Surfaces

Some Structural Properties of Discrete Surfaces

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Abstract. In the framework of combinatorial topology a surface is described as a set of faces which are linked by adjacency relations. This corresponds to a structural description of surfaces where we have some desirable properties: for example, any point is surrounded by a set of faces which constitute a "cycle". The notion of combinatorial surface extracts these "structural" properties of surfaces.
In this paper, we introduce a relation for points in Z^3 which is based on the notion of homotopy. This allows to propose a definition of a class of surfaces which are combinatorial surfaces. We then show that the main existing notions of discrete surfaces belong to this class of combinatorial surfaces.

Keywords: surfaces, discrete topology, homotopy, simple points

1 Introduction

In the three-dimensional discrete space Z^3, several approaches of surfaces have been proposed:
- a graph-theoretical approach: a surface is defined as a set of points linked by adjacency relations [16, 17, 20];
- a voxel approach: a surface is defined as a set of faces (surfels) between pairs of adjacent voxels [1, 8];
- a general topology approach [13];
- a combinatorial approach: a surface is defined as a structure [7, 9, 15].

In the framework of combinatorial topology a surface is described as a set of faces which are linked by adjacency relations. This corresponds to a structural description of surfaces where we have some desirable properties: for example, any point is surrounded by a set of faces which constitute a "cycle". The notion of combinatorial surface extracts these "structural" properties of surfaces.
The graph-theoretical definitions of closed surfaces are not based upon structural properties. In fact, the structural nature of these surfaces is difficult to extract. The major problem which arises for these surfaces is that the adjacency relation used for defining them does not induce a structural relation. For example, the neighborhood of a point does not constitute a simple closed curve under the adjacency relation.

In this paper, we make a link between the definitions of surfaces based on the graph-theoretical approach and the combinatorial approach. For that purpose, we introduce a relation for points in Z^3 which is based on the notion of homotopy. This allows to propose a definition of a class of surfaces which are combinatorial surfaces. We then show that the main existing notions of surfaces belong to this class of combinatorial surfaces.

2 Basic notions

We recall some basic notions of 3D discrete topology (see also [12]).
We denote $E = Z^3$, Z being the set of relative integers. A point $x \in E$ is defined by (x_1, x_2, x_3) with $x_i \in Z$. We consider the four neighborhoods:
$N_{124}(x) = \{x' \in E;\ Max[|x_1 - x'_1|, |x_2 - x'_2|, |x_3 - x'_3|] \leq 2\}$,
$N_{26}(x) = \{x' \in E;\ Max[|x_1 - x'_1|, |x_2 - x'_2|, |x_3 - x'_3|] \leq 1\}$,
$N_{18}(x) = \{x' \in E;\ |x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3| \leq 2\} \cap N_{26}(x)$,
$N_6(x) = \{x' \in E;\ |x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3| \leq 1\}$.
We define $N_k^*(x) = N_k(x) \setminus \{x\}$, with $k = 6$, 18, 26, 124.
Two points x and y are said to be *n-adjacent* (n = 6, 18, 26) if $y \in N_n^*(x)$; we also say that y is an *n-neighbor* of x.
We denote $N_{18}^+(x) = N_{18}^*(x) \setminus N_6^*(x)$ and $N_{26}^+(x) = N_{26}^*(x) \setminus N_{18}^*(x)$.
Two points x and y are said to be *strictly n-adjacent* (n = 18, 26) if $y \in N_n^+(x)$.
An *n-path* π is a (possibly empty) sequence of points $x_0..x_k$, with x_i n-adjacent to x_{i-1}, for $i = 1..k$. If π is not empty, the *length of* π is equal to k. If $x_0 = x_k$, π is *closed* and x_0 is called the *origin* of π.

Let X be a subset of E. We denote by $\overline{X}$ the complement of X.
Let $x \in X$ and $y \in X$. We say that x is *n-connected* to y if there is an n-path in X between x and y. The relation "is n-connected to" is an equivalence relation. The equivalence classes relative to this relation are the *n-connected components of* X (or simply the *n-components* of X).
A subset X of E is *n-connected* if it is made of exactly one n-connected component.
A subset X of E is a *simple closed n-curve* if X is n-connected and if each point of X is n-adjacent to exactly two points in X.

As in 2D, if we use an n-adjacency relation for X we have to use another $\overline{n}$-adjacency relation for $\overline{X}$, i.e. the 6-adjacency for X is associated to the 18- or the 26-adjacency for $\overline{X}$ (and vice versa). This is necessary for having a correspondence between the topology of X and the topology of $\overline{X}$. Furthermore, it is sometimes necessary to distinguish the 6-adjacency associated with the 18-adjacency and the 6-adjacency associated with the 26-adjacency. Whenever we will have to make this distinction, a 6^+-notion will indicate a 6-notion associated with the 18-adjacency. So, we can have $(n, \overline{n}) = (6, 26)$, $(26, 6)$, $(6^+, 18)$ or $(18, 6^+)$.
Note that, if X is finite, the infinite $\overline{n}$-connected component of $\overline{X}$ is the *background*, the other $\overline{n}$-connected components of $\overline{X}$ are the *cavities*.

The notion of deformation allows to detect the presence of a "hole" in a set X (see [10]).
Let $X \subset E$ and let $p \in X$ be a point, called the *base point*. Let γ and γ' be

two closed n-paths composed of points of X and which have p as origin. We say that γ' is *an elementary n-deformation* of γ, or $\gamma \sim \gamma'$, if there are two n-paths π_1, π_2, and two non-empty n-paths π, π', such that γ and γ' are of the form $\gamma = \pi_1 \pi \pi_2$, $\gamma' = \pi_1 \pi' \pi_2$, and such that all points of π and π' are included in a little portion P of E:
- for $n = 6$, P is a unit square (a 2×2 square);
- for $n = 6^+$, 18, 26, P is a unit cube (a $2 \times 2 \times 2$ cube).
We say that γ' is an *n-deformation* of γ or $\gamma \simeq \gamma'$ if there is a sequence of closed n-paths $\gamma_0..\gamma_k$ such that $\gamma = \gamma_0$, $\gamma' = \gamma_k$ and $\gamma_{i-1} \sim \gamma_i$ for $i = 1..k$.
Let $\gamma = px_0...x_ip$ and $\gamma' = px'_0...x'_jp$ be two closed n-paths composed of points of X and which have p as origin. The *product* of γ and γ' is the closed n-path $px_0...x_ipx'_0...x'_jp$ obtained by catenating γ and γ'.
Let us consider the classes of equivalence of the closed n-paths with origin p under the relation $\simeq$. We may define the product of two such classes as the equivalence class of the product of two closed n-paths corresponding to the classes.
Under the product operation, these classes constitute a group $\Pi_n(p, X)$ which is the *fundamental n-group* (or Poincaré group) with base point p. As in the continuous spaces, the fundamental group reflects the structure of the holes (or tunnels) in X. For example, the fundamental group of a hollow torus is a free abelian group on two generators. Note that if p and q belong to the same n-connected component of X, then $\Pi_n(p, X)$ is isomorphic to $\Pi_n(q, X)$.

3 Homotopy and strong homotopy

In this section, we recall some notions of homotopy and strong homotopy. The homotopy in a discrete grid may be defined through the notion of simple point (see also [10]).
Let $n \in \{6, 6^+, 18, 26\}$. Let $X \subset E$. A point $x \in E$ is said to be *n-simple (for X)* if its removal from X (if $x \in X$) or its addition to X (if $x \in \overline{X}$) does not "change the topology of the image", i.e., if:

1) There is a one to one correspondence between the n-connected components of $X \setminus \{x\}$ and the n-connected components of $X \cup \{x\}$; and

2) There is a one to one correspondence between the $\overline{n}$-connected components of $\overline{X} \setminus \{x\}$ and the $\overline{n}$-connected components of $\overline{X} \cup \{x\}$; and

3) For each point p of $X \setminus \{x\}$, the inclusion map i: $X \setminus \{x\} \rightarrow X \cup \{x\}$ induces a group isomorphism i^*: $\Pi_n(p, X \setminus \{x\}) \rightarrow \Pi_n(p, X \cup \{x\})$; and

4) For each point q of $\overline{X} \setminus \{x\}$, the inclusion map j: $\overline{X} \setminus \{x\} \rightarrow \overline{X} \cup \{x\}$ induces a group isomorphism j^*: $\Pi_{\overline{n}}(q, \overline{X} \setminus \{x\}) \rightarrow \Pi_{\overline{n}}(q, \overline{X} \cup \{x\})$.

The set $Y \subset X$ is *lower n-homotopic* to X if there exists a sequence of sets $Z_0, ..., Z_k$, with $Z_0 = Y$, $Z_k = X$, such that $Z_{i-1} \subset Z_i$ and $Z_i \setminus Z_{i-1}$ consists in a single point which is an n-simple point for Z_{i-1}, $i = 1, ..., k$. The set $S \subset X$ is called a *(lower) n-simple set for X*, if $X \setminus \{S\}$ is lower n-homotopic to X.

Thus, the set Y is lower n-homotopic to X, if X may be obtained from Y by iterative additions of n-simple points, or, equivalently, if Y may be obtained from X by iterative deletions of n-simple points.

Let $X \subset E$ and $x \in E$. We denote $|X|^x = N_{26}^*(x) \cap X$.
The *geodesic n-neighborhood of x inside X of order k* is the set $N_n^k(x, X)$ defined recursively by: $N_n^1(x, X) = N_n^*(x) \cap X$ and $N_n^k(x, X) = \cup\{N_n(y) \cap |X|^x, y \in N_n^{k-1}(x, X)\}$.
In other words $N_n^k(x, X)$ is the set composed of all points y of $|X|^x$ such that there exists an n-path π from x to y of length less than or equal to k, all points of π, except x, belonging to $|X|^x$. We give now a definition of topological numbers which leads to a characterization of simple points [2]:

Definition 1: Let $X \subset E$, $x \in E$ and $n \in \{6, 6^+, 18, 26\}$.
The *geodesic neighborhoods* $G_n(x, X)$ are defined by:

$$G_6(x, X) = N_6^2(x, X); \;\; G_{6^+}(x, X) = N_6^3(x, X);$$
$$G_{18}(x, X) = N_{18}^2(x, X); G_{26}(x, X) = N_{26}^1(x, X).$$

The *topological number* $T_n(x, X)$ is defined as the number of n-components in $G_n(x, X)$.

Note that the topological number depends only on the neighborhood $N_{26}^*(x) \cap X$, we have $T_n(x, X) = T_n(x, |X|^x)$. The evaluation of the topological number may be done by using classical graph-theoretic algorithms for searching connected components. We have [2]:

Theorem 2: *Let $X \subset E$ and $x \in E$: x is an n-simple point if and only if $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$.*

We introduce the notion of strong homotopy: see [3], see also the work of Kong [11] in which the notion of hereditarily simple set is introduced, this notion is equivalent to the notion of strongly simple set presented hereafter:

Definition 3: Let $X \subset E$ and $Y \subset X$. The set Y is *strongly (lower) n-homotopic to X* if, for each subset Z such that $Y \subset Z \subset X$, Z is lower n-homotopic to X.
If Y is strongly n-homotopic to X, we say that $X \setminus Y$ is a *strongly (lower) n-simple set.*

4 Existing notions of surfaces

We now present existing notions of surfaces. First of all, we give some general definitions.

Definition 4: Let X be a subset of E and let x be a point of E.
The set X is an *n-thin set* if, $\forall x \in X$, $|\overline{X}|^x$ has exactly two $\overline{n}$-components which are $\overline{n}$-adjacent to x.
The set X is an *n-Jordan set* if X is n-connected and if $\overline{X}$ has two $\overline{n}$-connected components.
If X is an n-Jordan set, we will denote by A and B the two components of $\overline{X}$. These components are called the *back-components* of X. The *closure* of a back-component is the union of this back-component and X.
An *n-separating set* is an n-Jordan set which is also an n-thin set.
An n-separating set X is a *strongly n-separating set* if, $\forall x \in X$, x is $\overline{n}$-adjacent to both A and B.

4.1 Morgenthaler's surfaces

Let us present the definition of surfaces introduced by Morgenthaler and Rosenfeld [20]:

Definition 5: Let $(n, \overline{n}) = (6, 26)$ or $(26, 6)$ and let X be a subset of E. A point x of X is a *Morgenthaler's (simple) n-surface point* if:

1) $|X|^x$ has exactly one n-component which is n-adjacent to x; and

2) $|\overline{X}|^x$ has exactly two $\overline{n}$-components which are $\overline{n}$-adjacent to x; we denote C^{xx} and D^{xx} these components; and

3) $\forall y \in N_n(x) \cap X$, $N_{\overline{n}}(y) \cap C^{xx} \neq \emptyset$ and $N_{\overline{n}}(y) \cap D^{xx} \neq \emptyset$.

Furthermore, if $N_{124}(x) \cap \overline{X}$ has exactly two $\overline{n}$-components which are $\overline{n}$-adjacent to x, we say that the n-surface point x is *orientable*.
A *Morgenthaler's (simple) closed n-surface* is a finite n-connected set X consisting entirely of orientable Morgenthaler's n-surface points.

In Fig. 1, the configuration (a) corresponds to a Morgenthaler's 6-surface point, the configurations (b), (e) correspond to Morgenthaler's 26-surface points. It was shown that [20]:

Theorem 6: *A Morgenthaler's closed n-surface is a strongly n-separating set.*

Furthermore, it was proved that the assumption of orientability is unnecessary for the 6-connectivity [22] and for the 26-connectivity [21].

4.2 Malgouyres' surfaces

Malgouyres' surfaces [17] are based on a generalization of the notion of a simple closed curve.

Definition 7: Let X be a subset of E. We say that a point x of X is an *n-corner* if x is n-adjacent to two and only two points y and z belonging to X such that y and z are themself n-adjacent; we say that the n-corner x is *simple* if y and z are not corners and if x is the only point n-adjacent to both y and z. We say that X is a *generalized simple closed n-curve*, or a *G_n-curve*, if the set obtained by removing all simple n-corners of X is a simple closed n-curve.

Definition 8: A finite subset X of E is called a *Malgouyres' (simple) closed 18-surface* if X is 18-connected and if, for each x of X, the set $|X|^x$ is a G_{18}-curve.

In Figure 1, (b), (c), (f), (g) are examples of Malgouyres' surface points. It was proved in [17] that:

Theorem 9: *Any Malgouyres' 18-surface is a strongly n-separating set, for $n = 18$ and $n = 26$.*

4.3 Strong surfaces

The definition of strong surfaces is based on the notion of strong homotopy (see [5]):

Definition 10: Let $X \subset E$ be an n-separating set. The set X is a *strong (closed) n-surface* if any back-component of X is strongly n-homotopic to its closure.

We have the following result [5]:

Theorem 11: *Let $X \subset E$ be an n-separating set.*
X is a strong n-surface if and only if, for each x of X, each of the four following conditions is satisfied:

1) $T_n(x, |A|^x) = 1$ *and* $T_n(x, |B|^x) = 1$;
2) $T_{\overline{n}}(x, |A|^x) = 1$ *and* $T_{\overline{n}}(x, |B|^x) = 1$;
3) $\forall y \in N_n^*(x) \cap X$, $T_n(x, |A|^x \cup \{y\}) = 1$ *and* $T_n(x, |B|^x \cup \{y\}) = 1$;
4) $\forall y \in N_{\overline{n}}^*(x) \cap X$, $T_{\overline{n}}(x, |A|^x \cup \{y\}) = 1$ *and* $T_{\overline{n}}(x, |B|^x \cup \{y\}) = 1$.

In Fig. 1, the central points of (b), (c), (d), (g), could satisfy the conditions of Th. 11 for strong 26-surfaces. The central points of (b), (c), (d), (e), (f), (g), (h), could satisfy the conditions of Th. 11 for strong 18-surfaces.

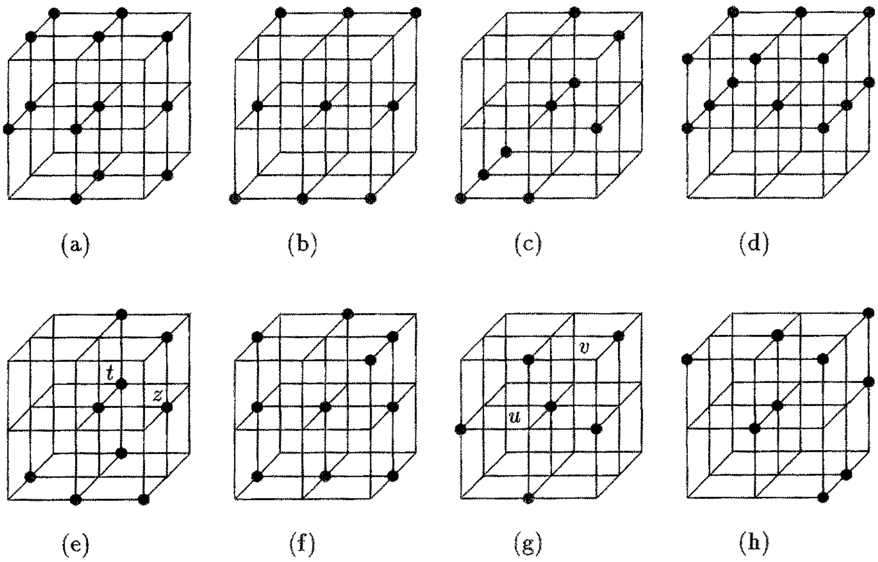


Fig. 1. Examples.

A fully local characterization of strong 26-surfaces was proposed [6], [19]. Furthermore, we have [5]:

Theorem 12: *Any Morgenthaler's closed 26-surface is a strong 26-surface.*

Theorem 13: *Any Malgouyres' closed 18-surface is a strong 18-surface.*

5 Combinatorial manifold

We introduce the notion of two-dimensional combinatorial manifold. We use the same definitions as in [7].

Definition 14: Let G be a graph. An *oriented loop* of G is a circular permutation $L = (v_0, v_1, .., v_{k-1})$, $k > 2$, of vertices of G such that , for all i, v_i is adjacent to v_{i+1} (indices taken modulo k) and $v_i \neq v_j$ if $i \neq j$; a *loop* of G is an oriented loop up to its orientation. A vertex v_i of a loop is called *adjacent* to the loop; the oriented edge (v_i, v_{i+1}) (resp. the edge $\{v_i, v_{i+1}\}$) is called *adjacent* to the oriented loop (resp. the loop). Two loops having a common edge are called adjacent.

Definition 15: A *two dimensional (closed) combinatorial manifold* $M = [G, F]$ is a graph G together with a set F of loops of G, called *faces* or *2-cells* of M, such that:
1) every edge of G is adjacent to exactly two faces, and
2) for every vertex v, the set of faces adjacent to v can be organized in a circular permutation $(f_0, f_1, ..., f_{k-1})$, $k > 1$, called the *umbrella* of v, such that, for all i, f_i is adjacent to f_{i+1} (indices taken modulo k).
The vertices (resp. edges) of a combinatorial manifold are also called the *0-cells* (resp. *1-cells*).

The notion of 2D combinatorial manifold corresponds to a structural description of a surface. It is then desirable to have such a description for surfaces in Z^3. The major problem is that the n-adjacency relation used to define these surfaces does not allow to recover this description. Let us see for example the simple configuration depicted Fig. 2 (a). It corresponds to a Morgenthaler's 26-surface point. We see that, under the 26-adjacency relation, the set of points of the surface which surround the central point x does not constitute a simple closed curve. For example the point 1 has four 26-neighbors (7, 8, 2, 3) in $N^*_{26}(x)$. The 26-adjacency relations for the configuration of Fig. 2 (a) are depicted Fig. 2 (b). We see that the elementary loops for the graph corresponding to the 26-adjacency does not satisfy the conditions of Def. 15: for example, the edge $\{x, 3\}$ is adjacent to the four loops $(x, 2, 3)$, $(x, 1, 3)$, $(x, 3, 4)$, $(x, 3, 5)$.

Thus, we have to consider another relation for extracting the structure of such surfaces. We consider a relation based upon the notion of homotopy and simple point. Let us consider again the configuration of Fig. 2 (a). We suppose that all the neighbors of the central point x have a 26-neighborhood which also corresponds to this configuration, i.e., the $5 \times 5 \times 5$ neighborhood of x is a digital plane. Let us first note that all points of a surface are non-simple points. Suppose now that the point x is removed; we see that there are four points 1, 3, 5, 7 which will appear as 26-simple points; the other points will not appear as simple, for example the neighborhood of the point 2 after deletion of x is depicted Fig. 2 (c), this does not correspond to a 26-simple point. Let us denote $S(x)$ the set of points which appear as simple after deletion of x. As already seen we have $S(x) = \{1, 3, 5, 7\}$, we also have $\{x, 2, 8\} \subset S(1)$, $\{1, 3\} \subset S(2)$... The restriction of the relation S inside the neighborhood of x is depicted Fig. 2 (d). We see that, if we consider a 2-cell as a closed path included in a unit cube, we have the structure of a 2D combinatorial manifold. In Fig. 2 (e), a configuration which could appear in a strong 18-surface is represented. We could expect that the restriction of the relation S inside the neighborhood of x be the one depicted

Fig. 2 (f): we see that the central point is surrounded by a simple closed curve $1,2,3,4,6,7,8$.

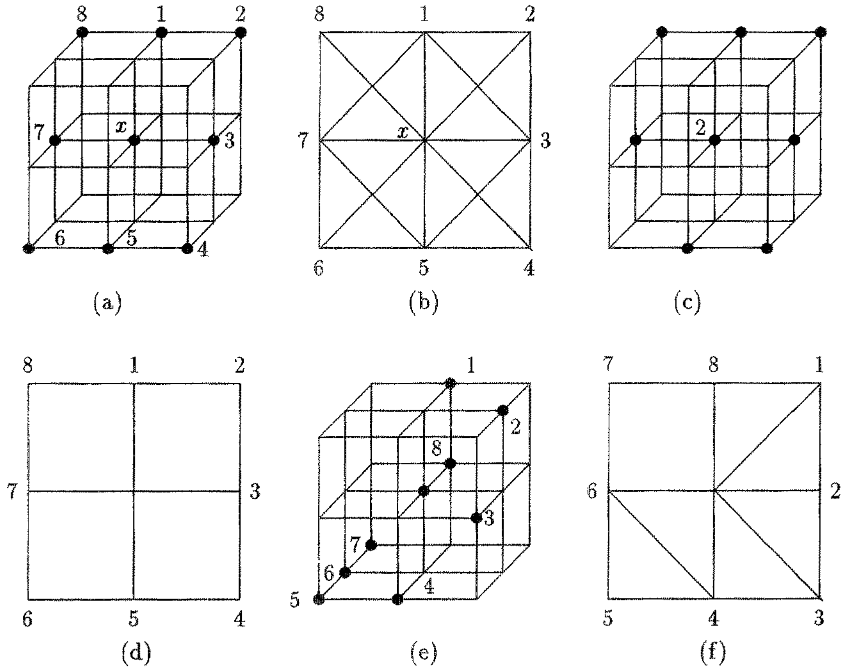


Fig. 2. Examples.

It follows the idea of considering the following graph for extracting a surface structure from a discrete surface in Z^3:

Definition 16: Let $X \subset E$. We define the graph $G_n(X)$ the vertices of which are the points of X and such that, for all x, y of X, x is adjacent to y if:
1) x and y are n-adjacent; and
2) x and y are not n-simple points for X; and
3) x is n-simple for $X \setminus \{y\}$; and
4) y is n-simple for $X \setminus \{x\}$.

Let G be a graph. A loop of G is *simple* if any vertex of the loop is adjacent to exactly two vertices in the loop.

Definition 17:
Let $X \subset E$. We define the set $F_n(X)$ composed of all simple loops for the graph $G_n(X)$ such that:
- for $n = 6$, these loops are included in a unit square;
- for $n = 6^+$, 18, 26, these loops are included in a unit cube.

The following theorem is the main result of this paper. It shows that the main existing notions of discrete surfaces in Z^3 are 2D combinatorial manifolds.

Theorem 18: *Let $X \subset E$.*
If X is a Morgenthaler n-surface, then $[G_n(X), F_n(X)]$ is a 2D combinatorial manifold, with $n = 6$ and $n = 26$.
If X is a Malgouyres 18-surface, then $[G_{18}(X), F_{18}(X)]$ is a 2D combinatorial manifold.
If X is a strong n-surface, then $[G_n(X), F_n(X)]$ is a 2D combinatorial manifold, with $n = 18$ and $n = 26$.

In Fig. 3 (a), a strong 26-surface X is depicted. The cavity of this surface is made of three points which constitute a "corner". The graph $G_{26}(X)$ is depicted Fig. 3 (b). It may be seen that $[G_{26}(X), F_{26}(X)]$ is a 2D combinatorial manifold.

6 Proof of the theorem

The proof of Th. 18 has been made with the help of a computer. Even with a computer this proof is not obvious. The reason is that proposed definition of combinatorial manifold involves the checking of the 125-neighborhood of a point: see the above discussion for the configuration of Fig. 2 (a) where some assumptions about the 125-neighborhood of the central point has been made for recovering the structural description of Fig. 2 (d). An exhaustive checking of all the 2^{125} configurations in this neighborhood is out of the reach of computers. This explains that we have to establish some intermediate lemmas in order to induce the properties of combinatorial manifolds from the 26-neighborhood of a point.

First of all, we introduce the notion of extensible configurations (see also [17]). Let us consider the configuration of Fig. 1 (e). It satisfies the conditions for Morgenthaler's 26-surface points (Def. 5). Nevertheless, it is impossible that such a configuration appear in a Morgenthaler's 26-surface. The reason is that this configuration is not extensible: it is not possible that all the points 26-adjacent to the central point could satisfy the conditions of Morgenthaler's 26-surface points. We give a precise definition of these cases:

Let $x \in E$. A *configuration of* x is a subset of $N_{26}(x)$ which contains x. Let $\mathcal{K}_x$ be a set of configurations of x. We say that $C_x \in \mathcal{K}_x$ is *extensible* if, for each point y of C_x there exists a configuration $C'_x \in \mathcal{K}_x$ such that C'_y, which is the translation of C'_x by the vector xy, satisfies $C'_y \cap N_{26}(x) = C_x \cap N_{26}(y)$.

We present now the way for proving Th. 18. By Th. 12 and 13, we have only to prove it for Morgenthaler's 6-surfaces and for strong n-surfaces ($n = 18, 26$).

6.1 Morgenthaler's 6-surfaces

First, a list of all possible configurations which satisfy the conditions for Morgenthaler's 6-surfaces has been made. Such an exhaustive checking is within the reach of computers since it involves only 2^{26} cases. Second all non-extensible configurations have been eliminated. Then, the following lemma was proved by checking all extensible configurations, (xG_ny means that x and y are adjacent for the graph $G_n(X)$):

Lemma 19: *Let X be a Morgenthaler's 6-surface. If $x \in X$ and $y \in X$ are 6-neighbors, then xG_6y.*

On the other hand, by Def. 16, if xG_6y, then x and y are necessarily 6-neighbors. Thus it is possible to prove the theorem by checking again all extensible configurations: with Lemma 19, it is possible to recover the graph $G_6(X)$ and the set of loops $F_6(X)$ which appear in the neighborhood of the central point. The conditions for 2D combinatorial manifolds were verified.

6.2 Strong 26-surfaces

We see that the characterization of Th. 11 for strong n-surfaces is not "fully local": the knowledge of $|X|^x$ is not sufficient to decide if x satisfies the four properties. For checking the characterization, we need to know $|X|^x$ but we also need to know the distribution of the points of $|\overline{X}|^x$ between $|A|^x$ and $|B|^x$. In fact, since the symmetry of the four conditions with respect to A and B, we see that it is sufficient to know this distribution up to a renaming of A and B. More precisely, it is sufficient to know, for each x of X, a labeling of $|\overline{X}|^x$:

Definition 20: Let $X \subset E$ be an n-separating set and let $x \in X$. A *labeling* of $|\overline{X}|^x$ is a map $f_x \colon |\overline{X}|^x \longrightarrow \{0, 1\}$ such that $\{f_x^{-1}(0), f_x^{-1}(1)\} = \{|A|^x, |B|^x\}$.

The knowledge of a labeling is necessary only if there is a component of $|\overline{X}|^x$ not $\overline{n}$-adjacent to x, (see, for example, the configuration depicted Fig. 1 (g)). The following lemmas allow to characterize these cases (see [6] for the proof).

Lemma 21:
Let X be a strong 26-surface and let $x \in X$. If $|\overline{X}|^x$ contains a 6-component not 6-adjacent to x, then this 6-component is composed solely of one point. Furthermore this point is necessarily strictly 26-adjacent to x.

Lemma 22:
Let X be a strong 26-surface. Let $x \in X$ and let $y \in N_{18}^+(x) \cap X$. If $N_6^*(x) \cap N_6^*(y)$ is a subset of the same 6-component of $|\overline{X}|^x$, then y is 6-adjacent to a one-point component of $|\overline{X}|^x$. Furthermore this one-point component and $N_6^*(x) \cap N_6^*(y)$ will belong to two different 6-components of $\overline{X}$.

For proving Th. 18, we first make a list of all possible configurations such that there exists a labeling for which the conditions of Th. 11 are satisfied. We eliminate all non-extensible configurations. Then, we prove the following lemma by an exhaustive checking of all remaining configurations:

Lemma 23: *Let X be a strong 26-surface and let $x \in X$. We have:*
- *$\forall y \in N_6^*(x) \cap X$, $xG_{26}y$; and*
- *$\forall y \in N_{18}^+(x) \cap X$, if one of the two common 6-neighbors of x and y belong to X, then we do not have $xG_{26}y$; and*
- *$\forall y \in N_{18}^+(x) \cap X$, if the two common 6-neighbors of x and y belong to two different 6-components of $\overline{X}$, then $xG_{26}y$; and*
- *$\forall y \in N_{18}^+(x) \cap X$, if the two common 6-neighbors of x and y belong to the same 6-component of $\overline{X}$, then we do not have $xG_{26}y$.*

On the other hand, by the definition of simple points, if $y \in N_{26}^+(x) \cap X$, we do not have $xG_{26}y$. Thus, as for Morgenthaler's 6-surfaces, it is possible to examine all extensible configurations and, with Lemmas 22 and 23, to recover the graph

$G_{26}(X)$ and the set of loops $F_{26}(X)$ which appear in the neighborhood of the central point. The conditions for 2D combinatorial manifolds were verified.

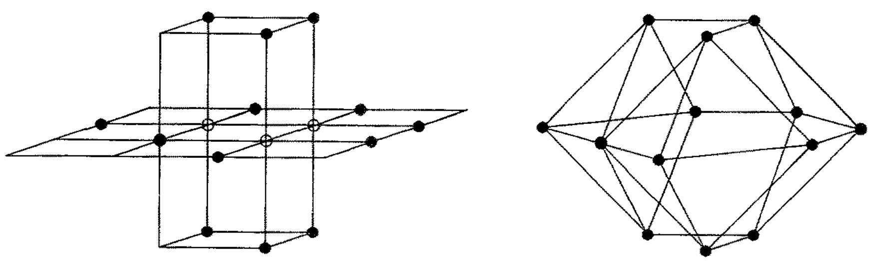
6.3 Strong 18-surfaces

As for strong 26-surfaces, we first make a list of all possible configurations such that there exists a labeling for which the conditions of Th. 11 are satisfied. We eliminate all non-extensible configurations. We then prove the following lemmas by an exhaustive checking of all these configurations:

Lemma 24:
If X is a strong 18-surface, then $\forall x \in X$, $|\overline{X}|^x$ admits an unique labeling.

Lemma 25: *Let X be a strong 18-surface and let x and y be two points of X. Let C be any back-component of X, i.e. $C = A$ or $C = B$. We say that a 6-path π from x to y is a C-path if all points of π, except x and y, belong to C. We say that x and y are C^k-connected if there is a C-path from x to y and if the minimal length of a C-path from x to y is equal to k.*
We have:
- $\forall y \in N_6^*(x) \cap X$, $xG_{18}y$; and
- $\forall y \in N_{18}^+(x) \cap X$, $xG_{18}y$ if and only if x and y are A^k-connected and B^l-connected, with $k + l \leq 6$; and
- $\forall y \in N_{18}^+(x) \cap X$, if the two common 6-neighbors of x and y belong to $\overline{X}$, then we have $xG_{18}y$ if and only if the two common 6-neighbors belong to two different 6-components of $\overline{X}$.

The problem with strong 18-surfaces is that, if a point y is such that $y \in N_{18}^+(x) \cap X$, and if only one of the two common 6-neighbors of x and y belong to X, then we may have $xG_{18}y$ but it is also possible that x and y are not adjacent under the G_{18} relation. It follows that it is not always possible to recover the G_{18} graph by examining the 26-neighborhood of a point. For these cases, we make two assumptions: we first suppose that $xG_{18}y$ and then we suppose that x and y are not adjacent under the G_{18} relation. With this exhaustive checking and with Lemmas 24 and 25, Th. 18 was proved.



(a) (b)

Fig. 3. A strong 26-surface X and the graph $G_{26}(X)$.

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