Intrinsic Stabilizers of Planar Curves

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Abstract. Regularization offers a powerful framework for signal reconstruction by enforcing weak constraints through the use of stabilizers. Stabilizers are functionals measuring the degree of smoothness of a surface. The nature of those functionals constrains the properties of the reconstructed signal. In this paper, we first analyze the invariance of stabilizers with respect to size, transformation and their ability to control scale at which the smoothness is evaluated. Tikhonov stabilizers are widely used in computer vision, even though they do not incorporate any notion of scale and may result in serious shape distortion. We first introduce an extension of Tikhonov stabilizers that offers natural scale control of regularity. We then introduce the intrinsic stabilizers for planar curves that apply smoothness constraints on the curvature profile instead of the parameter space.

1 Introduction

Most tasks in computer vision can be described as inferring geometric and physical properties of three dimensional objects from two dimensional images. A characteristic of those inverse problems is their *ill-posed* nature[PT84]. Assumptions about the scene, such as *smoothness* or *shape* must be made to retain the "best" solution within the range of prior knowledge. Regularization transforms an ill-posed problem into a well-posed minimization problem by constraining the solution to belong to a set of allowed functions. If the problem is formalized as $A\nu = d$, where A is an operator describing the image formation process and d is a function describing the data extracted from the image, then the regularized problem consists in minimizing a functional of the form[BTT87]:

$$E(\nu) = \lambda \cdot S(\nu) + D(\nu) = \lambda ||P\nu||_1^2 + ||A\nu - d||_2^2 \tag{1}$$

$||P\nu||_1^2$ evaluates the smoothness of the solution ν and is called a *stabilizing functional* or *stabilizers.* $||A\nu - d||_2^2$ evaluates the distance between the solution to the data. The regularizing parameter λ weights the relative importance of smoothness with respect to the closeness of fit.

Variational principles involving smoothness constraints are widely used in computer vision ranging from surface reconstruction[BK86], segmentation with active contours[KWT88] and surfaces[DHI91b]. Geometric modeling primitives such as splines under tension [Sch66], Beta-Spline[BT83] proposed in computer-aided-design are derived from variational principles similar to the interpolation approach of regularization.

In this paper, we first analyze the different smoothness measures with regard to five criteria of invariance. Then, we extend the notion of *stabilizing functionals* to *differential stabilizers* by transforming the variational principle of equation (1) into the problem of solving a differential equation. Finally, we propose a generalization of Tikhonov stabilizers that provides both spatial control of the smoothness constraint and intrinsic shape formulation.

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2 Smoothness Measures

2.1 Invariance

We have retained five criteria that characterize the notion of smoothness as it is generally conceived for the human perception of shape:

- **Invariance with rigid motion.** For all isometries T, a smoothness measure $S(\nu)$ should verify: $S(T\nu) = S(\nu)$.
- **Invariance with size.** The smoothness of an object is independent on how far the viewer is from the object, assuming an infinite perceptual resolution. Therefore, a smoothness measure should verify: $S(l\nu) = S(\nu), \forall l \in \mathbb{R}$.
- **Invariance with respect to parameterization.** Shape is clearly independent of the way a curve or surface is described but relies only on its intrinsic geometric parameters. We would therefore expect for every mapping $\mathcal{M}$ from $\Omega_w \subset \mathbb{R}^d$, $(d = 1$ or $2)$, to $\Omega_u \subset \mathbb{R}^d$, that $S(\nu(u)) = S(\nu(\mathcal{M}(w)))$.
- **Dependance with inner-scale.** Smoothness is clearly relative to the scale at which it is considered. A sensible smoothness measure should therefore be a function of scale.
- **Sphere Invariance.** This criterion states that circles and spheres should be among the curves or surfaces of least energy. Besides that spheres enclose the notion of ideal shape, this criterion ensures natural deformations against external constraints. For instance, if a stabilizer does not accept circles as optimum, the approximating spline minimizing equation 1 would be a circle, generally of smaller radius. Consequently, the spline will tend to consistently deform toward its center of curvature, especially where the curvature is high. This smoothing distortion is known as the "shrinking effect". Several methods have been proposed to overcome this undesirable effect of linear smoothing: Lowe[Low88] and Oliensis[Oli93] studied algorithms for compensating the shrinkage entailed by Gaussian smoothing.

2.2 Quadratic Smoothness Measure

Most regularized problems in computer vision, are based on a quadratic smoothness measure. The first advantage of quadratic measures is that functional analysis provides a solid theoretical framework for studying convexity, stability and convergence. The corresponding Euler-Lagrange equation is a quasi-linear differential equations and in the particular case of the interpolation and approximation surface reconstruction problem, the analytical form of solutions are known explicitly. Let $S(\nu) = \int_{\mathbb{R}^d} (P\nu)^2 du$ be a quadratic functional over a set of multidimensional function $\nu : \mathbb{R}^d \longrightarrow \mathbb{R}^p$. P is a linear, symmetric, and translation invariant operator and therefore the functional may be written as $S(\nu) = \int_{\mathbb{R}^d} |\tilde{p}(s)|^2 |\tilde{\nu}(s)|^2 ds$ where $\tilde{\nu}(s)$ is the Fourier transform of $\nu(u)$. The measure $S(\nu)$ can be interpreted as the power signal of the transformed signal in the frequency domain. When P is a high pass filter, and under unrestrictive conditions, $S(\nu)$ is a semi-norm over a well-defined class of functions $\mathcal{F}$, with a finite dimensional null space[GJP93].

Tikhonov and Arsenin[TA87] used the qth-order weighted Sobolev semi-norms restricted on Sobolev spaces as a stabilizing functional for regularizing an ill-posed problem. The qth-order weighted multivariate formulation generalized by Duchon[Duc77] writes as:

$$S(\nu) = \sum_{m=0}^{q} \int_{\mathbb{R}^d} w_m(u) \sum_{j_1+\ldots+j_d=m}^{d} \frac{m!}{j_1!\ldots j_d!} \left(\frac{\partial^m \nu(u)}{\partial u_1^{j_1} \ldots \partial u_d^{j_d}} \right)^2 du \qquad (2)$$

where $w_m(u)$are non negative functions that control the non-homogeneity or the continuity of the surface.

2.3 Harmonic Functions

Curves of surfaces minimizing the Tikhonov stabilizers are harmonic or iterated harmonic functions. Harmonic functions correspond to the "most conservative" interpolation possible in terms of parameterization. Harmonic functions have the unique property that the value at the center of a ball in the parameter space is equal to the mean value taken over the ball :

$$\forall R \in \mathbb{R}^+, \forall u \in \mathbb{R}^d \ \nu(u) = \frac{1}{A(\mathcal{B}_R^u)} \int_{\mathcal{B}_R^u} \nu(v) d\mathcal{B}_R^u \qquad (3)$$

where $\mathcal{B}_R^u$ is the ball of radius R centered on u. This mean value property uniquely characterizes harmonic functions and indeed corresponds to a highly desirable property for solving interpolation problems. The mean value property may be expressed too in terms of mean value over a sphere $\mathcal{S}_R^u$ centered on u rather than over a ball $\mathcal{B}_R^u$.

2.4 Invariance of Tikhonov Stabilizers

Tikhonov stabilizers have the following properties:

- **Invariance with rigid motion**. The multivariate Tikhonov stabilizers have been especially designed for their isometric invariance.
- **Dependence on size**. For all stabilizers $E(l\nu) = l^2 E(\nu)$. However, for a solution $\nu^*$of a given set of data constraints and end conditions, the scaled solution is solution of the scaled problem.
- **Dependence on parameterization**. Tikhonov stabilizers are not posed in terms of intrinsic parameters and consequently fairness of the reconstructed surfaces is not guaranteed.
- **Independence with inner-scale**. The smoothness measure is estimated on infinitely small neighborhood around each point of a surface. The regularization parameter λ weights the smoothing effect on the regularized surface and thereupon controls the scale at which the surface is smoothed. However, it couples both notion of "scale" and "closeness of fit" that are clearly distinct.
- **Spheres are not optimal**. Circles and Spheres do not minimize the Tikhonov smoothness measures. Furthermore, in [DHI91a], we have proved that none of the quadratic stabilizers accept circles as optimal curves. Consequently, shrinkage is inherent to linear filtering.

2.5 Physically-based Smoothness Functionals

Many natural phenomena may be modelled through variational principles and the energy of deformations of physical system may be used as smoothness measures. For instance, an elastic spanned between two points reaches its equilibrium when minimizing its length:

$$S(\nu) = \int_0^{u_0} \|\nu_u\| du$$

The first variation of this first order stabilizing functional is $\delta S(\nu) = \frac{d}{du}\mathbf{T}$ and curves of least energy are lines.

The mechanical spline energy is derived from the physical deformation of a thin beam attached at specified points:

$$S(\nu) = \int_0^{u_0} k^2(u)ds = \int_0^{u_0} \frac{(x_u y_{uu} - y_u x_{uu})^2}{(x_u^2 + y_u^2)^{5/2}} du \tag{4}$$

This energy was proposed by Blake and Zisserman[Bla87] to achieve a view-point invariant surface reconstruction. Curves minimizing the sum of their square curvature or *mechanical splines* have been studied by many authors including Horn[Hor83] and they verify the following intrinsic equation:

$$\delta S(\nu) = \frac{d}{du}\left[k^2\mathbf{T} + 2\frac{dk}{ds}\mathbf{N}\right] = \frac{ds}{du}\left(k^3 + 2\frac{d^2k}{ds^2}\right)\mathbf{N} = 0$$

This intrinsic smoothness functional does not accept circles as optimal curves and furthermore is not size invariant.

3 Differential Stabilizer

A *necessary* condition for ν to minimize $E(\nu) = \lambda \cdot S(\nu) + D(\nu)$ is the vanishing of the first variation $\delta E(\nu) = \lambda \cdot \delta S(\nu) + \delta D(\nu) = 0$. Since $E(\nu)$ is formulated as a variational principal, $\delta E(\nu)$ is derived through the Euler-Lagrange equation. In general, solutions of a variational problem are recovered by solving the associated Euler-Lagrange equation, hence making abstraction of the actual minimization problem. In practice, the energy to minimize in non-convex, and the solution of Euler-Lagrange equation leads local minima.

It is therefore natural to extend the framework of regularization by replacing the *necessary* condition $\lambda \cdot \delta S(\nu) + \delta D(\nu) = 0$ by the more general condition

$$\lambda \cdot \sigma(\nu) + \delta D(\nu) = 0 \tag{5}$$

where:

- $\sigma(\nu)$ is an operator from a specified functional space $\mathcal{F}$ into $\mathcal{F}$. We will call $\sigma(\nu)$ a *Differential Stabilizer* (DS).
- $\delta D(\nu)$ is the first variation of $D(\nu) = \|A\nu - d\|^2$.

We will call *stabilization* the transformation of the problem $A\nu = d$ into the following problem:

$$\begin{array}{c} \text{Among all } \nu \in \mathcal{F}, \text{ that verify} \lambda \cdot \sigma(\nu) + \delta D(\nu) = 0 \\ \text{Find } \nu^* \text{that minimizes :} \\ C(\nu) = \int_{\mathbb{R}^d} \sigma(\nu)\nu du + \int_{\mathbb{R}^d} |A\nu(u) - d(u)|^2 du \end{array} \quad (6)$$

Instead of solving a minimization problem , stabilization proposes to solve the differential equation $\lambda \cdot \sigma(\nu) + \delta D(\nu) = 0$, and then to discriminate among solutions by minimizing the cost function $C(\nu)$. In general, stabilization is not equivalent to minimizing the cost function $C(\nu)$. However, when the differential stabilizer $\sigma(\nu)$ is a linear, symmetric and positive operator on a Hilbert space, then $\sigma(\nu)$ corresponds to the first variation of the functional $S(\nu) = \int_{\mathbb{R}^d} \sigma(\nu)\nu du$ and hence stabilization is equivalent to regularization.

The incentive behind stabilization is to provide a wider range of smoothness functional for solving inverse problems. We can justify this approach with an analogy with mechanics theory. The laws of mechanics are based on the minimization of the Lagrangian $L = T - U$ where T is the kinetic energy and U the total potential energy of the system. The Euler-Lagrange equation corresponding to the minimization of L is the law of motion $m\Gamma = \mathbf{F}$. However, some forces do not derived from a potential field such as viscous or friction forces, such that it is not always possible to set the problem in terms of minimization of energy but only in terms of force equilibrium. Hence, the differential stabilizer $\sigma(\nu)$ may be seen as an internal force enforcing shape constraints while $\delta D(\nu)$ may be seen as an external force enforcing accuracy.

Several properties are desirable for a DS to render feasible and computable solutions. In addition to invariance with rigid motion, size, parameterization, we add the notion of sphere invariance as well as stability and convergence.

4 Intrinsic Polynomial Stabilizer

4.1 Controlled-Scale Extensions of Tikhonov Stabilizers

We now propose an extension of the Tikhonov functionals described in section 2.2 by introducing the notion of "scale-sensitive derivatives". For instance, we can evaluate the first derivative on a curve $\nu(u)$ at different scale with the ratio $(\nu(u+r) - \nu(u-r))/2r$ where r controls the scale at which we consider the curve geometry. A smoothness measure of the first order at scale r on closed curves then writes as:

$$S(\nu) = \int_{\Omega} \frac{(\nu(u+r) - \nu(u-r))^2}{4r^2} du$$
$$\delta S(\nu) = \frac{\nu(u)}{2r^2} - \frac{\nu(u+2r) + \nu(u-2r)}{4r^2}$$

The curves of least energy verifying $\nu(u) = \frac{\nu(u+2r)+\nu(u-2r)}{4r}$, are therefore harmonic, i.e. lines for a univariate function. we further extent the Tikhonov stabilizers by allowing the scale parameter r to vary spatially along the curve. In general, the scale parameter should be large at the center of the set Ω where the surface is defined and should be decreasing near the boundary $\partial\Omega$. Table 1 summarizes the different controlled-scale differential stabilizers generalizing Tikhonov functionals.

Controlled-Scale Weak String	$\sigma(\nu) = \frac{2}{r^2(u)}\left(\nu(u) - \frac{\nu(u+r(u)) + \nu(u-r(u))}{2}\right)$
Controlled-Scale Thin Rod	$\sigma(\nu) = -\frac{2}{r^2(u)}\left(\nu_{uu}(u) - \frac{\nu_{uu}(u+r(u)) + \nu_{uu}(u-r(u))}{2}\right)$
Controlled-Scale Membrane	$\sigma(\nu) = \frac{4}{r(u)}\left(\nu(u) - \frac{\int_{\mathcal{S}_u^{r(u)}} \nu(u)du}{2\pi r(u)}\right)$
Controlled-Scale Thin Plate	$\sigma(\nu) = -\frac{4}{r(u)}\left(\Delta\nu(u) - \frac{\int_{\mathcal{S}_u^{r(u)}} \Delta\nu(u)du}{2\pi r(u)}\right)$

Table 1. The controlled-scale extensions of Tikhonov stabilizers

Those *controlled-scale differential stabilizers* fully generalizes the Tikhonov stabilizers since they converge toward the Tikhonov stabilizers as $r(u)$ converges toward zero.

Using an analogy with mechanics, those "smoothing forces" can be interpreted as spring forces exerted between a surface point and the centroid of the curve $\nu(v), v \in \mathcal{S}_u^{r(u)}$. Instead of considering the centroid of the curve $\nu(v)$ surrounding a point, we can consider the centroid of the area it encloses. We then obtain another set of smoothing functionals that rely on the same notion of "scaled derivatives", but leads to smoother deformations because it averages over a larger extent. The *uniform controlled-scale differential stabilizers* are defined as:

Uniform Controlled-Scale Weak String	$\sigma(\nu) = \nu(u) - \frac{\int_{u-r(u)}^{u+r(u)} \nu(v)dv}{2r(u)}$
Uniform Controlled-Scale Thin Rod	$\sigma(\nu) = -\nu_{uu}(u) + \frac{\int_{u-r(u)}^{u+r(u)} \nu_{uu}(v)dv}{2r(u)}$
Uniform Controlled-Scale Membrane	$\sigma(\nu) = \nu(u) - \frac{\int_{\mathcal{B}_u^{r(u)}} \nu(u)du}{4\pi r^2(u)}$
Uniform Controlled-Scale Thin Plate	$-\sigma(\nu) = \Delta\nu(u) + \frac{\int_{\mathcal{B}_u^{r(u)}} \Delta\nu(u)du}{\pi r^2(u)}$

Table 2. The uniform controlled-scale extensions of Tikhonov stabilizers

Solutions of the differential equation 5 typically use finite differences or finite elements methods with iterative schemes such as Gauss-Seidel relaxation. Controlled-scale stabilizers involve inverting a banded positive definite matrix whose bandwidth depends on the scale parameter $r(u)$. The computational complexity for solving those systems is the same that for regular Tikhonov stabilizers but the rate of convergence is significantly increased since constraints propagate faster along the curve.

For sparse data approximation, smoothness should not be evaluated over the discontinuity entailed by each data constraint. For appropriate approximation over data points P_i, the scale parameters r_i should be picked such that smoothing does not occur across discontinuities (see Figure (1)).

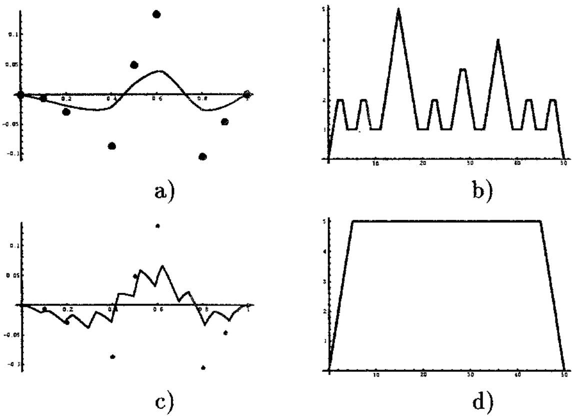


Fig. 1. a) Approximation of data points with the controlled-scale thin rod stabilizers and varying scale parameter; b) Distribution of the scale parameters along the curve. The parameter at each "attached" nodes is one, and vary linearly otherwise; c) Result of the same approximation with almost constant scale parameters ; d) Distribution of the scale parameters corresponding to c): $r_i = Min(5, i, N-i)$

4.2 Intrinsic Polynomial Stabilizer

The Intrinsic Polynomial Stabilizer[DHI91a] (IPS) are differential stabilizers acting on planar curves. They are invariant to rigid motion, parameterization, they are scale sensitive and they accept circles as optimal curves. Another interesting feature is their intrinsic nature which makes them sensitive to shape regardless of the parameterization. Our approach consists in linearly filtering the curvature space instead of linearly filtering the parameter space.

More precisely, given a curve $\nu(u)$, we choose to filter the derivative of the tangent polar angle $\frac{d\phi}{du} = k(u)\frac{ds}{du}$. Given a differential stabilizer $\sigma_1(\frac{d\phi}{du})$ applied on the rate of turn $\frac{d\phi}{du}(u)$, we define a differential stabilizer σ applied on the parametric equation:

$$\sigma(\nu)(u) = \frac{d^2 s}{du^2}\mathbf{T} + \frac{ds}{du}\sigma_1(\frac{d\phi}{du})\mathbf{N} \tag{7}$$

The *Intrinsic Polynomial Stabilizers* are derived directly from equation (7), with σ_1 corresponding to uniform controlled-scale differential stabilizers of different orders:

IPS order zero $$\sigma_{IPS0}(\nu) = \frac{d^2 s}{du^2}\mathbf{T} \tag{8}$$

IPS order one $$\sigma_{IPS1}(\nu) = \frac{d^2 s}{du^2}\mathbf{T} + \frac{ds}{du}\frac{d\phi}{du}\mathbf{N} \tag{9}$$

IPS order two $$\sigma_{IPS2}(\nu) = \frac{d^2 s}{du^2}\mathbf{T} + \frac{ds}{du}\left(\frac{d\phi}{du}(u) - \frac{\int_{u-r(u)}^{u+r(u)} \frac{d\phi}{du}(v)dv}{2r(u)}\right)\mathbf{N} \tag{10}$$

The IPS of order one correspond to the weak string differential stabilizer. The curves that nullify the IPS of order n verify both $\frac{d^2 s}{du^2} = 0$ and $\sigma_1(\frac{d\phi}{du}) = 0$

and therefore are curves whose curvature profile is a polynomial of degree $2n-3$ of the arc-length. For $n=0$, the "smoothest" curve verify only $\frac{d^2s}{du^2}=0$ which does not constraint the shape of a curve, only its parameterization. For second order stabilizers the curve of least energy are *Cornu's Spirals* or *Clothoids*.

Intrinsic Polynomial Stabilizers can be seen as merely scale-sensitive Tikhonov stabilizers regularizing the curvature profile instead of the parametric equation. They are circle-invariant which prevents any "shrinking effect" during filtering.

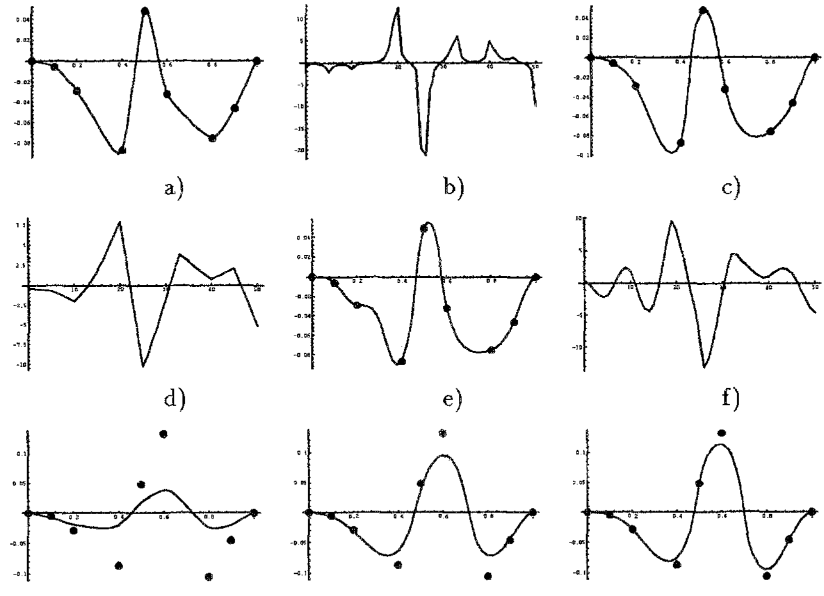


Fig. 2. a) Interpolation with the thin rod stabilizer; b) Its curvature profile; c) Interpolation with IPS of order two; the curve is C^2 continuous; d) Its curvature profile is piecewise linear; e) Interpolation with IPS of order three the curve is C^4; f) Its curvature is piecewise cubic; g) Approximation with a thin rod; h) Approximation with IPS of order one; i) Approximation with IPS of order two (same regularization parameter)

We use an explicit finite difference scheme for solving approximation, interpolation, and segmentation problem. The expression of the stabilizer is simple enough to render real-time deformations of an active contour on a Sun4 workstation. Figure (2) compares the interpolation and approximation solutions for the thin rod, IPS of order two and IPS of order three. The curvature profile shows clearly that IPS release smoother and natural-looking shapes than the linear thin rod stabilizer.

4.3 Shape constraints

Another interesting type of internal constraints for solving computer vision problems, is shape. For instance, in order to track deformable object, one would like to have a template with enough shape constraints for correctly matching the target but with enough flexibility to adapt to perspective distortion and target deformation[BACZ93]. Weighted Intrinsic Polynomial Stabilizers create complex-shaped deformable templates with controlled-rigidity. Those templates naturally converge toward their initial shape when not submitted to any external constraints.

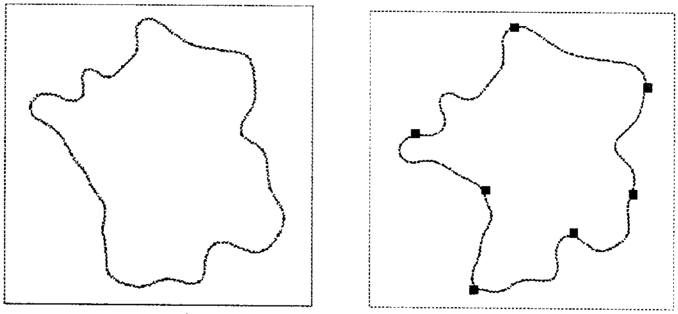
Given a curve and its curvature profile: $k = f(s)$, we first determine the extrema of curvature. If we compute the weight function as $w(u) = 1/|f'(u)|$, then solutions of the weighted weak membrane differential stabilizer $\sigma(\nu) = \frac{d}{du}[w(u)\nu_u] = 0$ between two extrema are the functions $\nu(u) = af(u) + b$. A stabilizer enforcing shape prior on a contour is defined as following:

$$\sigma(\nu)(u) = \frac{d^2 s}{du^2}\mathbf{T} + \frac{ds}{du}\sigma_1(\frac{d\phi}{du})\mathbf{N}$$

with $\sigma_1(\frac{d\phi}{du})$ equals to:

- $\frac{d}{du}[w(u)\frac{d^2\phi}{du^2}]$ with $w(u) = 1/|f'(u)|$ if $f(u)$ is between two extrema.
- $f(u) - \frac{d\phi}{du}$ if $f(u)$ is an extremum of curvature.

This method applies to any C^2 continuous contour. The previous stabilizer can be extended further by integrating for notion of scale at which the shape is defined. In Figure (3), we use the smoothed shape of France to illustrate the shape prior ability of intrinsic stabilizers. After constraining the position of seven nodes, the curve reaches a state of equilibrium with a trade-off between natural shape and closeness of fit.



a) b)

Fig. 3. a)Initial curve with its rest shape; b) Curve solution of an approximation problem under the influence of the weighted intrinsic polynomial stabilizer. The curve is constrained by seven springs attached to the black squares. Under the influence of the stabilizer, the curve shape is similar to its prior shape.

5 Conclusion

The controlled-scale stabilizers, on one hand, provide an additional set of parameters, the scale parameters, that influences both the convergence rate and the smoothness of the reconstructed signal. Intrinsic stabilizers on the other hand, provide a complete control of the curvature profile of a curve and consequently its shape. A promising application of shape-control is the creation of smoothly deformable templates for target tracking. Finally, intrinsic splines for which curvature is a polynomial function of arc-length are of great interest for computer-aided design because of their natural appearance.

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