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Cohomological Topics in Group Theory

Comments and Corrections

p. 88, end of §6.1. Add the sentence "This construction is independent of the chosen I for A".

p. 119. Chapter 8 has a natural continuation in Chapter 2 of Robert Bieri's Queen Mary College Mathematics Notes *Homological Dimension of Discrete Groups*, 2nd. edition (1981).

p. 121, line −6. After "all A" insert "and all $q > k$".

p. 129. A theorem by C.T.C. Wall and J.-P. Serre gives a cohomological condition that implies conjugacy conclusions on the finite subgroups. This gives an attractive proof of an old result about one-relator groups: *if G is one-relator with maximal finite cyclic subgroup C, then any finite subgroup of G is conjugate to a subgroup of C.* Neither Serre nor Wall published the result; but a recent good account is in the Appendix (p. 597) by C. Scheiderer to the paper by P. Lochak and L. Schneps in Inventiones 127 (1997).

p. 144. In exercise 5, line 2, before "prove that" add the phrase "and $n > 0$ or at least one $s_i = \infty$, or $m > 3$".
There is much more about polyhedral groups (Fuchsian groups of genus 0) in L.L. Scott *Matrices and cohomology*, Annals Math. 105 (1977) 473–492.

p. 155. The strict inequality ($\star$) is always true: cf. Bieri's Queen Mary College Mathematics Notes §8.3 (p. 120).

p. 173, Exercise. Magnus' result has very recently been generalised to its best possible form by J.S. Wilson (*On growth of groups with few relators*, Bull. London Math. Soc. 36 (2004) 1–2), based on earlier work of N.S. Romanovskii. The result is the following: *If G has a presentation with $r + s$ generators and r relations with $s > 0$, and S is an arbitrary generating set of G, then some subset of s elements of S freely generates a free subgroup of G.*

p. 175, end of §8.11. For more examples of groups with trivial cohomological dimension 0, cf. P. de la Harpe and D. McDuff in *Acyclic groups of automorphisms*, Comment. Math. Helvetici 58 (1983) 48–71. (They call a group G acyclic if $tcd\, G = 0$.)

p. 202. In line 3 of the definition add "and $A_1 \neq 0$ ".
Comment: $(0,0)$ is a projective pair in $\mathcal{Q}_G$ if, and only if, G is a free group.

p. 212, line −5. " ...and $(A, co(A|E))$ is ... "

p. 216, Problem. $\mathbb{Q}$ has no minimal projectives. Cf. J.S. Williams *Nielsen equivalence of presentations of some solvable groups*, Math. Z. 137 (1974) 351–362.

p. 227. In the definition of Heller module, it is better to allow KG-projectives to be Heller modules.
Proposition 4 is incorrect as it stands. KG must be assumed to have the following further property: if U, V are KG-lattices, then $(U \oplus V)' \simeq U' \oplus V'$. (If we allow KG-projectives to be Heller modules, then the conclusion of Proposition 4 is just that C is a Heller module if, and only if, C' is indecomposable.)
Comment. That the first property of KG (in Proposition 4) is insufficient for the conclusion is due to the existence of groups G with minimal relation modules $\bar{R}$ having the property that $\bar{R}^{(n)}$ has $\mathbb{Z}G$ as a direct summand for some $n > 1$. If n is the smallest such number and $K = \mathbb{Z}_{(G)}$, then $U = \bar{R}^{(n-1)}_{(G)}$ and $V = \bar{R}_{(G)}$ have $U' = U$ and $V' = V$, but $(U \oplus V)' < U \oplus V$. Cf. J.S. Williams *Trace ideals of relation modules of finite groups*, Math. Z. 163 (1978) 261–274.

p. 228, Proposition 5. The proof really gives a bit more: *C is a Heller module if, and only if, A is a Heller module.*

p. 229. Misprint in line 7: $KG \simeq K^*$

p. 230. Replace "Problem" with: "Note that $\mathbb{Z}$ is a Heller module if, and only if, $\mathbb{Z}_{(G)}$ is a Heller module."
Comment. The question which groups have $\mathbb{Z}$ as Heller module has led to a substantial body of work. It is now known (modulo CFSG) that $\mathbb{Z}$ is a Heller module for G if, and only if, the prime graph of G is connected: cf. J.S. Williams *Prime graph components of finite groups*, J. Alg. 69 (1981) 487–513. My survey in London Math. Soc. Lecture Note Series 36 (1979) discusses the links between prime graphs and Heller properties. These and related questions were taken impressively further (for not necessarily finite groups) by P.A. Linnell in *Decomposition of augmentation ideals and relation modules*, Proc. London Math. Soc. 47 (1983) 83–127. In another direction, the representation-theoretic significance of Heller-type properties for general lattices is studied by S.N. Aloneftis in *Decomposition modulo projectives of lattices over finite groups*, J. Alg. 223 (2000) 1–14.

p. 237. After line 1 add: "Strictly, $K_{(G)}$ is undefined if $|G|$ is a unit in K. When this is so, set $K_{(G)}$ to be the field of fractions of K."

p. 240. Definition should read: "A ring L is called semi-local if $L/Jac(L)$ has descending chain condition on right ideals (and so is semi-simple)".

Exercise 1 should read: "The commutative ring L is semi-local if, and only if, L has only a finite number of maximal ideals."

p. 242. The last sentence of the Corollary should read: "Then KG has the first of the two properties required in Proposition 4 of §10.2 (p. 227). (Cf. the comment above for p. 227.)"

p. 243. Delete the line immediately before equation (3) (i.e., "and only if").

p. 252. For Theorem 1 one needs to know that minimal projectives exist. They do: this is a consequence of the fact that if KG is semi-perfect, then every finitely generated KG-module satisfies the ascending chain condition on projective direct summands (use (i) KG/J is semi-simple, where $J = Jac(KG)$, and (ii) if P is KG-projective, then $P/PJ \neq 0$; cf. Bass' paper cited on p. 274).

p. 258, line 4. A should be A^G.

p. 264, line 4. Theorem 5 does not remain true over $\mathbb{Z}$. Here is an almost trivial example: Let $G = \mathbb{Z}/p\mathbb{Z}$, where p is a prime > 3; let $\varphi, \psi : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ be the homomorphisms $\varphi(1) = 1$, $\psi(1) = 2$. Both give minimal projective objects but they are not isomorphic.
On a distinctly non-trivial level, modules in minimal projectives need not be isomorphic. This was first proved by Dyer and Sieradski (J. Pure and Appl. Algebra 15 (1979) 199–217) and refined by P.J. Webb in *The minimal relation modules of a finite abelian group*, J. Pure and Appl. Algebra 21 (1981) 205–232. This paper is also relevant for Theorem 6 on this page.

p. 268, last line. This soon ceased to be an open problem. A free extension is minimal if, and only if, the relation module is core-equal (has no non-zero projective direct summand). This is known for large classes of groups (cf. my article in London Math. Soc. Lecture Note Series 36, 1979).

p. 270. In the statement of Theorem 9 delete "KG has the property of Proposition 4 of §10.2 (p. 227) and"
In the paragraph following Theorem 9 keep sentence 2 and delete the rest.
In the proof of Theorem 9 delete the first paragraph and retain the rest.

p. 271. The Corollary is false (since $\mathbb{Z}_{(G)}$ need not be Heller). This has led to a series of papers (and more errors! - but all finally resolved satisfactorily):

1. *Decomposition of the augmentation ideal and of the relation modules of a finite group*, Proc. London Math. Soc. 31 (1975) 149–166;
2. *Decomposition of the relation modules of a finite group*, J. London Math. Soc. 12 (1976) 262–266;
3. *The decomposition of relation modules: a corrrection*, Proc. London Math. Soc. 45 (1982) 89–96.

   These 3 papers are by K.W. Gruenberg and K.W. Roggenkamp.

4. R.M. Guralnick and W. Kimmerle *On the cohomology of alternating and symmetric groups and decomposition of relation modules*, J. Pure and Appl. Algebra 69 (1990) 135–140.

p. 272. In the statement of Theorem 10, delete the reference to Proposition 4, §10.2.

p. 273, line 3. Minimal projectives need not be isomorphic: cf. the comments (above) to p. 264.
Concerning the various questions raised on this page, I mention two surveys of the theory published a few years after publication of this book: K.W. Roggenkamp *Integral representations and presentations of finite groups* Lecture Notes in Mathematics 744 (Springer) 1979; and my article in London Math. Soc. Lecture Note Series 36 (1979).
In the theorem stated at the end of the page, it is implicitly assumed that G is a p-group. This is a very special case of the main result of my paper in Math. Z. 118 (1970) 30–33 (mentioned at the end of the comments on p. 274).