# Optimal Estimation of Three-Dimensional Rotation and Reliability Evaluation*

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**Abstract.** We discuss optimal rotation estimation from two sets of 3-D points in the presence of anisotropic and inhomogeneous noise. We first present a theoretical accuracy bound and then give a method that attains that bound, which can be viewed as describing the reliability of the solution. We also show that an efficient computational scheme can be obtained by using quaternions and applying renormalization. Using real stereo images for 3-D reconstruction, we demonstrate that our method is superior to the least-squares method and confirm the theoretical predictions of our theory by applying the bootstrap procedure.

## 1 Introduction

Determining a rotational relationship between two sets of 3-D points is an important task for 3-D object reconstruction and recognition. For example, if we use stereo vision or range sensing, the 3-D shape can be reconstructed only for visible surfaces. Hence, we need to fuse separately reconstructed surfaces into one object [13]. For this task, we need to determine the rigid transformation between two sets of points. If one set is translated so that its centroid coincides with that of the other, the problem reduces to estimating a rotation.

Let $\{\boldsymbol{r}_\alpha\}$ and $\{\boldsymbol{r}'_\alpha\}$, $\alpha = 1, ..., N$, be the sets of three-dimensional vectors before and after a rotation, respectively. A conventional method for determining the rotation is the following least squares method:

$$\sum_{\alpha=1}^{N} \|\boldsymbol{r}'_\alpha - \boldsymbol{R}\boldsymbol{r}_\alpha\|^2 \to \min. \tag{1}$$

In this paper, $\|\boldsymbol{a}\|$ denotes the norm of a vector $\boldsymbol{a}$.

The solution of the minimization (1) can be obtained analytically: Horn [3] proposed a method using quaternions; Arun et al. [1] used the singular value decomposition; Horn et al. [4] used the polar decomposition. The method of Horn [3] is guaranteed to yield a rotation matrix, while the methods of Arun et al. [1] and Horn et al. [4] may yield an orthogonal matrix of determinant $-1$.

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This drawback was later remedied by Umeyama [14] by introducing a Lagrange multiplier for that constraint; Kanatani [8] restated it from a group-theoretical viewpoint.

From a statistical point of view, the above least-squares method implicitly assumes the following noise model:

- Points $\{\boldsymbol{r}_\alpha\}$ are observed without noise, while the rotated points $\{\boldsymbol{R}\boldsymbol{r}_\alpha\}$ are observed with noise $\{\Delta\boldsymbol{r}'_\alpha\}$.
- The noise $\{\Delta\boldsymbol{r}'_\alpha\}$ is subject to an isotropic, identical, and independent Gaussian distribution of zero mean.

The least-squares solution is optimal for this model. However, this model is not realistic in many situations:

- The noise is often neither isotropic nor identical [10, 11]. 3-D points measured by stereo vision or range sensing usually have errors that are very large along the viewing direction as compared with the directions perpendicular to it. In addition, the noise characteristics differ from point to point; usually, points near the sensor are more accurate than points far away.
- The 3-D points in both sets suffer noise. If the 3-D points are measured by a sensor before and after a rotation, it is unreasonable to assume that noise exists only in one set.

In this paper, we first introduce a realistic noise model and present a *theoretical accuracy bound*, which can be evaluated *independently* of particular solution techniques involved. Then, we describe an estimation method that attains the accuracy bound; such a method alone can be called "optimal".

Since the solution attains the accuracy bound, we can view it as quantitatively describing the reliability of the solution; in the past, the reliability issue has attracted little attention.

The optimal method turns out to be highly nonlinear. However, we show that an efficient computational scheme can be obtained by using quanternions and applying the *renormalization* technique proposed by Kanatani [9]. Using real stereo images for 3-D reconstruction, we demonstrate that our method is indeed superior to the least-squares method and confirm the theoretical predictions of our theory by applying the bootstrap procedure [2].

## 2 Noise Model

Let $\bar{\boldsymbol{r}}_\alpha$ and $\bar{\boldsymbol{r}}'_\alpha$, $\alpha = 1, ..., N$, denote the true 3-D positions before and after a rotation, respectively, and let $\boldsymbol{r}_\alpha$ and $\boldsymbol{r}'_\alpha$ be their respective positions observed in the presense of noise. We write

$$\boldsymbol{r}_\alpha = \bar{\boldsymbol{r}}_\alpha + \Delta\boldsymbol{r}_\alpha, \qquad \boldsymbol{r}'_\alpha = \bar{\boldsymbol{r}}'_\alpha + \Delta\boldsymbol{r}'_\alpha, \tag{2}$$

and assume that $\Delta\boldsymbol{r}_\alpha$ and $\Delta\boldsymbol{r}'_\alpha$ are independent Gaussian random variables of mean zero. Their covariance matrices are defined by

$$V[\boldsymbol{r}_\alpha] = E[\Delta\boldsymbol{r}_\alpha\Delta\boldsymbol{r}_\alpha^\top], \qquad V[\boldsymbol{r}'_\alpha] = E[\Delta\boldsymbol{r}'_\alpha\Delta\boldsymbol{r}'^\top_\alpha], \tag{3}$$

where $E[\cdot]$ denotes expectation and the superscript $\top$ denotes transpose. The problem is formally stated as follows:

*Problem 1.* Estimate the rotation matrix $\boldsymbol{R}$ that satisfies

$$\bar{\boldsymbol{r}}'_\alpha = \boldsymbol{R}\bar{\boldsymbol{r}}_\alpha, \qquad \alpha = 1, ..., N, \tag{4}$$

from the noisy data $\{\boldsymbol{r}_\alpha\}$ and $\{\boldsymbol{r}'_\alpha\}$.

In practice, it is often very difficult to predict the covariance matrices $V[\boldsymbol{r}_\alpha]$ and $V[\boldsymbol{r}'_\alpha]$ precisely. In many cases, however, we can estimate their relative scales. If the 3-D positions are computed by stereo vision for example, the distribution of errors can be computed up to scale from the geometry of the camera configuration [10]. In view of this, we decompose the covariance matrices into an unknown constant $\epsilon$ and known matrices $V_0[\boldsymbol{r}_\alpha]$ and $V_0[\boldsymbol{r}'_\alpha]$ in the form

$$V[\boldsymbol{r}_\alpha] = \epsilon^2 V_0[\boldsymbol{r}_\alpha], \qquad V[\boldsymbol{r}'_\alpha] = \epsilon^2 V_0[\boldsymbol{r}'_\alpha]. \tag{5}$$

We call $\epsilon$ the *noise level*, and $V_0[\boldsymbol{r}_\alpha]$ and $V_0[\boldsymbol{r}'_\alpha]$ the *normalized covariance matrices*.

## 3 Theoretical Accuracy Bound

The reliability of an estimator can be evaluated by its covariance matrix if the set of parameters to be estimated can be identified with a point in a Euclidean space. However, a rotation is an element of the group of rotations $SO(3)$, which is a three-dimensional Lie group. Hence, we cannot define the covariance matrix of a rotation in the usual sense.

Let $\hat{\boldsymbol{R}}$ be an estimator of the true rotation $\bar{\boldsymbol{R}}$. Let $\boldsymbol{l}_r$ and $\Delta\Omega$ be, respectively, the axis (unit vector) and the angle of the relative rotation $\hat{\boldsymbol{R}}\bar{\boldsymbol{R}}^\top$. We define a three-dimensional vector

$$\Delta\boldsymbol{\Omega} = \Delta\Omega \boldsymbol{l}_r, \tag{6}$$

and regard this as the measure of deviation of the estimator $\hat{\boldsymbol{R}}$ from the true rotation $\bar{\boldsymbol{R}}$. We define the covariance matrix of $\hat{\boldsymbol{R}}$ by

$$V[\hat{\boldsymbol{R}}] = E[\Delta\boldsymbol{\Omega}\Delta\boldsymbol{\Omega}^\top]. \tag{7}$$

The group of rotations $SO(3)$ has the topology of the three-dimensional projective space $P^3$, which is locally homeomorphic to a 3-sphere $S^3$ [6]. If the noise is small, the deviation $\Delta\boldsymbol{\Omega}$ is also small and identified with an element of the Lie algebra $so(3)$, which can be viewed as a Euclidean space. This is equivalent to regarding errors as occurring in the tangent space to the 3-sphere $S^3$ at $\bar{\boldsymbol{R}}$.

With this definition of the covariance matrix, we can apply the theory of Kanatani (Sect. 14.4.3 of [9]) to obtain a theoretical accuracy bound, which he called the *Cramer-Rao lower bound* in analogy with the corresponding bound in traditional statistics. In the present case, it reduces to

$$V[\hat{\boldsymbol{R}}] \succ \epsilon^2 \left( \sum_{\alpha=1}^{N} (\bar{\boldsymbol{R}}\bar{\boldsymbol{r}}_\alpha) \times \bar{\boldsymbol{W}}_\alpha \times (\bar{\boldsymbol{R}}\bar{\boldsymbol{r}}_\alpha) \right)^{-1}, \tag{8}$$

$$\bar{\boldsymbol{W}}_\alpha = \left( \bar{\boldsymbol{R}} V_0[\boldsymbol{r}_\alpha] \bar{\boldsymbol{R}}^\top + V_0[\boldsymbol{r}'_\alpha] \right)^{-1}. \tag{9}$$

Here, $\boldsymbol{A} \succ \boldsymbol{B}$ means that $\boldsymbol{A} - \boldsymbol{B}$ is a positive semi-definite symmetric matrix. The product $\boldsymbol{v} \times \boldsymbol{A} \times \boldsymbol{v}$ of a vector $\boldsymbol{u} = (u_i)$ and a matrix $\boldsymbol{A} = (A_{ij})$ is the matrix whose $(ij)$ element is $\sum_{k,l,m,n=1}^{3} \varepsilon_{ikl}\varepsilon_{jmn} v_k v_m A_{ln}$, where $\varepsilon_{ijk}$ is the *Eddington epsilon*, taking 1 when $(ijk)$ is an even permutation of (123), $-1$ when it is an odd permutation of (123), and 0 otherwise.

If the noise is isotropic and identical, we have $V_0[\boldsymbol{r}_\alpha] = V_0[\boldsymbol{r}'_\alpha] = \boldsymbol{I}$ (unit matrix). In this case, Eq. (8) corresponds to the result obtained by Oliensis [12].

## 4 Optimal Estimation

Applying the general theory of Kanatani [9] (Sect. 14.5.2 of [9]), we can obtain a computational scheme for solving Problem 1 in such a way that the resulting solution attains the accuracy bound (8) in the first order (i.e., ignoring terms of $O(\epsilon^4)$): we minimize the sum of squared *Mahalanobis distances*, i.e.,

$$J = \sum_{\alpha=1}^{N} (\boldsymbol{r}_\alpha - \bar{\boldsymbol{r}}_\alpha, V_0[\boldsymbol{r}_\alpha]^{-1}(\boldsymbol{r}_\alpha - \bar{\boldsymbol{r}}_\alpha)) + \sum_{\alpha=1}^{N} (\boldsymbol{r}'_\alpha - \bar{\boldsymbol{r}}'_\alpha, V_0[\boldsymbol{r}'_\alpha]^{-1}(\boldsymbol{r}'_\alpha - \bar{\boldsymbol{r}}'_\alpha)) \to \min, \tag{10}$$

subject to the constraint (4). Throughout this paper, $(\boldsymbol{a}, \boldsymbol{b})$ denotes the inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Note that Eq. (10) involves the normalized covariance matrices $V_0[\boldsymbol{r}_\alpha]$ and $V_0[\boldsymbol{r}'_\alpha]$ alone; no knowledge of the noise level $\epsilon$ is required.

If $\boldsymbol{R}$ is fixed, the values of $\bar{\boldsymbol{r}}_\alpha$ and $\bar{\boldsymbol{r}}'_\alpha$ that minimize $J$ subject to Eq. (4) can be obtained analytically. Introducing Lagrange multipliers, we obtain

$$\bar{\boldsymbol{r}}_\alpha = \boldsymbol{r}_\alpha + V_0[\boldsymbol{r}_\alpha]\boldsymbol{R}^\top \boldsymbol{W}_\alpha (\boldsymbol{r}'_\alpha - \boldsymbol{R}\boldsymbol{r}_\alpha), \tag{11}$$

$$\bar{\boldsymbol{r}}'_\alpha = \boldsymbol{r}'_\alpha - V_0[\boldsymbol{r}'_\alpha] \boldsymbol{W}_\alpha (\boldsymbol{r}'_\alpha - \boldsymbol{R}\boldsymbol{r}_\alpha), \tag{12}$$

where

$$\boldsymbol{W}_\alpha = \left( \boldsymbol{R} V_0[\boldsymbol{r}_\alpha] \boldsymbol{R}^\top + V_0[\boldsymbol{r}'_\alpha] \right)^{-1}. \tag{13}$$

The resulting minimum is then minimized with respect to $\boldsymbol{R}$. If Eqs. (11) and (12) are substituted into Eq. (10), the problem reduces to

$$J = \sum_{\alpha=1}^{N} (\boldsymbol{r}'_\alpha - \boldsymbol{R}\boldsymbol{r}_\alpha, \boldsymbol{W}_\alpha (\boldsymbol{r}'_\alpha - \boldsymbol{R}\boldsymbol{r}_\alpha)) \to \min. \tag{14}$$

If $V_0[\boldsymbol{r}_\alpha] = V_0[\boldsymbol{r}'_\alpha] = \boldsymbol{I}$, Eq. (14) reduces to Eq. (1). This proves that the least-squares method (1) is optimal for isotropic and identical noise even if $\boldsymbol{r}_\alpha$ and $\boldsymbol{r}'_\alpha$ both contain noise. This corresponds to the result of Goryn and Hein [5].

## 5 Reliability of the Solution

The unknown noise level $\epsilon$ can be estimated *a posteriori*. Let $\hat{J}$ be the *residual*, i.e., the minimum of $J$. Since $\hat{J}/\epsilon^2$ is subject to a $\chi^2$ distribution with $3(N-1)$ degrees of freedom in the first order (Sect. 7.1.4 of [9]), we obtain an unbiased estimator of the squared noise level $\epsilon^2$ in the following form:

$$\hat{\epsilon}^2 = \frac{\hat{J}}{3(N-1)}. \tag{15}$$

Because the solution $\hat{\boldsymbol{R}}$ of (14) attains the accuracy bound (8) in the first order, we can evaluate its covariance matrix $V[\hat{\boldsymbol{R}}]$ by optimally estimating the true positions $\{\bar{\boldsymbol{r}}_\alpha\}$ (we discuss this in Section 9) and substituting the solution $\hat{\boldsymbol{R}}$ and the estimator (15) for their true values $\bar{\boldsymbol{R}}$ and $\epsilon^2$ in Eq. (8).

The minimization (14) must be conducted subject to the constraint that $\boldsymbol{R}$ be a rotation matrix. This means we need to parameterize $\boldsymbol{R}$ appropriately and do numerical search in the parameter space. Such a technique is often inefficient. Kanatani [9] proposed an efficient computational scheme called *renormalization* for maximum likelihood estimation with *linear* constraints. Here, the constraint is nonlinear, so Kanatani's technique cannot be applied directly. However, we can show that the constraint can be converted into a *linear* equation in terms of *quaternions*.

## 6 Quaternion Optimization

Consider a rotation by angle $\Omega$ around axis $\boldsymbol{l}$ (unit vector). Define a scalar $q_0$ and a three-dimensional vector $\boldsymbol{q}_l$ by

$$q_0 = \cos\frac{\Omega}{2}, \qquad \boldsymbol{q}_l = \boldsymbol{l}\sin\frac{\Omega}{2}. \tag{16}$$

Note that $q_0^2 + \|\boldsymbol{q}_l\|^2 = 1$ by definition. Conversely, a scalar $q_0$ and a three-dimensional vector $\boldsymbol{q}_l$ such that $q_0^2 + \|\boldsymbol{q}_l\|^2 = 1$ uniquely determine a rotation $\boldsymbol{R}$ around axis $\boldsymbol{l}$ by angle $\Omega$ $(0 \le \Omega < \pi)$ in the form

$$\boldsymbol{R} = (q_0^2 - \|\boldsymbol{q}_l\|^2)\boldsymbol{I} + 2(\boldsymbol{q}_l\boldsymbol{q}_l^\top + q_0\boldsymbol{q}_l \times \boldsymbol{I}), \tag{17}$$

where the product $\boldsymbol{a} \times \boldsymbol{A}$ of a vector $\boldsymbol{a}$ and a matrix $\boldsymbol{A}$ is the matrix whose columns are vector products of $\boldsymbol{a}$ and the corresponding columns of $\boldsymbol{A}$. Hence, a rotation is uniquely represented by a pair $\{q_0, \boldsymbol{q}_l\}$, which is called a *quaternion* [6].

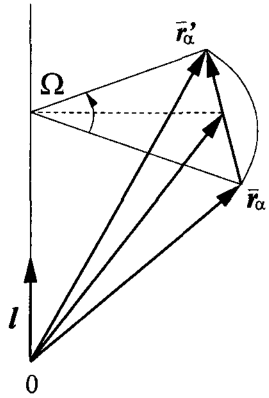



**Fig. 1.** 3-D rotation by angle $\Omega$ around axis $l$.

Suppose a point $\bar{\boldsymbol{r}}_\alpha$ undergoes a rotation $\boldsymbol{R}$ by angle $\Omega$ around axis $\boldsymbol{l}$ and moves to a new position $\bar{\boldsymbol{r}}'_\alpha$. From Fig. 1, we can see that the displacement $\bar{\boldsymbol{r}}'_\alpha - \bar{\boldsymbol{r}}_\alpha$ and the midpoint $(\bar{\boldsymbol{r}}_\alpha + \bar{\boldsymbol{r}}'_\alpha)/2$ are related by

$$\bar{\boldsymbol{r}}'_\alpha - \bar{\boldsymbol{r}}_\alpha = 2\tan\frac{\Omega}{2}\boldsymbol{l} \times \frac{\bar{\boldsymbol{r}}_\alpha + \bar{\boldsymbol{r}}'_\alpha}{2}. \tag{18}$$

Solving this for $\bar{\boldsymbol{r}}'_\alpha$ in terms of $\bar{\boldsymbol{r}}_\alpha$, we can obtain a relation equivalent to Eq. (4) expressed in terms of the angle $\Omega$ and axis $\boldsymbol{l}$ of rotation $\boldsymbol{R}$. Hence, Eq. (18) is equivalent to Eq. (4). Multiplying Eq. (18) by $\cos(\Omega/2)$ on both sides, we obtain after some manipulations

$$q_0(\bar{\boldsymbol{r}}'_\alpha - \bar{\boldsymbol{r}}_\alpha) + (\bar{\boldsymbol{r}}'_\alpha + \bar{\boldsymbol{r}}_\alpha) \times \boldsymbol{q}_l = \boldsymbol{0}. \tag{19}$$

Define a $3 \times 4$ matrix $\boldsymbol{X}_\alpha$ and a four-dimensional unit vector $\boldsymbol{q}$ by

$$\boldsymbol{X}_\alpha = \Big(\boldsymbol{r}'_\alpha - \boldsymbol{r}_\alpha \;\; (\boldsymbol{r}'_\alpha + \boldsymbol{r}_\alpha) \times \boldsymbol{I}\Big), \qquad \boldsymbol{q} = \begin{pmatrix} q_0 \\ \boldsymbol{q}_l \end{pmatrix}. \tag{20}$$

Let $\bar{\boldsymbol{X}}_\alpha$ be the value of $\boldsymbol{X}_\alpha$ obtained by replacing $\boldsymbol{r}_\alpha$ and $\boldsymbol{r}'_\alpha$ by $\bar{\boldsymbol{r}}_\alpha$ and $\bar{\boldsymbol{r}}'_\alpha$, respectively, in the first of Eqs. (20). Then, Eq. (19) can be expressed as a *linear* equation in $\boldsymbol{q}$ in the form

$$\bar{\boldsymbol{X}}_\alpha \boldsymbol{q} = \boldsymbol{0}. \tag{21}$$

Now the problem is to minimize Eq. (10) subject to the constraint (21). Introducing Lagrange multipliers for this constraint and eliminating $\bar{\boldsymbol{r}}_\alpha$ and $\bar{\boldsymbol{r}}'_\alpha$, we can reduce the problem to the following minimization with respect to $\boldsymbol{q}$:

$$J = (\boldsymbol{q}, \boldsymbol{M}\boldsymbol{q}) \to \min. \tag{22}$$

Here, $\boldsymbol{M}$ is a $4 \times 4$ matrix defined by

$$\boldsymbol{M} = \sum_{\alpha=1}^{N} \boldsymbol{X}_\alpha^\top \boldsymbol{W}_\alpha \boldsymbol{X}_\alpha, \tag{23}$$

where $\boldsymbol{W}_\alpha$ is a $3 \times 3$ matrix given by

$$\boldsymbol{W}_\alpha = \Big( {q_0}^2 \left(V_0[\boldsymbol{r}_\alpha] + V_0[\boldsymbol{r}'_\alpha]\right) - 2q_0 S[\boldsymbol{q_l} \times (V_0[\boldsymbol{r}_\alpha] - V_0[\boldsymbol{r}'_\alpha])] \\ + \boldsymbol{q_l} \times (V_0[\boldsymbol{r}_\alpha] + V_0[\boldsymbol{r}'_\alpha]) \times \boldsymbol{q_l} \Big)^{-1}. \tag{24}$$

Here, the operation $S[\,\cdot\,]$ designates symmetrization: $S[\boldsymbol{A}] = (\boldsymbol{A} + \boldsymbol{A}^\top)/2$.

If noise is isotropic and identical, Eq. (22) reduces to the method implied by Zhang and Faugeras [16] and Weng et al. [15]. In this sense, Eq. (22) can also be viewed as an extension of their methods to cope with anisotropic noise.

## 7 Renormalization

Since the constraint (21) is linear, the renormalization technique of Kanatani [9] can be applied to the optimization (22). In order to do so, we first evaluate the statistical bias of the moment matrix $\boldsymbol{M}$ defined by Eq. (23).

Let $\bar{\boldsymbol{X}}_\alpha$ be the true value of $\boldsymbol{X}_\alpha$, and write $\boldsymbol{X}_\alpha = \bar{\boldsymbol{X}}_\alpha + \Delta\boldsymbol{X}_\alpha$. From the first of Eqs. (20), we see that the error term $\Delta\boldsymbol{X}_\alpha$ is given by

$$\Delta\boldsymbol{X}_\alpha = \Big( \Delta\boldsymbol{r}'_\alpha - \Delta\boldsymbol{r}_\alpha \;\; (\Delta\boldsymbol{r}'_\alpha + \Delta\boldsymbol{r}_\alpha) \times \boldsymbol{I} \Big). \tag{25}$$

Similarly, let $\bar{\boldsymbol{M}}$ be the true value of $\boldsymbol{M}$, and write $\boldsymbol{M} = \bar{\boldsymbol{M}} + \Delta\boldsymbol{M}$. From Eq. (23), we see that the error term $\Delta\boldsymbol{M}$ has the following expression:

$$\Delta\boldsymbol{M} = \sum_{\alpha=1}^{N} \Big( \Delta\boldsymbol{X}_\alpha^\top \boldsymbol{W}_\alpha \boldsymbol{X}_\alpha + \boldsymbol{X}_\alpha^\top \boldsymbol{W}_\alpha \Delta\boldsymbol{X}_\alpha + \Delta\boldsymbol{X}_\alpha^\top \boldsymbol{W}_\alpha \Delta\boldsymbol{X}_\alpha \Big). \tag{26}$$

It follows that the moment matrix $\boldsymbol{M}$ has the following statistical bias.

$$\begin{aligned} E[\Delta\boldsymbol{M}] &= \sum_{\alpha=1}^{N} E[\Delta\boldsymbol{X}_\alpha^\top \boldsymbol{W}_\alpha \Delta\boldsymbol{X}_\alpha] \\ &= \sum_{\alpha=1}^{N} E[\begin{pmatrix} (\Delta\boldsymbol{r}'_\alpha - \Delta\boldsymbol{r}_\alpha, \boldsymbol{W}_\alpha(\Delta\boldsymbol{r}'_\alpha - \Delta\boldsymbol{r}_\alpha)) \\ -(\Delta\boldsymbol{r}'_\alpha + \Delta\boldsymbol{r}_\alpha) \times \boldsymbol{W}_\alpha(\Delta\boldsymbol{r}'_\alpha - \Delta\boldsymbol{r}_\alpha) \end{pmatrix} \\ &\qquad \begin{pmatrix} ((\Delta\boldsymbol{r}'_\alpha + \Delta\boldsymbol{r}_\alpha) \times \boldsymbol{W}_\alpha(\Delta\boldsymbol{r}'_\alpha - \Delta\boldsymbol{r}_\alpha))^\top \\ (\Delta\boldsymbol{r}'_\alpha + \Delta\boldsymbol{r}_\alpha) \times \boldsymbol{W}_\alpha \times (\Delta\boldsymbol{r}'_\alpha + \Delta\boldsymbol{r}_\alpha) \end{pmatrix}]. \end{aligned} \tag{27}$$

Define a $4 \times 4$ matrix $\boldsymbol{N}$ by

$$\boldsymbol{N} = \begin{pmatrix} n_0 & \boldsymbol{n}^\top \\ \boldsymbol{n} & \boldsymbol{N}' \end{pmatrix}, \tag{28}$$

where

$$n_0 = \sum_{\alpha=1}^{N} (\boldsymbol{W}_\alpha; V_0[\boldsymbol{r}_\alpha] + V_0[\boldsymbol{r}'_\alpha]), \tag{29}$$

$$\boldsymbol{n} = -2\sum_{\alpha=1}^{N} t_3[A[\boldsymbol{W}_\alpha(V_0[\boldsymbol{r}_\alpha] - V_0[\boldsymbol{r}'_\alpha])]], \tag{30}$$

$$\boldsymbol{N}' = \sum_{\alpha=1}^{N} [\boldsymbol{W}_\alpha \times (V_0[\boldsymbol{r}_\alpha] + V_0[\boldsymbol{r}'_\alpha])]. \tag{31}$$

The inner product $(\boldsymbol{A}; \boldsymbol{B})$ of matrices $\boldsymbol{A} = (A_{ij})$ and $\boldsymbol{B} = (B_{ij})$ is defined by $(\boldsymbol{A}; \boldsymbol{B}) = \sum_{i,j=1}^{3} A_{ij}B_{ij}$. The exterior product $[\boldsymbol{A} \times \boldsymbol{B}]$ is the matrix whose $(ij)$ element is $\sum_{k,l,m,n=1}^{3} \varepsilon_{ikl}\varepsilon_{jmn}A_{km}B_{ln}$. The operation $A[\,\cdot\,]$ designates antisymmetrization: $A[\boldsymbol{A}] = (\boldsymbol{A} - \boldsymbol{A}^\top)/2$. For an antisymmetric matrix $\boldsymbol{C} = (C_{ij})$, we define $t_3[\boldsymbol{C}] = (C_{32}, C_{13}, C_{21})^\top$. Then, the statistical bias $E[\Delta \boldsymbol{M}]$ is expressed as follows:

$$E[\Delta \boldsymbol{M}] = \epsilon^2 \boldsymbol{N}. \tag{32}$$

Applying the recipe of Kanatani [9], we obtain the following renormalization procedure:

1. From the data $\{\boldsymbol{r}_\alpha\}$ and $\{\boldsymbol{r}'_\alpha\}$, compute $\boldsymbol{X}_\alpha$, $\alpha = 1, ..., N$, by the first of Eqs. (20).
2. Set $c = 0$ and $\boldsymbol{W}_\alpha = \boldsymbol{I}$, $\alpha = 1, ..., N$.
3. Compute the moment matrix $\boldsymbol{M}$ by Eq. (23).
4. Compute the matrix $\boldsymbol{N}$ by Eq. (28).
5. Compute the smallest eigenvalue $\lambda$ of matrix

$$\hat{\boldsymbol{M}} = \boldsymbol{M} - c\boldsymbol{N} \tag{33}$$

   and the corresponding unit eigenvector $\boldsymbol{q} = (q_0\ q_1\ q_2\ q_3)^\top$.
6. If $|\lambda| \approx 0$, return $\boldsymbol{q}$ and stop. Otherwise, update $c$ and $\boldsymbol{W}_\alpha$ as follows and go back to Step 3:

$$c \leftarrow c + \frac{\lambda}{(\boldsymbol{q}, \boldsymbol{N}\boldsymbol{q})}, \tag{34}$$

$$\begin{aligned} \boldsymbol{W}_\alpha \leftarrow \Big(& {q_0}^2 (V_0[\boldsymbol{r}_\alpha] + V_0[\boldsymbol{r}'_\alpha]) - 2q_0 S[\boldsymbol{q}_l \times (V_0[\boldsymbol{r}_\alpha] - V_0[\boldsymbol{r}'_\alpha])] \\ & + \boldsymbol{q}_l \times (V_0[\boldsymbol{r}_\alpha] + V_0[\boldsymbol{r}'_\alpha]) \times \boldsymbol{q}_l \Big)^{-1}. \end{aligned} \tag{35}$$

   Here, we put $\boldsymbol{q}_l = (q_1\ q_2\ q_3)^\top$.

**Table 1.** Estimated rotations.

| | Axis | Angle |
|---|---|---|
| Renormalization | $(0.9999, 0.0003, 0.0123)$ | $29.769^\circ$ |
| Least squares | $(0.9985, -0.0545, 0.0040)$ | $26.790^\circ$ |
| True values | $(1.0000, 0.0000, 0.0000)$ | $30.000^\circ$ |

## 8 Rotation Estimation from Stereo Images

We conducted experiments for 3-D data obtained by stereo vision. Figures 2(a) and (b) are pairs of stereo images of an object before and after a rigid rotation around a vertical axis. We manually selected the feature points marked by black dots and computed their 3-D positions $\boldsymbol{r}_\alpha$ and normalized covariance matrices $V_0[\boldsymbol{r}_\alpha]$ by the method described in [10], assuming that image noise was isotropic and homogeneous (but the resulting errors in the reconstructed 3-D positions were highly anisotropic and inhomogeneous). We thus obtained two sets of 3-D points.

After translating one set so that its centroid coincides with that of the other, we computed the rotation by renormalization. As a comparison, we also tried the conventional least-squares method (the schemes described in [1, 3, 4, 8, 14, 15, 16] all yield the same solution). Table 1 lists the computed values together with the true values. We can see from this that our method considerably improves accuracy as compared with the least-squares method. However, this result is for just one occurrence of noise. In order to assert the superiority of our method, we need to examine the reliability of the solution for *all possible occurrences* of noise.

## 9 Reliability Analysis

We evaluated the reliability of the computed solution $\hat{\boldsymbol{R}}$ in the following two ways:

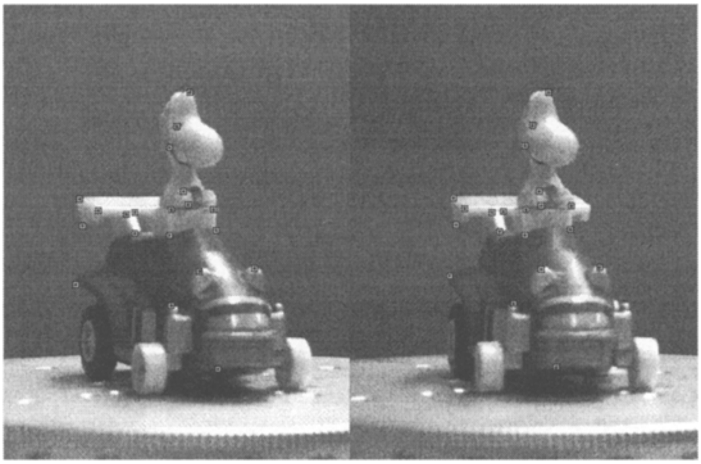
- Theoretical analysis.
- Random noise simulation.

The former is straightforward: since our method attains the theoretical accuracy bound (8) in the first order, we can evaluate the reliability of the solution by approximating the true values by their estimates in Eq. (8).

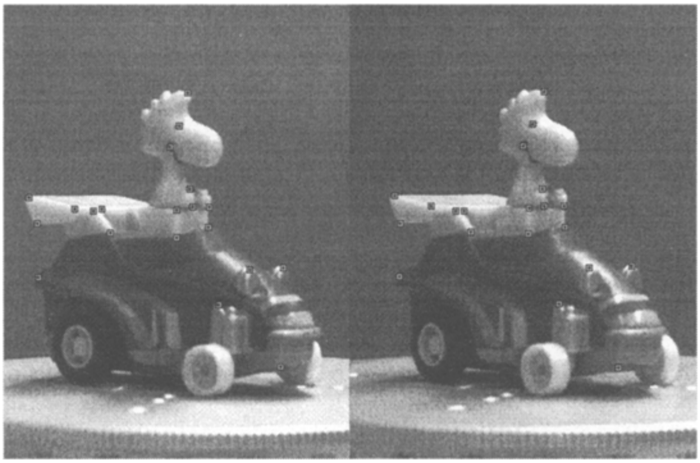
A well known method to the latter is (parametric) *bootstrap* [2], which can be applied to any solution method. In the present case, we do not know the true positions $\{\bar{\boldsymbol{r}}_\alpha\}$, but we know the true rotation $\bar{\boldsymbol{R}}$ (see Table 1). So, we first estimate $\{\bar{\boldsymbol{r}}_\alpha\}$ by optimally correcting the data $\{\boldsymbol{r}_\alpha\}$ and $\{\boldsymbol{r}'_\alpha\}$ into $\{\hat{\boldsymbol{r}}_\alpha\}$ and $\{\hat{\boldsymbol{r}}'_\alpha\}$, respectively, so that the constraint $\hat{\boldsymbol{r}}'_\alpha = \bar{\boldsymbol{R}}\hat{\boldsymbol{r}}_\alpha$ is exactly satisfied. From Eqs. (11) and (12), this optimal correction is done as follows[2] [9]:

$$\hat{\boldsymbol{r}}_\alpha = \boldsymbol{r}_\alpha + V_0[\boldsymbol{r}_\alpha]\bar{\boldsymbol{R}}^\top \bar{\boldsymbol{W}}_\alpha(\boldsymbol{r}'_\alpha - \bar{\boldsymbol{R}}\boldsymbol{r}_\alpha), \tag{36}$$

[2] If the true value $\bar{\mathbf{R}}$ is not known, its estimate $\hat{\mathbf{R}}$ is used.



(a) Before rotation.



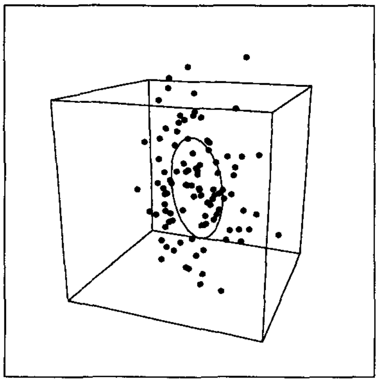
(b) After rotation.

**Fig. 2.** Stereo images.

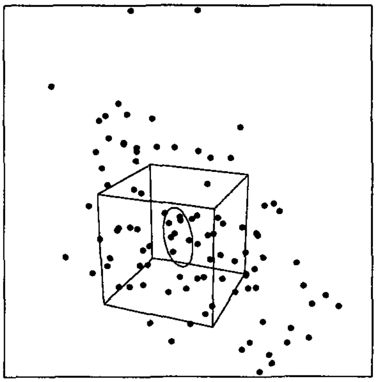
$$\hat{\boldsymbol{r}}'_\alpha = \boldsymbol{r}_\alpha - V_0[\boldsymbol{r}'_\alpha]\bar{\boldsymbol{W}}_\alpha(\boldsymbol{r}'_\alpha - \bar{\boldsymbol{R}}\boldsymbol{r}_\alpha), \tag{37}$$

$$\bar{\boldsymbol{W}}_\alpha = \left(\bar{\boldsymbol{R}}V_0[\boldsymbol{r}_\alpha]\bar{\boldsymbol{R}}^\top + V_0[\boldsymbol{r}'_\alpha]\right)^{-1}. \tag{38}$$

Estimating the variance $\epsilon^2$ by Eq. (15), we generated random independent Gaussian noise that has the estimated variance $\hat{\epsilon}^2$ and added it to the projections of the corrected positions $\{\hat{\boldsymbol{r}}_\alpha\}$ and $\{\hat{\boldsymbol{r}}'_\alpha\}$ $(= \{\bar{\boldsymbol{R}}\hat{\boldsymbol{r}}_\alpha\})$ on the image planes of the left and the right cameras independently. Then, we computed the rotation $\boldsymbol{R}^*$ and the error vector $\Delta\boldsymbol{\Omega}^*$ in the form given by Eq. (6).



(a) Renormalization.



(b) Least squares.

**Fig. 3.** Error distribution.

Figure 3(a) shows three-dimensional plots of the error vector $\Delta\boldsymbol{\Omega}^*$ for 100 trials. The ellipsoid in the figure is defined by

$$(\Delta\boldsymbol{\Omega}^*, \hat{V}[\hat{\boldsymbol{R}}]^{-1}\Delta\boldsymbol{\Omega}^*) = 1, \tag{39}$$

where $\hat{V}[\hat{\boldsymbol{R}}]$ is the covariance matrix computed by approximating $\bar{\boldsymbol{R}}$, $\{\bar{\boldsymbol{r}}_\alpha\}$, and $\epsilon^2$ by $\hat{\boldsymbol{R}}$, $\{\hat{\boldsymbol{r}}_\alpha\}$, and $\hat{\epsilon}^2$, respectively, on the right-hand side of Eq. (8). This ellipsoid indicates the standard deviation of the errors in each orientation [9]; the cube in the figure is displayed merely as a reference. Figure 3(b) is the corresponding figure for the least-squares method (the ellipsoid and the cube are the same as in Fig. 3(a)).

Comparing Figs. 3(a) and (b), we can confirm that our method improves the accuracy of the solution considerably as compared with the least-squares method. We can also see that errors for our method distribute around the ellipsoid defined by Eq. (39), indicating that our method already attains the theretical accuracy bound; no further improvement is possible.

The above visual observation can be given quantitative measures. We define the *bootstrap mean* $\boldsymbol{m}_\Omega^*$ and the *bootstrap covariance matrix* $V[\hat{\boldsymbol{R}}^*]$ by

$$\boldsymbol{m}_\Omega^* = \frac{1}{B}\sum_{b=1}^{B}\Delta\boldsymbol{\Omega}_b^*, \tag{40}$$

$$V[\hat{\boldsymbol{R}}^*] = \frac{1}{B}\sum_{b=1}^{B}(\Delta\boldsymbol{\Omega}_b^* - \boldsymbol{m}_\Omega^*)(\Delta\boldsymbol{\Omega}_b^* - \boldsymbol{m}_\Omega^*)^\top, \tag{41}$$

where $B$ is the number of bootstrap samples and $\Delta\boldsymbol{\Omega}_b^*$ is the error vector for the $b$th sample. The *bootstrap mean error* $E_\Omega^*$ and the *bootstrap standard deviation* $S_\Omega^*$ are defined by

$$E_\Omega^* = \|\boldsymbol{m}_\Omega^*\|, \qquad S_\Omega^* = \sqrt{\mathrm{tr}V[\hat{\boldsymbol{R}}^*]}, \tag{42}$$

**Table 2.** Bootstrap errors and the theoretical lower bound.

| | $E^*_\Omega$ | $S^*_\Omega$ |
|---|---|---|
| Renormalization | 0.0277° | 1.1445° |
| Least squares | 0.0468° | 3.0868° |
| Lower bound | 0° | 1.1041° |

where $\mathrm{tr}\boldsymbol{A}$ denotes the trace of matrix $\boldsymbol{A}$. The corresponding standard deviation for the (estimated) theoretical lower bound $\hat{V}[\hat{\boldsymbol{R}}]$ is $\sqrt{\mathrm{tr}\hat{V}[\hat{\boldsymbol{R}}]}$. Table 2 lists the values of $E^*_\Omega$ and $S^*_\Omega$ for our method and the least-squares method ($B = 2000$) together with their theoretical lower bounds. We see from this that although the mean errors are very small for both methods, the standard deviation of our solution is almost 1/3 that of the least-squares solution and very close to the theoretical lower bound.

This observation confirms that the reliability of the solution computed by our method can indeed be evaluated by (approximately) computing the theoretical accuracy bound given by Eq. (8).

## 10 Concluding Remarks

We have discussed optimal rotation estimation from two sets of 3-D points in the presence of anisotropic and inhomogeneous noise. We have first presented a theoretical accuracy bound defined independently of solution techniques and then given a method that attains it; our method is truly "optimal" in that sense. This optimal method is highly nonlinear, but we have shown that an efficient computational scheme can be obtained by using quaternions and applying the renormalization technique.

Since the solution attains the accuracy bound, we can view it as describing the reliability of the solution; the computation does not require any knowledge about the noise magnitude. Using real stereo images for 3-D reconstruction, we have demonstrated that our method is considerably more accurate than the conventional least-squares method. We have also confirmed the theoretical predictions of our theory by applying bootstrap procedure.

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