From Principles to Applications

Homotopy in 2-dimensional Digital Images

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Abstract. We recall the basic definitions concerning homotopy in 2D Digital Topology, and we set and prove several results concerning homotopy of subsets. Then we introduce an explicit isomorphism between the fundamental group and a free group. As a consequence, we provide an algorithm for deciding whether two closed path are homotopic.

Key words: Digital topology, homotopy, fundamental group.

Introduction

Homotopy in the framework of Digital Topology is an important question in the field of Image Analysis. In particular, T. Y. Kong introduced a notion of the digital fundamental group in the 3−dimensionnal digital Euclidian space ([1]), and in a more genaral framework ([2]). The purpose of this paper is to study the corresponding notion in the 2−dimensional digital space. First we recall the basic definitions concerning homotopy in 2D Digital Topology. Then we set and prove several results which are required in the sequel concerning homotopy of subsets. Finally we introduce an explicit isomorphism between the fundamental group of any object with m holes and and the free group with m generators. As a consequence, we provide an algorithm for deciding whether two closed path are homotopic or not in a given arbitrary object. The computational complexity of this algorithm is the sum of the lengths of the two considered paths.

1 Basic definitions and notations

If X is a subset of $\mathbb{Z}^2$, we denote $\overline{X} = \mathbb{Z}^2 \backslash X$ the complement of X. In this paper, we shall consider only finite subsets X of $\mathbb{Z}^3$. For $x = (i, j) \in \mathbb{Z}^2$, we consider the two following neighborhoods:
$N_4(x) = \{y = (i', j') \in \mathbb{Z}^2 \ / \ |i - i'| + |j - j'| = 1\}$;
$N_8(x) = \{y = (i', j') \in \mathbb{Z}^2 \ / \ max(|i - i'|, |j - j'|) = 1\}$.
Let $n \in \{4, 8\}$. Two points x and y of $\mathbb{Z}^2$ are said to be $n-$adjacent if $y \in N_n(x)$. This $n-$adjacency relation defines a graph structure on $\mathbb{Z}^2$, called the $n-$adjacency graph. For any subset X of $\mathbb{Z}^2$, $n-$*connected components* of X are connected components of the subgraph of the $n-$adjacency graph induced by X. The set X is said to be $n-$*connected* if it has a single $n-$connected component. As usual, when we analyze a set $X \subset \mathbb{Z}^3$ using an $n-$connectivity type with

$n \in \{4,8\}$, we analyze $\overline{X}$ with a different $\overline{n}$-connectivity with $\overline{n} = 12-n$. In the sequel, we consider $(n,\overline{n}) \in \{(4,8),(8,4)\}$. An n-hole in $X \subset \mathbb{Z}^2$ is a bounded $\overline{n}$-connected component of $\overline{X}$. A *finite n-path* is a finite sequence $(x_0,\ldots,x_p)$ such that for $i \in \{1,\ldots,p\}$ the point x_{i-1} is n-adjacent or equal to x_i. Such a finite n-path is said to be *closed* if $x_0 = x_p$. An *infinite n-path* is a sequence $(x_i)_{i\in\mathbb{N}}$ such that for $i \in \mathbb{N}^*$ the point x_{i-1} is n-adjacent or equal to x_i. Such an infinite n-path is called *simple* if $i \neq j \Longrightarrow x_i \neq x_j$. If π is a finite or infinite n-path of $\mathbb{Z}^2$, we denote by π^* the set of the points of π. We also denote by $\pi * \pi'$ the concatenation of two finite n-paths π and π'. Given an n-path $\pi = (x_0,\ldots,x_p)$, we denote by π^{-1} the n-path $(x_p,\ldots,x_0)$.

Now we need to introduce the *n-homotopy relation* between n-paths. Let us consider $X \subset \mathbb{Z}^2$ and two points $B \in X$ and $B' \in X$. We also consider $A^n_{B,B'}(X)$ the set of all closed n-paths $\pi = (x_0,\ldots,x_p)$ which are included in X and such that $x_0 = B$ and $x_p = B'$. First we introduce the notion of an *elementary deformation.* Two finite n-paths $\pi \in A^n_{B,B'}(X)$ and $\pi' \in A^n_{B,B'}(X)$ are said to be *the same up to an elementary deformation (with fixed extremities)* if they are of the form $\pi = \pi_1 * \gamma * \pi_2$ and $\pi' = \pi_1 * \gamma' * \pi_2$, the n-paths γ and γ' having the same extremities and being both included in a common unit square. Now, the two n-paths $\pi \in A^n_{B,B'}(X)$ and $\pi' \in A^n_{B,B'}(X)$ are said to be *n-homotopic (with fixed extremities) in X* if there exists a finite sequence of n-paths $\pi = \pi_0,\ldots,\pi_m = \pi'$ of $A^n_{B,B'}(X)$ such that for $i = 1,\ldots,m$ the n-paths π_{i-1} and π_i are the same up to an elementary deformation (with fixed extremities).

We denote $A^n_B = A^n_{B,B}$ The homotopy relation defines an equivalence relation on $A^n_B(X)$, and we denote by $\Pi^n_1(X)$ the set of equivalence classes of this equivalence relation. The concatenation of closed n-paths is compatible with the homotopy relation, hence it defines an operation on $\Pi^n_1(X)$, and this operation provides $\Pi^n_1(X)$ with a group structure. We call this group the *n-fundamental group of X*. The n-fundamental group defined using a point B' as the based point is isomorphic to the n-fundamental group defined using a point B as the based point.

Now we consider n-connected sets $X \subset Y \subset \mathbb{Z}^2$. First we observe that a closed n-path in X is also a closed n-path in Y. Moreover, two n-homotopic closed n-paths in X are also n-homotopic in Y. These two properties enables us to define a canonical morphism $i_* : \Pi^n_1(X) \longrightarrow \Pi^n_1(Y)$ which is called the morphism induced by the inclusion $i : X \longrightarrow Y$.

Now we must introduce an algebraic notion called *the free group with m generators.* Let $\{a_1,\ldots,a_m\} \cup \{a_1^{-1},\ldots,a_m^{-1}\}$ be an alphabet with $2m$ distinct letters, and let L_m be the set of the all words over this alphabet (i.e. finite sequences of letters of the alphabet). We say that two words $w \in L_m$ and $w' \in L_m$ are *the same up to an elementary simplification* if, either w can be obtained from w' by inserting in w' a sequence of the form $a_i a_i^{-1}$ or a sequence of the form $a_i^{-1} a_i$ with $i \in \{1,\ldots,m\}$, or w' can be obtained from w by inserting in w a sequence of the form $a_i a_i^{-1}$ or a sequence of the form $a_i^{-1} a_i$ with $i \in \{1,\ldots,m\}$. Now, two words $w \in L_m$ and $w' \in L_m$ are said to be *free equivalent* if there is a finite sequence $w = w_1, \dots, w_k = w'$ of words of L_m such that for $i = 2, \dots, k$ the word w_{i-1} and w_i are the same up to an elementary simplification. This defines an equivalence relation on L_m, and we denote by $\mathcal{F}_n$ the set of equivalence classes of this equivalence relation. If $w \in L_m$, we denote by $\overline{w}$ the class of w under the free equivalence relation. The concatenation of words defines an operation on $\mathcal{F}_n$ which provides $\mathcal{F}_n$ with a group structure. The group thus defined is called the *free group with m generators.* We denote by 1_m the unit element of $\mathcal{F}_m$, which is equal to $\overline{w}$ where w is the empty word. The only result which we shall admit on the free group is the classical result that if a word $w \in L_n$ is such that $\overline{w} = 1_m$ and w is not the empty word, then there exists in w two successive letters $a_i a_i^{-1}$ or $a_i^{-1} a_i$ with $i \in \{1, \dots, m\}$. This remark leads to an immediate algorithm to decide whether a word $w \in L_n$ is such that $\overline{w} = 1_m$.

2 On the fundamental group of subsets

In this section, we state and prove some results relative to inclusion of sets and the fundamental group. First we set a definition :

Definition 1. Let $X \subset \mathbb{Z}^2$ and $x \in X$. The point x is called n*-simple* if the number of n-connected components of $N_8(x) \cap X$ which are n-adjacent to x is equal to 1, and $N_{\overline{n}}(x) \cap \overline{X} \neq \emptyset$.

Observe that if $x \in X$ is such that $N_n(x) \cap X$ is nonempty such that $N_{\overline{n}}(x) \cap \overline{X} \neq \emptyset$, then x is n-simple if and only if the number of $\overline{n}$-connected components of $N_8(x) \cap \overline{X}$ which are $\overline{n}$-adjacent to x is equal to 1.

Let $X \subset Y \subset \mathbb{Z}^2$. The set X is said to be *lower* n*-homotopic to* Y if X can be obtained from Y be deleting sequentially n-simple points. In this case the set Y is called *upper* n*-homotopic to* X. Finally, the set X and Y are called n*-homotopic* if there exists a finite sequence $X_0, \dots, X_m \subset \mathbb{Z}^3$ of sets such that $X = X_0$ and $Y = X_m$ and for $i = 1, \dots, m$ the set X_{i-1} is either lower n-homotopic or upper n-homotopic to X_i.

Lemma 2. *Let $X \subset \mathbb{Z}^2$, let $B, B' \in X$ and $x \in X$ an n-simple point which is distinct from B and B'. Then if two n-paths π and π' of $A^n_{B,B'}(X \setminus \{x\})$ are n-homotopic (with fixed extremities) in X, they are n-homotopic in $X \setminus \{x\}$.*

Proof: First, if $c = (x_0, \dots, x_p)$ is an n-path in X such that $x_0 \neq x$ and $x_p \neq x$, we define an n-path $P(c)$ as follows: For any maximal sequence $\sigma = (x_k, \dots, x_l)$ with $0 \leq k \leq l \leq p$ of points of c such that for $i = k, \dots, l$ we have $x_i \neq x$, we define $c(\sigma) = \sigma$. For any maximal sequence $\sigma = (x_k, \dots, x_l)$ with $1 \leq k \leq l < p$ of points of c such that for $i = k, \dots, l$ we have $x_i = x$, we define $c(\sigma)$ as equal to the shortest n-path in $N_8(x) \cap X$ from x_{l-1} to x_{k+1}. Now, $P(c)$ is the concatenation of all $c(\sigma)$ for all maximal sequence $\sigma = (x_k, \dots, x_l)$ of points of c such that either for $i = k, \dots, l$ we have $x_i \neq x$ or for $i = k, \dots, l$ we have $x_i = x$.

Now, it is sufficient to prove that if π and π' are two elements of $A^n_{B,B'}(X)$ and are the same up to an elementary deformation, the two $n-$paths $P(\pi)$ and $P(\pi')$ also are the same up to an elementary deformation. Hence we assume π and π' are of the form $\pi = \pi_1 * \gamma * \pi_2$ and $\pi' = \pi_1 * \gamma' * \pi_2$, the $n-$paths γ and γ' having the same extremities and being both included in a common unit square S. Without loss of generality, we assume that $x \in S$. Let $\pi_1 = (x_{1,0}, \dots, x_{1,k_1})$ and $\pi_2 = (x_{2,0}, \dots, x_{2,k_2})$. We denote by α_1 the shortest $n-$path in $N_8(x) \cap X$ from the last point of π_1 to S, and we denote by α_2 the shortest $n-$path in $N_8(x) \cap X$ from S to the first point of π_2. We denote $\alpha_1 = (y_{1,0}, \dots, y_{1,k_1})$ and $\alpha_2 = (y_{2,0}, \dots, y_{2,k_2})$. Finally, we define $\delta = (y_{1,k_1}) * \gamma * (y_{2,0})$, and $\delta' = (y_{1,k_1}) * \gamma' * (y_{2,0})$. Now we have $P(\pi) = (P(\pi_1) * \alpha_1) * P(\delta) * (\alpha_2 * P(\pi_2))$ and $P(\pi') = (P(\pi_1) * \alpha_1) * P(\delta') * (\alpha_2 * P(\pi_2))$. Since $P(\delta)$ and $P(\delta')$ have the same extremities and are both included in the unit square S, the $n-$paths $P(\pi)$ and $P(\pi')$ are the same up to an elementary deformation. □

Corollary 3. *Let $X \subset Y \subset \mathbb{Z}^2$ be such that X is lower $n-$homotopic to Y. Let $B, B' \in X$. Then if two closed $n-$paths π and π' of $A^n_{B,B'}(X)$ are $n-$homotopic (with fixed extremities) in Y, they are $n-$homotopic in X.*

Lemma 4. *Let $X \subset \mathbb{Z}^2$, let $B, B' \in X$ and $x \in X$ an $n-$simple point distinct from B and B'. Then any $n-$path c of $A^n_{B,B'}(X)$ is $n-$homotopic (with fixed extremities) to an $n-$path contained in $X \setminus \{x\}$.*

Proof: Let $P(c)$ be the $n-$path as defined in the proof of Lemma 2. It is easy to see that c is $n-$homotopic (with fixed extremities) to $P(C)$. □

Corollary 5. *Let $X \subset Y \subset \mathbb{Z}^2$ be such that X is lower $n-$homotopic to Y, and let $B, B' \in X$. Then any $n-$path c of $A^n_{B,B'}(Y)$ is $n-$ homotopic (with fixed extremities) to an $n-$path contained in X.*

Corollary 6. *Let $X \subset Y \subset \mathbb{Z}^2$ be such that X is lower $n-$homotopic to Y. The morphism $i_* : \Pi^n_1(X) \longrightarrow \Pi^n_1(Y)$ induced by the inclusion map is a group isomorphism.*

The following result is folklore:

Theorem 7. *Let $X \subset Y \subset \mathbb{Z}^2$ be two $n-$connected sets. Suppose that any $\overline{n}-$connected component of $\overline{X}$ contains exactly one $\overline{n}-$connected component of $\overline{Y}$. Then X is lower $n-$homotopic to Y.*

3 The noncommutative winding number

In this section, for any $X \subset \mathbb{Z}^2$ with m $n-$holes and $B \in X$, for any $c \in A^n_B(X)$, we define a word $W \in L_m$. The corresponding element $\overline{W}$ of $\mathcal{F}_m$ is called the *noncommutative winding number of c*. The idea is the following: first we chose a point P_i in the i^{th} $n-$hole of X. Then we consider a particular infinite simple 4−path π which contains all points $P_1, \dots, P_m$. By remaining $P_1, \dots, P_m$ if necessary, we may assume that the order in which the P_i's appear in π is the order of increasing i's. Then we construct the word w following c, adding a symbol a_i or a_i^{-1} to the word we construct each time c crosses the section of π between P_i and P_{i+1}, depending on how c crosses π. For instance, in Figure 1, the noncommutative winding number is equal to $a_2^{-1}a_1a_2a_1^{-1}$.

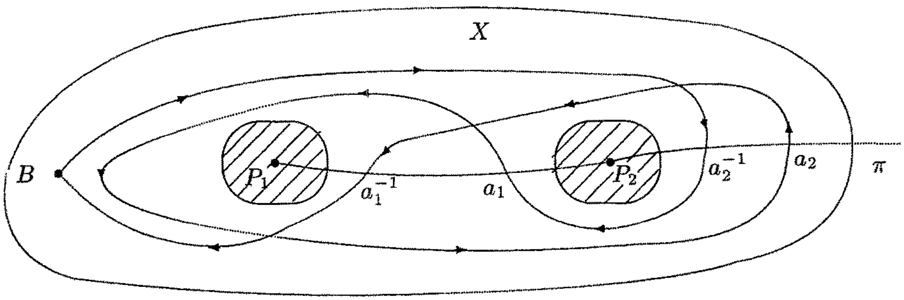

Fig. 1. Example with non commutative winding number equal to $a_2^{-1}a_1a_2a_1^{-1}$.

Let us now consider all this more precisely. In the following, X is a subset of $\mathbb{Z}^2$ with m n−holes. We chose a point $P_i = (\alpha_i, \beta_i)$ in the i^{th} n−hole of X. Let $R = [a, b] \times [a', b']$ be a rectangle such that X is contained in R. We denote $X_1 = R \backslash \{P_1, \dots, P_m\}$.

We construct a particlular infinite simple 4−path $\pi = (y_i)_{i \in \mathbb{N}}$ (see Figure 2) as follows: Let $k_1 = b + 1 - \alpha_1$, and for $j = 2, \dots, \beta_m - \beta_1 + 1$, let $k_j = k_{j-1} + b - a + 3$. For convenience, we set $k_{\beta_m - \beta_1 + 2} = +\infty$. For $i \in \{0, \dots k_1\}$ we set $y_i = (\alpha_1 + i, \beta_1)$, and for $k_j < i \leq k_{j+1}$ with $j \geq 2$, we set $y_i = (a + i - k_j, \beta_1 + j - 1)$ if j is even and $y_i = (b - (i - k_j), \beta_1 + j - 1)$ otherwise.

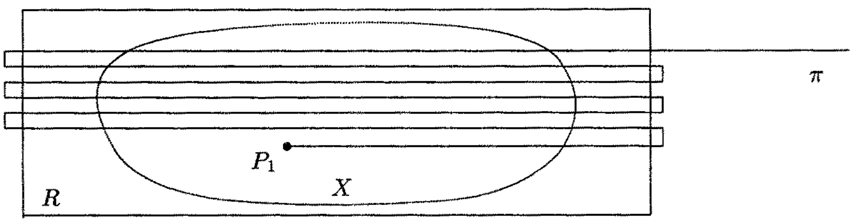

Fig. 2. The 4−path π.

In other words, π is the concatenation of strait line segments, $[(a-1, \beta_1 + j - 1), (b+1, \beta_1 + j - 1)]$ or $[(b+1, \beta_1 + j - 1), (a-1, \beta_1 + j - 1)]$, depending on the parity of j (except for $j = 1$ for which we have a segment $[(\alpha_1, \beta_1), (b+1, \beta_1)]$).

By remaining the P_i's if necessary, we may assume that the order in which they appear in the 4−path π is the order of increasing i's.

Now, for $k = 1, \dots, m$, we denote by e_k the unique integer such that $y_{e_k} = P_k$. For convenience, we denote $e_{m+1} = +\infty$. We denote by I_k the interval of integers $\{e_k, \dots, e_{k+1}\}$. We also denote $\pi(I_k) = \{y_{e_k}, \dots, y_{e_{k+1}}\}$.

In the sequel, B is the base point of X and we assume without loss of generality that it has its second coordinate less than β_1. In particular, $B \notin \pi^*$. In the sequel of this section, $c = (x_0, \dots, x_p)$ is an element of $A^n_B(X_1)$. For $i = 0, \dots, p$, we denote $x_i = (x_{i,1}, x_{i,2})$. For $k = 1, \dots, m$, we call *maximal sequence of indices of points* $c^* \cap \pi(I_k)$ any interval $\{i, \dots, j\}$ of integers such that $\{x_i, \dots, x_j\} \subset \pi(I_k)$, $x_{i-1} \notin \pi(I_k)$ and $x_{j+1} \notin \pi(I_k)$.

Let $\{i, \dots, j\}$ be a maximal sequence of indices of points $c^* \cap \pi(I_k)$. Observe that $x_{i,2} - x_{i-1,2} = \pm 1$ and $x_{j+1,2} - x_{j,2} = \pm 1$. We denote $W_{b,i}(c, P_1, \dots, P_m) = (x_{i,2} - x_{i-1,2}).(-1)^{(x_{i,2}-\beta_1)}$ and $W_{e,j}(c, P_1, \dots, P_m) = (x_{j+1,2} - x_{j,2}).(-1)^{(x_{j,2}-\beta_1)}$ We define the *contribution of i to the noncommutative winding number of c relative to* $\{P_1, \dots, P_m\}$, the element of $\mathcal{F}_m$ defined by:

$$W_i(c, P_1, \dots, P_m) = a_k^{\frac{W_{b,i}(c,P_1,\dots,P_m)+W_{e,j}(c,P_1,\dots,P_m)}{2}}$$

$W_i(c, P_1, \dots, P_m)$ is an element of $\mathcal{F}_m$ which is equal either to a_k, or to a_k^{-1}, or to 1_m. For convenience, we denote $W_i(c, P_1, \dots, P_m) = 1_m$ if i is not the smallest element of a maximal sequence of indices of points of some $c^* \cap \pi(I_k)$ with $k \in \{1, \dots, m\}$, so that $W_i(c, P_1, \dots, P_m)$ is defined for any $i \in \{0, \dots, p\}$.

Definition 8. We call *noncommutative winding number of c relative to* $\{P_1, \dots, P_m\}$ the element of $\mathcal{F}_m$ defined by:

$$\overline{W(c, P_1, \dots, P_m)} = \prod_{i=0}^{p} W_i(c, P_1, \dots, P_m)$$

Observe that, since $\mathcal{F}_m$ is not an abelian group, the order in which the product is defined in this last definition must be respected.

We also define a *word* $W(c, P_1, \dots, P_m)$ of L_m by the sequence of letters obtained by replacing the element $W_i(c, P_1, \dots, P_m)$ of $\mathcal{F}_m$ by the corresponding letter a_k or a_k^{-1} if $W_i(c, P_1, \dots, P_m) \neq 1_m$ in the product of the last definition.

4 Main results

The purpose of this section is to prove that $\overline{W(c, P_1, \dots, P_m)}$, the noncommutative winding number, depends on the n−path c only up to homotopy, so that a map from $\Pi_1^n(X)$ to $\mathcal{F}_m$ is defined. Then we prove that this map is a group isomorphism.

Theorem 9. *If two n−paths $c, c' \in A^n_B(X_1)$ are n−homotopic (with fixed extremities), then* $\overline{W(c, P_1, \dots, P_m)} = \overline{W(c', P_1, \dots, P_m)}$

Proof: We only need to prove this result when c and c' are the same up to an elementary deformation. So, let $c = c_1 * \gamma * c_2$ and $c' = c_1 * \gamma' * c_2$, the n−paths

γ and γ' having the same extremities and being both included in a common unit square S. If γ and γ' are both included in $\pi(I_k)$ with $k \in \{1, \ldots, m\}$, the result is obvious. Hence we may assume that S meets two sets $\pi(I_k)$ and $\pi(I_{k'})$ with $k \neq k'$. Moreover, it is easily seen that the unit square S meets at most two sets of the form $\pi(I_j)$ with $j \in \{1, \ldots, m\}$. Hence the two lower points of S are, say, in $\pi(I_k) \cup \{P_1, \ldots, P_m\}$ and the two upper points of S then are in $\pi(I_{k'}) \cup \{P_1, \ldots, P_m\}$.

We denote $c = (x_0, \ldots, x_p)$, $c' = (x'_0, \ldots, x'_p)$, for $i \in \{0, \ldots, p\}$ $x_i = (x_{i,1}, x_{i,2})$ and for $i \in \{0, \ldots, p'\}$ $x'_i = (x'_{i,1}, x'_{i,2})$. We also denote $\gamma = (x_{i_0}, \ldots, x_{i_1})$ and $\gamma' = (x'_{i_0}, \ldots, x'_{i'_1})$.

Let us consider the case when $x_{i_0} = x'_{i_0} \in \pi(I_k)$ and $x_{i_1} = x'_{i'_1} \in \pi(I_{k'})$. Let $d = min(\{j \geq i_0 \ / \ x_j \in \pi(I_{k'})\})$ and $d' = min(\{j \geq i_0 \ / \ x'_j \in \pi(I_{k'})\})$. Let $f = max(\{j \leq i_1 \ / \ x_j \in \pi(I_k)\})$ and $f' = max(\{j \leq i'_1 \ / \ x'_j \in \pi(I_k)\})$. If $(i, \ldots, j)$ is a maximal sequence of indices of points of $c^* \cap \pi(I_k)$ which is included in $\{i_0, \ldots, i_1\}$, if $j \neq d-1$ and $i \neq f+1$ we have $W_{b,i}(c, P_1, \ldots P_m) = -W_{e,j}(c, P_1, \ldots P_m)$, hence $W_i(c, P_1, \ldots, P_m) = 1_m$. Similarly, for $(i, \ldots, j)$ a maximal sequence of indices of points of $c'^* \cap \pi(I_k)$ which is included in $\{i_0, \ldots, i'_1\}$, if $j \neq d'-1$ and $i \neq f'+1$ we have $W_i(c', P_1, \ldots, P_m) = 1_m$. Hence we consider $(i, \ldots, d-1)$ the maximal sequence of indices of points of $c^* \cap \pi(I_k)$ which contains $d-1$, and we consider $(i, \ldots, d'-1)$ the maximal sequence of indices of points of $c'^* \cap \pi(I_k)$ which contains $d'-1$. We have $W_{e,d-1}(c, P_1, \ldots, P_m) = (-1)^{(x_{d-1,2} - \beta_1)} = (-1)^{(x'_{d'-1,2} - \beta_1)} = W_{e,d'-1}(c', P_1, \ldots, P_m)$. On the other hand, we have $W_{b,i}(c, P_1, \ldots, P_m) = W_{b,i}(c', P_1, \ldots, P_m)$, and therefore $W_i(c, P_1, \ldots, P_m) = W_i(c', P_1, \ldots, P_m)$. Similarly, we can obtain $W_f(c, P_1, \ldots, P_m) = W_{f'}(c', P_1, \ldots, P_m)$ so that $W(c, P_1, \ldots, P_m) = W(c', P_1, \ldots, P_m)$.

The case when x_{i_0} and x_{i_1} lie on the same horizontal line is similar, except in the case, say, γ is included in $\pi(I_k)$ and γ' is not. In this case, we prove that, either $W(c, P_1, \ldots, P_m) = W(c', P_1, \ldots, P_m)$, or the word $W(c', P_1, \ldots, P_m)$ is obtained from $W(c, P_1, \ldots, P_m)$ by inserting $a_k a_k^{-1}$ or $a_k^{-1} a_k$ in $W(c, P_1, \ldots, P_m)$. In both cases, $\overline{W(c, P_1, \ldots, P_m)} = \overline{W(c', P_1, \ldots, P_m)}$. □

Remark 10. From Theorem 9 follows that the map $c \longmapsto \overline{W(c, P_1, \ldots, P_m)}$ from $A^n_B(X)$ to $\mathcal{F}_m$ induces a map $\varphi_X : \Pi^n_1(X) \longrightarrow \mathcal{F}_m$. This map φ_X is a group morphism.

In order to prove that the map φ_X is a group isomorphism, we first need some technical lemmas.

Lemma 11. *Let $c \in A^n_B(X_1)$ be such that $W(c, P_1, \ldots, P_m)$ is the empty word. Then c is $n-$homotopic in X_1 to an $n-$path which contains only the point B.*

Proof: For convenience, we call $\pi(I_{m+1})$ and $\pi(I_{m+2})$ the $n-$connected components of $X_1 \backslash \pi^*$. We also denote $m' = m + 2$. First of all, since π^* has no $n-$hole, is not bounded, and contains all $n-$holes of X_1, the set $X_1 \backslash \pi^*$ has no $n-$hole.

Hence if we assume that $c \cap \pi^* = \emptyset$, it follows from Theorem 7 and Corollary 5 that c is $n-$homotopic in $X_1 \backslash \pi^*$ to an $n-$path which contains only the point B. Now, we shall prove that c is $n-$homotopic in X_1 to an $n-$path which contains only the point B by induction on the number $\mathcal{I}(c)$ of maximal intervals $\{i, \ldots, j\}$ such that there exists $k \in \{1, \ldots, m'\}$ with $\{x_i, \ldots, x_j\} \subset \pi(I_k)$. Suppose that the result is true for any $c' \in A_B^n(X_1)$ with $\mathcal{I}(c') \leq h$ with $h \geq 0$, and $W(c, P_1, \ldots, P_m)$ is the empty word. Suppose that c is such that $\mathcal{I}(c) = h + 1$. We distinguish two case :

First case: for all maximal interval $\{i, \ldots, j\}$ of points of $c^* \cap \pi(I_k)$ with $k \in \{1, \ldots, m'\}$ we have: $x_{i-1,2} \neq x_{j+1,2}$ or $\exists k' \in \{1, \ldots, m'\}$ such that $\{x_{i-1,2}, x_{j+1,2}\} \subset \pi(I_{k'})$. We consider the first maximal interval $\{i, \ldots, j\}$ of indices of points of $c^* \cap \pi(I_k)$ with $k \in \{1, \ldots, m\}$ such that $x_{j+1,2} = x_{j,2} - 1$. Since $W_i(c, P_1, \ldots, P_m) = 1_m$ and $x_{i,2} = x_{i-1,2} + 1$, $x_{i,2}$ and $x_{j,2}$ have the same parity, hence they are equal. Therefore, from the assumptions of the first case we are dealing with, there exists $k' \in \{1, \ldots, m'\}$ such that $\{x_{i-1,2}, x_{j+1,2}\} \subset \pi(I_{k'})$. It is easily seen that the horizontal strait line segment $S = [x_{i-1}, x_{j+1}]$ is included in $\pi(I_{k'})$ and that $S \cup \pi(I_k)$ has no $n-$hole. Hence, from Theorem 7 and Corollary 5 follows that the $n-$path $c_1 = (x_{i-1}, \ldots, x_{j+1})$ is $n-$homotopic with fixed extremities to an $n-$path contained in S. Therefore, c is $n-$homotopic to an $n-$path $c' \in A_B^n(X_1)$ with $\mathcal{I}(c') = h$. Furthermore, it is easy to see that $W(c', P_1, \ldots, P_m)$ is the empty word, hence from our induction hypothesis c' is $n-$homotopic in X_1 to a constant $n-$path. This completes the proof in the first case.

Second case: there exists a maximal interval $\{i, \ldots, j\}$ of indices of points such that $\{x_i, \ldots, x_j\} \subset \pi(I_k)$ with $k \in \{1, \ldots, m'\}$ such that $x_{i-1,2} = x_{j+1,2}$ and $\exists k' \neq k''$ with $x_{i-1,2} \in \pi(I_{k'})$ and $x_{j+1,2} \in \pi(I_{k''})$. Since $x_{i,2} \neq x_{i-1,2} = x_{j+1,2}$, we have either $k < k'$ and $k < k''$ or $k > k'$ and $k > k''$. We assume for instance that $k' < k'' < k$. let $\{i', \ldots, j'\}$ be the last maximal interval of indices of points such that $\{x_{i'}, \ldots, x_{j'}\} \subset \pi(I_{k'''})$ with $k''' \in \{1, \ldots, m'\}$ such that for all $j \leq i'' \leq j'$ $x_{i''} \in \pi(I_{k''}) \cup \cdots \cup \pi(I_{k-1})$. We denote by S_1 and S_2 respectively the two horizontal strait line segments $[(a, x_{j,2}), (b, x_{j,2})]$ and $[(a, x_{j+1,2}), (b, x_{j+1,2})]$. For any $i'' \in \{i' - 1, \ldots, j'\}$ we have : $x_{i''} \in S_1 \cup S_2$. Since $x_{i'-1} \in S_1 \cup S_2$ and $W_{i'}(c, P_1, \ldots, P_m) = 1_m$, we must have $x_{j'+1} \in S_1 \cup S_2$. It is easily seen that the only possibility is that $x_{j'+1,2} = x_{j,2}$ and $x_{j'+1} \in \pi(I_k)$. Hence we can proceed as in the first case : The strait line segment $S = [x_j, x_{j'+1}]$ is contained in $\pi(I_k)$ and in S_1. Since $(S_1 \cup S_2) \cap X_1$ has no $n-$hole, the $n-$path $c_1 = (x_j, \ldots, x_{j'+1})$ is $n-$homotopic with fixed extremities to an $n-$path contained in S, hence, as in the first case, c is $n-$homotopic to an $n-$path $c' \in A_B^n(X_1)$ with $\mathcal{I}(c') = h$ and such that $W(c', P_1, \ldots, P_m)$ is the empty word. □

Now we observe that the definition of $W_i(c, P_1, \ldots, P_m)$ makes sense for a non-closed $n-$path $c = (x_0, \ldots, x_p)$ and for any maximal sequence $(i, \ldots, j)$ of indices of points of some $c^* \cap \pi(I_k)$ such that $j \neq p$ and $i \neq 0$.

Lemma 12. *Let $k \in \{1, \ldots, m\}$ and $\varepsilon \in \{-1, 1\}$. Then there exists an $n-$path $c = (x_0, \ldots, x_p)$ from B to a point of $\pi(I_k)$ such that (denoting $x_i = (x_{i,1}, x_{i,2})$ for $i \in \{0, \ldots, p\}$):*

1. *For any maximal sequence $(i, \dots, j)$ of indices of points of some $c^* \cap \pi(I_{k'})$ such that $j \neq p$ and $i \neq 0$ we have: $W_i(c, P_1, \dots, P_m) = 0$;*
2. *If $(i, \dots, p)$ is the maximal sequence of indices of points of $c^* \cap \pi(I_k)$ which contains p we have: $(x_{i,2} - x_{i-1,2}).(-1)^{x_{i,2}-\beta_1} = \varepsilon$.*

Proof: We distinguish two cases depending on whether $\pi(I_k)$ is included in a strait line segment or not. We treat for instance the first case, assuming that $\pi(I_k)$ is included in a strait line segment S at the height h. Let S_1 and S_2 be respectively the set of the points of R of height $h-1$ and $h+1$. We assume for instance that $h - \beta_1$ is even. Let a_1 be the first coordinate of the leftmost $P_{k'}$ with $k' \in \{1, \dots, m\}$. we construct an $n-$path c'_1 by following a horizontal line from B to a point M_1 having $a_1 - 1$ as its first coordinate, and then a vertical line from the point M_1 to the point $M_2 = (a_1 - 1, h+1) \in S_2$. Since $(S \cup S_2) \cap X_1$ is $n-$connected, there is an $n-$path c'_2 contained in $(S \cup S_2) \cap X_1$ from M_2 to a point of $\pi(I_k)$. Let $c_1 = (x_0, \dots, x_p)$ be the concatenation of c'_1 and c'_2. We denote $x_i = (x_{i,1}, x_{i,2})$ for $i \in \{0, \dots, p\}$. Then it is clear that for any maximal sequence $(i, \dots, j)$ of indices of points of some $c_1^* \cap \pi(I_{k'})$ such that $j \neq p$ and $i \neq 0$ we have: $W_i(c_1, P_1, \dots, P_m) = 0$. Moreover, if $(i, \dots, p)$ is the maximal sequence of indices of points of $c_1^* \cap \pi(I_k)$ which contains p we have: $x_{i-1} \in S_2$ so that $(x_{i,2} - x_{i-1,2}).(-1)^{x_{i,2}-\beta_1} = (-1).(h - \beta_1) = -1$.

Now, by considering b_1 the first coordinate of the rightmost point P_i for $i = 1, \dots, m$, we construct an $n-$path c_2 satisfying the conditions *1.* and *2.* with $\varepsilon = 1$. □

Lemma 13. *Let $c = (x_0, \dots, x_p)$ be a closed $n-$path with $x_0 = x_p \in \pi(I_k)$ with $k \in \{1, \dots, m\}$, such that or any maximal sequence $(i, \dots, j)$ of indices of points of some $c^* \cap \pi(I_{k'})$ such that $j \neq p$ and $i \neq 0$ we have: $W_i(c, P_1, \dots, P_m) = 0$. Moreover we assume that (denoting $x_i = (x_{i,1}, x_{i,2})$ for $i \in \{0, \dots, p\}$) if $j' = -1 + min\{0 \le i \le p \ / \ x_i \notin \pi(I_k)\}$ and $i' = 1 + max\{0 \le i \le p \ / \ x_i \notin \pi(I_k)\}$, $(x_{i',2} - x_{i'-1,2}).(-1)^{x_{i',2}-\beta_1} = -(x_{j'+1,2} - x_{j',2}).(-1)^{x_{j',2}-\beta_1}$.*
Then c is $n-$homotopic (with fixed extremities) in X_1 to a constant $n-$path.

Proof: We denote $\varepsilon = (x_{i',2} - x_{i'-1,2}).(-1)^{x_{i',2}-\beta_1} = -(x_{j'+1,2} - x_{j',2}).(-1)^{x_{j',2}-\beta_1}$. Let $c' = (x'_0, \dots, x'_{p'}$ be the $n-$path from B to a point $x'_{p'}$ of $\pi(I_k)$ given by Lemma 12. Since $\pi(I_k)$ is $n-$connected, we may assume that $x'_{p'} = x_0$. The $n-$path c is $n-$homotopic in X_1 to $c'^{-1} * c' * c * c'^{-1} * c'$. Now, from Lemma 11, the $n-$path $c' * c * c'^{-1}$ is $n-$homotopic in X_1 to a constant $n-$path. Hence c is $n-$homotopic in X_1 to a constant $n-$path. □

Theorem 14. *The map $\varphi_X : \Pi_1^n(X) \longrightarrow \mathcal{F}_m$ defined in Remark 10 is a group isomorphism.*

Proof: First we prove that φ_X is one to one, i.e. that $\overline{W(c, P_1, \dots, P_m)} = 1_m$ implies that c is homotopic in X to a constant $n-$path. We already have proved it (Lemma 11) in the case $W(c, P_1, \dots, P_m)$ is the empty word. Now, we prove our result by induction on the length of the word $W(c, P_1, \ldots, P_m)$. Let us assume that $c = (x_0, \ldots, x_p) \in A_B^n(X)$ is such that $W(c, P_1, \ldots, P_m)$ contains a sequence $a_k a_k^{-1}$ or $a_k^{-1} a_k$, say $a_k a_k^{-1}$, and that the result is true for shorter $n-$paths. Let $(i, \ldots, j)$ and $(i' \ldots, j')$ be the corresponding respective maximal sequences of indices of points of $c^* \cap \pi(I_k)$ such that $W_i(c, P_1, \ldots, P_m) = a_k$ and $W_{i'}(c, P_1, \ldots, P_m) = a_k^{-1}$. Necessarily, $W_{e,j}(c, P_1, \ldots, P_m) = 1$ and $W_{b,i'}(c, P_1, \ldots, P_m) = -1$. Let c' be an $n-$path from $x_{i'}$ to x_j which is contained in $\pi(I_k)$. We denote $c_1 = (x_0, \ldots, x_j)$, $c_2 = (x_j, \ldots, x_{i'})$ and $c_3 = (x_{i'}, \ldots, x_p)$ Then $c = c_1 * c_2 * c_3$ is $n-$homotopic in X_1 to $c_1 * c_2 * c' * c'^{-1} * c_3$. Now, from Lemma 13, $c_2 * c'$ is $n-$homotopic with fixed extremities to a constant $n-$path, so that c is $n-$homotopic to $c_1 * c'^{-1} * c_3$. Since $W(c_1 * c'^{-1} * c_3, P_1, \ldots, P_m)$ is the word obtained from $W(c, P_1, \ldots, P_m)$ by removing a sequence $a_k a_k{-1}$, we apply our induction hypothesis to $c_1 * c'^{-1} * c_3$, so that it is $n-$homotopic in X_1 to a constant $n-$path. Therefore c also is. From Theorem 7 and Corollary 5, c is then $n-$homotopic in X to a constant $n-$path.

Now, for proving that φ_X is onto, we only observe that by applying twice Lemma 12 we obtain, for any $k \in \{1, \ldots, m\}$, an $n-$path c such that $W(c, P_1, \ldots, P_m) = a_k$. □

Corollary 15. *There is an algorithm for deciding whether two $n-$paths c and c' of $A_B^n(X)$ are $n-$homotopic in X whose complexity is the sum of the lengths of c and c'.*

Conclusion

Now, besides the characterization of low homotopy of sets using the fundamental group, we have a complete presentation of the fundamental group of any object in a 2−dimensionnal digital image. Moreover, since the word problem (i.e. the problem of knowing whether a word in the generators is trivial or not) has a simple solution in a free group, we can decide whether two closed path are homotopic or not.

Of course, the same problem exists in three dimensions, but it seems rather more complicated. First of all, we do not have a good characterization of low homotopy of sets, and this is probably the first problem to solve. Then we know that the word problem is not decidable in general ([3]). Hence a first step is may be to treat the fundamental group of surfaces since we know that, in the continuous framework, the word problem is decidable for the surface groups.

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