# Area and Length Preserving Geometric Invariant Scale-Spaces

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**Abstract.** In this paper, area preserving geometric multi-scale representations of planar curves are described. This allows *geometric smoothing without shrinkage* at the same time preserving all the scale-space properties. The representations are obtained deforming the curve via invariant geometric heat flows while simultaneously magnifying the plane by a homethety which keeps the enclosed area constant. The flows are geometrically intrinsic to the curve, and exactly satisfy all the basic requirements of scale-space representations. In the case of the Euclidean heat flow for example, it is completely local as well. The same approach is used to define length preserving geometric flows. The geometric scale-spaces are implemented using an efficient numerical algorithm.

## 1 Introduction

Multi-scale representations and smoothing of signals have been studied now for several years since the basic work of Witkin [30] (see for example [5, 14, 15, 17, 21, 31]). In this work we deal with multi-scale representations of closed planar curves, that is, the boundaries of bounded planar shapes. We show how to derive a smoothing operation which is geometric, sometimes local, and which satisfies all the standard properties of scale-spaces *without shrinkage.*

An important example of a (linear) scale-space is the one obtained filtering the initial curve $\mathcal{C}_0$ with the Gaussian kernel $\mathcal{G}(\cdot, \sigma)$, where $\sigma$, the Gaussian-variance, controls the scale [5, 8, 14, 31]. It has a number of interesting properties, one of them being that the family of curves $\mathcal{C}(\sigma)$ obtained from it, is the solution of the heat equation (with $\mathcal{C}_0$ as initial condition). From the Gaussian example we see that the scale-space can be obtained as the solution of a partial differential equation called an *evolution equation.* This idea was developed in a number of different papers [1, 2, 13, 21, 23, 25]. We describe below a number of scale-spaces for planar curves which are obtained as solutions of nonlinear evolution equations.

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The Gaussian kernel also has several undesirable properties, principally when applied to planar curves. One of these is that the filter is not intrinsic to the curve. This can be remedied by replacing the linear heat equation by *geometric heat flows*, invariant to a given transformation group [10, 11, 24, 25, 27]. Geometric heat flows are presented in forthcoming sections.

Another problem with the Gaussian kernel is that the smoothed curve shrinks when $\sigma$ increases. Several approaches, discussed in Section 2.1, have been proposed in order to partially solve this problem for Gaussian-type kernels (or linear filters). These approaches violate basic scale-space properties. In this paper, we show that this problem can be completely solved using a variation of the geometric heat flow methodology, which keeps the area enclosed by the curve constant. The flows which we obtain, precisely satisfy all the basic scale-space requirements. In the Euclidean case for example, the flow is local as well. The same approach can be used for deriving length preserving heat flows. In this case, the similarity flow exhibits locality. In short, *we can get geometric smoothing without shrinkage.*

## 2 Curve Evolution: The Euclidean Geometric Heat Flow

We consider now planar curves deforming in time, where "time" represents "scale." Let $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$ denote a family of closed embedded curves, where $t$ parametrizes the family, and $p$ the curves ($\mathcal{C}(p,t) = [x(p,t), y(p,t)]^T$). We assume throughout this paper that all of our mappings are periodic and sufficiently smooth. We should add that these results may be generalized to non-smooth curves based on the theory of viscosity solutions or the results in [3, 4].

For the case of the classical heat equation, the curves deform via

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \frac{\partial^2 \mathcal{C}}{\partial p^2} = \begin{bmatrix} x_{pp} \\ y_{pp} \end{bmatrix}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p). \end{cases} \tag{1}$$

As pointed out in the Introduction, $\mathcal{C}(p,t) = [x(p,t), y(p,t)]^T$, satisfying (1), can be obtained from the convolution of $x(p,0), y(p,0)$ with the Gaussian $\mathcal{G}(p,t)$.

In order to separate the geometric concept of a planar curve from its formal algebraic description, it is useful to refer to the planar curve described by $\mathcal{C}(p,t)$ as the image (trace) of $\mathcal{C}(p,t)$, denoted by $\text{Img}[\mathcal{C}(p,t)]$ [25]. Therefore, if the curve $\mathcal{C}(p,t)$ is parametrized by a new parameter $w$ such that $w = w(p,t)$, $\frac{\partial w}{\partial p} > 0$, we obtain $\text{Img}[\mathcal{C}(p,t)] = \text{Img}[\mathcal{C}(w,t)]$.

We see that different parametrizations of the curve, will give different results in (1), i.e, different Gaussian multi-scale representations. This is an undesirable property, since parametrizations are in general arbitrary, and may not be connected with the geometry of the curve. We can attempt to solve this problem choosing a parametrization which is intrinsic to the curve, i.e., that can be computed when only $\text{Img}[\mathcal{C}]$ is given. A natural parametrization is the

*Euclidean arc-length* $v$, which means that the curve is traveled with constant velocity, $\| \mathcal{C}_v \| \equiv 1$. The initial curve $\mathcal{C}_0(p)$ can be re-parametrized as $\mathcal{C}_0(v)$, and the Gaussian filter $\mathcal{G}(v,t)$, or the corresponding heat flow, is applied using this parameter. The problem is that the arc-length is a time-dependent parametrization, i.e., $v(p)$ depends on time. Also, with this kind of re-parametrization, some of the basic properties of scale-spaces are violated. For example, the order between curves is not preserved. Also, the semi-group property, which is one of the most important requirements of a scale-space, can be violated with this kind of re-parametrization. The theory described below solves these problems.

Assume now that the family $\mathcal{C}(p,t)$ evolves (changes) according to the following general flow:

$$\frac{\partial \mathcal{C}}{\partial t} = \beta \mathcal{N}, \tag{2}$$

where $\mathcal{N}$ is the inward Euclidean unit normal and $\beta$ the normal curve velocity component. If $\beta$ is a geometric function of the curve, then the "geometric" curve Img[$\cdot$] is only affected by this normal component [7]. The tangential component affects only the parametrization. Therefore, (2) is the most general geometric flow.

The evolution (2) was studied by different researchers for different functions $\beta$. A key evolution equation is the one obtained for $\beta = \kappa$, where $\kappa$ is the Euclidean curvature [29]. In this case, the flow is given by

$$\frac{\partial \mathcal{C}}{\partial t} = \kappa \mathcal{N}. \tag{3}$$

Equation (3) has its origins in physical phenomena [3, 9]. Gage and Hamilton [10] proved that a planar embedded convex curve converges to a round point when evolving according to (3). Grayson [11] proved that a planar embedded smooth non-convex curve, remains smooth and simple, and converges to a convex one. Next note that if $v$ denotes the Euclidean arc-length, then $\kappa\mathcal{N} = \frac{\partial^2 \mathcal{C}}{\partial v^2}$ [29]. Therefore, equation (3) can be written as

$$\mathcal{C}_t = \mathcal{C}_{vv}. \tag{4}$$

Equation (4) is not linear, since $v$ is a time-dependent parametrization. Equation (4) is called the *(Euclidean) geometric heat flow.* This flow has been proposed for defining a multi-scale representation of closed curves [1, 13, 17]. Note that in contrast with the classical heat flow, the Euclidean geometric one defines an intrinsic, geometric, multi-scale representation. In order to complete the theory, we must prove that all the basic properties required for a scale-space hold for the flow (4). This is obtained directly from [10, 11] on the Euclidean geometric heat flow, and [3] on more general curvature dependent flows [28].

## 2.1 Euclidean Geometric Heat Flow without Shrinkage

In the previous section, we described the Euclidean geometric heat flow, which can be used to replace the classical heat flow or Gaussian filtering in order to obtain an intrinsic scale-space for planar curves. We show now how to modify this flow in order to keep the area enclosed by the evolving curve constant.

A curve deforming according to the classical heat flow shrinks. This is due to the fact that the Gaussian filter also affects low frequencies of the curve coordinate functions [18]. Oliensis [18] proposed to change the Gaussian kernel by a filter which is closer to the ideal low pass filter. This way, low frequencies are less affected, and less shrinkage is obtained. With this approach, which is also non-intrinsic, the semi-group property holds just approximately. Note that in [1, 5, 31] it was proved that filtering with a Gaussian kernel is the unique linear operation for which the causality criterion holds, i.e., zero-crossings (or maxima) are not created at non-zero scales. Therefore, the approach presented in [18], which is closed related to wavelet approaches, violates this important principle.

Lowe [16] proposes to estimate the amount of shrinkage and to compensate for it. The estimate is based on the amount of smoothing ($\sigma$) and the curvature. This approach, which only reduces the shrinkage problem, is again non-intrinsic, since it is based on Gaussian filtering, and works only for small rates of change. The semi-group property is violated as well.

Horn and Weldon [12] also investigated the shrinkage problem, but only for convex curves. In their approach, the curve is represented by its extended circular image, which is the radius of curvature of the given curve as a function of the curve orientation. The scale-space is obtained by filtering this representation.

We now show how to solve the shrinkage problem with the Euclidean geometric heat flow. It is important to know that in the approach proposed below, the enclosed area is conserved exactly.

When a closed curve evolves according to (2), it is easy to prove [9] that the enclosed area $\mathbf{A}$ evolves according to

$$\frac{\partial \mathbf{A}}{\partial t} = - \oint \beta dv. \tag{5}$$

Therefore, in the case of the Euclidean geometric heat flow we obtain ($\beta = \kappa$)

$$\frac{\partial \mathbf{A}}{\partial t} = -2\pi \quad , \quad \mathbf{A}(t) = \mathbf{A}_0 - 2\pi t, \tag{6}$$

where $\mathbf{A}_0$ is the area enclosed by the initial curve $\mathcal{C}_0$. As pointed out in [9, 10, 11], curves evolving according to (3) can be normalized in order to keep constant area. The normalization process is given by a change of the time scale, from $t$ to $\tau$, such that a new curve is obtained via

$$\tilde{\mathcal{C}}(\tau) := \psi(t)\,\mathcal{C}(t), \tag{7}$$

where $\psi(t)$ represents the normalization factor (time scaling). (The equation can be normalized so that the point $\mathcal{P}$ to which $\mathcal{C}(t)$ shrinks is taken as the origin.) In the Euclidean case, $\psi(t)$ is selected such that $\psi^2(t) = \frac{\partial \tau}{\partial t}$.

The new time scale $\tau$ must be chosen to obtain $\tilde{\mathbf{A}}_\tau \equiv 0$. Define the collapse time $T$, such that $\lim_{t\to T} \mathbf{A}(t) \equiv 0$. Then, $T = \frac{\mathbf{A}_0}{2\pi}$. Let

$$\tau(t) = -T \ln(T - t). \tag{8}$$

Then, since the area of $\tilde{\mathcal{C}}$ and $\mathcal{C}$ are related by the square of the normalization factor $\psi(t) = \left(\frac{\partial\tau}{\partial t}\right)^{1/2}$, $\tilde{\mathbf{A}}_\tau \equiv 0$ for the time scaling given by (8). The evolution of $\tilde{\mathcal{C}}$ is obtained from the evolution of $\mathcal{C}$ and the time scaling given by (8). Taking partial derivatives in (7) we have

$$\frac{\partial\tilde{\mathcal{C}}}{\partial\tau} = \frac{\partial t}{\partial\tau}\frac{\partial\tilde{\mathcal{C}}}{\partial t} = \psi^{-2}(\psi_t\mathcal{C} + \psi\mathcal{C}_t) = \psi^{-2}\psi_t\mathcal{C} + \psi^{-1}\kappa\mathcal{N} = \psi^{-3}\psi_t\tilde{\mathcal{C}} + \tilde{\kappa}\mathcal{N}.$$

From previous Section we know that the flow above is geometric equivalent to

$$\frac{\partial\tilde{\mathcal{C}}}{\partial\tau} = \psi^{-3}\psi_t < \tilde{\mathcal{C}}, \mathcal{N} > \mathcal{N} + \tilde{\kappa}\mathcal{N}. \tag{9}$$

Define the *support function* as $\rho := - < \mathcal{C}, \mathcal{N} >$. Then, it is easy to show that $\mathbf{A} = \frac{1}{2}\oint \rho dv$. Therefore, applying the general area evolution equation (5) to the flow (9), together with the constraint $\tilde{\mathbf{A}}_\tau \equiv 0$ ($\tilde{\mathbf{A}}(\tau) \equiv \mathbf{A}_0$), we obtain

$$\frac{\partial\tilde{\mathcal{C}}}{\partial\tau}(p,\tau) = \left(\tilde{\kappa} - \frac{\pi\tilde{\rho}}{\mathbf{A}_0}\right)\mathcal{N}, \tag{10}$$

which gives a local, area preserving, flow. Note that the flow exists for all $0 \leq \tau < \infty$. Since $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are related by dilations, the flows (3) and (10) have the same geometric properties [9, 10, 11, 28]. In particular, since a curve evolving according to the Euclidean heat flow satisfies all the required properties of a multi-scale representation, so does the normalized flow. See Figure 1, where the flow is implemented using the algorithm proposed in [20] for curve evolution.

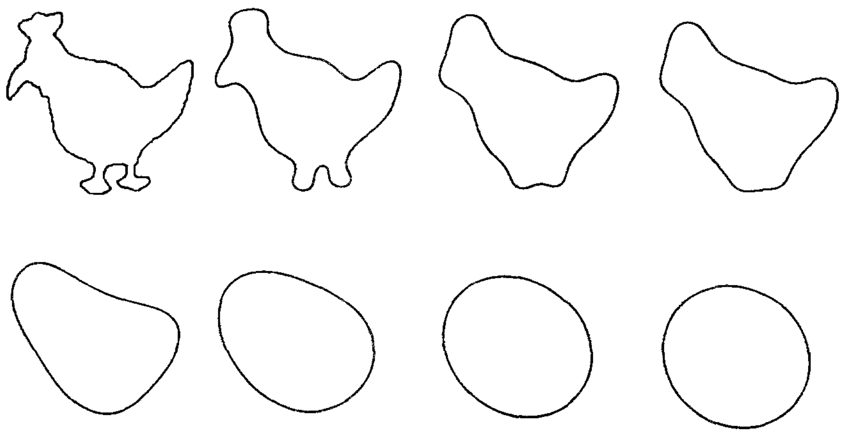

**Figure 1. Example of the area preserving Euclidean heat flow.**

## 3 Affine Geometric Heat Flow

We present now the affine invariant evolution analogue of (3) (or (4)). For details see [24, 25, 27].

Following [24], we first consider the affine analogue of the Euclidean heat flow for convex initial curves. Let $s$ be the *affine arc-length* [6], i.e., the simplest affine invariant parametrization. In this case, with $\mathcal{C}_0$ as initial curve, $\mathcal{C}(p,t)$ satisfies the following evolution equation (compare with equation (4)):

$$\frac{\partial \mathcal{C}(p,t)}{\partial t} = \mathcal{C}_{ss}(p,t). \tag{11}$$

Since the affine normal $\mathcal{C}_{ss}$ is affine invariant, so is the flow (11). This flow was first presented and analyzed in [24]. We proved that, in analogy with the Euclidean heat flow, any convex curve converges to an elliptical point when evolving according to it (the curve remains convex as well). For other properties of the flow, see [24].

To complete the analogy between the Euclidean geometric heat flow (4), and the affine one given by (11), the theory must be extended to non-convex curves. In order to perform the extension, we have to overcome the problem of the "non-existence" of affine differential geometry for non-convex curves. We carry this out now. See [25, 27] for details. Assume now that the family of curves $\mathcal{C}(p,t)$ evolves according to the flow

$$\frac{\partial \mathcal{C}(p,t)}{\partial t} = \begin{cases} 0 & p \text{ inflection point}, \\ \mathcal{C}_{ss} & p \text{ non-inflection point}, \end{cases} \tag{12}$$

with the corresponding initial condition $\mathcal{C}(p,0) = \mathcal{C}_0(p)$. Since $\mathcal{C}_{ss}$ exists for all non-inflection points [6], (12) is well defined also for non-convex curves. Also, due to the affine invariance property of the inflection points, (12) is affine invariant.

We already know that if we are interested only in the geometry of the curve, i.e., Img[$\mathcal{C}$], we can consider just the Euclidean normal component of the velocity in (11). In [24], it was proved that the Euclidean normal component of $\mathcal{C}_{ss}$ is equal to $\kappa^{1/3}$. Then, for a convex initial curve, Img[$\mathcal{C}(p,t)$] = Img[$\hat{\mathcal{C}}(w,t)$], where $\mathcal{C}(p,t)$ is the solution of (11), and $\hat{\mathcal{C}}(w,t)$ is the solution of $\hat{\mathcal{C}}_t = \kappa^{1/3}\mathcal{N}$. Since for an inflection point $q \in \mathcal{C}$, we have $\kappa^{1/3}(q) = 0$, the evolution given by (12) is the natural extension of the affine curve flow of convex curves given by equation (11). Then, equation (12) is transformed into

$$\mathcal{C}_t = \kappa^{1/3}\mathcal{N}. \tag{13}$$

If $\mathcal{C}$ is the solution of (12) and $\hat{\mathcal{C}}$ is the one of (13), Img[$\mathcal{C}$] = Img[$\hat{\mathcal{C}}$], and Img[$\hat{\mathcal{C}}$] is an affine invariant solution of the evolution (13). Note that the image of the curve is affine invariant, not the curve itself.

In [4, 27], we have proved that any smooth and simple non-convex curve evolving according to (13) (or (12)), remains smooth and simple, and becomes convex. From there, it converges into an ellipse from the results described above.

In [1], the authors showed that under certain assumptions, equation (13), when regarded as the flow of the level sets of a 3D image, is unique in its affine invariance property. The uniqueness was also proved by us in [19], based on symmetry groups. In [4], among other results, we also extended the flow to initial Lipschitz curves.

We have showed that the flow given by (12) (or (13)) is the (unique) affine analogue of the Euclidean geometric heat flow given by (4). This evolution is called the *affine geometric heat flow.* It defines an intrinsic, geometric, affine invariant multi-scale representation for planar curves. In [25], we analyzed this flow and showed that the multi-scale representation which we obtained, satisfies all the required scale-space properties. Affine invariant smoothing examples can be found in [25] as well. See also [26] for applications of this flow to image processing.

### 3.1 Affine Geometric Heat Flow Without Shrinkage

From the general evolution equation for areas (5) we have that when a curve evolves according to (13), the evolution of the enclosed area is given by $\mathbf{A}_t = -\oint \kappa^{1/3} dv$. Define the *affine perimeter* as $\mathbf{L} := \oint [\mathcal{C}_p, \mathcal{C}_{pp}]^{1/3} dp$ [6]. Then it is easy to show that $\mathbf{L} = \oint \kappa^{1/3} dv$ [24], and

$$\mathbf{A}_t = -\mathbf{L}. \tag{14}$$

As in the Euclidean case, we define a normalized curve $\tilde{\mathcal{C}}(\tau) := \psi(t)\,\mathcal{C}(t)$, such that when $\mathcal{C}$ evolves according to (13), $\tilde{\mathcal{C}}$ encloses a constant area. In this case, the time scaling is chosen such that

$$\frac{\partial \tau}{\partial t} = \psi^{4/3}. \tag{15}$$

(We see from the Euclidean and affine examples that in general, the exponent $\lambda$ in $\frac{\partial \tau}{\partial t} = \psi^{\lambda}$ is chosen such that $\tilde{\beta} = \psi^{1-\lambda}\beta$.) Taking partial derivatives, using the relations (5), (14), and (15), and constraining $\tilde{\mathbf{A}}_\tau \equiv 0$ ($\tilde{\mathbf{A}}(\tau) \equiv \mathbf{A}_0$), we obtain the following geometric affine invariant, area preserving, flow:

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau} = \left( \tilde{\kappa}^{1/3} - \frac{\tilde{\rho}\tilde{\mathbf{L}}}{2\mathbf{A}_0} \right) \mathcal{N}. \tag{16}$$

Note that in contrast with the Euclidean area preserving flow given by equation (10), the affine one is not local. This is due to the fact that the rate of area change in the Euclidean case is constant, but in the affine case it depends on the affine perimeter (which is global).

As in the Euclidean case, the flow (16) satisfies the same geometric properties as the affine geometric heat flow (13). Therefore, it defines a geometric, affine invariant, area preserving multi-scale representation.

Again, based on the theory of viscosity solutions, or in the new results in [4], the flow (13), as well as its normalized version (16), are well defined also for non-smooth curves. Based on the same concepts described above, we showed how to derive invariant geometric heat flows for any Lie group in [27]. In [19] we give the characterization of all invariant flows for subgroups of the projective group and show that the heat flows are the simplest possible. These results are based on classical Lie theory and symmetry groups. The similarity group is also analyzed in detail, including convergence results, in [28].

## 4 Length Preserving Geometric Flows

Similar techniques to those presented in previous sections, can be used in order to keep fixed other curve characteristics, e.g., the Euclidean length $\mathbf{P}$. In this case, when $\mathcal{C}$ evolves according to the general geometric flow $\frac{\partial \mathcal{C}}{\partial t} = \beta \mathcal{N}$, and $\tilde{\mathcal{C}}(\tau) := \psi(t)\,\mathcal{C}(t)$, we obtain the following length preserving geometric flow:

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(\tilde{\beta} - \frac{\oint \tilde{\beta}\tilde{\kappa}}{\mathbf{P}_0}\tilde{\rho}\right)\mathcal{N}. \tag{17}$$

The computation of (17) is performed again taking partial derivatives and using the relations $\mathbf{P}_t = -\oint \beta\kappa dv$, $\mathbf{P} = \oint \kappa\rho dv$, together with the constraint $\tilde{\mathbf{P}}_\tau \equiv 0$.

Since the similarity flow (scale invariant) is given by $\mathcal{C}_t = \kappa^{-1}\mathcal{N}$ [28], its length preserving analogue is $\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(\tilde{\kappa}^{-1} - \tilde{\rho}\right)\mathcal{N}$, and the flow is completely local. Another local, length preserving flow may be obtained for the Euclidean constant motion given by $\mathcal{C}_t = \mathcal{N}$. This flow models morphological dilation with a disk [23]. In this case, the rate of change of length is constant and the length preserving flow is given by $\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(1 - \frac{2\pi\tilde{\rho}}{\mathbf{P}_0}\right)\mathcal{N}$, see Figure 2. A smooth initial curve evolving with constant motion can develop singularities [1, 13, 23, 27], and the physically correct weak solution of the flow is the viscosity (or *entropy*) one [1, 23].

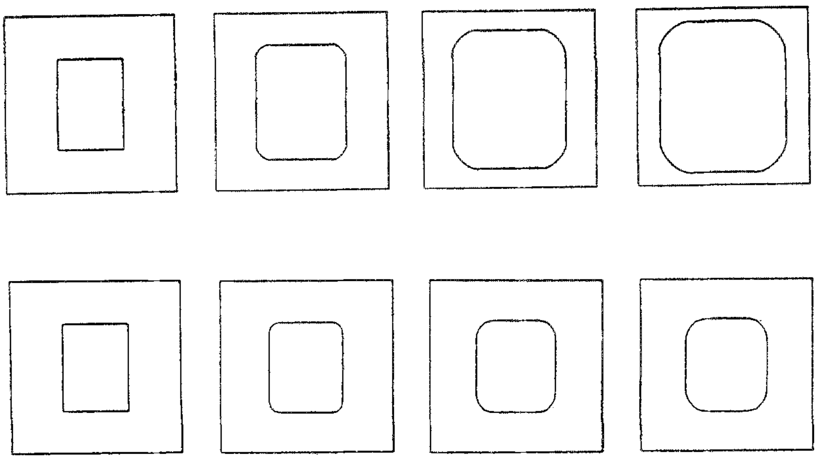

**Figure 2. Euclidean constant motion and area preserving form.**

## 5 Concluding Remarks

In this paper, area preserving multi-scale representations for planar shapes were described. The representations are obtained deforming the curve via the invariant geometric heat flows while simultaneously magnifying the plane by a homethety which keeps the enclosed area constant. The flow is geometrically intrinsic to the curve, and exactly satisfies all the required properties of scale-spaces. For the Euclidean case for example, the flow is local as well.

The same approach was used to derive length preserving geometric flows. In this case, locality is obtained for example for the similarity heat flow and the Euclidean constant motion. Similar techniques can be used in order to keep other curve characteristics constant, and to transform other geometric scale-spaces [19, 27], into analogous area or length preserving ones.

Different area or length preserving flows can be proposed. In [9, 22], non-local preserving flows are presented motivated by physical phenomena models. The advantage of the approach here described is that the non-shrinking curve is obtained by a homothety, and the resulting normalized flow keeps all the geometric properties of the original one. The flow is also local in some cases.

In [2], the importance of the Euclidean geometric heat flow for image enhancement was demonstrated. This was extended for the affine geometric heat flow in [1, 26]. We are currently investigating the use of the corresponding area (or length) preserving flows for this application as well.

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