Relative Undecidability in the Termination Hierarchy of Single Rewrite Rules

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Abstract. For a hierarchy of properties of term rewriting systems, related to termination, we prove *relative undecidability* even in the case of single rewrite rules: for implications $X \Rightarrow Y$ in the hierarchy the property X is undecidable for rewrite rules satisfying Y.

1 Introduction

A fundamental problem in the theory of term rewriting is the detection of termination: for a fixed system of rewrite rules, determine whether there are infinite rewrite sequences. Besides termination a number of related properties are of interest, linearly ordered by implication:

$$\begin{aligned} &\text{polynomial termination} \Rightarrow \omega\text{-termination} \Rightarrow \text{total termination} \\ &\quad \Rightarrow \text{simple termination} \Rightarrow \text{non-self-embeddingness} \Rightarrow \text{termination} \\ &\qquad \Rightarrow \text{non-loopingness} \Rightarrow \text{acyclicity} \end{aligned}$$

We call this the *termination hierarchy.* Apart from polynomial termination, all properties in the termination hierarchy are known to be undecidable ([11, 15, 13, 18, 8, 9]). In [9] we showed the stronger result of *relative undecidability*: for all implications $X \Rightarrow Y$ in the termination hierarchy except one—polynomial termination $\Rightarrow$ ω-termination—the property X is undecidable for term rewriting systems (TRSs for short) satisfying property Y.

In this paper we address the question of relative undecidability for TRSs consisting of a single rewrite rule. We show that for all implications $X \Rightarrow Y$ in the termination hierarchy except two—polynomial termination $\Rightarrow$ ω-termination

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$\Rightarrow$ total termination—the property X is undecidable for one-rule TRSs satisfying property Y.

Dauchet [1] was the first to prove undecidability of termination for one-rule TRSs, by means of a reduction to the uniform halting problem for Turing machines. Middeldorp and Gramlich [13] reduced the undecidability of simple termination, non-self-embeddingness, and non-loopingness for one-rule TRSs to the uniform halting problem for linear bounded automata. Lescanne [12] showed that Dauchet's result can also be obtained by a reduction to Post's Correspondence Problem (PCP). The results presented in this paper are stronger because (1) we obtain the same undecidability results for (much) smaller classes of one-rule TRSs, and (2) we show the undecidability of total termination for one-rule (simply terminating) TRSs. The latter solves problem 87 in [4] and rectifies a conjecture in [18].

The relative undecidability results in [9] are obtained by using PCP in the following way: for the lower five implications $X \Rightarrow Y$ in the termination hierarchy and for all PCP instances P a TRS is constructed that always satisfies Y and satisfies X if and only if P admits no solution. In this paper we present a more uniform approach. First we construct a TRS $\mathcal{U}(P, \mathcal{Q})$ parameterized by a PCP instance P and a TRS $\mathcal{Q}$. The TRS $\mathcal{U}(P, \mathcal{Q})$ has the following properties: (1) the left-hand sides of its rewrite rules are the same, (2) if P admits no solution then $\mathcal{U}(P, \mathcal{Q})$ is totally terminating, and (3) if P admits a solution then $\mathcal{U}(P, \mathcal{Q})$ simulates $\mathcal{Q}$. Because of property (1) every $\mathcal{U}(P, \mathcal{Q})$ can be compressed into a one-rule TRS $\mathcal{S}(P, \mathcal{Q})$ without affecting the termination behaviour. In particular, if P admits no solution then $\mathcal{S}(P, \mathcal{Q})$ is totally terminating. Finally, for the lower five implications $X \Rightarrow Y$ in the termination hierarchy we define a suitable TRS $\mathcal{Q}$ such that $\mathcal{S}(P, \mathcal{Q})$ satisfies Y if and only if P admits no solution. The advantage of this approach is that the complicated part—the construction and properties of the TRS $\mathcal{U}(P, \mathcal{Q})$—is independent of the involved level in the termination hierarchy.

The remainder of this paper is organized as follows. In the next section we briefly recall the definitions of the properties in the termination hierarchy and PCP. In Section 3 we define the TRS $\mathcal{U}(P, \mathcal{Q})$ and show that it simulates $\mathcal{Q}$ whenever P admits a solution. In Section 4 we define the one-rule TRS $\mathcal{S}(P, \mathcal{Q})$ and show that it inherits the termination behaviour from $\mathcal{U}(P, \mathcal{Q})$. In Section 5 we instantiate $\mathcal{S}(P, \mathcal{Q})$ by suitable TRSs $\mathcal{Q}$ in order to conclude the desired relative undecidability results. For reasons of space, the difficult proof of total termination of $\mathcal{U}(P, \mathcal{Q})$ in the case that P admits no solution has been omitted. It can be found in the full version of this paper [10].

2 Preliminaries

For preliminaries on rewriting and termination we refer to [2, 3]. Let $\mathcal{F}$ be a signature containing at least one constant. We write $\mathcal{T}(\mathcal{F})$ for the set of ground terms over $\mathcal{F}$; for a set $\mathcal{X}$ of variable symbols we write $\mathcal{T}(\mathcal{F}, \mathcal{X})$ for the set of open terms. A (strict partial) order $>$ on $\mathcal{T}(\mathcal{F})$ is called *monotonic* if for all

$f \in \mathcal{F}$ and $t, u \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ with $t > u$ we have $f(\ldots, t, \ldots) > f(\ldots, u, \ldots)$. A TRS $\mathcal{R}$ over $\mathcal{F}$ and an order $>$ on $\mathcal{T}(\mathcal{F})$ are called *compatible* if $t > u$ for all rewrite steps $t \to_{\mathcal{R}} u$. For compatibility with a monotonic order it suffices to check that $l\sigma > r\sigma$ for all rules $l \to r$ in $\mathcal{R}$ and all ground substitutions σ. It is well-known that a TRS is terminating if and only if it is compatible with a monotonic well-founded order. An $\mathcal{F}$-algebra consists of a set A and for every $f \in \mathcal{F}$ a function $f_A \colon A^n \to A$, where n is the arity of f. A *monotone* $\mathcal{F}$-algebra $(A, >)$ is an $\mathcal{F}$-algebra A for which the underlying set is provided with an order $>$ such that every algebra operation is monotonic in all of its arguments. More precisely, for all $f \in \mathcal{F}$ and $a, b \in A$ with $a > b$ we have $f_A(\ldots, a, \ldots) > f_A(\ldots, b, \ldots)$. A monotone $\mathcal{F}$-algebra $(A, >)$ is called *well-founded* if $>$ is a well-founded order. Every monotone $\mathcal{F}$-algebra $(A, >)$ induces an order $>_A$ on the set of terms $\mathcal{T}(\mathcal{F}, \mathcal{X})$ as follows: $t >_A u$ if and only if $[\alpha](t) > [\alpha](u)$ for all assignments $\alpha \colon \mathcal{X} \to A$. Here $[\alpha]$ denotes the homomorphic extension of α, i.e., $[\alpha](x) = \alpha(x)$ and $[\alpha](f(t_1, \ldots, t_n)) = f_A([\alpha](t_1), \ldots, [\alpha](t_n))$ for $x \in \mathcal{X}$, $f \in \mathcal{F}$, and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. A TRS $\mathcal{R}$ and a monotone algebra $(A, >)$ are called *compatible* if $\mathcal{R}$ and $>_A$ are compatible. It is well-known that a TRS is terminating if and only if it is compatible with a well-founded monotone algebra. The set of rewrite rules $f(x_1, \ldots, x_n) \to x_i$ for all $f \in \mathcal{F}$ and all $i = 1, \ldots, n$, where $n \geq 1$ is the arity of f, is denoted by $\mathcal{E}mb(\mathcal{F})$, or simply by $\mathcal{E}mb$ when the signature $\mathcal{F}$ can be inferred from the context.

The properties in the hierarchy are defined as follows. A TRS is called *terminating* if it does not allow an infinite reduction. A TRS $\mathcal{R}$ over a signature $\mathcal{F}$ is called *simply terminating* if $\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})$ is terminating, or, equivalently, $\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})$ has no cycle. A well-known sufficient condition for simple termination of terminating TRSs is *length-preservingness*, which means that $|l\sigma| = |r\sigma|$ for all rules $l \to r$ and all ground substitutions σ. Here $|t|$ denotes the number of function symbols in t. A TRS over a signature $\mathcal{F}$ is called *totally terminating* if it is compatible with a monotonic well-founded total order on $\mathcal{T}(\mathcal{F})$, or, equivalently, it is compatible with $>_A$ for some well-founded monotone $\mathcal{F}$-algebra $(A, >)$ in which the order $>$ is total. A TRS over a signature $\mathcal{F}$ is called *ω-terminating* if it is compatible with $>_A$ for some well-founded monotone $\mathcal{F}$-algebra $(A, >)$ in which $A = \mathbb{N}$ and $>$ is the usual order on $\mathbb{N}$. A TRS over a signature $\mathcal{F}$ is called *polynomially terminating* if it is compatible with $>_A$ for some well-founded monotone $\mathcal{F}$-algebra $(A, >)$ in which $A = \mathbb{N}$, $>$ is the usual order on $\mathbb{N}$ and for which all functions f_A are polynomials. A TRS $\mathcal{R}$ is called *looping* if it admits a reduction $t \to_{\mathcal{R}}^{+} C[t\sigma]$ for some term t, some context C and some substitution σ. A TRS $\mathcal{R}$ is called *cyclic* if it admits a reduction $t \to_{\mathcal{R}}^{+} t$ for some term t. A TRS $\mathcal{R}$ over a signature $\mathcal{F}$ is called *self-embedding* if it admits a reduction $t \to_{\mathcal{R}}^{+} u \to_{\mathcal{E}mb(\mathcal{F})}^{*} t$ for some terms t, u. Recent investigations of these notions include [5, 7, 8, 14, 19].

For the proofs we use Post's Correspondence Problem (PCP), which can be described as follows:

> given a finite alphabet Γ and a finite set $P \subset \Gamma^{+} \times \Gamma^{+}$, is there some natural number $n > 0$ and $(\alpha_i, \beta_i) \in P$ for $i = 1, \ldots, n$ such that $\alpha_1 \alpha_2 \cdots \alpha_n = \beta_1 \beta_2 \cdots \beta_n$?

This problem is known to be undecidable even in the case of a two-letter alphabet ([16]). The set P is called an *instance* of PCP, the string $\alpha_1\alpha_2\cdots\alpha_n = \beta_1\beta_2\cdots\beta_n$ a *solution* for P. We use a fixed two-letter alphabet $\Gamma = \{0, 1\}$.

We encode PCP instances P and, for each layer $X \Rightarrow Y$ of the hierarchy, a characteristic TRS $\mathcal{Q}$ into a one-rule TRS $\mathcal{S}(P, \mathcal{Q})$ such that $\mathcal{S}(P, \mathcal{Q})$ is in Y for all P, and in X if and only if P has no solution. Thus we reduce PCP to the relative decision problem in each layer.

3 The Encoding

We are now going to encode a PCP instance P and a TRS $\mathcal{Q}$ with the property that all left-hand sides coincide in a TRS $\mathcal{U}(P, \mathcal{Q})$ with the same property.

The signature $\mathcal{F}_{\mathcal{U}}$ we add for our TRSs consists of constants 0, 1, \$, and ε, binary symbols cons and $\overline{\mathsf{cons}}$, and a symbol A the arity of which will depend on the size of the PCP instance P.

The binary symbols cons and $\overline{\mathsf{cons}}$ as well as the constant ε build lists of terms. Usually we drop the cons and $\overline{\mathsf{cons}}$ symbols, and write only the appended terms and barred terms, respectively. Formally, we define the notation $\zeta(t)$ for any term t and mixed sequence $\zeta \in \{t, \bar{t} \mid t \in \mathcal{T}(\mathcal{F}, \mathcal{X})\}^*$ of barred and unbarred terms as follows:

$$\begin{aligned}
\zeta(t) &= t && \text{if } \zeta = \varepsilon,\\
\zeta(t) &= \mathsf{cons}(t', \zeta'(t)) && \text{if } \zeta = t'\zeta',\\
\zeta(t) &= \overline{\mathsf{cons}}(t', \zeta'(t)) && \text{if } \zeta = \overline{t'}\zeta'.
\end{aligned}$$

Moreover, with any sequence $\alpha = t_1 t_2 \ldots t_n$ of unbarred terms we associate the sequence $\overline{\alpha} = \overline{t_n} \ldots \overline{t_2}\, \overline{t_1}$ of barred terms. Hence

$$\begin{aligned}
\alpha(t) &= \mathsf{cons}(t_1, \mathsf{cons}(t_2, \ldots \mathsf{cons}(t_n, t) \ldots)),\\
\overline{\alpha}(t) &= \overline{\mathsf{cons}}(t_n, \overline{\mathsf{cons}}(t_{n-1}, \ldots \overline{\mathsf{cons}}(t_1, t) \ldots)).
\end{aligned}$$

In order to avoid confusion, we will use the latter abbreviation only when the appended terms are in the set $\{0, 1, \$\} \cup \mathcal{X}$. For instance, $0\overline{0}\$(\varepsilon)$ stands for $\mathsf{cons}(0, \overline{\mathsf{cons}}(0), \mathsf{cons}(\$, \varepsilon)))$, $\overline{x}y1(\varepsilon)$ for $\overline{\mathsf{cons}}(x, \mathsf{cons}(y, \mathsf{cons}(1, \varepsilon)))$, $\overline{01}0(z)$ for $\overline{\mathsf{cons}}(1, \overline{\mathsf{cons}}(0, \mathsf{cons}(0, z)))$, and $z(x)$ for $\mathsf{cons}(z, x)$. Note that $\overline{01}0(z)$ differs from $\overline{0}\,\overline{1}0(z) = \overline{\mathsf{cons}}(0, \overline{\mathsf{cons}}(1, \mathsf{cons}(0, z)))$.

Before we give the technical definition of $\mathcal{U}(P, \mathcal{Q})$ let us explain the intuition behind its architecture. The system $\mathcal{U}(P, \mathcal{Q})$ is a modification of the following system from [18]:

$$\mathcal{S}(P) = \left\{ \begin{array}{lll} F(x, \overline{a}(y), x, \overline{a}(y)) \to F(a(x), y, a(x), y) & \text{for all } a \in \Gamma,\\ F(\alpha(x), y, \beta(z), w) \to F(x, \overline{\alpha}(y), z, \overline{\beta}(w)) & \text{for all } (\alpha, \beta) \in P. \end{array} \right.$$

The system $\mathcal{S}(P)$ admits a reduction

$$F(\gamma(x), y, \gamma(x), y) \to^+ F(\gamma(x), y, \gamma(x), y) \tag{1}$$

if and only if γ is a solution of the PCP P. If P has no solution then $\mathcal{S}(P)$ is totally terminating. The use of barred symbols in the second and fourth argument is essential for total termination.

It is now straightforward to change the cyclic behaviour (1) to any desired behaviour that can be expressed by some rewrite system $\mathcal{Q}$. To this end an argument is added to F. This last argument is left unchanged, except for the step completing the cycle in which it is rewritten by a rule in $\mathcal{Q}$.

To avoid unintended rewrite steps, we refine control: we distinguish two states, exhibited by function symbols G and H, which enable only steps of the first and second shape, respectively, in $\mathcal{S}(P)$. A change from state G to state H is possible only if the second and the fourth argument equals ε. Vice versa, a change of state from H to G requires that the first and third fourth argument equals ε. This gives the rewrite system consisting of the rule

$$G(x,\varepsilon,z,\varepsilon,\mathsf{LHS}) \to H(x,\varepsilon,z,\varepsilon,\mathsf{LHS}), \tag{2}$$

the rules

$$H(\alpha(x),y,\beta(z),w,\mathsf{LHS}) \to H(x,\overline{\alpha}(y),z,\overline{\beta}(w),\mathsf{LHS}) \tag{3}$$

for each $(\alpha,\beta) \in P$, and the rules

$$H(\varepsilon,\overline{a}(y),\varepsilon,\overline{a}(w),\mathsf{LHS}) \to G(a(\varepsilon),y,a(\varepsilon),w,\mathsf{RHS}_j) \tag{4}$$
$$G(x,\overline{a}(y),z,\overline{a}(w),\mathsf{LHS}) \to G(a(x),y,a(z),w,\mathsf{LHS}) \tag{5}$$

for each $a \in \Gamma$ and each rule $(\mathsf{LHS} \to \mathsf{RHS}_j) \in \mathcal{Q}$.

In view of the one-rule construction, finally, there is the need to have equal left-hand sides. For this reason $\mathcal{Q}$ has to have this property, too. The two states G and H in the previous definition are encoded by argument pairs $(0,1)$ and $(1,0)$, respectively, hence one function symbol, A, can replace both G and H. Finally, the end of a sequence may not be ε because sequences of various lengths have to match. Instead the end is marked by a special symbol, \$.

In this way, one gets four left-hand sides which can be regarded as instances of one pattern. The match to the pattern can be delayed by the same trick as in Lescanne [12]: One extends the argument vector (to the left) by a vector of terms to match, and exchanges variables with the terms they should match.

Definition 1. Let $P = \{(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\} \subseteq \Gamma^+ \times \Gamma^+$ be a PCP instance[5] and let $\mu = \max\{|\alpha|,|\beta| \mid (\alpha,\beta) \in P\}$. Let $\mathcal{Q} = \{\mathsf{LHS} \to \mathsf{RHS}_1,\ldots,\mathsf{LHS} \to \mathsf{RHS}_m\}$ be a TRS over a signature $\mathcal{F}_{\mathcal{Q}}$ disjoint from $\mathcal{F}_{\mathcal{U}}$. We assign to P and $\mathcal{Q}$ a TRS $\mathcal{U}(P,\mathcal{Q})$ over the signature $\mathcal{F}_{\mathcal{U}} \cup \mathcal{F}_{\mathcal{Q}}$ where A has arity $2n+15$. It consists of the rules $l \to r_i$, $1 \le i \le n+2m+3$, where l and r_i are defined as follows:

$$\begin{aligned} l = A(&0,1,0,1,\$,\alpha_1(\varepsilon),\ldots,\alpha_n(\varepsilon),0,1,\$,\beta_1(\varepsilon),\ldots,\beta_n(\varepsilon),\\ &u,v,w_1\ldots w_\mu(w),\overline{x_1}(x),y_1\ldots y_\mu(y),\overline{z_1}(z),\mathsf{LHS}), \end{aligned}$$

[5] Presenting PCP instances as ordered lists instead of sets entails no loss of generality.

$$r_1 = A(u,v,0,1,x_1,\alpha_1(\varepsilon),\ldots,\alpha_n(\varepsilon),0,1,z_1,\beta_1(\varepsilon),\ldots,\beta_n(\varepsilon), \\ 1,0,w_1\ldots w_\mu(w),\overline{\$}(x),y_1\ldots y_\mu(y),\overline{\$}(z),\mathsf{LHS}), \tag{2}$$

$$r_{i+1} = A(v,u,0,1,\$,\alpha_1(\varepsilon),\ldots,\alpha_{i-1}(\varepsilon),w_1\ldots w_{|\alpha_i|}(\varepsilon),\alpha_{i+1}(\varepsilon),\ldots,\alpha_n(\varepsilon), \\ 0,1,\$,\beta_1(\varepsilon),\ldots,\beta_{i-1}(\varepsilon),y_1\ldots y_{|\beta_i|}(\varepsilon),\beta_{i+1}(\varepsilon),\ldots,\beta_n(\varepsilon), \\ 1,0,w_{|\alpha_i|+1}\ldots w_\mu(w),\overline{\alpha_i}\,\overline{x_1}(x),y_{|\beta_i|+1}\ldots y_\mu(y),\overline{\beta_i}\overline{z_1}(z),\mathsf{LHS}) \tag{3}$$

for all $1 \le i \le n$,

$$r_{n+1+j} = A(v,u,x_1,1,w_1,\alpha_1(\varepsilon),\ldots,\alpha_n(\varepsilon),z_1,1,y_1,\beta_1(\varepsilon),\ldots,\beta_n(\varepsilon), \\ 0,1,0\$w_2\ldots w_\mu(w),x,0\$y_2\ldots y_\mu(y),z,\mathsf{RHS}_j) \tag{4}$$

$$r_{n+1+m+j} = A(v,u,0,x_1,w_1,\alpha_1(\varepsilon),\ldots,\alpha_n(\varepsilon),0,z_1,y_1,\beta_1(\varepsilon),\ldots,\beta_n(\varepsilon), \\ 0,1,1\$w_2\ldots w_\mu(w),x,1\$y_2\ldots y_\mu(y),z,\mathsf{RHS}_j)$$

for all $1 \le j \le m$, and finally

$$r_{n+2m+2} = A(u,v,x_1,1,\$,\alpha_1(\varepsilon),\ldots,\alpha_n(\varepsilon),z_1,1,\$,\beta_1(\varepsilon),\ldots,\beta_n(\varepsilon), \\ 0,1,0w_1\ldots w_\mu(w),x,0y_1\ldots y_\mu(y),z,\mathsf{LHS}) \tag{5}$$

$$r_{n+2m+3} = A(u,v,0,x_1,\$,\alpha_1(\varepsilon),\ldots,\alpha_n(\varepsilon),0,z_1,\$,\beta_1(\varepsilon),\ldots,\beta_n(\varepsilon), \\ 0,1,1w_1\ldots w_\mu(w),x,1y_1\ldots y_\mu(y),z,\mathsf{LHS}).$$

In the following we denote $0,1,0,1,\$,\alpha_1(\varepsilon),\ldots,\alpha_n(\varepsilon),0,1,\$,\beta_1(\varepsilon),\ldots,\beta_n(\varepsilon)$, i.e., the first $2n+8$ arguments of l, by V.

We are now going to show that in case P has a solution, reductions in $\mathcal{Q}$ mirror reductions in $\mathcal{U}(P,\mathcal{Q})$. That is, if P is a PCP instance that has a solution then we get the following particular form of reduction in $\mathcal{U}(P,\mathcal{Q})$.

Proposition 2. *If the PCP instance P has a solution, $\gamma' a$, then for every rewrite rule* LHS $\to$ RHS *in $\mathcal{Q}$ we have*

$$A(V,W,\mathsf{LHS}) \quad \to^+_{\mathcal{U}(P,\mathcal{Q})} \quad A(V,W,\mathsf{RHS})$$

where W denotes the sequence $0,1,a\$w_2\ldots w_\mu(w),\overline{\$\gamma'}(x),a\$y_2\ldots y_\mu(y),\overline{\$\gamma'}(z)$.

Proof. Let $\gamma = \alpha_1\ldots\alpha_n = \beta_1\ldots\beta_n = \gamma' a$ be a solution of the PCP instance P. Let LHS $\to$ RHS be a rule in $\mathcal{Q}$ and abbreviate the terms $\$w_2\ldots w_\mu(w)$ and $\$y_2\ldots y_\mu(y)$ by w' and y', respectively. We have the following reduction in $\mathcal{U}(P,\mathcal{Q})$:

$$\begin{aligned}
& A(V,0,1,aw',\overline{\$\gamma'}(x),ay',\overline{\$\gamma'}(z),\mathsf{LHS}) \\
\to^*_{(5)}\ & A(V,0,1,\gamma w',\overline{\$}(x),\gamma y',\overline{\$}(z),\mathsf{LHS}) \\
\to_{(2)}\ & A(V,1,0,\gamma w',\overline{\$}(x),\gamma y',\overline{\$}(z),\mathsf{LHS}) \\
\to_{(3)}\ & A(V,1,0,\alpha_2\ldots\alpha_n w',\overline{\$\alpha_1}(x),\beta_2\ldots\beta_n y',\overline{\$\beta_1}(z),\mathsf{LHS}) \\
\to^*_{(3)}\ & A(V,1,0,w',\overline{\$\gamma}(x),y',\overline{\$\gamma}(z),\mathsf{LHS}) \\
\to_{(4)}\ & A(V,0,1,aw',\overline{\$\gamma'}(x),ay',\overline{\$\gamma'}(z),\mathsf{RHS}).
\end{aligned}$$

First, using rules (5), γ' in the $2n+12$-th ($2n+14$-th) argument is shifted to the $2n+11$-th ($2n+13$-th, resp.) argument character by character. Note that $\overline{\$\gamma'(x)} = \overline{\gamma'}\ \overline{\$}(x)$. Next by rule (2), there is a change of state from $0,1$ to $1,0$. Then, since γ is a solution of P, it can be shifted back by using rules (3). Finally, with rule (4), the state is changed back to $0,1$. □

Conversely, a reduction in $\mathcal{U}(P,\mathcal{Q})$ gives rise either to an underlying reduction in $\mathcal{Q}$ or to a reduction in $\mathcal{U}(P,\mathcal{Q})$ without the $2m$ rules (4). We will denote the latter system by $\mathcal{U}(P,\emptyset)$.

Proposition 3. *If W and t contain no A symbols then $A(V,W,t) \rightarrow_{\mathcal{U}(P,\mathcal{Q})} A(V,W',t')$ implies $t \rightarrow_{\mathcal{Q}} t'$ or $t = t'$ and $A(V,W,t) \rightarrow_{\mathcal{U}(P,\emptyset)} A(V,W',t)$.*

Proof. Since there is only one A symbol in $A(V,W,t)$, the reduction must take place at the root position. If a rule (4) has been applied, then $t \rightarrow_{\mathcal{Q}} t'$. Otherwise, $A(V,W,t) \rightarrow_{\mathcal{U}(P,\emptyset)} A(V,W',t')$. Obviously, this implies $t = t'$ by the form of the rules in $\mathcal{U}(P,\emptyset)$. □

Proposition 4. *The TRS $\mathcal{U}(P,\emptyset)$ is simply terminating, for any P.*

Proof. Since $\mathcal{U}(P,\emptyset)$ is length-preserving, it is sufficient to show termination. We show termination by semantic labelling [20]. Let the model be $\{0,1\}$, and let 1 be interpreted by 1, and every other symbol by constant 0. Label the symbol A by $2x_2 + x_{2n+10}$ where x_i denotes the value of A's i-th argument. In the labelled system, $\mathcal{U}(P,\mathcal{Q}_3)$ obtained this way the symbol A carries the label $2+v$ at the left hand side, and the labels $2v$, $2u$, $2v+1$ at the right hand sides r_1, r_{i+1}, and r_{n+2m+2}, respectively. Taking into account that $u, v \in \{0,1\}$ one finds that the label decreases for all rules except in case $u = 1$, $v = 0$ for type (2) rules, and case $v = 1$ for type (5) rules, where it stays equal. Termination of the labelled system is now shown by recursive path order with precedence $A_{i+1} > A_i$ and A_i greater than any other function symbol, and A_{2i} having status lexicographic first $2n+11$ then $2n+13$ then the other arguments, A_{2n+1} having status first $2n+12$ then $2n+14$ then the other arguments, and cons and $\overline{\mathsf{cons}}$ having status right-to-left. □

If P has no solution then $\mathcal{U}(P,\mathcal{Q})$ can be ordered by a total reduction order, for any $\mathcal{Q}$.

Theorem 5. *If P has no solution then $\mathcal{U}(P,\mathcal{Q})$ is totally terminating.* □

The complicated proof can be found in the full version [10] of this paper.

4 One-Rule Systems

Transforming $\mathcal{U}(P,\mathcal{Q})$ into a single-rule TRS $\mathcal{S}(P,\mathcal{Q})$ is easy: we define $\mathcal{S}(P,\mathcal{Q})$ as the rule

$$l \quad \rightarrow \quad B(r_1,\ldots,r_{n+2m+3})$$

where B is a new function symbol of arity $n + 2m + 3$. The symbol B is called a *dummy* because it only appears in the right-hand sides of the rules, hence it acts as a barrier for rewrite steps. So the transition from $\mathcal{S}(P, Q)$ to $\mathcal{U}(P, Q)$ is a particular form of *dummy elimination* [6], a method to support proofs of termination by decomposing right-hand sides.

Proposition 6. *Let $\mathcal{R}$ be a one-rule TRS $l \to B(r_1, \ldots, r_k)$ where B is a symbol that does not occur in l nor in any of the r_i, and let $E(\mathcal{R})$ denote the system $\{l \to r_i \mid 1 \leq i \leq k\}$. Suppose $E(\mathcal{R})$ is linear.*[6]

1. *If $\mathcal{R}$ is looping then $E(\mathcal{R})$ is looping.*
2. *If $E(\mathcal{R})$ is terminating then $\mathcal{R}$ is terminating.*
3. *If $\mathcal{R}$ is self-embedding then $E(\mathcal{R})$ is self-embedding.*
4. *$E(\mathcal{R})$ is simply terminating if and only if $\mathcal{R}$ is simply terminating.*
5. *$E(\mathcal{R})$ is totally terminating if and only if $\mathcal{R}$ is totally terminating.*

The converse of statements 1, 2, and 3 does not hold, as the counterexample $\mathcal{R} = \{f(g(x)) \to B(f(f(x)), g(g(x)))\}$ shows. Here $E(\mathcal{R})$ is looping, but $\mathcal{R}$ is non-self-embedding.

Proof. A proof of statement 1 for the case $k = 2$ can be found in [19]. It easily extends to the general case. Proofs of statements 2, 4, and 5 appear in [17]. It remains to prove statement 3.

We call a position an *inner* position of t if it is a function symbol position of t not at the top. Call a position p in a term t *touched by the rewrite step* $t \xrightarrow[l \to r]{u} t'$ if p is of the form $p = u.v$ where v is an inner position in l. Now a position p may be called *touched during the reduction* $t \to_R^+ t'$ if the reduction is of the form $t \to_R^* t'' \to_R t''' \to_R^* t'$ and a residual p'' in t'' of p by $t \to_R^* t''$ is touched in the step $t'' \to_R t'''$.

Assume a self-embedding reduction $t \to_{\mathcal{R}}^+ t' \to_{\mathcal{E}mb}^* t$. If an inner position, q, of t remains untouched during this reduction, the reduction may be split into the reduction steps above and those below the (unique) residual of q:

$$t[z]_q \to_{\mathcal{R}}^* t'[z]_{q'} \to_{\mathcal{E}mb}^* t[z]_{q''}, \qquad t|_q \to_{\mathcal{R}}^* t'|_{q'} \to_{\mathcal{E}mb}^* t|_{q''}$$

If q'' is below q then $t[z]_q \to_{\mathcal{R}}^+ t'[z]_{q'} \to_{\mathcal{E}mb}^* t[z]_{q''} \to_{\mathcal{E}mb}^* t[z]_q$ is a self-embedding reduction. If $q'' = q$ then one of the two reductions must be nonempty; it forms a self-embedding reduction. Otherwise $t|_q \to_{\mathcal{R}}^+ t'|_{q'} \to_{\mathcal{E}mb}^* t|_{q''} \to_{\mathcal{E}mb}^* t|_q$ is a self-embedding reduction. By induction, all untouched inner positions of t can be eliminated.

One may so assume that every inner position of t is touched during the self-embedding reduction. Then t cannot contain B symbols except one B symbol at the top. By a counting argument no B symbols occur in t at all. All B symbols that are created by $\mathcal{R}$ steps must therefore be cancelled by an $\mathcal{E}mb$ step later. One may commute the $\mathcal{E}mb$ step, $B(t_1, \ldots, t_k) \to t_i$, with all preceding steps until the $\mathcal{R}$ step that created the corresponding B symbol. The pair

[6] The proposition also holds without $E(\mathcal{R})$ right-linear.

$c[l\sigma] \to_{\mathcal{R}} c[B(r_1\sigma, \ldots, r_k\sigma)] \to_{\mathcal{E}mb} c[r_i\sigma]$ of steps can be replaced by an $E(\mathcal{R})$ step $c[l\sigma] \to c[r_i\sigma]$. Each such replacement reduces the number of B symbols in the intermediate term, t'. Repeating this procedure removes all B symbols from t' hence the reduction contains no more $\mathcal{R}$ steps. We have thus obtained a self-embedding reduction for $E(\mathcal{R})$.

□

Proposition 7. *If there are no A symbols in the sequence W of terms then $A(V, W) \to^+_{\mathcal{U}(P,\mathcal{Q})} A(V, W')$ if and only if $A(V, W) \to^+_{\mathcal{S}(P,\mathcal{Q})} C[A(V, W')]$ for some context C.* □

5 The Termination Hierarchy

In this section we apply the construction $\mathcal{S}(P, \mathcal{Q})$ to the following TRSs $\mathcal{Q}$.

Definition 8. The TRSs $\mathcal{Q}_1, \ldots, \mathcal{Q}_5$ are defined as follows:

$$\mathcal{Q}_1 = \{\, d \to d \,\}$$

$$\mathcal{Q}_2 = \left\{ \begin{array}{l} g(d, b(x'), y') \to g(d, x', b(y')) \\ g(d, b(x'), y') \to g(x', y', b(b(d))) \end{array} \right\}$$

$$\mathcal{Q}_3 = \{\, g(d) \to g(h(d)) \,\}$$

$$\mathcal{Q}_4 = \left\{ \begin{array}{l} g(d, e, x') \to g(x', h(e), e) \\ g(d, e, x') \to g(h(d), x', d) \end{array} \right\}$$

$$\mathcal{Q}_5 = \left\{ \begin{array}{l} g(d, e) \to g(e, e) \\ g(d, e) \to g(d, d) \end{array} \right\}$$

Observe that in each $\mathcal{Q}_i$ the left-hand sides coincide and that each $\mathcal{Q}_i$ is linear and uses no variables from Defn. 1. Hence $\mathcal{U}(P, \mathcal{Q}_i)$ is linear, too.

Now we have all the ingredients to complete the relative undecidability results for single rule systems.

Proposition 9. *The TRS $\mathcal{S}(P, \mathcal{Q}_1)$ is acyclic. It is non-looping if and only if P admits no solution.*

Proof. Acyclicity is obvious. If P has a solution then $\mathcal{U}(P, \mathcal{Q}_1)$ is cyclic by Prop. 2. According to Prop. 7 $\mathcal{S}(P, \mathcal{Q}_1)$ is looping. Conversely, if P has no solution then $\mathcal{S}(P, \mathcal{Q}_1)$ is totally terminating and hence non-looping by Theorem 5 and Prop. 6.

□

Proposition 10. *The TRS $\mathcal{S}(P, \mathcal{Q}_2)$ is non-looping. It is terminating if and only if P admits no solution.*

Proof. Assume $\mathcal{S}(P, \mathcal{Q}_2)$ admits a loop. By Prop. 6 one obtains a loop, say $t \to^+ C[t\sigma]$, in $\mathcal{U}(P, \mathcal{Q}_2)$. Define the linear interpretation ψ by $\psi(b(t)) = \psi(t)$ and $\psi(f(t_1, \ldots, t_k)) = \psi(t_1) + \cdots + \psi(t_k) + 1$. for every other function symbol f of arity k. Clearly, $s \to_{\mathcal{U}(P,\mathcal{Q}_2)} s'$ implies $\psi(s) \geq \psi(s')$ for all terms s and s', hence

C consists of b symbols only. Define another linear interpretation ϕ by $\phi(b(t)) = \phi(t) + 1$ and $\phi(f(t_1, \ldots, t_k)) = 0$ for every other function symbol of arity k. For all terms s and s', if $s \to_{\mathcal{U}(P,\mathcal{Q}_2)} s'$ then $\phi(s) = \phi(s')$, hence C is empty. Now the loop must be of the shape $D[A(V, W, u)] \to^+ D[A(V\sigma, W\sigma, u\sigma)]$ where D is a context not containing any A symbol. Then $A(V, W, u) \to^+ A(V\sigma, W\sigma, u\sigma)$. Since $A(V, W, u) \to^+_{\mathcal{U}(P,\emptyset)} A(V\sigma, W\sigma, u\sigma)$ would contradict Prop. 4, we obtain $u \to^+_{\mathcal{Q}_2} u\sigma$ by Prop. 3. This is impossible since $\mathcal{Q}_2$ is non-looping [19].

Now let P have a solution. There exists an infinite $\mathcal{Q}_2$-reduction $t_1 \to t_2 \to t_3 \to \cdots$ in which all steps take place at the root position. With help of Props. 2 and 7 this sequence is transformed into an infinite $\mathcal{S}(P, \mathcal{Q}_2)$-reduction

$$A(V, W, t_1) \to^+ C_1[A(V, W, t_2)] \to^+ C_2[A(V, W, t_3)] \to \cdots$$

Conversely, if P has no solution then $\mathcal{S}(P, \mathcal{Q}_2)$ is totally terminating and therefore terminating by Theorem 5 and Prop. 6. □

Proposition 11. *The TRS $\mathcal{S}(P, \mathcal{Q}_3)$ is terminating. It is non-self-embedding if and only if P admits no solution.*

Proof. We prove that $\mathcal{U}(P, \mathcal{Q}_3)$ is terminating, from which termination of $\mathcal{S}(P, \mathcal{Q}_3)$ follows by Prop. 6. We use semantic labelling ([20]). As a model we choose $\{0, 1\}$, where g is interpreted as the identity, h as being constant 0, and all other symbols as being constant 1. Label the symbol A by the value of its last argument. According to the main result of semantic labelling then $\mathcal{U}(P, \mathcal{Q}_3)$ is terminating if and only if $\overline{\mathcal{U}(P, \mathcal{Q}_3)}$ is terminating, where $\overline{\mathcal{U}(P, \mathcal{Q}_3)}$ is obtained from $\mathcal{U}(P, \mathcal{Q}_3)$ by replacing the A symbols in the right hand sides of the type (4) rules by A_0 and all other A symbols by A_1. Now the number of A_1 symbols strictly decreases by applying a type (4) rule from $\overline{\mathcal{U}(P, \mathcal{Q}_3)}$, while it remains constant by applying any other rule. Hence an infinite $\overline{\mathcal{U}(P, \mathcal{Q}_3)}$-reduction gives rise to an infinite $\overline{\mathcal{U}(P, \mathcal{Q}_3)}$-reduction without application of type (4) rules. By omitting the labels this gives an infinite $\mathcal{U}(P, \emptyset)$-reduction, contradicting Prop. 4.

If P has a solution then we obtain $A(V, W, g(d)) \to^+_{\mathcal{S}(P,\mathcal{Q}_3)} C[A(V, W, g(h(d)))]$ from Props. 2 and 7. Since $A(V, W, g(d))$ is embedded in $C[A(V, W, g(h(d)))]$ this shows that $\mathcal{S}(P, \mathcal{Q}_3)$ is self-embedding. Conversely, if P has no solution then $\mathcal{S}(P, \mathcal{Q}_3)$ is totally terminating and thus non-self-embedding by Theorem 5 and Prop. 6. □

Proposition 12. *The TRS $\mathcal{S}(P, \mathcal{Q}_4)$ is non-self-embedding. It is simply terminating if and only if P admits no solution.*

Proof. We prove that $\mathcal{U}(P, \mathcal{Q}_4)$ is non-self-embedding, non-self-embeddingness of $\mathcal{S}(P, \mathcal{Q}_4)$ follows then by Prop. 6. Suppose to the contrary that $\mathcal{U}(P, \mathcal{Q}_4)$ is self-embedding. Using a standard minimality argument we obtain

$$t = A(V, W, g(d, e, t')) \to^+_{\mathcal{U}(P,\mathcal{Q}_4)} u = A(V, W', v) \to^*_{\mathcal{E}mb} t$$

such that t contains only one A symbol. Hence rules in $\mathcal{E}mb(\{A\})$ are not applied. So $W' \to^*_{\mathcal{E}mb} W$ and $v \to^*_{\mathcal{E}mb} g(d, e, t')$ must hold. By Prop. 3 either

$g(d,e,t') \rightarrow^+_{Q_4} v$ or $A(V,W,g(d,e,t')) \rightarrow^+_{\mathcal{U}(P,\emptyset)} A(V,W',g(d,e,t'))$. The former contradicts the non-self-embeddingness of Q_4 and the latter simple termination of $\mathcal{U}(P,\emptyset)$ (Prop. 4).

If P has a solution then with help of Props. 2 and 7 we obtain the cyclic $\mathcal{S}(P,Q_4) \cup \mathcal{E}mb(\mathcal{F}_{\mathcal{U}} \cup \mathcal{F}_Q)$-reduction

$$\begin{aligned} A(V,W,g(d,e,d)) &\rightarrow^+ C_1[A(V,W,g(d,h(e),e))] \rightarrow^+ A(V,W,g(d,e,e)) \\ &\rightarrow^+ C_2[A(V,W,g(h(d),e,d))] \rightarrow^+ A(V,W,g(d,e,d)). \end{aligned}$$

So in this case $\mathcal{S}(P,Q_4)$ is not simply terminating. Conversely, if P has no solution then $\mathcal{S}(P,Q_4)$ is totally terminating and hence simply terminating by Theorem 5 and Prop. 6. □

Proposition 13. *The TRS $\mathcal{S}(P,Q_5)$ is simply terminating. It is totally terminating if and only if P admits no solution.*

Proof. If P has no solution then total termination of $\mathcal{S}(P,Q_5)$ follows from Theorem 5 in conjunction with Prop. 6. It remains to show that $\mathcal{S}(P,Q_5)$ is simply terminating but not totally terminating whenever P has a solution. By Prop. 6, it is sufficient to show this for $\mathcal{U}(P,Q)$.

Let P have a solution. Any infinite $\mathcal{U}(P,Q_5)$-reduction would by Proposition 3 imply an infinite Q_5-reduction, contradicting termination of Q_5. So $\mathcal{U}(P,Q_5)$ is terminating and, since it is length preserving, even simply terminating. Suppose $\mathcal{U}(P,Q_5)$ is totally terminating. With help of Prop. 2 we conclude the existence of a total reduction order $>$ such that both $A(V,W,g(d,e)) > A(V,W,g(e,e))$ and $A(V,W,g(d,e)) > A(V,W,g(d,d))$. By the truncation rule for total reduction orders $>$ in Zantema [17] one may remove the context C from an inequation $C[t] > C[t']$. By doing this for the contexts $A(V,W,g(\_,e))$ and $A(V,W,g(d,\_))$ we get $d > e$ and $e > d$, which contradicts the irreflexivity of $>$. So $\mathcal{U}(P,Q_5)$ cannot be totally terminating. □

Of course the question emerges whether the next implication — ω-termination $\Longrightarrow$ total termination — is undecidable even for single rule TRSs. It is not hard to encode the implication in a suitable TRS Q_6, but one needs the stronger result of ω-termination in Theorem 5. In the full version [10], we present a proof in ω^4. Trying hard we have also established a termination proof in ω^2 but no proof in ω. So the question remains open.

Conclusion

We have shown that the lower five levels of the termination hierarchy are relatively undecidable even for single rules. These results shows how difficult it is in general to detect one of the properties in the termination hierarchy. A consequence of our work is the impossibility of extending methods for establishing total termination, like recursive path orders and Knuth-Bendix orders, to a level where total termination can always be detected. This even holds if only simply terminating single rewrite rules are allowed as input for the method.

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