A New Characterization of the Trifocal Tensor*

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Abstract. This paper deals with the problem of characterizing and parametrizing the manifold of trifocal tensors that describe the geometry of three views like the fundamental matrix characterizes the geometry of two. The paper contains two new results. First a new, simpler, set of algebraic constraints that characterizes the set of trifocal tensors is presented. Second, we give a new parametrization of the trifocal tensor based upon those constraints which is also simpler than previously known parametrizations. Some preliminary experimental results of the use of these constraints and parametrization to estimate the trifocal tensor from image correspondences are presented.

1 Introduction

This article deals with the problem of finding a minimal representation of the trifocal tensor and using the representation to estimate the tensor from feature correspondences in three views.

Given three views, it has been shown originally by Shashua [Sha94] that the coordinates of three corresponding points satisfied a set of algebraic relations of degree 3 called the trilinear relations. It was later on pointed out by Hartley [Har94] that those trilinear relations were in fact arising from a tensor that governed the correspondences of lines between three views which he called the trifocal tensor. Hartley also correctly pointed out that this tensor had been used, if not formally identified as such, by researchers working on the problem of the estimation of motion and structure from line correspondences [SA90]. Given three views, there are of course three such tensors, depending upon which view is selected as the one one wants to predict to.

The trinocular tensors play the same role in the analysis of scenes from three views as the fundamental matrix play in the two-view case, therefore the question of their estimation from feature correspondences arise naturally. The question of estimating the fundamental matrix between two views has received considerable attention in the last few years and robust algorithms have been proposed by a number of researchers [DZLF94,ZDFL95,TZ97b,Har95]. The main difficulty of the estimation arises from the fact that the fundamental matrix must satisfy one nonlinear constraint, i.e. that its determinant is equal to 0, which prevents the straightforward application of quadratic least-squares methods.

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The related question for the trinocular tensor has received much less attention except for the obvious application of quadratic least-squares methods [Har94,Sha95]. What makes the use of these methods even more questionable in the case of the trinocular tensor is the fact that it is much more constrained than the fundamental matrix: even though it superficially seems to depend upon 26 parameters (27 up to scale), these 26 parameters are not independent since the number of degrees of freedom of three views has been shown to be equal to 18 in the projective framework (33 parameters for the 3 perspective projection matrices minus 15 for an unknown projective transformation) [LV94]. Therefore the trifocal tensor can depend upon at most 18 independent parameters and therefore its 27 components must satisfy a number of algebraic constraints, some of them have been elucidated [SW95,Hey95,AS96].

The contributions of this paper are three-folds. First we give a new set of algebraic constraints that characterize the set of trifocal constraints. Those constraints are simpler than the ones we derived in [FP97] and used in [FP98]. Second we derive a new minimal parametrization of the trifocal tensor that does not suffer from the problems of previous ones. Third we present some preliminary experimental results where we use this new parametrization to estimate the trifocal tensor from image correspondences.

We assume that the reader is familiar with elementary Grassmann-Cayley algebra since the necessary ingredients have already been presented to the Computer Vision community in a number of publications such as, for example, [Car94,FM95].

2 A New Formulation of the Trifocal Tensor Constraints

2.1 The Trifocal Tensors

Let us consider three views, with projection matrices $\mathcal{P}_n, n = 1,2,3$, a 3D line L with images l_n. We denote by $\boldsymbol{\Gamma}_n$, $\boldsymbol{\Lambda}_n$, $\boldsymbol{\Theta}_n$ the row vectors of $\mathcal{P}_n$. They define three planes called the *principal planes* of camera n. The three lines of intersection $\boldsymbol{\Lambda}_n \triangle \boldsymbol{\Theta}_n$, $\boldsymbol{\Theta}_n \triangle \boldsymbol{\Gamma}_n$, $\boldsymbol{\Gamma}_n \triangle \boldsymbol{\Lambda}_n$ of those three planes are three optical rays, i.e. lines going through the optical center C_n, called the *principal rays*.

Given two images l_j and l_k of L, L can be defined as the intersection (the meet) of the two planes $\mathcal{P}_j^T \mathbf{l}_j$ and $\mathcal{P}_k^T \mathbf{l}_k$: $\mathbf{L} \simeq \mathcal{P}_j^T \mathbf{l}_j \triangle \mathcal{P}_k^T \mathbf{l}_k$, where the vector $\mathbf{L}$ is the 6×1 vector of Plücker coordinates of the line L.

Let us write the right-hand side of this equation explicitly in terms of the row vectors of the matrices $\mathcal{P}_j$ and $\mathcal{P}_k$ and the coordinates of $\mathbf{l}_j$ and $\mathbf{l}_k$:

$$\mathbf{L} \simeq (l_j^1 \boldsymbol{\Gamma}_j + l_j^2 \boldsymbol{\Lambda}_j + l_j^3 \boldsymbol{\Theta}_j) \triangle (l_k^1 \boldsymbol{\Gamma}_k + l_k^2 \boldsymbol{\Lambda}_k + l_k^3 \boldsymbol{\Theta}_k)$$

By expanding the meet operator and applying the matrix $\tilde{\mathcal{P}}_i$ defined in [FP97,FP98] to the Plücker coordinates of L, we obtain the coordinates of the image l_i of L:

$$\mathbf{l}_i \simeq \tilde{\mathcal{P}}_i(\mathcal{P}_j^T \mathbf{l}_j \triangle \mathcal{P}_k^T \mathbf{l}_k) \tag{1}$$

which is valid for $i \neq j \neq k$. Note that if we exchange view j and view k, we just change the sign of $\mathbf{l}_i$ and therefore we do not change l_i. A geometric interpretation of this is shown in Fig. 1. For convenience, we rewrite (1) in a more compact form:

$$\mathbf{l}_i \simeq \mathcal{T}_i(\mathbf{l}_j, \mathbf{l}_k) \ .$$

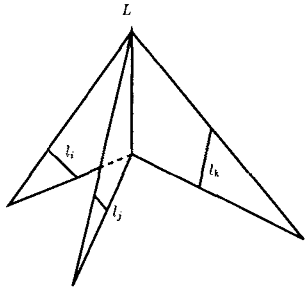

Fig. 1. The line l_i is the image by camera i of the 3D line L intersection of the planes defined by the optical centers of the cameras j and k and the lines l_j and l_k, respectively.

This expression can be also put in a slightly less compact form with the advantage of making the dependency on the projection planes of the matrices $\mathcal{P}_n, n = 1,2,3$ explicit: $\mathbf{l}_i \simeq \left[\mathbf{l}_j^T \mathbf{G}_i^1 \mathbf{l}_k \;\; \mathbf{l}_j^T \mathbf{G}_i^2 \mathbf{l}_k \;\; \mathbf{l}_j^T \mathbf{G}_i^3 \mathbf{l}_k\right]^T$. This is, in the projective framework, the exact analog of the equation used in the work of Spetsakis and Aloimonos [SA90] to study the structure from motion problem from line correspondences.

The three 3×3 matrices $\mathbf{G}_i^n, n = 1,2,3$ will play an important role in the sequel. We do not give their explicit forms which will not be needed in this paper [FP97,FP98].

Note that (1) allows us to predict the coordinates of a line l_i in image i given two images l_j and l_k of an unknown 3D line in images j and k, except in two cases where $\mathcal{T}_i(\mathbf{l}_j, \mathbf{l}_k) = \mathbf{0}$ detailed in [FP98]. Except in those cases, we have defined an application $\mathcal{T}_i$ from $\mathbb{P}^{*2} \times \mathbb{P}^{*2}$, the Cartesian product of two duals of the projective plane, into $\mathbb{P}^{*2}$. A pictorial view is shown in Fig. 2: the tensor is represented as a 3×3 cube, the three horizontal planes representing the matrices $\mathbf{G}_i^n, n = 1,2,3$. It can be thought of as a black box which takes as its input two lines, l_j and l_k and outputs a third one, l_i. Hartley

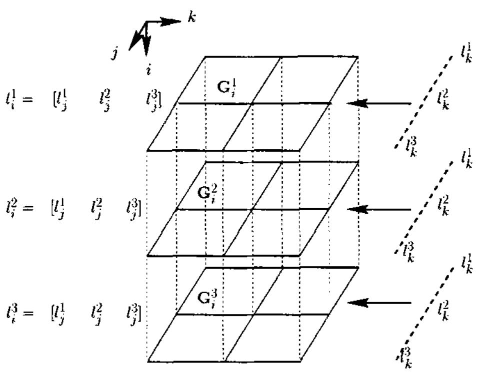

Fig. 2. A three-dimensional representation of the trifocal tensor.

has shown [Har94,Har97] that the trifocal tensors can be very simply parameterized by the perspective projection matrices $\mathcal{P}_n$, $n = 1, 2, 3$ of the three cameras. This result is summarized in the following proposition:

Proposition 1 (Hartley). *Let $\mathcal{P}_n$, $n = 1, 2, 3$ be the three perspective projection matrices of three cameras in general viewing position. After a change of coordinates, those matrices can be written, $\mathcal{P}_1 = [\mathbf{I}_3\ \mathbf{0}]$, $\mathcal{P}_2 = [\mathbf{X}\ \mathbf{e}_{2,1}]$ and $\mathcal{P}_3 = [\mathbf{Y}\ \mathbf{e}_{3,1}]$ and the matrices $\mathbf{G}_1^n$ can be expressed as:*

$$\mathbf{G}_1^n = \mathbf{e}_{2,1}\mathbf{Y}^{(n)T} - \mathbf{X}^{(n)}\mathbf{e}_{3,1}^T \quad n = 1, 2, 3 \tag{2}$$

where the vectors $\mathbf{X}^{(n)}$ and $\mathbf{Y}^{(n)}$ are the column of the matrices $\mathbf{X}$ and $\mathbf{Y}$, respectively.

We use this proposition as a definition:

Definition 1. *Any tensor of the form (2) is a trifocal tensor.*

2.2 Algebraic and Geometric Properties of the Trifocal Tensors

The matrices $\mathbf{G}_i^n, n = 1, 2, 3$ have interesting properties which are closely related to the epipolar geometry of the views j and k. We start with the following proposition, which was proved for example in [Har97]. The proof hopefully gives some more geometric insight of what is going on:

Proposition 2 (Hartley). *The matrices $\mathbf{G}_i^n$ are of rank 2 and their nullspaces are the three epipolar lines, noted l_k^n in view k of the three projection rays of camera i. These three lines intersect at the epipole $e_{k,i}$. The corresponding lines in view i are represented by $\mathbf{e}_n \times \mathbf{e}_{i,k}$ and can be obtained as $\mathcal{T}_i(\mathbf{l}_j, \mathbf{l}_k^n), n = 1, 2, 3$ for any l_j not equal to l_j^n (see proposition 3).*

Proof. The nullspace of $\mathbf{G}_i^n$ is the set of lines l_k^n such that $\mathcal{T}_i(\mathbf{l}_j, \mathbf{l}_k^n)$ has a zero in the n-th coordinate for all lines l_j. The corresponding lines l_i such that $\mathbf{l}_i = \mathcal{T}_i(\mathbf{l}_j, \mathbf{l}_k^n)$ all go through the point represented by $\mathbf{e}_n, n = 1, 2, 3$ in the i-th retinal plane. This is true if and only if l_k^n is the image in the k-th retinal plane of the projection ray $\Lambda_i \vartriangle \Theta_i\ (n = 1)$, $\Theta_i \vartriangle \Gamma_i\ (n = 2)$ and $\Gamma_i \vartriangle \Lambda_i\ (n = 3)$: l_k^n is an epipolar line with respect to view i. Moreover, it is represented by $\mathbf{e}_n \times \mathbf{e}_{i,k}$. □

A similar reasoning applies to the matrices $\mathbf{G}_i^{nT}$:

Proposition 3 (Hartley). *The nullspaces of the matrices $\mathbf{G}_i^{nT}$ are the three epipolar lines, noted $l_j^n, n = 1, 2, 3$, in the j-th retinal plane of the three projection rays of camera i. These three lines intersect at the epipole $e_{j,i}$, see Fig. 3. The corresponding lines in view i are represented by $\mathbf{e}_n \times \mathbf{e}_{i,j}$ and can be obtained as $\mathcal{T}_i(\mathbf{l}_j^n, \mathbf{l}_k), n = 1, 2, 3$ for any l_k not equal to l_k^n.*

This provides a geometric interpretation of the matrices $\mathbf{G}_i^n$: they represent mappings from the set of lines in view k to the set of points in view j located on the epipolar line l_j^n defined in proposition 3. This mapping is geometrically defined by taking the intersection of the plane defined by the optical center of the kth camera and any line

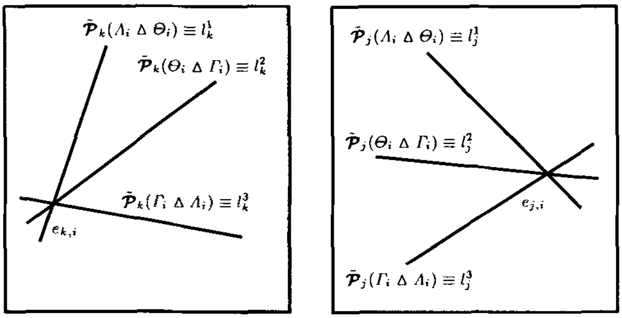

Fig. 3. The lines l_j^n (resp. l_k^n), $n = 1, 2, 3$ in the nullspaces of the matrices $\mathbf{G}_i^{nT}$ (resp. $\mathbf{G}_i^n$) are the images of the three projection rays of camera i. Hence, they intersect at the epipole $e_{j,i}$ (resp. $e_{k,i}$). The corresponding epipolar lines in camera i are obtained as $\mathcal{T}_i(l_j^n, l_k)$ (resp. $\mathcal{T}_i(l_j, l_k^n)$) for $l_k \neq l_k^n$ (resp. $l_j \neq l_j^n$).

of its retinal plane with the nth projection ray of the ith camera and forming the image of this point in the jth camera. This point does not exist when the plane contains the projection ray. The corresponding line in the kth retinal plane is the epipolar line l_k^n defined in proposition 2. Moreover, the three columns of $\mathbf{G}_i^n$ represent three points which all belong to the epipolar line l_j^n.

We can go a little further in the interpretation of the matrices $\mathbf{G}_i^n$. They also define each a collineation from the line l_k^n to the line l_j^n: given a point a_k of l_k^n, the pencil of lines going through a_k defines a pencil of planes whose axis is the 3D line (C_k, A_k) (A_k is the point of the corresponding optical ray of the ith camera). The corresponding point b_k on l_j^n is the image of A_k. The correspondence (a_k, b_k) is the analog for $\mathbf{G}_i^n$ of the correspondence between two epipolar lines for the fundamental matrix. Indeed, there is a very strong analogy between the matrices $\mathbf{G}_i^n$ and fundamental matrices: both are of rank two and both, as we just showed, define collineations between $\mathbb{P}^1$.

Since each matrix $\mathbf{G}_i^n$, $n = 1, 2, 3$ defines a collineation from l_k^n to l_j^n, we have three such collineations that we denote by h^n. Those three collineations are not independent and we have the following proposition:

Proposition 4. *The three collineations h^n satisfy the relations:*

$$h^n(e_{k,i}) = e_{j,i} \quad n = 1, 2, 3 \tag{3}$$

Proof. We know that the three lines l_k^n (respectively l_j^n) go through the epipole ek, i (respectively $e_{j,i}$) and from the definition of h^n, when the point a_k coincides with the epipole $e_{k,i}$, the point b_k coincides with the epipole $e_{j,i}$, hence (3). □

Similarly, the matrices $\mathbf{G}_i^{nT}$ represent mappings from the set of lines in view j to the set of points in view k located on the epipolar line l_k^n. From the previous discussion we deduce that:

Proposition 5. *The rank of the matrices $\left[\mathbf{l}_k^1\, \mathbf{l}_k^2\, \mathbf{l}_k^3\right]$ and $\left[\mathbf{l}_j^1\, \mathbf{l}_j^2\, \mathbf{l}_j^3\right]$ is 2 in general.*

Algebraically, this implies that the three determinants $det(\mathbf{G}_i^n)$, $n = 1, 2, 3$ are equal to 0. Another constraint implied by proposition 5 is that the 3×3 determinants formed with the three vectors in the nullspaces of the $\mathbf{G}_i^n, n = 1, 2, 3$ (resp. of the $\mathbf{G}_i^{nT}, n = 1, 2, 3$) are equal to 0. It turns out that the applications $\mathcal{T}_i$, $i = 1, 2, 3$ satisfy other algebraic constraints which are relevant for this paper.

To simplify a bit the notations, we assume in the sequel that $i = 1, j = 2, k = 3$ and ignore the ith index everywhere, e.g. denote $\mathcal{T}_1$ by $\mathcal{T}$.

We have already seen several such constraints when we studied the matrixes $\mathbf{G}^n$. Let us summarize those constraints in the following proposition:

Proposition 6. *Under the general viewpoint assumption, the trifocal tensor $\mathcal{T}$ satisfies the three constraints, called the* rank *constraints:*

$$rank(\mathbf{G}^n) = 2 \Rightarrow det(\mathbf{G}^n) = 0 \;\; n = 1, 2, 3$$

The trifocal tensor $\mathcal{T}$ satisfies the two constraints, called the epipolar *constraints:*

$$rank(\left[\mathbf{l}_2^1\, \mathbf{l}_2^2\, \mathbf{l}_2^3\right]) = rank(\left[\mathbf{l}_3^1\, \mathbf{l}_3^2\, \mathbf{l}_3^3\right]) = 2 \Rightarrow \mid \mathbf{l}_2^1\, \mathbf{l}_2^2\, \mathbf{l}_2^3 \mid = \mid \mathbf{l}_3^1\, \mathbf{l}_3^2\, \mathbf{l}_3^3 \mid 0$$

Those five constraints on the form of the matrices $\mathbf{G}^n$ are clearly algebraically independent since the rank constraints say nothing about the way the kernels are related.

There is a further set of constraints that are satisfied by any trifocal tensor and are of great interest for this paper. They are described in the next proposition.

Proposition 7. *The trifocal tensor $\mathcal{T}$ satisfies the ten algebraic constraints, called the* extended rank constraints*:*

$$rank(\sum_{n=1}^{3} \lambda_n \mathbf{G}^n) \leq 2 \;\; \forall \lambda_n \; n = 1, 2, 3$$

Proof. Notice that for fixed values (not all zero) of the λ_n's, and for a given line l_3 in view 3, the point which is the image in view 2 of line l_3 by $\sum_{n=1}^{3} \lambda_n \mathbf{G}^n$ is the image of the point defined by: $\lambda_1 \mathcal{P}_2^T \mathbf{l} \vartriangle (\boldsymbol{\Lambda} \vartriangle \boldsymbol{\Theta}) + \lambda_2 \mathcal{P}_2^T \mathbf{l} \vartriangle (\boldsymbol{\Theta} \vartriangle \boldsymbol{\Gamma}) + \lambda_3 \mathcal{P}_2^T \mathbf{l} \vartriangle (\boldsymbol{\Gamma} \vartriangle \boldsymbol{\Lambda})$ This expression can be rewritten as:

$$\mathcal{P}_2^T \mathbf{l} \vartriangle (\lambda_1 \boldsymbol{\Lambda} \vartriangle \boldsymbol{\Theta} + \lambda_2 \boldsymbol{\Theta} \vartriangle \boldsymbol{\Gamma} + \lambda_3 \boldsymbol{\Gamma} \vartriangle \boldsymbol{\Lambda}) \tag{4}$$

The line $\lambda_1 \boldsymbol{\Lambda} \vartriangle \boldsymbol{\Theta} + \lambda_2 \boldsymbol{\Theta} \vartriangle \boldsymbol{\Gamma} + \lambda_3 \boldsymbol{\Gamma} \vartriangle \boldsymbol{\Lambda}$ is an optical ray of the first camera, and when l varies in view 3, the point defined by (4) is well defined except when l is the image of that line in view 3. In that case the meet in (4) is zero and the image of that line is in the nullspace of $\sum_{n=1}^{3} \lambda_n \mathbf{G}^n$. □

Proposition 7 is equivalent to the vanishing of the 10 coefficients of the homogeneous polynomial of degree 3 in the three variables $\lambda_n, n = 1, 2, 3$ equal to $det(\sum_{n=1}^{3} \lambda_n \mathbf{G}^n)$. The coefficients of the terms λ_n^3, $n = 1, 2, 3$ are the determinants $\det(\mathbf{G}^n)$, $n = 1, 2, 3$. Therefore the extended rank constraints contain the rank constraints.

To be complete, we give the expressions of the seven extended rank constraints which are different from the three rank constraints:

Proposition 8. *The seven extended rank constraints are given by:*

$$\lambda_1^2\lambda_2 \quad | \mathbf{G}_1^1 \mathbf{G}_2^1 \mathbf{G}_3^2 | + | \mathbf{G}_1^1 \mathbf{G}_2^2 \mathbf{G}_3^1 | + | \mathbf{G}_1^2 \mathbf{G}_2^1 \mathbf{G}_3^1 | = 0 \tag{5}$$

$$\lambda_1^2\lambda_3 \quad | \mathbf{G}_1^1 \mathbf{G}_2^1 \mathbf{G}_3^3 | + | \mathbf{G}_1^1 \mathbf{G}_2^3 \mathbf{G}_3^1 | + | \mathbf{G}_1^3 \mathbf{G}_2^1 \mathbf{G}_3^1 | = 0 \tag{6}$$

$$\lambda_2^2\lambda_1 \quad | \mathbf{G}_1^2 \mathbf{G}_2^2 \mathbf{G}_3^1 | + | \mathbf{G}_1^2 \mathbf{G}_2^1 \mathbf{G}_3^2 | + | \mathbf{G}_1^1 \mathbf{G}_2^2 \mathbf{G}_3^2 | = 0 \tag{7}$$

$$\lambda_2^2\lambda_3 \quad | \mathbf{G}_1^2 \mathbf{G}_2^2 \mathbf{G}_3^3 | + | \mathbf{G}_1^2 \mathbf{G}_2^3 \mathbf{G}_3^2 | + | \mathbf{G}_1^3 \mathbf{G}_2^2 \mathbf{G}_3^2 | = 0 \tag{8}$$

$$\lambda_3^2\lambda_1 \quad | \mathbf{G}_1^3 \mathbf{G}_2^3 \mathbf{G}_3^1 | + | \mathbf{G}_1^3 \mathbf{G}_2^1 \mathbf{G}_3^3 | + | \mathbf{G}_1^1 \mathbf{G}_2^3 \mathbf{G}_3^3 | = 0 \tag{9}$$

$$\lambda_3^2\lambda_2 \quad | \mathbf{G}_1^3 \mathbf{G}_2^3 \mathbf{G}_3^2 | + | \mathbf{G}_1^3 \mathbf{G}_2^2 \mathbf{G}_3^3 | + | \mathbf{G}_1^2 \mathbf{G}_2^3 \mathbf{G}_3^3 | = 0 \tag{10}$$

$$\lambda_1\lambda_2\lambda_3 \quad | \mathbf{G}_1^1 \mathbf{G}_2^2 \mathbf{G}_3^3 | + | \mathbf{G}_1^1 \mathbf{G}_2^3 \mathbf{G}_3^2 | + | \mathbf{G}_1^2 \mathbf{G}_2^1 \mathbf{G}_3^3 | + | \mathbf{G}_1^2 \mathbf{G}_2^3 \mathbf{G}_3^1 | + | \mathbf{G}_1^3 \mathbf{G}_2^1 \mathbf{G}_3^2 | + | \mathbf{G}_1^3 \mathbf{G}_2^2 \mathbf{G}_3^1 | = 0 \tag{11}$$

2.3 New Constraints that Characterize the Tensor

We now show the new result that the ten extended constraints and the epipolar constraints characterize the trifocal tensors:

Theorem 1. *Let $\mathcal{T}$ be a bilinear mapping from $\mathbb{P}^{*2} \times \mathbb{P}^{*2}$ to $\mathbb{P}^{*2}$ which satisfies the twelve extended rank and epipolar constraints. Then this mapping is a trifocal tensor, i.e. it satisfies definition 1. Those twelve algebraic equations are another set of implicit equations of the manifold of trifocal tensors.*

The proof of this theorem is the subject of the rest of this section. We start with a proposition that will be used to prove that the three rank constraints and the two epipolar constraints are not sufficient to characterize the set of trifocal tensors. Its proof that can be found in [FP97].

Proposition 9. *If a tensor $\mathcal{T}$ satisfies the three rank constraints and the two epipolar constraints, then its matrices $\mathbf{G}^n, n = 1, 2, 3$ can be written:*

$$\mathbf{G}^n = a_n \mathbf{X}^{(n)} \mathbf{Y}^{(n)T} + \mathbf{X}^{(n)} \mathbf{e}_{3,1}^T + \mathbf{e}_{2,1} \mathbf{Y}^{(n)T}, \tag{12}$$

where $e_{2,1}$ (resp. $e_{3,1}$) is a fixed point of image 2 (resp. of image 3), the three vectors $\mathbf{X}^{(n)}$ represent three points of image 2, and the three vectors $\mathbf{Y}^{(n)}$ represent three points of image 3.

What about the ten extended rank constraints: Are they sufficient to characterize the trilinear tensor? the following proposition answers this question negatively.

Proposition 10. *The ten extended rank constraints do not imply the epipolar constraints.*

Proof. The proof consists in exhibiting a counterexample. The tensor $\mathcal{T}$ defined by:

$$\mathbf{G}^1 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{G}^2 = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{G}^3 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

satisfies the ten extended rank constraints but the corresponding three left nullspaces do not satisfy the left epipolar constraint. □

We are now ready to prove Theorem 1:

Proof. The proof consists in showing that any bilinear application $\mathcal{T}$ that satisfies the five rank and epipolar constraints, i.e. whose matrices $\mathbf{G}^n$ can be written as in (12) and the remaining seven extended rank constraints (5-11) can be written as in (2), i.e. is such that $a_n = 0,\ n = 1, 2, 3$.

If we use the parametrization (12) and evaluate the constraints (5-10), we find:

$$-a_2 \mid \mathbf{e}_{2,1}\, \mathbf{X}^{(1)}\, \mathbf{X}^{(2)} \mid\mid \mathbf{e}_{3,1}\, \mathbf{Y}^{(1)}\, \mathbf{Y}^{(2)} \mid \tag{13}$$
$$-a_3 \mid \mathbf{e}_{2,1}\, \mathbf{X}^{(1)}\, \mathbf{X}^{(3)} \mid\mid \mathbf{e}_{3,1}\, \mathbf{Y}^{(1)}\, \mathbf{Y}^{(3)} \mid \tag{14}$$
$$-a_1 \mid \mathbf{e}_{2,1}\, \mathbf{X}^{(1)}\, \mathbf{X}^{(2)} \mid\mid \mathbf{e}_{3,1}\, \mathbf{Y}^{(1)}\, \mathbf{Y}^{(2)} \mid \tag{15}$$
$$-a_3 \mid \mathbf{e}_{2,1}\, \mathbf{X}^{(2)}\, \mathbf{X}^{(3)} \mid\mid \mathbf{e}_{3,1}\, \mathbf{Y}^{(2)}\, \mathbf{Y}^{(3)} \mid \tag{16}$$
$$-a_1 \mid \mathbf{e}_{2,1}\, \mathbf{X}^{(1)}\, \mathbf{X}^{(3)} \mid\mid \mathbf{e}_{3,1}\, \mathbf{Y}^{(1)}\, \mathbf{Y}^{(3)} \mid \tag{17}$$
$$-a_2 \mid \mathbf{e}_{2,1}\, \mathbf{X}^{(2)}\, \mathbf{X}^{(3)} \mid\mid \mathbf{e}_{3,1}\, \mathbf{Y}^{(2)}\, \mathbf{Y}^{(3)} \mid \tag{18}$$

In those formulas, our attention is drawn to determinants of the form $\mid \mathbf{e}_{2,1}\, \mathbf{X}^{(i)}\, \mathbf{X}^{(j)} \mid$, $i \neq j$ (type 2) and $\mid \mathbf{e}_{3,1}\, \mathbf{Y}^{(i)}\, \mathbf{Y}^{(j)} \mid$, $i \neq j$ (type 3). The nullity of a determinant of type 2 (resp. type 3) implies that the epipole $e_{2,1}$ (resp. $e_{3,1}$) is on the line defined by the two points $X^{(i)}$, $X^{(j)}$ (resp. $Y^{(i)}$, $Y^{(j)}$), if the corresponding points are distinct. If all determinants are non zero, the constraints (13-18) imply that all a_n's are zero. Things are slightly more complicated if some of the determinants are equal to 0.

We prove that if the matrices $\mathbf{G}^n$ are of rank 2, no more than one of the three determinants of each of the two types can equal 0. We consider several cases. • The first case is when all points of one type are different. Suppose that the three points represented by the three vectors $\mathbf{X}^{(n)}$ are not aligned. Having two of the determinants of type 2 equal to 0 implies that the point $e_{2,1}$ is identical to one of the points $X^{(n)}$ since it is at the intersection of two of the lines they define. But, according to (12), this implies that the corresponding matrix $\mathbf{G}^n$ is of rank 1, contradicting the hypothesis that this rank is 2. Similarly, if the three points $X^{(n)}$ are aligned, if one determinant is equal to 0, the epipole $e_{2,1}$ belongs to the line $(X^{(1)}, X^{(2)}, X^{(3)})$ which means that the three epipolar lines l_2^1, l_2^2, l_2^3 are identical contradicting the hypothesis that they form a matrix of rank 2. Therefore, in this case, all three determinants are non null. • The second case is when two of the points are equal, e.g. $\mathbf{X}^{(1)} \simeq \mathbf{X}^{(2)}$. The third point must then be different, otherwise we would only have one epipolar line contradicting the rank 2 assumption on those epipolar lines, and, if it is different, the epipole $e_{2,1}$ must not be on the line defined by the two points for the same reason. Therefore in this case also at most one of the determinants is equal to 0.

Having at most one determinant of type 2 and one of type 3 equal to 0 implies that at least two of the a_n are 0. This is seen by inspecting the constraints (13-18). If we now express the seventh constraint:

$$
\begin{aligned}
&a_1a_2a_3 \mid \mathbf{Y}^{(1)}\mathbf{Y}^{(2)}\mathbf{Y}^{(3)} \mid\mid \mathbf{X}^{(1)}\mathbf{X}^{(2)}\mathbf{X}^{(3)} \mid \\
&-(\mid \mathbf{e}_{2,1}\,\mathbf{X}^{(1)}\mathbf{X}^{(2)} \mid\mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(1)}\mathbf{Y}^{(3)} \mid + \mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(1)}\mathbf{Y}^{(2)} \mid\mid \mathbf{e}_{2,1}\,\mathbf{X}^{(1)}\mathbf{X}^{(3)} \mid)a_1 \\
&+(\mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(1)}\mathbf{Y}^{(2)} \mid\mid \mathbf{e}_{2,1}\,\mathbf{X}^{(2)}\mathbf{X}^{(3)} \mid + \mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(2)}\mathbf{Y}^{(3)} \mid\mid \mathbf{e}_{2,1}\,\mathbf{X}^{(1)}\mathbf{X}^{(2)} \mid)a_2 \\
&-(\mid \mathbf{e}_{2,1}\,\mathbf{X}^{(2)}\mathbf{X}^{(3)} \mid\mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(1)}\mathbf{Y}^{(3)} \mid + \mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(2)}\mathbf{Y}^{(3)} \mid\mid \mathbf{e}_{2,1}\,\mathbf{X}^{(1)},\mathbf{X}^{(3)} \mid)a_3 \\
&+(\mid \mathbf{e}_{2,1}\,\mathbf{X}^{(1)}\mathbf{X}^{(2)} \mid\mid \mathbf{Y}^{(1)}\mathbf{Y}^{(2)}\mathbf{Y}^{(3)} \mid + \mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(1)}\mathbf{Y}^{(2)} \mid\mid \mathbf{X}^{(1)}\mathbf{X}^{(2)}\mathbf{X}^{(3)} \mid)a_1a_2 \\
&+(\mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(2)}\mathbf{Y}^{(3)} \mid\mid \mathbf{X}^{(1)}\mathbf{X}^{(2)}\mathbf{X}^{(3)} \mid + \mid \mathbf{e}_{2,1}\,\mathbf{X}^{(2)}\mathbf{X}^{(3)} \mid\mid \mathbf{Y}^{(1)}\mathbf{Y}^{(2)}\mathbf{Y}^{(3)} \mid)a_2a_3 \\
&-(\mid \mathbf{e}_{2,1}\,\mathbf{X}^{(1)}\mathbf{X}^{(3)} \mid\mid \mathbf{Y}^{(1)}\mathbf{Y}^{(2)}\mathbf{Y}^{(3)} \mid + \mid \mathbf{X}^{(1)}\mathbf{X}^{(2)}\mathbf{X}^{(3)} \mid\mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(1)}\mathbf{Y}^{(3)} \mid)a_1a_3,
\end{aligned}
$$

we find that it is equal to the third a_n multiplied by two of the nonzero determinants, implying that the third a_n is null and completing the proof.

Let us give a few examples of the various cases. Assume first that $\mid \mathbf{e}_{2,1}\,\mathbf{X}^{(1)}\mathbf{X}^{(2)} \mid = \mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(1)}\mathbf{Y}^{(2)} \mid = 0$. Then (17), (18) and (16) imply $a_1 = a_2 = a_3 = 0$. The second situation occurs if we assume for example $\mid \mathbf{e}_{2,1}\,\mathbf{X}^{(1)}\mathbf{X}^{(2)} \mid = \mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(1)}\mathbf{Y}^{(3)} \mid = 0$. Then (18) and (16) imply $a_2 = a_3 = 0$. Equation (11) takes then the form $- \mid \mathbf{e}_{2,1}\,\mathbf{X}^{(1)}\mathbf{X}^{(3)} \mid\mid \mathbf{e}_{3,1}\,\mathbf{Y}^{(1)}\mathbf{Y}^{(2)} \mid a_1$, and implies $a_1 = 0$. □

Practically, Theorem 1 provides a simpler set of sufficient constraints than those used in [FP98]: The ten extended constraints are of degree 3 in the elements of $\mathcal{T}$ whereas the other constraints are of degree 6 as are the two epipolar constraints.

3 A New Parameterization

Having minimal parameterizations of the trifocal tensor is very useful:
• first, the canonical elements emphasized by such parameterizations may help to have a better geometrical understanding of the mathematical concept of the trifocal tensor (some results of this articles were obtained in this way).
• if we want to use general non-constrained optimization techniques to estimate trifocal tensors that are optimal with respect to the data. In the absence of such parameterizations, one has either to rely on constrained optimization, or to optimize only over a subset of the parameters defining the trifocal tensor [Har97]. However, finding a good minimal parameterization of the trifocal tensor is not an easy task and to our knowledge only two have been proposed to date [TZ97a,FP98].
This section presents a new minimal parameterization of the trifocal tensor, that overcomes some problems that arise with those.

3.1 A Few Notations

There is a common point to the objects introduced to deal with multiple view geometry such as fundamental matrices or trifocal tensor: they all involve at some point matrices of rank 2. Furthermore, in most cases, the left and right kernels of these matrices are attached to geometrical properties of the system of cameras. This suggests that the set

$\mathcal{M}(\mathbf{L}, \mathbf{R})$ of all the matrices that have a given left kernel $\mathbf{L} = [l_1, l_2, l_3]^T \neq \mathbf{0}$ and right kernel $\mathbf{R} = [r_1, r_2, r_3]^T \neq \mathbf{0}$ is of some importance, and it indeed received a lot of attention with the study and parameterization of the fundamental matrix. Most of this section is just a slightly different formulation of a well-known parameterization of the fundamental matrix, which uses the 2 epipoles and 4 coefficients of the original matrix that describe the homography relating the epipolar lines in the two images [LF96].

Obviously, $\mathcal{M}(\mathbf{L}, \mathbf{R})$ is a linear space of dimension 4. Thinking in terms of linear spaces, a basis can be found and the coordinates of a given matrix of $\mathcal{M}(\mathbf{L}, \mathbf{R})$ in that basis correspond to the 4 coefficients of the fundamental matrix parameterization. Unfortunately, there is no systematic way to define a basis for $\mathcal{M}(\mathbf{L}, \mathbf{R})$ that would be valid for all choices of $\mathbf{L}$ and $\mathbf{R}$: different maps cannot be avoided as long as we want a minimal basis. To simplify the presentation, we assume that the highest components in magnitude of both $\mathbf{L}$ and $\mathbf{R}$ are in first position: in this case, the four matrices of rank 1 $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ and $\mathbf{M}_4$ constitute a basis of $\mathcal{M}(\mathbf{L}, \mathbf{R})$.

$$\mathbf{M}_1 = \begin{bmatrix} r_3 l_3 & 0 & -r_1 l_3 \\ 0 & 0 & 0 \\ -r_3 l_1 & 0 & r_1 l_1 \end{bmatrix}, \mathbf{M}_2 = \begin{bmatrix} -r_2 l_3 & r_1 l_3 & 0 \\ 0 & 0 & 0 \\ r_2 l_1 & -r_1 l_1 & 0 \end{bmatrix},$$
$$\mathbf{M}_3 = \begin{bmatrix} -r_3 l_2 & 0 & r_1 l_2 \\ r_3 l_1 & 0 & -r_1 l_1 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{M}_4 = \begin{bmatrix} r_2 l_2 & -r_1 l_2 & 0 \\ -r_2 l_1 & r_1 l_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (19)$$

This basis is valid as long as $l_1 \neq 0$ and $r_1 \neq 0$. This explains our choice of having maximal magnitudes for those coefficients. The 8 other maps that correspond to different choices for the positions of the highest components are obtained similarly.

The coordinates of a matrix $\mathbf{M}$ of $\mathcal{M}(\mathbf{L}, \mathbf{R})$ in this basis are called a_1, a_2, a_3 and a_4: $\mathbf{M} = a_1\mathbf{M}_1 + a_2\mathbf{M}_2 + a_3\mathbf{M}_3 + a_4\mathbf{M}_4$. $\mathbf{M}$ is of rank 1 iff $a_2 a_3 - a_1 a_4 = 0$.

3.2 The Extended Rank Constraints Revisited

The 3 matrices $\mathbf{G}^i, i = 1 \ldots 3$ being rank-2, we note $\mathbf{L}^i$ and $\mathbf{R}^i$ their respective left and right kernels. Each $\mathbf{G}^i$ can thus be represented by its coordinates a_1^i, a_2^i, a_3^i and a_4^i in a basis of $\mathcal{M}(\mathbf{L}^i, \mathbf{R}^i)$. We assume hereafter that the first coordinates of $\mathbf{L}^i$ and $\mathbf{R}^i$ are those of highest magnitude, so that we work in the basis described by (19).

Since $\mathbf{L}^i, i = 1 \ldots 3$ and $\mathbf{R}^i, i = 1 \ldots 3$ are orthogonal to $\mathbf{e}_{2,1}$ and $\mathbf{e}_{3,1}$ respectively, we can write $\mathbf{L}^i = \mathbf{e}_{2,1} \times \mathbf{X}^{(i)}, i = 1 \ldots 3$ and $\mathbf{R}^i = \mathbf{e}_{3,1} \times \mathbf{Y}^{(i)}, i = 1 \ldots 3$, where $\mathbf{X}^{(i)}$ and $\mathbf{Y}^{(i)}$ are the vectors of formula 2. Plugging these values into the extended rank constraints 5–11 and using computer algebra to factorize the results leads to the following result which is an algebraic translation of proposition 4:

Theorem 2. *Assuming that the first coordinates of* $\mathbf{L}^i$ *and* $\mathbf{R}^i$ *are those of highest magnitude, The four coefficients* $a_j^i, j = 1 \ldots 4$ *representing* $\mathbf{G}^i$ *in* $\mathcal{M}(\mathbf{L}^i, \mathbf{R}^i)$ *satisfy the following* linear *relation:*

$$e_{2,1}^2 e_{3,1}^2 a_1^i + e_{2,1}^2 e_{3,1}^3 a_2^i + e_{2,1}^3 e_{3,1}^2 a_3^i + e_{2,1}^3 e_{3,1}^3 a_4^i = 0 , \qquad (20)$$

where $\mathbf{e}_{2,1} = \left[e^1_{2,1}, e^2_{2,1}, e^3_{2,1}\right]^T$ *and* $\mathbf{e}_{3,1} = \left[e^1_{3,1}, e^2_{3,1}, e^3_{3,1}\right]^T$.

Remarks:

• Equations 20 are only factors that appear in the extended rank constraints, but it can be shown that it is the only one that must generically vanish.

• In general, the first coordinates of $\mathbf{L}^i$ and $\mathbf{R}^i$ are not those of highest magnitude, but (20) remains essentially the same with the substitution of $e^j_{2,1}$ and $e^k_{3,1}$ by $e^{\lambda^i(j)}_{2,1}$ and $e^{\rho^i(k)}_{3,1}$, where λ^i and ρ^i are the circular permutations that bring respectively the coordinates of highest magnitude of $\mathbf{L}^i$ and $\mathbf{R}^i$ into first position.

• It is not difficult to show that (20) are never degenerated provided that the proper permutations λ^i and ρ^i have been made.

3.3 A Minimal Parameterization of the Trifocal Tensor

Let us first assume that the 6 vectors $\mathbf{L}^i, i = 1\ldots3$ and $\mathbf{R}^i, i = 1\ldots3$ are given, we show that the three matrices $\mathbf{G}^i, i = 1\ldots3$ can be parameterized by 8 coefficients. To do so, consider the 12 coordinates $a^i_j, i = 1\ldots3, j = 1\ldots4$. Since for each i the $a^i_j, j = 1\ldots4$ are satisfying (20), it is possible to drop one of those four coordinates: for numerical stability, the best choice is to drop the coordinate which has the highest coefficient in magnitude in (20). Moreover, since the $\mathbf{G}^i$ are only defined up to a global scale factor, we can drop one more of the 9 remaining coordinates by normalizing it to 1. This leaves us with 8 coefficients that completely describe the $\mathbf{G}^i, i = 1\ldots3$, given the $\mathbf{L}^i, i = 1\ldots3$ and $\mathbf{R}^i, i = 1\ldots3$. Since 8 parameters have been used, only 10 parameters remain to parameterize the $\mathbf{L}^i, i = 1\ldots3$ and $\mathbf{R}^i, i = 1\ldots3$.

We can assume without loss of generality that $\|\mathbf{L}^i\| = 1, i = 1\ldots3$. These 3 vectors are orthogonal to the epipole $\mathbf{e}_{2,1}$ which can be parameterized by 2 coordinates by normalizing its biggest coordinate to 1 (there are thus 3 maps). The vectors $\mathbf{L}^i$ are conveniently represented by 3 angles in a canonical basis of the plane orthogonal to the direction defined by $\mathbf{e}_{2,1}$. All the $\mathbf{L}^i$ can thus be represented by 5 parameters. A similar parameterization can be obtained for the $\mathbf{R}^i, i = 1\ldots3$, which gives the desired result.

As a consequence, we have obtained a minimal parameterization of the trifocal tensor, i.e by 18 parameters. As the reader may have noticed, the number of maps of this parameterization is very large ($9 \times 3^2 \times 3^6$) but it is nonetheless easy to define a general routine that chooses the best map and computes the parameterization.

3.4 Relationship with Projection Matrices (Hartley Parameterization)

Following Hartley's work as described in proposition 1, we choose without loss of generality a projective basis of the 3D space such that the projection matrix of the first camera $\mathbf{P}_1 = [\mathbf{I}|\mathbf{0}]$. In this basis, we note $\mathbf{P}_2 = \left[\alpha^i_j\right]$ and $\mathbf{P}_3 = \left[\beta^i_j\right]$ the projection matrices of the two other cameras. With these notations, the matrices $\mathbf{G}^i, i = 1\ldots3$ of the trifocal tensor T_1 can be written as $G^i_{jk} = \alpha^j_i\beta^k_4 - \alpha^j_4\beta^k_i$. Although not minimal (22 parameters), this parameterization is interesting because it establishes a link between the trifocal tensors and the projection matrices. We now give expressions for the parameters introduced in the previous section in terms of those projection matrices.

$$\mathbf{L}^i = \left[\alpha_4^3\alpha_i^2 - \alpha_4^2\alpha_i^3, \alpha_4^1\alpha_i^3 - \alpha_4^3\alpha_i^1, \alpha_4^2\alpha_i^1 - \alpha_4^1\alpha_i^2\right]^T ,$$
$$\mathbf{R}^i = \left[\beta_4^3\beta_i^2 - \beta_4^2\beta_i^3, \beta_4^1\beta_i^3 - \beta_4^3\beta_i^1, \beta_4^2\beta_i^1 - \beta_4^1\beta_i^2\right]^T .$$

Using the permutations λ^i and ρ^i of Sect. 3.2, we can express the coordinates $a_j^i, j = 1 \ldots 4$ of $\mathbf{G}^i$ in the basis of $\mathcal{M}(\mathbf{L}^i, \mathbf{R}^i)$ defined by those permutations:

$$a_1^i = \frac{\alpha_i^{\lambda^i(3)}\beta_4^{\rho^i(3)} - \alpha_4^{\lambda^i(3)}\beta^{\rho^i(3)_i}}{D}, a_2^i = \frac{\alpha_i^{\lambda^i(3)}\beta_4^{\rho^i(2)} - \alpha_4^{\lambda^i(3)}\beta^{\rho^i(2)_i}}{D},$$
$$a_3^i = \frac{\alpha_i^{\lambda^i(2)}\beta_4^{\rho^i(3)} - \alpha_4^{\lambda^i(2)}\beta^{\rho^i(3)_i}}{D}, a_4^i = \frac{\alpha_i^{\lambda^i(2)}\beta_4^{\rho^i(2)} - \alpha_4^{\lambda^i(2)}\beta^{\rho^i(2)_i}}{D},$$

where $D = L^i_{\lambda^i(1)} R^i_{\rho^i(1)} = (\alpha_4^{\lambda^i(3)}\alpha_i^{\lambda^i(2)} - \alpha_4^{\lambda^i(2)}\alpha_i^{\lambda^i(3)})(\beta_4^{\lambda^i(3)}\beta_i^{\lambda^i(2)} - \beta_4^{\lambda^i(2)}\beta_i^{\lambda^i(3)})$ is the product of the two highest coordinates in magnitude of $\mathbf{L}^i$ and $\mathbf{R}^i$ respectively. Since the parameters of the parameterization are ratios of coordinates of $\mathbf{L}^i$ and $\mathbf{R}^i$ and of $a_j^i, j = 1 \ldots 4$, they are projective invariants of the original projection matrices.

3.5 Comparison with Previous Minimal Parameterizations

Previously, only two minimal parameterizations have been proposed: in [TZ97a], Torr and Zisserman propose a parameterization **TZ** that is based on 6 corresponding points in the 3 images, in [FP98], Faugeras and Papadopoulo use a minimal parameterization **FP** that is based directly on the coefficients of the trifocal tensor.

In both cases, the parameterization is not one to one, i.e. one vector parameter parameterizes more than one trifocal tensor (up to three with **TZ** and up to two with **FP**). This arises because a polynomial equations of degree 3 for **TZ** and degree 2 for **FP** has to be solved in order to recover the trifocal tensor. This has two practical consequences:
• The multiple trifocal tensors parameterized by a single vector of 18 values can be distinguished only by using the image data. Although, the authors never experienced such a behavior, it might be possible to have situations for which this distinction is difficult.
• With a minimization process that updates "blindly" the vector of parameters, it is possible to "lose" the trifocal tensor when the solution designated by the data "disappears" in the complex plane leaving only one potential candidate with **TZ** and none with **FP**. In both cases, this is bad and there is no good solution to the problem. This should be a rare event but could result in a really bad solution when it happens. The authors experienced this problem in one case with the **FP** parameterization.

Since the new parameterization is one to one (one parameter code for only one trifocal tensor), such problems should never happen. Finally, none of the two parameterizations **TZ** or **FP** deals very well with map problems: in both case, there is no clear way to choose the map to minimize the numerical problems (i.e. what is the best choice of point for **TZ** apart from being in general position, and what columns of the trifocal tensor to take as parameters with **FP**). On the contrary, with our new parameterization, there is always a well-defined way to choose the best map for a given trifocal tensor.

4 Experimental Results

We have used our new parameterization in a minimisation process, similar to the one described in [FP98]. We start with a set of point of three corresponding images for which triplets of corresponding points have been extracted. From an initial linear estimate that, in general, does not verify the trifocal constraint, we compute an initial tensor that satisfies them[1]. Then, this initial trifocal tensor is parameterized and this minimal description of the tensor is used in a non-linear optimization process that refines the tensor by minimizing a criterion defined as the sum of the squared euclidean distances between the predicted and measured points in all the three images. For more details on this criterion see [FP98].

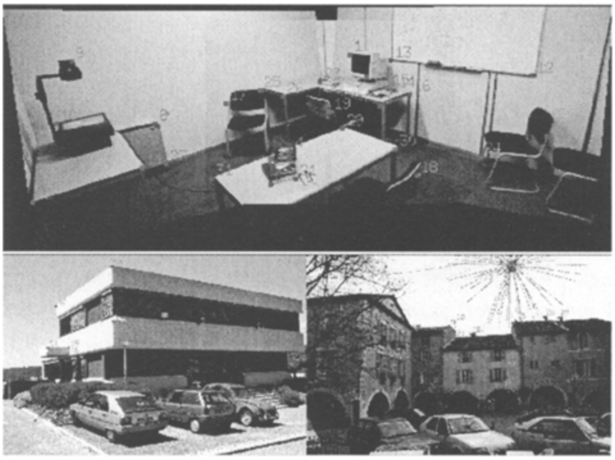

Fig. 4. One image excerpted of each triplet used for the experiments. In each case, the point matches used for the experiments are shown. Notice that the first of these triplets is actually a triplet of mosaics made from different pictures. Starting from top to bottom and from left to right these experimental sets are called **Triplet 1**, **Triplet 2** and **Triplet 3** respectively.

Figure 4 shows one image of each of the three triplets **Triplet 1**, **Triplet 2** and **Triplet 3** of real images that we used for our experiments. Each triplet contains about 30 point matches that are used to estimate the trifocal tensor. Those points were obtained using an interactive tool and are of good quality (reasonable accuracy, of the order of .5 pixels, and no false matches). To check the behavior of the minimizations in less perfect situations, we have also rounded to the closest integer value the pixels coordinates for the second and third triplets which are referred to as **Triplet 2'** and **Triplet 3'**.

Table 1 show the residual errors after minimization. For comparison purposes, we have also included the results obtained with the **FP** parameterization. As it can be seen,

[1] There are many reliable ways to achieve this step: we can use either a parameterization or a minimisation process over the trifocal constraints. The quality of the result varies with the method but this is not the topic of this paper.

	Triplet 1		Triplet 2		Triplet 2'		Triplet 3		Triplet 3'	
	Average	Max.	Average	Max.	Average	Max.	Average	Max.	Average	Max.
FP	$3.9e^{-3}$	2.2	$4.5e^{-4}$	0.1	$6.9e^{-3}$	1.7	$3.3e^{-4}$	0.1	$6.2e^{-4}$	0.2
	$3.6e^{-3}$	1.6	$9.5e^{-4}$	0.2	$8.1e^{-3}$	1.9	$1.7e^{-4}$	0.1	$6.8e^{-4}$	0.7
	$8.4e^{-3}$	3.6	$7.4e^{-4}$	0.1	$3.0e^{-3}$	0.6	$2.4e^{-4}$	0.1	$7.3e^{-4}$	0.3
New	$4.7e^{-3}$	1.3	$7.4e^{-4}$	0.2	$7.9e^{-3}$	2.0	$4.3e^{-4}$	0.1	$1.0e^{-3}$	0.3
	$2.1e^{-3}$	0.6	$8.2e^{-4}$	0.2	$1.1e^{-2}$	2.4	$2.9e^{-4}$	0.1	$6.0e^{-4}$	0.3
	$5.2e^{-3}$	2.2	$6.5e^{-4}$	0.1	$2.7e^{-3}$	0.5	$3.5e^{-4}$	0.2	$1.5e^{-3}$	0.8

Table 1. Prediction errors in pixels for all experiments. The rows labelled **FP** corresponds to the minimization process using the **FP** parameterization, whereas **New** corresponds to the one using the new parameterization. For each experiment, both average and maximal errors for each image are shown.

the results are quite comparable even though those obtained with the new method seem to be be generally of slightly lower quality. It should be said that during the tunning of the whole minimization process (test with different initializations and with different setup for the minimization processes), the results were consistently of lower quality (sometimes slightly worse than what is happening with the options that have been used for the results shown here). However, we have also seen among the hundreds of tests, at least three cases for which the **FP** based method fails miserably with NaN values for the trifocal tensors coefficients. This is because of the phenomenon explained in Sect. 3.5. On the contrary, the **New** method has always given a plausible result.

5 Conclusion

We have given a new set of algebraic constraints that characterize the set of trifocal constraints (Theorem 1). Those constraints are simpler than the ones we derived in [FP97] and used in [FP98]. We have used those constraints to derive a new minimal parametrization of the trifocal tensor that does not suffer from the problems of previous ones. Finally we have presented some experimental results where we use this new parametrization to estimate the trifocal tensor from image correspondences.

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