishes when D_η acts, the same computation as (2.51) can be applied to show $(-)^d d \Psi_\eta[\widetilde{V}] = D_\eta \Psi_d[\widetilde{V}]$.

Thus, the action is written in the WZW-like form,

$$S_{\text{EKS}}[\widetilde{V}] = \int_0^1 dt \langle \Psi_t[\widetilde{V}(t)], Q \Psi_\eta[\widetilde{V}(t)] \rangle. \quad (\text{C.11})$$

Its variation and the on-shell condition can be taken by parallel computations in section 2.3.

Gauge transformations. The variation of the action $S_{\text{EKS}}[\widetilde{V}]$ in the form of (C.1) can be taken as

$$\delta S_{\text{EKS}}[\widetilde{V}] = \langle \delta \widetilde{V}, \pi_1 (\mathbf{L}(e^{\wedge \eta \widetilde{V}})) \rangle, \quad (\text{C.12})$$

and one can find the action $S_{\text{EKS}}[\widetilde{V}]$ is invariant under the gauge transformations¹³

$$\delta \widetilde{V} = -\eta \xi \omega - \pi_1 \mathbf{L}(\xi \lambda \wedge e^{\wedge \eta \widetilde{V}}), \quad (\text{C.13})$$

where ω and λ are the gauge parameters belonging to the small Hilbert space, which carry ghost numbers 1 and 1, and picture numbers 0 and -1 , respectively. The minus sign is a convention in which $\eta \delta \widetilde{V}$ becomes $\delta \Phi$ of (2.24). In the following, we will discuss how this gauge symmetry can be written in the WZW-like form:

$$\Psi_\delta[\widetilde{V}] = D_\eta \Omega[\omega, \widetilde{V}] + Q \Lambda[\lambda, \widetilde{V}]. \quad (\text{C.14})$$

Inserting the gauge transformation $\delta \widetilde{V}$, $\Psi_\delta[\widetilde{V}]$ becomes

$$\begin{aligned} \Psi_\delta[\widetilde{V}] &= \pi_1 \widehat{\mathbf{G}} (\delta \widetilde{V} \wedge e^{\wedge \eta \widetilde{V}}) \\ &= -\pi_1 \widehat{\mathbf{G}} ((\eta \xi \omega) \wedge e^{\wedge \eta \widetilde{V}}) - \pi_1 \widehat{\mathbf{G}} ((\pi_1 \mathbf{L}(\xi \lambda \wedge e^{\wedge \eta \widetilde{V}})) \wedge e^{\wedge \eta \widetilde{V}}) \\ &= -\pi_1 \mathbf{L}^\eta (\pi_1 \widehat{\mathbf{G}} (\xi \omega \wedge e^{\wedge \eta \widetilde{V}}) \wedge e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \eta \widetilde{V}})}) - Q \pi_1 \widehat{\mathbf{G}} (\xi \lambda \wedge e^{\wedge \eta \widetilde{V}}) - \Delta_T[\lambda, \eta \widetilde{V}] \\ &= D_\eta \Omega[\omega, \widetilde{V}] + Q \Lambda[\lambda, \widetilde{V}] - \Delta_T[\lambda, \eta \widetilde{V}]. \end{aligned} \quad (\text{C.15})$$

¹³More generally, the second term can be written by $-\pi_1 \boldsymbol{\xi}_Q (\lambda \wedge e^{\wedge \eta \widetilde{V}})$. The representation above is only a choice of ξ_d . However, the difference of the choices becomes η -exact term and it can be absorbed into $-\eta \xi \omega$.

The computation here is parallel to (2.65). Since Δ_T is a trivial gauge transformation as in section 2.4,

$$\Delta_T[\lambda, \eta\tilde{V}] = \pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge \pi_1 \mathbf{L}(e^{\wedge\eta\tilde{V}}) \wedge e^{\wedge\eta\tilde{V}}), \quad (\text{C.16})$$

we conclude that the gauge transformations of the trivial embedded theory (C.13) are written in the WZW-like form (C.14), where the gauge parameters Ω and Λ are parameterized as

$$\Omega[\omega, \tilde{V}] = -\pi_1 \widehat{\mathbf{G}}(\xi\omega \wedge e^{\wedge\eta\tilde{V}}), \quad (\text{C.17})$$

$$\Lambda[\lambda, \tilde{V}] = -\pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge\eta\tilde{V}}). \quad (\text{C.18})$$

In particular, one can find that the gauge transformations generated by η and Q do not mix: the gauge parameters λ and ω of (C.13) are related to Λ and Ω of (C.14) respectively, as seen in (C.17) and (C.18).

D Open NS superstrings with stubs

In this section, we construct a new action for generic open NS string field theory, which we call alternative WZW-like action S_η . After defining the dual A_∞ products and explaining its WZW-like structure, we give two realizations of this type of WZW-like action using two different dynamical string fields Φ and Ψ in the large and small Hilbert spaces, respectively.

Dual A_∞ -products and derivation properties. Let η be the coderivation constructed from η , which is nilpotent $\eta^2 = 0$, and let \mathbf{a} be a nilpotent coderivation satisfying $\mathbf{a}\eta = -(-)^{a\eta}\eta\mathbf{a}$ and $\mathbf{a}^2 = 0$. Then, we assume that $\widehat{\mathbf{G}}^{-1} : (\mathcal{H}, \mathbf{a}) \rightarrow (\mathcal{H}_S, \mathbf{D}_a)$ is an A_∞ -morphism, where \mathcal{H} is the large Hilbert space, \mathcal{H}_S is the small Hilbert space, and $\mathbf{D}_a \equiv \widehat{\mathbf{G}}^{-1}\mathbf{a}\widehat{\mathbf{G}}$. Note that \mathbf{D}_a is nilpotent: $\mathbf{D}_a^2 = (\widehat{\mathbf{G}}^{-1}\mathbf{a}\widehat{\mathbf{G}})(\widehat{\mathbf{G}}^{-1}\mathbf{a}\widehat{\mathbf{G}}) = \widehat{\mathbf{G}}^{-1}\mathbf{a}^2\widehat{\mathbf{G}} = 0$. For example, one can use Q , $Q + \mathbf{m}_2|_2$, and so on for \mathbf{a} , and various $\widehat{\mathbf{G}}$ appearing in [6–8] for $\widehat{\mathbf{G}}$. Suppose that the coderivation \mathbf{D}_a also commutes with η , which means

$$(\mathbf{D}_a)^2 = 0, \quad [[\mathbf{D}_a, \eta]] = 0. \quad (\text{D.1})$$

Then, we can introduce a dual A_∞ products \mathbf{D}_η defined by

$$\mathbf{D}_\eta \equiv \widehat{\mathbf{G}}\eta\widehat{\mathbf{G}}^{-1}. \quad (\text{D.2})$$

Note that the pair of nilpotent maps $(\mathbf{D}_\eta, \mathbf{a})$ have the same properties as (\mathbf{D}_a, η) :

$$(\mathbf{D}_\eta)^2 = 0, \quad [[\mathbf{D}_\eta, \mathbf{a}]] = 0. \quad (\text{D.3})$$

We can quickly find when the A_∞ products \mathbf{D}_a commutes with the coderivation η as (D.1), its dual A_∞ product \mathbf{D}_η and coderivation \mathbf{a} also satisfies (D.3) as follows

$$\begin{aligned} \mathbf{a}\mathbf{D}_\eta &= (\widehat{\mathbf{G}}\widehat{\mathbf{G}}^{-1})\mathbf{a}(\widehat{\mathbf{G}}\eta\widehat{\mathbf{G}}^{-1}) = \widehat{\mathbf{G}}\mathbf{D}_a\eta\widehat{\mathbf{G}}^{-1} \\ &= (-)^{a\eta}\widehat{\mathbf{G}}\eta\mathbf{D}_a\widehat{\mathbf{G}}^{-1} = (-)^{a\eta}\widehat{\mathbf{G}}\eta\widehat{\mathbf{G}}^{-1}\mathbf{a}\widehat{\mathbf{G}}\widehat{\mathbf{G}}^{-1} = (-)^{a\eta}\mathbf{D}_\eta\mathbf{a}. \end{aligned}$$

In this paper, as these coderivation \mathbf{a} and A_∞ -morphism $\widehat{\mathbf{G}}$, we always use $\mathbf{a} \equiv \mathbf{Q}$ and a gauge product $\widehat{\mathbf{G}}$ given by [7]. Therefore, the original A_∞ product is equal to the Neveu-Schwarz products \mathbf{M} of open strings with stubs given [7],

$$\mathbf{D}_\mathbf{a} = \mathbf{D}_\mathbf{Q} = \mathbf{M} \equiv \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}},$$

and the dual A_∞ products is always given by

$$\mathbf{D}^\eta \equiv \widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1} = \eta - \mathbf{m}_2 + \dots \quad (\text{D.4})$$

The symbol \mathbf{D}^η always denotes (D.4) in the rest. The dual A_∞ products \mathbf{D}^η consists of a linear map η and a set of multilinear products $\{D_n^\eta\}_{n=2}^\infty$. Now the bilinear product D_2^η just equals to the star product $-\mathbf{m}_2$ because we take $\widehat{\mathbf{G}}$ of [7]. We write

$$[B_1, \dots, B_n]^\eta \equiv D_n^\eta(B_1 \otimes \dots \otimes B_n) \quad (\text{D.5})$$

for higher products of \mathbf{D}^η . Then, the Maurer-Cartan element of $\mathbf{D}^\eta = \eta - \mathbf{m}_2 + \dots$ is given by

$$\begin{aligned} \mathbf{D}^\eta \frac{1}{1-A} &= \frac{1}{1-A} \otimes \pi_1 \left(\mathbf{D}^\eta \frac{1}{1-A} \right) \otimes \frac{1}{1-A} \\ &= \frac{1}{1-A} \otimes \left(\eta A - \mathbf{m}_2(A, A) + \dots \right) \otimes \frac{1}{1-A}, \end{aligned}$$

where A is a state of the large Hilbert space \mathcal{H} and π_1 is an natural 1-state projection onto \mathcal{H} . Hence, the solution of the Maurer-Cartan equation $\mathbf{D}^\eta(1-A)^{-1} = 0$ is given by a state \widetilde{A}_η satisfying $\eta A_\eta - \mathbf{m}_2(A_\eta, A_\eta) + \dots = 0$, or equivalently,

$$\eta A_\eta - \mathbf{m}_2(A_\eta, A_\eta) + \sum_{n=3}^{\infty} [\overbrace{A_\eta, \dots, A_\eta}^n]^\eta = 0.$$

Shift of the dual A_∞ products \mathbf{D}^η . We introduce the A_η -shifted products $[B_1, \dots, B_n]_{A_\eta}^\eta$ defined by

$$[B_1, \dots, B_n]_{A_\eta}^\eta \equiv \pi_1 \mathbf{D}^\eta \left(\frac{1}{1-A_\eta} \otimes B_1 \otimes \frac{1}{1-A_\eta} \otimes \dots \otimes \frac{1}{1-A_\eta} \otimes B_n \otimes \frac{1}{1-A_\eta} \right). \quad (\text{D.6})$$

Note that higher shifted products all vanish $[B_1, \dots, B_{n>2}]_{A_\eta}^\eta = 0$ when higher products of $\mathbf{D}^\eta = \eta - \mathbf{m}_2 + D_3^\eta + D_4^\eta \dots$ vanish: $D_{n>2}^\eta = 0$. In particular, we write $D_\eta B$ for $[B]_\eta$:

$$\begin{aligned} D_\eta B &\equiv \pi_1 \mathbf{D}^\eta \left(\frac{1}{1-A_\eta} \otimes B \otimes \frac{1}{1-A_\eta} \right) \\ &= \eta B - \llbracket A_\eta, B \rrbracket + \sum_{\text{cyclic } n=2}^{\infty} \sum [\overbrace{A_\eta, \dots, A_\eta}^n, B]^\eta. \end{aligned}$$

When \tilde{A}_η gives a solution on the Maurer-Cartan equation of \mathbf{D}_η , these A_η -shifted products also satisfy A_∞ -relations, which implies that the linear operator D_η becomes nilpotent. We find

$$\begin{aligned} (D_\eta)^2 B &= \pi_1 \mathbf{D}^\eta \left(\frac{1}{1-A_\eta} \otimes \pi_1 \mathbf{D}^\eta \left(\frac{1}{1-A_\eta} \otimes B \otimes \frac{1}{1-A_\eta} \right) \otimes \frac{1}{1-A_\eta} \right) \\ &= \pi_1 (\mathbf{D}^\eta)^2 \frac{1}{1-A_\eta} - \left[\left[\pi_1 \mathbf{D}^\eta \left(\frac{1}{1-A_\eta} \right), B \right] \right]_{A_\eta}^\eta = 0, \end{aligned}$$

where the A_η -shifted commutator $\llbracket A, B \rrbracket_{A_\eta}$ of A and B is defined by

$$\llbracket A, B \rrbracket_{A_\eta}^\eta \equiv [A, B]_{A_\eta}^\eta - (-)^{AB} [B, A]_{A_\eta}^\eta.$$

D.1 Alternative WZW-like relations and action $S_\eta[\varphi]$

We call a state A_η a pure-gauge-like (functional) field when A_η satisfies

$$\eta A_\eta - m_2(A_\eta, A_\eta) + \sum_{n=2}^{\infty} \left[\overbrace{A_\eta, \dots, A_\eta}^n \right]^\eta = 0 \quad (\text{D.7})$$

and has ghost-and-picture number $(1|-1)$, odd Grassmann parity, and even degree. Let d is a derivation operator commuting with η and let $(d_g|d_p)$ be its ghost-and-picture number. We call a state A_d a associated (functional) field when A_d satisfies

$$(-)^d d A_\eta = D_\eta A_d \quad (\text{D.8})$$

and has ghost-and-picture number $(d_g - 1|d_p + 1)$.

Let φ be a dynamical string field, $t \in [0, 1]$ be a real parameter, and $\varphi(t)$ be a path satisfying $\varphi(0) = 0$ and $\varphi(1) = \varphi$. Once a pure-gauge-like (functional) field $A_\eta = A_\eta[\varphi]$ and an associated (functional) field $A_d = A_d[\varphi]$ are constructed as functionals of given dynamical string field φ , one can construct a gauge invariant action

$$S_\eta[\varphi] = - \int_0^1 dt \langle A_t[\varphi(t)], Q A_\eta[\varphi(t)] \rangle. \quad (\text{D.9})$$

We write $A_t[\varphi]$ for the associated field $A_d[\varphi]$ of the parameter differential $d = \partial_t$. One can check that the t -dependence is ‘‘topological’’. Namely, the variation of the action is given by

$$\delta S_\eta[\varphi] = \langle A_\delta[\varphi], Q A_\eta[\varphi] \rangle,$$

which is independent of the real parameter t . Since $Q A_\eta = -D_\eta A_Q$ and $(D_\eta)^2 = Q^2 = 0$, we find that $\delta S = 0$ with the gauge transformations

$$A_\delta[\varphi] = D_\eta \Omega + Q \Lambda, \quad (\text{D.10})$$

where gauge parameters Ω and Λ have ghost-and-picture number $(-1|1)$ and $(-1|0)$ respectively, and belong to the large Hilbert space. Since $A_\delta[\varphi]$ is an invertible functional of $\delta\varphi$, the explicit form of the gauge transformation of the dynamical string field $\delta\phi$ is obtained from $A_\delta[\varphi]$.

In the rest, we give two realisations of this action by using two different dynamical string fields $\varphi = \Psi$ and $\varphi = \Phi$: small-space parametrisation $S_\eta[\Psi]$ and large-space parametrisation $S_\eta[\Phi]$. By construction, the A_∞ action proposed in [7] is equivalent to the small-space parametrisation $S_\eta[\Psi]$. Since the large-space parametrisation $S_\eta[\Phi]$ has the same (alternative) WZW-like structure as $S_\eta[\Psi]$, our new actions $S_\eta[\Psi]$ and $S_\eta[\Phi]$ both are equivalent to that of A_∞ formulation.

D.2 Small-space parametrization: $\varphi = \Psi$

Let us consider an NS dynamical string field Ψ which is a Grassmann odd, ghost number 1, and picture number -1 state of the small Hilbert space: $\eta\Psi = 0$. We show that one can construct WZW-like ingredients $A_\eta[\Psi]$ and $A_d[\Psi]$ as functionals of this dynamical string field Ψ . Once we obtain these WZW-like ingredients $A_\eta[\Psi], A_d[\Psi]$ parametrized by Ψ , one can construct the WZW-like action $S[\Psi]$ parametrized by Ψ ,

$$S_\eta[\Psi] = - \int_0^1 ds \langle A_s[\Psi(s)], QA_\eta[\Psi(s)] \rangle, \tag{D.11}$$

where s is a real parameter $s \in [0, 1]$ and $\Psi(s)$ is the path satisfying $\Psi(s = 0) = 0$ and $\Psi(s = 1) = \Psi$. We wrote $A_s[\Psi]$ for the associated field $A_d[\Psi]$ of $d = \partial_s$.

Pure-gauge-like field $A_\eta[\Psi]$. When the η -complex (\mathcal{H}, η) is exact, there exist ξ such that $[\eta, \xi] = 1$ and \mathcal{H} , the large Hilbert space, is decomposed into the direct sum of η -exact and ξ -exact $\mathcal{H} = P_\eta\mathcal{H} \oplus P_\xi\mathcal{H}$, where P_η and P_ξ are projector onto η -exact and ξ -exact states respectively.¹⁴ Note that since the small Hilbert space \mathcal{H}_S is defined by $\mathcal{H}_S \equiv P_\eta\mathcal{H}$ and satisfies $\mathcal{H}_S \subset P_\eta\mathcal{H}_S$, all the states Ψ belonging to \mathcal{H}_S satisfy $P_\eta\Psi = \Psi$ and $P_\xi\Psi = 0$, or simply,

$$\eta\Psi = 0.$$

Using this fact, we can construct a desired pure-gauge-like (functional) fields $A_\eta[\Psi]$, namely, a solution of the Maurer-Cartan equation of the dual A_∞ products $D^\eta = \eta - m_2 + \dots$

A pure-gauge-like (functional) field $A_\eta[\Psi]$ is given by

$$A_\eta[\Psi] \equiv \pi_1 \widehat{\mathbf{G}} \frac{1}{1 - \Psi}$$

because it becomes a trivial solution of the Maurer-Cartan equation as follows

$$\begin{aligned} D^\eta \frac{1}{1 - A_\eta[\Psi]} &= D^\eta \frac{1}{1 - \pi_1 \widehat{\mathbf{G}} \frac{1}{1 - \Psi}} = D^\eta \widehat{\mathbf{G}} \frac{1}{1 - \Psi} = \widehat{\mathbf{G}} \eta \frac{1}{1 - \Psi} \\ &= \widehat{\mathbf{G}} \left(\frac{1}{1 - \Psi} \otimes \eta\Psi \otimes \frac{1}{1 - \Psi} \right) = 0. \end{aligned}$$

Recall that $\pi_1 D^\eta (1 - A_\eta[\Psi])^{-1} = 0$ is equal to

$$\eta A_\eta[\Psi] - m_2(A_\eta[\Psi], A_\eta[\Psi]) + \sum_{n=2}^{\infty} \overbrace{[A_\eta[\Psi], \dots, A_\eta[\Psi]]}^n = 0.$$

¹⁴These satisfy $P_\eta^2 = P_\eta, P_\xi^2 = P_\xi, P_\eta P_\xi = P_\xi P_\eta = 0$, and $P_\eta + P_\xi = 1$ on \mathcal{H} .

Associated fields $A_d[\Psi]$. Let d be a coderivation constructed from a derivation d of the dual A_∞ products D^η , which implies that the d -derivation property $[[d, D^\eta]] = 0$ holds. Then, we obtain $[[D_d, \eta]] = 0$ with $D_d \equiv \widehat{\mathbf{G}}^{-1} d \widehat{\mathbf{G}}$, which means that D_d is “ η -exact” and there exists a coderivation ξ_d such that

$$D_d = \widehat{\mathbf{G}}^{-1} d \widehat{\mathbf{G}} = (-)^d [[\eta, \xi_d]].$$

Using this coderivation ξ_d , we can construct an associated (functional) field $A_d[\Psi]$. Note that the response of d acting on the group-like element of $A_\eta[\Psi] = \widehat{\mathbf{G}}(1 - \Psi)^{-1}$ is given by

$$\begin{aligned} (-)^d d \frac{1}{1 - A_\eta[\Psi]} &= (-)^d \widehat{\mathbf{G}} \widehat{\mathbf{G}}^{-1} d \widehat{\mathbf{G}} \frac{1}{1 - \Psi} = \widehat{\mathbf{G}} \eta \xi_d \frac{1}{1 - \Psi} = D^\eta \widehat{\mathbf{G}} \left(\xi_d \frac{1}{1 - \Psi} \right) \\ &= D^\eta \left(\frac{1}{1 - A_\eta[\Psi]} \otimes \pi_1 \widehat{\mathbf{G}} \left(\xi_d \frac{1}{1 - \Psi} \right) \otimes \frac{1}{1 - A_\eta[\Psi]} \right). \end{aligned}$$

An associated (functional) field of d is given by

$$A_d[\Psi] \equiv \pi_1 \widehat{\mathbf{G}} \left(\xi_d \frac{1}{1 - \Psi} \right)$$

because one can directly check

$$\begin{aligned} (-)^d d \frac{1}{1 - A_\eta[\Psi]} &= D^\eta \left(\frac{1}{1 - A_\eta[\Psi]} \otimes \pi_1 \widehat{\mathbf{G}} \left(\xi_d \frac{1}{1 - \Psi} \right) \otimes \frac{1}{1 - A_\eta[\Psi]} \right) \\ &= D^\eta \left(\frac{1}{1 - A_\eta[\Psi]} \otimes A_d[\Psi] \otimes \frac{1}{1 - A_\eta[\Psi]} \right). \end{aligned}$$

Picking up the relation on \mathcal{H} , or equivalently acting π_1 on this relation on $T(\mathcal{H})$, we obtain

$$(-)^d d A_\eta[\Psi] = \eta A_d[\Psi] - [[A_\eta[\Psi], A_d[\Psi]]]_{A_\eta[\Psi]}^\eta,$$

which is the simplest case of $(-)^d d A_\eta[\Psi] = \pi_1 D^\eta$ -exact term.

D.3 Large-space parametrization: $\varphi = \Phi$

Let us consider an NS dynamical string field Φ which is a Grassmann even, ghost number 0, and picture number 0 state of the large Hilbert space: $\eta \Phi \neq 0$. We show that one can construct WZW-like ingredients $A_\eta[\Phi]$ and $A_d[\Phi]$ as functionals of this dynamical string field Φ . Using these WZW-like ingredients $A_\eta[\Phi], A_d[\Phi]$ parametrized by Φ , we obtain the WZW-like action $S_\eta[\Phi]$ parametrized by Φ ,

$$S_\eta[\Phi] = - \int_0^1 dt \langle A_t[\Phi(t)], Q A_\eta[\Phi(t)] \rangle, \quad (\text{D.12})$$

where t is a real parameter $t \in [0, 1]$ and $\Phi(t)$ is the path satisfying $\Phi(t = 0) = 0$ and $\Phi(t = 1) = \Phi$. We wrote $A_t[\Psi]$ for the associated field $A_d[\Psi]$ of $d = \partial_t$.

Pure-gauge-like field $A_\eta[\Phi]$. Let us consider a functional $A_\eta[\tau; \Phi]$ defined by the differential equation

$$\begin{aligned} \frac{\partial}{\partial \tau} A_\eta[\tau; \Phi] &= \eta \Phi + \sum_{k=1}^{\infty} \sum_{\text{cyclic}} \left[\overbrace{A_\eta[\tau; \Phi], \dots, A_\eta[\tau; \Phi]}^k, \Phi \right]^\eta \\ &\equiv D_\eta(\tau) \Phi \end{aligned} \quad (\text{D.13})$$

with the initial condition $A_\eta[\tau = 0; \Phi] = 0$, where τ is a real parameter. A few terms of $A_\eta[\tau; \Phi]$ is given by

$$A_\eta[\tau; \Phi] = \tau \eta \Phi + \frac{\tau^2}{2} \llbracket \eta \Phi, \Phi \rrbracket + \frac{\tau^3}{3} \left(\sum_{\text{cyclic}} [\eta \Phi, \eta \Phi, \Phi]^\eta + \llbracket \llbracket \eta \Phi, \Phi \rrbracket^\eta, \Phi \rrbracket^\eta \right) + \dots$$

We write $A_\eta[\Phi]$ for the $\tau = 1$ value of the solution $A_\eta[\tau; \Phi]$:

$$A_\eta[\Phi] \equiv A_\eta[\tau = 1; \Phi].$$

We quickly find that this $A_\eta[\Phi]$, a functional on the dynamical string field Φ , satisfies (D.7) and gives a pure-gauge-like (functional) field. For brevity, we introduce

$$F(\tau) \equiv \eta A_\eta[\tau; \Phi] - m_2(A_\eta[\tau; \Phi], A_\eta[\tau; \Phi]) + \sum_{n=2}^{\infty} \left[\overbrace{A_\eta[\tau; \Phi], \dots, A_\eta[\tau; \Phi]}^n \right]^\eta.$$

The statement “ $A_\eta[\Phi]$ is a pure-gauge-like (functional) field” is equivalent to the equation $F(1) = 0$. By definition, $F(\tau)$ satisfies $F(\tau = 0) = 0$ and the following relation holds:

$$\begin{aligned} \frac{\partial}{\partial \tau} F(\tau) &= \eta \frac{\partial}{\partial \tau} A_\eta[\tau; \Phi] + \sum_{n=1}^{\infty} \sum_{\text{cyclic}} \left[\overbrace{A_\eta[\tau; \Phi], \dots, A_\eta[\tau; \Phi]}^n, \frac{\partial}{\partial \tau} A_\eta[\tau; \Phi] \right]^\eta \\ &= D_\eta(\tau) D_\eta(\tau) \Phi = - \llbracket F(\tau), \Phi \rrbracket_{A_\eta[\tau; \Phi]}^\eta. \end{aligned} \quad (\text{D.14})$$

The solution of the differential equation (D.14) with the initial condition $F(0) = 0$ is given by $F(\tau) = 0$ for any τ . Hence, $F(1) = 0$ holds and $A_\eta[\Phi]$ indeed gives a pure-gauge-like (functional) field.

Associated field $A_d[\Phi]$. Then, we consider a functional $A_d[\tau; \Phi]$ defined by the differential equation

$$\frac{\partial}{\partial \tau} A_d[\tau; \Phi] = d\Phi + \llbracket \Phi, A_d[\tau; \Phi] \rrbracket_{A_d[\tau; \Phi]}^\eta \quad (\text{D.15})$$

with the initial condition $A_d[\tau = 0; \Phi] = 0$. A few terms of $A_d[\tau; \Phi]$ is given by

$$A_d[\tau; \Phi] = \tau d\Phi + \frac{\tau^2}{2} \llbracket d\Phi, \Phi \rrbracket + \frac{\tau^3}{3} \left(\sum_{\text{cyclic}} [\eta \Phi, d\Phi, \Phi]^\eta + \frac{1}{2} \llbracket \llbracket d\Phi, \Phi \rrbracket^\eta, \Phi \rrbracket^\eta \right) + \dots$$

The $\tau = 1$ value of the functional $A_d[\tau; \Phi]$ of the dynamical string field Φ , which we write

$$A_d[\Phi] \equiv A_d[\tau = 1; \Phi],$$

satisfies (D.8) and gives an associated (functional) field. To prove this fact, we introduce a function of τ

$$\mathcal{I}(\tau) \equiv D_\eta(\tau)A_d[\tau; \Phi] - (-)^d A_\eta[\tau; \Phi],$$

whose zeros would provide the WZW-like relation that associated (functional) field must satisfy. By definition, $\mathcal{I}(\tau)$ satisfies $\mathcal{I}(\tau = 0) = 0$ and

$$\begin{aligned} \frac{\partial}{\partial \tau} \mathcal{I}(\tau) &= D_\eta(\tau) (\partial_\tau A_d[\tau; \Phi]) + [\partial_\tau A_\eta[\tau; \Phi], A_d[\tau; \Phi]]_{A_\eta[\tau; \Phi]}^\eta - (-)^d d(\partial_\tau A_\eta[\tau; \Phi]) \\ &= [\Phi, \mathcal{I}(\tau)]_{A_\eta[\tau; \Phi]}^\eta + D_\eta \left(\frac{\partial}{\partial \tau} A_d[\tau; \Phi] - d\Phi - [[\Phi, A_d[\tau; \Phi]]_{A_\eta[\tau; \Phi]}^\eta \right). \end{aligned}$$

Hence, when $A_d[\tau; \Phi]$ is defined by (D.15), the function $\mathcal{I}(\tau)$ gives the solution of the differential equation

$$\frac{\partial}{\partial \tau} \mathcal{I}(\tau) = [[\Phi, \mathcal{I}(\tau)]_{A_\eta[\tau; \Phi]}^\eta]$$

with the initial condition $\mathcal{I}(0) = 0$, namely, $\mathcal{I}(\tau) = 0$ for any τ : we obtain $\mathcal{I}(1) = 0$.

E NS-NS sector

In this section, starting from the NS-NS superstring L_∞ -products, we clarify a WZW-like structure and give alternative WZW-like form $S_{\eta\tilde{\eta}}[\varphi]$ of the action for NS-NS string field theory.

Dual L_∞ products. Using path-ordered exponential map $\widehat{\mathbf{G}}$, the NS-NS superstring products \mathbf{L} of the small-space L_∞ action of [7] is given by $\mathbf{L} = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}$. By construction, it satisfies $[[\boldsymbol{\eta}, \mathbf{L}] = 0$ and $[[\tilde{\boldsymbol{\eta}}, \mathbf{L}] = 0$. (See [7] or [6] for details.) We start with its dual L_∞ products:

$$\mathbf{L}^\boldsymbol{\eta} \equiv \widehat{\mathbf{G}} \boldsymbol{\eta} \widehat{\mathbf{G}}^{-1}, \tag{E.1a}$$

$$\mathbf{L}^{\tilde{\boldsymbol{\eta}}} \equiv \widehat{\mathbf{G}} \tilde{\boldsymbol{\eta}} \widehat{\mathbf{G}}^{-1}. \tag{E.1b}$$

They satisfy L_∞ -relations $(\mathbf{L}^\alpha)^2 = 0$, commutativity $[[\mathbf{L}^\boldsymbol{\eta}, \mathbf{L}^{\tilde{\boldsymbol{\eta}}}] = 0$, and Q -derivation properties

$$\mathbf{Q} \mathbf{L}^\alpha = \widehat{\mathbf{G}} (\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}) \alpha \widehat{\mathbf{G}}^{-1} = -\widehat{\mathbf{G}} \alpha (\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}) \widehat{\mathbf{G}}^{-1} = -\mathbf{L}^\alpha \mathbf{Q}$$

for $\alpha = \boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}$, which give extensions of η -constraints. In the rest, we write

$$[A_1, \dots, A_n]^\alpha := \pi_1 \widehat{\mathbf{G}} \alpha \widehat{\mathbf{G}}^{-1} (A_1 \wedge \dots \wedge A_n), \quad (\alpha = \boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}).$$

By definition, these dual L_∞ products satisfy $(\mathbf{L}^\alpha)^2 = 0$ for $\alpha = \boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}$, namely L_∞ -relations,

$$\sum_\sigma \sum_{k=1}^n (-)^{|\sigma|} [[A_{i_{\sigma(1)}}, \dots, A_{i_{\sigma(k)}}]^\alpha, A_{i_{\sigma(k+1)}}, \dots, A_{i_{\sigma(n)}}]^\alpha = 0. \tag{E.2a}$$

Note that the commutation relation $[[\mathbf{L}^\boldsymbol{\eta}, \mathbf{L}^{\tilde{\boldsymbol{\eta}}}] = 0$ can be written as

$$\sum_{\alpha_1, \alpha_2 = \boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}} \sum_\sigma \sum_{k=1}^n (-)^{|\sigma|} [[A_{i_{\sigma(1)}}, \dots, A_{i_{\sigma(k)}}]^{\alpha_1}, A_{i_{\sigma(k+1)}}, \dots, A_{i_{\sigma(n)}}]^{\alpha_2} = 0. \tag{E.2b}$$

For $d = Q, \partial_t, \delta$, these products satisfy d -derivation properties:

$$d [B_1, \dots, B_n]^\alpha + \sum_{i=1}^{n-1} (-)^{d(B_1+\dots+B_{i-1})} [B_1, \dots, d B_i, \dots, B_n]^\alpha = 0. \quad (\text{E.2c})$$

E.1 Alternative Wess-Zumino-Witten-like relations

Let $\Psi_{\eta\tilde{\eta}} = \Psi_{\eta\tilde{\eta}}[\varphi]$ be a Grassmann even, ghost number 2, left-moving picture number -1 , and right-moving picture number -1 state in the left-and-right large Hilbert space. When this $\Psi_{\eta\tilde{\eta}}$ satisfies the Maurer-Cartan equations for the both dual products (E.1a) and (E.1b),

$$\alpha \Psi_{\eta\tilde{\eta}} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \overbrace{[\Psi_{\eta\tilde{\eta}}, \dots, \Psi_{\eta\tilde{\eta}}]^{n+1}}^\alpha = 0, \quad (\alpha = \eta, \tilde{\eta}), \quad (\text{E.3a})$$

we call $\Psi_{\eta\tilde{\eta}}[\varphi]$ as a *pure-gauge-like (functional) field*. As we will see, in addition to this $\Psi_{\eta\tilde{\eta}}[\varphi]$, if one can obtain a state $\Psi_d = \Psi_d[\varphi]$ which satisfy the *WZW-like relation*,

$$(-)^d d \Psi_{\eta\tilde{\eta}} = -D_\eta D_{\tilde{\eta}} \Psi_d, \quad (\text{E.3b})$$

then, one can always find a gauge invariant action. We call $\Psi_d[\varphi]$ as a *large associated (functional) field*. Here, the linear operator D_α for $\alpha = \eta, \tilde{\eta}$ is given by

$$D_\alpha B \equiv \alpha B + \sum_{n=1}^{\infty} \frac{1}{n!} \overbrace{[\Psi_{\eta\tilde{\eta}}, \dots, \Psi_{\eta\tilde{\eta}}, B]^n}^\alpha, \quad (\alpha = \eta, \tilde{\eta}),$$

and d is a derivation operator satisfying (E.2c). For example, one can take $d = Q, \partial_t$, or δ . Note that $(D_\eta)^2 = (D_{\tilde{\eta}})^2 = 0$ and $D_\eta D_{\tilde{\eta}} B = -D_{\tilde{\eta}} D_\eta B$ hold because $\Psi_{\eta\tilde{\eta}}$ satisfies (E.3a).

Although it is sufficient to consider the above (E.3a) and (E.3b), it would be helpful to consider their *small associated (functional) fields* $\Psi_{\eta d} = \Psi_{\eta d}[\varphi]$ and $\Psi_{d\tilde{\eta}} = \Psi_{d\tilde{\eta}}[\varphi]$ which are defined by

$$\Psi_{\eta d} \equiv D_\eta \Psi_d, \quad \Psi_{d\tilde{\eta}} \equiv -D_{\tilde{\eta}} \Psi_d. \quad (\text{E.3c})$$

While $\Psi_d[\varphi]$ has the same ghost, left-moving picture, and right-moving picture numbers as d , this $\Psi_{\eta d}[\varphi]$ or $\Psi_{d\tilde{\eta}}[\varphi]$ has the same as “ d plus η ” or “ d plus $\tilde{\eta}$ ” respectively.

E.2 Alternative WZW-like action $S_{\eta\tilde{\eta}}[\varphi]$

Let φ be a dynamical NS-NS string field and $\varphi(t)$ be a path satisfying $\varphi(0) = 0$ and $\varphi(1) = \varphi$, where $t \in [0, 1]$ is a real parameter. Once we construct WZW-like ingredients $\Psi_{\eta\tilde{\eta}}[\varphi]$ and $\Psi_d[\varphi]$ as functionals of given dynamical string field φ , we can obtain a new gauge invariant action

$$S_{\eta\tilde{\eta}}[\varphi] = \int_0^1 dt \langle \Psi_t[\varphi(t)], Q \Psi_{\eta\tilde{\eta}}[\varphi(t)] \rangle, \quad (\text{E.4})$$

whose gauge transformations are given by

$$\Psi_\delta[\varphi] = D_\eta \Omega + D_{\tilde{\eta}} \tilde{\Omega} + Q \Lambda. \quad (\text{E.5})$$

Here, we write $\Psi_t[\varphi(t)]$ for $\Psi_d[\varphi(t)]$ with $d = \partial_t$, and $\Psi_d[\varphi]$ for $\Psi_d[\varphi]$ with $d = \delta$. The equation of motion is given by t -independent form

$$Q \Psi_{\eta\tilde{\eta}}[\varphi] = -D_\eta \Psi_{Q\tilde{\eta}}[\varphi] = -D_{\tilde{\eta}} \Psi_{\eta Q}[\varphi] = D_\eta D_{\tilde{\eta}} \Psi_Q[\varphi] = 0. \quad (\text{E.6})$$

One can find these facts by using WZW-like relations (E.3a), (E.3b), and (E.3c) only, which we explain. Note that computations are almost parallel to the conventional WZW-like case [21].

Variation of the action. To derive (E.5) and (E.6) from (E.4), we compute the variation of (E.4):

$$\delta S_{\eta\tilde{\eta}}[\varphi] = \int_0^1 dt \left(\langle \delta \Psi_t[\varphi(t)], Q \Psi_{\eta\tilde{\eta}}[\varphi(t)] \rangle + \langle \Psi_t[\varphi(t)], \delta(Q \Psi_{\eta\tilde{\eta}}[\varphi(t)]) \rangle \right).$$

For this purpose, we define two bilinear maps, so-called shifted L_∞ -products,

$$[A, B]_{\Psi_{\eta\tilde{\eta}}}^\alpha \equiv \sum_{n=0}^{\infty} \frac{1}{n!} [\overbrace{\Psi_{\eta\tilde{\eta}}, \dots, \Psi_{\eta\tilde{\eta}}}^n, A, B]^\alpha, \quad (\alpha = \eta, \tilde{\eta}). \quad (\text{E.7})$$

With D_α , it satisfies $D_\alpha[A, B]_{\Psi_{\eta\tilde{\eta}}}^\alpha + [D_\alpha A, B]_{\Psi_{\eta\tilde{\eta}}}^\alpha + (-)^A[A, D_\alpha B]_{\Psi_{\eta\tilde{\eta}}}^\alpha = 0$. For $d = \partial_t, \delta$, or Q , because of the derivation properties (E.2c) of \mathbf{L}^α , we find

$$(-)^d d(D_\alpha A) - D_\alpha(dA) - [d\Psi_{\eta\tilde{\eta}}, A]_{\Psi_{\eta\tilde{\eta}}}^\alpha = 0. \quad (\text{E.8a})$$

By considering $[[d_1, d_2]\Psi_{\eta\tilde{\eta}}] = 0$ with this formula (E.8a), for example, we quickly find

$$D_{\tilde{\eta}} \left(d_1 \Psi_{\eta d_2} - (-)^{d_1 d_2} d_2 \Psi_{\eta d_1} - (-)^{d_1} [\Psi_{\eta d_1}, \Psi_{\eta d_2}]_{\Psi_{\eta\tilde{\eta}}}^{\tilde{\eta}} \right) = 0. \quad (\text{E.8b})$$

For brevity, we omit $\varphi(t)$ -dependence of functionals. Note that the inner product $\langle A, B \rangle$ includes the c_0^- -insertion: we have $\langle dA, B \rangle = (-)^{dA} \langle A, dB \rangle$ for $d = D_\eta, D_{\tilde{\eta}}, Q$, and we use $\langle A, [B, C]_{\Psi_{\eta\tilde{\eta}}}^\alpha \rangle = (-)^{AB} \langle B, [A, C]_{\Psi_{\eta\tilde{\eta}}}^\alpha \rangle$ for $\alpha = \eta, \tilde{\eta}$. Using (E.8a) with (E.3b) and (E.3c), we find that the second term can be rewritten as $\langle \Psi_\delta, \partial_t(Q\Psi_{\eta\tilde{\eta}}) \rangle$ plus extra terms:

$$\begin{aligned} \langle \Psi_t, \delta(Q \Psi_{\eta\tilde{\eta}}) \rangle &= \langle \Psi_t, Q D_{\tilde{\eta}} \Psi_{\eta\delta} \rangle \\ &= -\langle \Psi_t, D_{\tilde{\eta}} Q \Psi_{\eta\delta} \rangle - \langle \Psi_t, [Q \Psi_{\eta\tilde{\eta}}, \Psi_{\eta\delta}]_{\Psi_{\eta\tilde{\eta}}}^{\tilde{\eta}} \rangle \\ &= -\langle \Psi_{\eta\delta}, Q \Psi_{t\tilde{\eta}} \rangle + \langle \Psi_{\eta\delta}, [\Psi_t, Q \Psi_{\eta\tilde{\eta}}]_{\Psi_{\eta\tilde{\eta}}}^{\tilde{\eta}} \rangle \\ &= \langle \Psi_\delta, Q D_\eta \Psi_{t\tilde{\eta}} \rangle + \langle \Psi_\delta, [Q \Psi_{\eta\tilde{\eta}}, \Psi_{t\tilde{\eta}}]_{\Psi_{\eta\tilde{\eta}}}^{\tilde{\eta}} \rangle + \langle \Psi_{\eta\delta}, [\Psi_t, Q \Psi_{\eta\tilde{\eta}}]_{\Psi_{\eta\tilde{\eta}}}^{\tilde{\eta}} \rangle \\ &= \langle \Psi_\delta, \partial_t(Q \Psi_{\eta\tilde{\eta}}) \rangle + \langle \Psi_{Q\tilde{\eta}}, D_\eta([\Psi_{t\tilde{\eta}}, \Psi_\delta]_{\Psi_{\eta\tilde{\eta}}}^{\tilde{\eta}} - [\Psi_t, \Psi_{\eta\delta}]_{\Psi_{\eta\tilde{\eta}}}^{\tilde{\eta}}) \rangle. \end{aligned} \quad (\text{E.9})$$

Likewise, we find the first term of the variation becomes $\langle \partial_t \Psi_\delta, Q \Psi_{\eta\tilde{\eta}} \rangle$ plus extra terms:

$$\begin{aligned}
\langle \delta \Psi_t, Q \Psi_{\eta\tilde{\eta}} \rangle &= -\langle D_\eta \delta \Psi_t, \Psi_{Q\tilde{\eta}} \rangle & (E.10) \\
&= -\langle \delta (D_\eta \Psi_t), \Psi_{Q\tilde{\eta}} \rangle + \langle [\delta \Psi_{\eta\tilde{\eta}}, \Psi_t]_{\Psi_{\eta\tilde{\eta}}}^\eta, \Psi_{Q\tilde{\eta}} \rangle \\
&= -\langle \delta \Psi_{\eta t}, \Psi_{Q\tilde{\eta}} \rangle + \langle [D_{\tilde{\eta}} \Psi_{\eta\delta}, \Psi_t]_{\Psi_{\eta\tilde{\eta}}}^\eta, \Psi_{Q\tilde{\eta}} \rangle \\
&= \langle \partial_t \Psi_{\eta\delta}, \Psi_{Q\tilde{\eta}} \rangle + \langle [\Psi_{\eta t}, \Psi_{\eta\delta}]_{\Psi_{\eta\tilde{\eta}}}^\eta, \Psi_{Q\tilde{\eta}} \rangle + \langle [D_{\tilde{\eta}} \Psi_{\eta\delta}, \Psi_t]_{\Psi_{\eta\tilde{\eta}}}^\eta, \Psi_{Q\tilde{\eta}} \rangle \\
&= \langle D_\eta \partial_t \Psi_\delta + \langle [\partial_t \Psi_{\eta\tilde{\eta}}, \Psi_\delta]_{\Psi_{\eta\tilde{\eta}}}^{\tilde{\eta}}, \Psi_{Q\tilde{\eta}} \rangle + \langle \Psi_{Q\tilde{\eta}}, [\Psi_{\eta t}, \Psi_{\eta\delta}]_{\Psi_{\eta\tilde{\eta}}}^\eta + [\Psi_t, D_{\tilde{\eta}} \Psi_{\eta\delta}]_{\Psi_{\eta\tilde{\eta}}}^\eta \rangle \\
&= \langle \partial_t \Psi_\delta, Q \Psi_{\eta\tilde{\eta}} \rangle + \langle \Psi_{Q\tilde{\eta}}, [D_\eta \Psi_{t\tilde{\eta}}, \Psi_\delta]_{\Psi_{\eta\tilde{\eta}}}^{\tilde{\eta}} + [\Psi_{\eta t}, \Psi_{\eta\delta}]_{\Psi_{\eta\tilde{\eta}}}^\eta + [\Psi_t, D_{\tilde{\eta}} \Psi_{\eta\delta}]_{\Psi_{\eta\tilde{\eta}}}^\eta \rangle.
\end{aligned}$$

From the third line to the fourth line, we used (E.8b) with (E.3c). If and only if the sum of these extra terms vanishes, the action has a topological t -dependence. However, (E.2b) provides the cancellation of these extra terms. Therefore, using $\varphi(0) = 0$ and $\varphi(1) = \varphi$, we obtain

$$\delta S_{\eta\tilde{\eta}}[\varphi] = \int_0^1 dt \left[(E.9) + (E.10) \right] = \langle \Psi_\delta[\varphi], Q \Psi_{\eta\tilde{\eta}}[\varphi] \rangle.$$

We proved that when we have WZW-like functional fields $\Psi_{\eta\tilde{\eta}}[\varphi]$ and $\Psi_d[\varphi]$ which satisfy (E.3b), our NS-NS action $S_{\eta\tilde{\eta}}[\varphi]$ has topological t -dependence: $\delta S_{\eta\tilde{\eta}}[\varphi] = \langle \Psi_\delta[\varphi], Q \Psi_{\eta\tilde{\eta}}[\varphi] \rangle$. As a result, this form of the variation of (E.4) ensures that it is invariant under (E.5) and the equations of motion is given by (E.6) because of (E.3b).

In the rest, we give two realisations of this action: small-space parametrisation $S_{\eta\tilde{\eta}}[\Phi]$ and large-space parametrisation $S_{\eta\tilde{\eta}}[\Psi]$. By construction, the L_∞ action proposed in [7] is equivalent to $S_{\eta\tilde{\eta}}[\Phi]$. Since $S_{\eta\tilde{\eta}}[\Psi]$ has the same (alternative) WZW-like structure as $S_{\eta\tilde{\eta}}[\Phi]$, our new actions $S_{\eta\tilde{\eta}}[\Phi]$ and $S_{\eta\tilde{\eta}}[\Psi]$ both are equivalent to that of L_∞ formulation.

E.3 Small-space parametrisation: $\varphi = \Phi$

Let Φ be a NS-NS dynamical string field belonging to the small Hilbert space, which is a Grassmann even, ghost number 2, left-moving picture number -1 , and right-moving picture number -1 state. We show that the pure-gauge-like field $\Psi_{\eta\tilde{\eta}}$ is given by

$$\Psi_{\eta\tilde{\eta}}(t) = \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}), \quad (E.11)$$

and the linear operator D_α for $\alpha = \eta, \tilde{\eta}$ becomes

$$D_\alpha = \pi_1 \mathbf{L}^\alpha (\mathbb{I} \wedge e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi})}), \quad (\alpha = \eta, \tilde{\eta}).$$

Then, the left associated field $\Psi_{d\tilde{\eta}}$ and the right associated field $\Psi_{\eta d}$ are given by

$$\begin{aligned}
\Psi_{d\tilde{\eta}}(t) &= \pi_1 \widehat{\mathbf{G}}(\xi_d e^{\wedge \Phi(t)}), \\
\Psi_{\eta d}(t) &= \pi_1 \widehat{\mathbf{G}}(\tilde{\xi}_d e^{\wedge \Phi(t)}),
\end{aligned}$$

respectively. The large associated field Ψ_d , which we call the large associated field, is given by

$$\Psi_d = \pi_1 \widehat{\mathbf{G}}(\xi \tilde{\xi} d \Phi(t) \wedge e^{\wedge \Phi(t)}) \quad (E.12)$$

Proofs of properties. Since the group-like element given by (E.11) satisfies

$$\mathbf{L}^\eta(e^{\pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})}) = (\widehat{\mathbf{G}} \boldsymbol{\eta} \widehat{\mathbf{G}}^{-1}) \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) = \widehat{\mathbf{G}} \boldsymbol{\eta}(e^{\wedge \Phi(t)}) = 0,$$

we obtain the desired equation $\pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta(t)}) = 0$, or equivalently,

$$\eta \Psi_{\eta\tilde{\eta}}(t) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \left[\overbrace{\Psi_{\eta\tilde{\eta}}(t), \dots, \Psi_{\eta\tilde{\eta}}(t)}^{n+1} \right]^\eta = 0.$$

Similarly, provided that $d\widehat{\mathbf{G}} = \widehat{\mathbf{G}} d$ and $[[d, \eta]] = 0$, after the following computation

$$\begin{aligned} (-)^X X \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) &= (-)^d \widehat{\mathbf{G}}(d\Phi(t) \wedge e^{\wedge \Phi(t)}) = \widehat{\mathbf{G}} \boldsymbol{\eta}(\xi_d \Phi(t) \wedge e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \widehat{\mathbf{G}}(\xi_d \Phi(t) \wedge e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \left(\pi_1 \widehat{\mathbf{G}}(\xi_d \Phi(t) \wedge e^{\wedge \Phi(t)}) \wedge e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})} \right), \end{aligned}$$

we find that the second Wess-Zumino-Witten-like relation holds:

$$(-)^d d \Psi_{\eta\tilde{\eta}}(t) = D_\eta(t) \Psi_{d\tilde{\eta}}(t).$$

Note that $[[Q, \eta]] = 0$ but $\mathbf{Q} \widehat{\mathbf{G}} = \widehat{\mathbf{G}} \mathbf{L}$. Utilizing the coderivation $\boldsymbol{\xi}_Q$ such that $\mathbf{L} = -[[\boldsymbol{\eta}, \boldsymbol{\xi}_Q]]$, we can check that the field

$$\Psi_{Q\tilde{\eta}}(t) = \pi_1 \widehat{\mathbf{G}}(\boldsymbol{\xi}_Q e^{\wedge \Phi(t)})$$

satisfies the second Wess-Zumino-Witten-like relation $-Q \Psi_{\eta\tilde{\eta}} = D_\eta \Psi_{Q\tilde{\eta}}$ as follows

$$\begin{aligned} -\mathbf{Q} \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) &= -\widehat{\mathbf{G}} \mathbf{L}(e^{\wedge \Phi(t)}) = \widehat{\mathbf{G}} \boldsymbol{\eta} \boldsymbol{\xi}_Q(e^{\wedge \Phi(t)}) = \mathbf{L}^\eta \widehat{\mathbf{G}} \boldsymbol{\xi}_Q(e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \left(\pi_1 \widehat{\mathbf{G}} \boldsymbol{\xi}_Q(e^{\wedge \Phi(t)}) \wedge e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})} \right). \end{aligned}$$

As the almost same way, one can check $\pi_1 \mathbf{L}^{\tilde{\eta}} e^{\wedge \Psi_{\eta\tilde{\eta}}} = 0$ and $(-)^d d \Psi_{\eta\tilde{\eta}} = D_{\tilde{\eta}} \Psi_{\eta d}$. Then, we find

$$[[D_\eta, D_{\tilde{\eta}}]] B = -\pi_1 \mathbf{L}^\eta(\pi_1 \mathbf{L}^{\tilde{\eta}}(e^{\Psi_{\eta\tilde{\eta}}}) \wedge B \wedge e^{\Psi_{\eta\tilde{\eta}}}) - \pi_1 \mathbf{L}^{\tilde{\eta}}(\pi_1 \mathbf{L}^\eta(e^{\Psi_{\eta\tilde{\eta}}}) \wedge B \wedge e^{\Psi_{\eta\tilde{\eta}}})$$

and thus, (E.12) gives an appropriate large associated (functional) field: (E.3c) holds.

E.4 Large-space parametrization: $\varphi = \Psi$

We write Ψ for a dynamical NS-NS string field which has ghost-and-picture numbers $(0|0, 0)$ and belongs to the left-large and right-large Hilbert space: $\eta\Psi \neq 0$, $\tilde{\eta}\Psi \neq 0$, and $\eta\tilde{\eta}\Psi \neq 0$. Let us consider the solution $\Psi_{\eta\tilde{\eta}}[\tau; \Psi]$ of the following differential equation,

$$\frac{\partial}{\partial \tau} \Psi_{\eta\tilde{\eta}}[\tau; \Psi] = D_\eta D_{\tilde{\eta}} \Psi \tag{E.13}$$

with the initial condition $\Psi_{\eta\tilde{\eta}}[\tau = 0; \Psi] = 0$, where for any state $A \in \mathcal{H}$ and for $\alpha = \eta, \tilde{\eta}$,

$$D_\alpha A \equiv \alpha A + \sum_{n=0}^{\infty} \frac{1}{n!} \left[\overbrace{\Psi_{\eta\tilde{\eta}}[\tau; \Psi], \dots, \Psi_{\eta\tilde{\eta}}[\tau; \Psi]}^n, A \right]^\alpha.$$

A pure-gauge-like (functional) field $\Psi_{\eta\tilde{\eta}}[\Psi]$ is obtained as the $\tau = 1$ value solution

$$\Psi_{\eta\tilde{\eta}}[\Psi] \equiv \Psi_{\eta\tilde{\eta}}[\tau = 1; \Psi].$$

One can check that this $\Psi_{\eta\tilde{\eta}}[\Psi]$ satisfies (E.3a) by the same (but double) way as NS theory, and then $\llbracket D_\eta, D_{\tilde{\eta}} \rrbracket A = -[\pi_1 \mathbf{L}^\eta e^{\Psi_{\eta\tilde{\eta}}}, A]_{\Psi_{\eta\tilde{\eta}}}^\eta - [\pi_1 \mathbf{L}^{\tilde{\eta}} e^{\Psi_{\eta\tilde{\eta}}}, A]_{\Psi_{\eta\tilde{\eta}}}^{\tilde{\eta}} = 0$ holds for any state $A \in \mathcal{H}$.

We consider the solutions $\Psi_{d\tilde{\eta}}[\tau; \Psi]$ and $\Psi_{\eta d}[\tau; \Psi]$ of the following differential equations

$$\frac{\partial}{\partial \tau} \Psi_{d\tilde{\eta}}[\tau; \Psi] = d D_{\tilde{\eta}} \Psi + [D_{\tilde{\eta}} \Psi, \Psi_{d\tilde{\eta}}[\tau; \Psi]]_{\Psi_{\eta\tilde{\eta}}[\tau; \Psi]}^\eta, \quad (\text{E.14a})$$

$$-\frac{\partial}{\partial \tau} \Psi_{\eta d}[\tau; \Psi] = d D_\eta \Psi + [D_\eta \Psi, \Psi_{\eta d}[\tau; \Psi]]_{\Psi_{\eta\tilde{\eta}}[\tau; \Psi]}^{\tilde{\eta}}, \quad (\text{E.14b})$$

with the initial conditions $\Psi_{d\tilde{\eta}}[\tau = 0; \Psi] = 0$ and $\Psi_{\eta d}[\tau = 0; \Psi] = 0$. Here, we used (E.7). Associated (functional) fields $\Psi_{d\tilde{\eta}}[\Psi]$ and $\Psi_{\eta d}[\Psi]$ are $\tau = 1$ values of these solutions,

$$\Psi_{d\tilde{\eta}}[\Psi] \equiv \Psi_{d\tilde{\eta}}[\tau = 1; \Psi], \quad \Psi_{\eta d}[\Psi] \equiv \Psi_{\eta d}[\tau = 1; \Psi].$$

One can also check these satisfy (E.3b) by the same way as NS theory. The minus sign of (E.14b) comes from the ordering of D_η and $D_{\tilde{\eta}}$ in the definition of (E.13).

Note that we can only specify the large associated (functional) field Ψ_d^* up to D_η - and $D_{\tilde{\eta}}$ -exact terms, and these ambiguities do not contribute in the action. The situation is parallel with that of the conventional WZW-like NS-NS theory [21]. Thus one may consider a differential equation defining Ψ_d^* up to D_η - and $D_{\tilde{\eta}}$ -exact terms by mimicking that of [21], but now we can take more economical way: we have operators $F\xi$ and $\tilde{F}\tilde{\xi}$ defined by

$$F\xi \equiv \sum_{n=0}^{\infty} [\xi(\eta - D_\eta)]^n \xi, \quad \tilde{F}\tilde{\xi} \equiv \sum_{n=0}^{\infty} [\tilde{\xi}(\tilde{\eta} - D_{\tilde{\eta}})]^n \tilde{\xi},$$

which satisfy $\llbracket F\xi, D_\eta \rrbracket = 1$ and $\llbracket \tilde{F}\tilde{\xi}, D_{\tilde{\eta}} \rrbracket = 1$, respectively. These $F\xi$ and $\tilde{F}\tilde{\xi}$ consist of the pure-gauge-like (functional) field $\Psi_{\eta\tilde{\eta}}[\Psi]$, which is already constructed, and operators ξ and $\tilde{\xi}$. Hence, using these pieces, one can quickly obtain the desired $\Psi_d = \Psi_d[\Psi]$ as follows,

$$\Psi_d[\Psi] \equiv F\xi \Psi_{d\tilde{\eta}}[\Psi] = -\tilde{F}\tilde{\xi} \Psi_{\eta d}[\Psi].$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] A. Sen, *Gauge invariant 1PI effective superstring field theory: inclusion of the Ramond sector*, *JHEP* **08** (2015) 025 [[arXiv:1501.00988](https://arxiv.org/abs/1501.00988)] [[INSPIRE](https://inspirehep.net/literature/1501009)].
- [2] H. Kunitomo and Y. Okawa, *Complete action for open superstring field theory*, *Prog. Theor. Exp. Phys.* **2016** (2016) 023B01 [[arXiv:1508.00366](https://arxiv.org/abs/1508.00366)] [[INSPIRE](https://inspirehep.net/literature/1508003)].

- [3] A. Sen, *BV master action for heterotic and type II string field theories*, *JHEP* **02** (2016) 087 [[arXiv:1508.05387](#)] [[INSPIRE](#)].
- [4] H. Matsunaga, *Comments on complete actions for open superstring field theory*, *JHEP* **11** (2016) 115 [[arXiv:1510.06023](#)] [[INSPIRE](#)].
- [5] A. Sen, *Covariant action for type IIB supergravity*, *JHEP* **07** (2016) 017 [[arXiv:1511.08220](#)] [[INSPIRE](#)].
- [6] T. Erler, S. Konopka and I. Sachs, *Resolving Witten's superstring field theory*, *JHEP* **04** (2014) 150 [[arXiv:1312.2948](#)] [[INSPIRE](#)].
- [7] T. Erler, S. Konopka and I. Sachs, *NS-NS sector of closed superstring field theory*, *JHEP* **08** (2014) 158 [[arXiv:1403.0940](#)] [[INSPIRE](#)].
- [8] T. Erler, S. Konopka and I. Sachs, *Ramond equations of motion in superstring field theory*, *JHEP* **11** (2015) 199 [[arXiv:1506.05774](#)] [[INSPIRE](#)].
- [9] S. Konopka, *The S-matrix of superstring field theory*, *JHEP* **11** (2015) 187 [[arXiv:1507.08250](#)] [[INSPIRE](#)].
- [10] E. Witten, *Interacting field theory of open superstrings*, *Nucl. Phys. B* **276** (1986) 291 [[INSPIRE](#)].
- [11] C. Wendt, *Scattering amplitudes and contact interactions in Witten's superstring field theory*, *Nucl. Phys. B* **314** (1989) 209 [[INSPIRE](#)].
- [12] I. Ya. Arefeva, P.B. Medvedev and A.P. Zubarev, *New representation for string field solves the consistency problem for open superstring field theory*, *Nucl. Phys. B* **341** (1990) 464 [[INSPIRE](#)].
- [13] C.R. Preitschopf, C.B. Thorn and S.A. Yost, *Superstring field theory*, *Nucl. Phys. B* **337** (1990) 363 [[INSPIRE](#)].
- [14] R. Saroja and A. Sen, *Picture changing operators in closed fermionic string field theory*, *Phys. Lett. B* **286** (1992) 256 [[hep-th/9202087](#)] [[INSPIRE](#)].
- [15] B. Jurčo and K. Muenster, *Type II superstring field theory: geometric approach and operadic description*, *JHEP* **04** (2013) 126 [[arXiv:1303.2323](#)] [[INSPIRE](#)].
- [16] N. Berkovits, *Super-Poincaré invariant superstring field theory*, *Nucl. Phys. B* **450** (1995) 90 [*Erratum ibid.* **B 459** (1996) 439] [[hep-th/9503099](#)] [[INSPIRE](#)].
- [17] N. Berkovits, *A new approach to superstring field theory*, *Fortsch. Phys.* **48** (2000) 31 [[hep-th/9912121](#)] [[INSPIRE](#)].
- [18] Y. Okawa and B. Zwiebach, *Heterotic string field theory*, *JHEP* **07** (2004) 042 [[hep-th/0406212](#)] [[INSPIRE](#)].
- [19] N. Berkovits, Y. Okawa and B. Zwiebach, *WZW-like action for heterotic string field theory*, *JHEP* **11** (2004) 038 [[hep-th/0409018](#)] [[INSPIRE](#)].
- [20] H. Matsunaga, *Construction of a gauge-invariant action for type II superstring field theory*, [arXiv:1305.3893](#) [[INSPIRE](#)].
- [21] H. Matsunaga, *Nonlinear gauge invariance and WZW-like action for NS-NS superstring field theory*, *JHEP* **09** (2015) 011 [[arXiv:1407.8485](#)] [[INSPIRE](#)].
- [22] N. Berkovits and C.T. Echevarria, *Four point amplitude from open superstring field theory*, *Phys. Lett. B* **478** (2000) 343 [[hep-th/9912120](#)] [[INSPIRE](#)].

- [23] Y. Iimori, T. Noumi, Y. Okawa and S. Torii, *From the Berkovits formulation to the Witten formulation in open superstring field theory*, *JHEP* **03** (2014) 044 [[arXiv:1312.1677](#)] [[INSPIRE](#)].
- [24] N. Berkovits, *The Ramond sector of open superstring field theory*, *JHEP* **11** (2001) 047 [[hep-th/0109100](#)] [[INSPIRE](#)].
- [25] Y. Michishita, *A covariant action with a constraint and Feynman rules for fermions in open superstring field theory*, *JHEP* **01** (2005) 012 [[hep-th/0412215](#)] [[INSPIRE](#)].
- [26] H. Kunitomo, *The Ramond sector of heterotic string field theory*, *Prog. Theor. Exp. Phys.* **2014** (2014) 043B01 [[arXiv:1312.7197](#)] [[INSPIRE](#)].
- [27] H. Kunitomo, *First-order equations of motion for heterotic string field theory*, *Prog. Theor. Exp. Phys.* **2014** (2014) 093B07 [[arXiv:1407.0801](#)] [[INSPIRE](#)].
- [28] H. Kunitomo, *Symmetries and Feynman rules for the Ramond sector in open superstring field theory*, *Prog. Theor. Exp. Phys.* **2015** (2015) 033B11 [[arXiv:1412.5281](#)] [[INSPIRE](#)].
- [29] T. Erler, Y. Okawa and T. Takezaki, *A_∞ structure from the Berkovits formulation of open superstring field theory*, [arXiv:1505.01659](#) [[INSPIRE](#)].
- [30] T. Erler, *Relating Berkovits and A_∞ superstring field theories; small Hilbert space perspective*, *JHEP* **10** (2015) 157 [[arXiv:1505.02069](#)] [[INSPIRE](#)].
- [31] T. Erler, *Relating Berkovits and A_∞ superstring field theories; large Hilbert space perspective*, *JHEP* **02** (2016) 121 [[arXiv:1510.00364](#)] [[INSPIRE](#)].
- [32] K. Goto and H. Matsunaga, *On-shell equivalence of two formulations for superstring field theory*, [arXiv:1506.06657](#) [[INSPIRE](#)].
- [33] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys. B* **271** (1986) 93 [[INSPIRE](#)].
- [34] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys. B* **268** (1986) 253 [[INSPIRE](#)].
- [35] M.R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories. 1: foundations*, *Nucl. Phys. B* **505** (1997) 569 [[hep-th/9705038](#)] [[INSPIRE](#)].
- [36] B. Zwiebach, *Closed string field theory: quantum action and the B-V master equation*, *Nucl. Phys. B* **390** (1993) 33 [[hep-th/9206084](#)] [[INSPIRE](#)].
- [37] I.A. Batalin and G.A. Vilkovisky, *Gauge algebra and quantization*, *Phys. Lett. B* **102** (1981) 27 [[INSPIRE](#)].
- [38] I.A. Batalin and G.A. Vilkovisky, *Quantization of gauge theories with linearly dependent generators*, *Phys. Rev. D* **28** (1983) 2567 [Erratum *ibid.* **D 30** (1984) 508] [[INSPIRE](#)].
- [39] J. Gomis, J. Paris and S. Samuel, *Antibracket, antifields and gauge theory quantization*, *Phys. Rept.* **259** (1995) 1 [[hep-th/9412228](#)] [[INSPIRE](#)].
- [40] T. Lada and J. Stasheff, *Introduction to SH Lie algebras for physicists*, *Int. J. Theor. Phys.* **32** (1993) 1087 [[hep-th/9209099](#)] [[INSPIRE](#)].
- [41] H. Kajiura, *Noncommutative homotopy algebras associated with open strings*, *Rev. Math. Phys.* **19** (2007) 1 [[math/0306332](#)] [[INSPIRE](#)].
- [42] E. Getzler, *Lie theory for nilpotent L_∞ -algebras*, [math/0404003](#).