A SOLUTION

An Exemple as requested in the problem of R. Thom, D 4c

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Consider in the plane $\mathbb{R}^2$ with euclidean metric $ds^2 = dx^2 + dy^2$ and unit sphere $x^2 + y^2 = 1$ the polynomial function $f = x(2x^2 + 3ty^2)$, $t > 0$, with covectorfield $df = 3[(2x^2 + ty^2)dx + 2txydy]$. The set of periods at which df is linearly dependent on $d(x^2 + y^2) = 2[xdx + ydy]$ is given by $(2x^2 + ty^2)y - 2txy.x = y[ty^2 - 2(t-1)x^2] = 0$. Then it is easily seen that the set of points that converge to $(0, 0)$ when following the gradientline of f downwards is:
for $0 < t \leqq 1$, the half line $\{(x, y): x \geqq 0, \ y = 0\}$
and for $t > 1$, the solid angle $\{(x, y): x \geqq 0, \ ty^2 - 2(t-1)x^2 \leqq 0\}$.
With the substitution $X = x$, $Y = \sqrt{t}\, y$ we can conclude:
The set of points in the (X, Y)-plane that converge to $(0, 0)$ downwards when following the gradient lines of the function $X(2X^2 + 3Y^2)$ with respect to the euclidean metric $ds^2 = tdX^2 + dY^2$ with unit sphere $tX^2 + Y^2 = 1$, is:
for $0 < t \leqq 1$, the half line $\{(X, Y): X \geqq 0, \ Y = 0\}$,
and for $t > 1$, the solid angle $\{(X, Y): X \geqq 0, \ Y^2 - 2(t-1)X^2 \leqq 0\}$.
Hence the topological type of this set varies with the metric depending on t.