On a Visibility Representation of Graphs

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Abstract. We give a visibility representation of graphs which extends
some very well-known representations considered extensively in the lite-
rature. Concretely, the vertices are represented by a collection of parallel
hyper-rectangles in $\mathbf{R}^n$ and the visibility is orthogonal to those hyper-
rectangles. With this generalization, we can prove that each graph ad-
mits a visibility representation. But, it arises the problem of determining
the minimum Euclidean space where such representation is possible. We
consider this problem for concrete well-known families of graphs such as
planar graphs, complete graphs and complete bipartite graphs.

1 Introduction

The problem of determining a visibility representation of a graph has been stu-
died extensively in the literature due to the large number of applications (as in
VLSI design, CASE tools, hidden-surface elimination problem, etc., [7, 8, 11, 13,
15]) and, also, by the combinatorial properties of those graphs.

In a visibility representation of a graph, the vertices map to objects in Eu-
clidean space and the edges are determined by certain visibility relations.

Of course, both, the objects and the visibility used play an important role
in characterizing the types of graphs that admit visibility representations. But,
in any case, given a certain class of objects and a concrete visibility, there exist
always graphs that are not representable, in this way, Tamassia & Tollis [14] and
Wismath [16] proved that a graph is a *bar visibility graph* (where the vertices
represent horizontal line segments in the plane and two nodes are connected
by an edge if their two horizontal rectangles can see each other vertically and
non-degenerately) if and only if it admits a planar embedding with all cutpoints
in the exterior face. And Bose et al. [3] proved that K_n is not VR-representable
for $n \geq 103$ (a graph is said to be *VR-representable* if each vertex of the graph
maps to a closed rectangle in $\mathbf{R}^3$ such that the rectangles are disjoint, the planes
determined by the rectangles are perpendicular to the z-axis, and the sides are
parallel to the x or y axes. And, again, two nodes are connected by an edge if
their two horizontal bars can see each other vertically and non-degenerately).

On the other hand, from a more theoretical point of view and since Kura-
towski's Theorem [10], several measures of the planarity and/or dimension of a
graph have been considered. But, few of these measures, notably, Boxicity, Grid

intersection graphs [1, 9, 2], are related with visibility representations in the line of the approachs mentioned above.

In this paper, we prove that the representations studied by Tamassia & Tollis [14] and Wismath [16] in $\mathbf{R}^2$ and by Bose et al. [3] in $\mathbf{R}^3$ can be easily generalized to any dimension, and, if we consider for each graph the minimum n where it is possible such a representation we obtain, in this way, a new measure of the complexity of the graph. Thus, we say that a graph is *representable* in $\mathbf{R}^n$ if it can be represented in such a way that each vertex maps to a hyper-rectangle in $\mathbf{R}^n$ (where the hyper-rectangles that we consider are a cartesian product of $n-1$ closed intervals in $\mathbf{R}$ and a number in the last coordinate; i.e., $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$) and two nodes are connected by an edge if there exists a closed cylinder in $\mathbf{R}^n$, orthogonal to the rectangles, of non-zero length and radius such that the ends of the cylinder are contained in each of the hyper-rectangles and it does not intersect any of the other hyper-rectangles. We say that a graph G has *representation index* equal to n (or $RI(G) = n$ for short) if $\mathbf{R}^n$ is the minimum where such a representation is possible.

This paper is organized in the following way, in Sect.2 we prove that any graph has a finite RI, and we see that it is convenient to extend that index. Section 3 is devoted to the study of the RI of planar graphs. And in Sect.4 and 5 we deal with the RI of complete graphs and bipartite graphs respectively.

2   Representation Index

In this section we are going to prove that each graph is representable in some $\mathbf{R}^n$. First, we need the following lemma.

Lemma 1. *Every graph representable in $\mathbf{R}^n$ is representable in $\mathbf{R}^{n+1}$.*

Proof. It is easy to check that of the configuration

$$R_i = [a_1^i, b_1^i] \times \cdots \times [a_{n-1}^i, b_{n-1}^i] \times \{a_i\} \quad i = 1, 2, \ldots p$$

represent G, then the configuration

$$R_i = [a_1^i, b_1^i] \times \cdots \times [a_{n-1}^i, b_{n-1}^i] \times [0, 1] \times \{a_i\} \quad i = 1, 2, \ldots p$$

also represents G. □

Theorem 2. *Given a graph G, there exists $n \in \mathbf{N}$ such that G is representable in $\mathbf{R}^n$.*

Proof. We will prove the theorem by induction on k the number of vertices of G. Obviously the statement is true for small values of k. We now assume that G has $k+1$ vertices and that the statement is true for graphs with k vertices. Given a vertex p of G, we split a representation $\mathcal{R}$ of $G - p$ into two subset

$$\mathcal{R} = \{R_{t_1}, R_{t_2}, \ldots, R_{t_{n_p}}\} \cup \{Q_{u_1}, Q_{u_2}, \ldots, Q_{u_{(k-n_p)}}\}$$
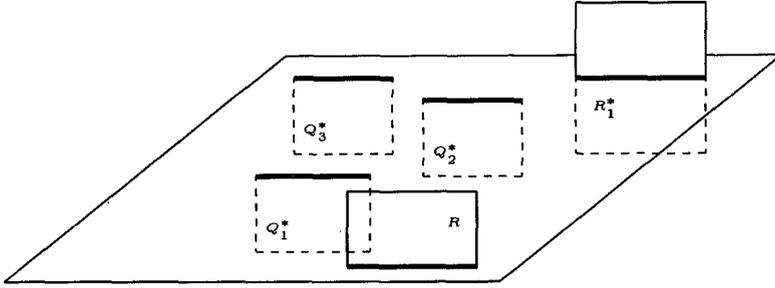
Fig. 1. G is representable in $\mathbf{R}^n$.

In such a way that the hyper-rectangles $R_{t_i} = I_{t_i\,1} \times \cdots \times I_{t_i\,n-1} \times \{t_i\}$ ($1 \le i \le n_p$) correspond with the vertices p_{t_i} adjacent to p in G. Now, it is easy to see that the configuration $\mathcal{R}^* = \{R^*_{t_1}, R^*_{t_2}, \ldots, R^*_{t_{n_p}}\} \cup \{Q^*_{u_1}, Q^*_{u_2}, \ldots, Q^*_{u_{(k-n_p)}}\} \cup \{R\}$ represents G (being p represented by R), where

$$Q^*_{u_j} = \mathcal{I}_{u_j\,1} \times \cdots \times \mathcal{I}_{u_j\,n-1} \overbrace{\times[-1,0] \times \cdots \times [-1,0]}^{n_p} \times \{u_j\}$$

$$R^*_{t_1} = I_{t_1\,1} \times \cdots \times I_{t_1\,n-1} \overbrace{\times[-1,1] \times \cdots \times [-1,0] \times \cdots \times [-1,0]}^{n_p} \times \{t_1\}$$

$$\vdots$$

$$R^*_{t_s} = I_{t_s\,1} \times \cdots \times I_{t_s\,n-1} \overbrace{\times[-1,0] \times \cdots \times [-1,1] \times \cdots \times [-1,0]}^{n_p} \times \{t_s\}$$

$$\vdots$$

$$R^*_{t_{n_p}} = I_{t_{n_p}\,1} \times \cdots \times I_{t_{n_p}\,n-1} \overbrace{\times[-1,0] \times \cdots \times [-1,0] \times \cdots \times [-1,1]}^{n_p} \times \{t_{n_p}\}$$

and $R = I_1 \times \cdots \times I_{n-1} \overbrace{\times[0,1] \times \cdots \times [0,1] \times \cdots \times [0,1]}^{n_p} \times \{t_{k+1}\}$ with $I_i = [\min_j\{x : x \in I_{j\,i}\}, \max_j\{x : x \in I_{j\,i}\}]$. $\qquad\Box$

Observe, that in Lemma 1 the last interval in all hyper-rectangles in the configuration is always $[0, 1]$, that means that we are not using the whole $\mathbf{R}^n$ for our representation but only a half ($x_{n-1} \ge 0$) and that all hyper-rectangles are lying on the hyper-plane of equation $x_{n-1} = 0$. Thus, we can give a finer definition of RI saying that a graph G of (old) representation index equal to n has actually $RI(G) = (n-1) + 1/2$ if it admits a represprentation in $\mathbf{R}^n$ such that any hyper-rectangle is of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [0, b_{n-1}] \times \{a_n\}$.

This new definition will allow us to get a better view of the problem, and we will get our main results using it. Moreover, there are some other additional reasons to consider this more general concept of representation index. For
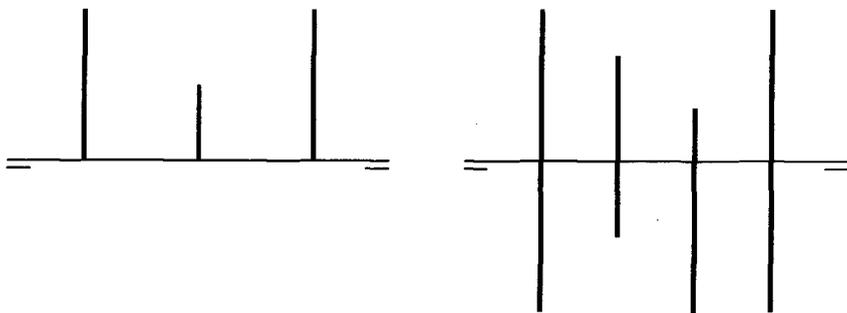
Fig. 2. $RI(K_3) = 1 + 1/2$ and $RI(K_4) = 2$.

instance, observe that if a graph has index $1 + 1/2$, then we can associate it an n-tuple of integer numbers in such a way that each vertex maps to one of the number of the n-tuple and two vertices are joined by an edge if all numbers between them in the n-tuple are smaller. Note that graphs with index $1 + 1/2$ are not the same of those of index 2 (K_4 has index 2 and K_3 has index $1 + 1/2$).

3   Planar Graphs

Bose et al. [3] proved that if G is planar then $RI(G) \leq 3$. On the other hand, Tamassia & Tollis [14] and Wismath [16] gave the following theorem characterizing those graphs with representation index smaller or equal to 2.

Theorem 3. *[14, 16] A graph G has representation index smaller or equal to 2 if and only if there is a planar embedding of G with all cutpoints on the exterior face.*

   We complete now this theorem, characterizing those graphs with representation index $1 + 1/2$. For that characterization we say that a graph G is *outerhamiltonian* if it has a path (possibly open) containing all vertices of G such that there exists a planar embedding of G with all edges of that path on the exterior face. Observe that Mitchell's algorithm to determine if a graph is outerplanar [12] with some modifications allows to get a linear-time algorithm to determine if a given graph is outerhamiltonian.

Theorem 4. *A graph has representation index $1 + 1/2$ if and only if it is outerhamiltonian.*

Proof. It is obvious that if $RI(G) = 1 + 1/2$ then G is outerhamiltonian.
   For the sufficiency there are two cases to consider.
   In the first case, we suppose that G is 2-connected. In this case, G is a polygon of vertices $\{v_1, v_2, \ldots, v_n\}$ (where that ordering if one of the two possible orderings of the vertices of the polygon) with some of its non-intersecting diagonals.
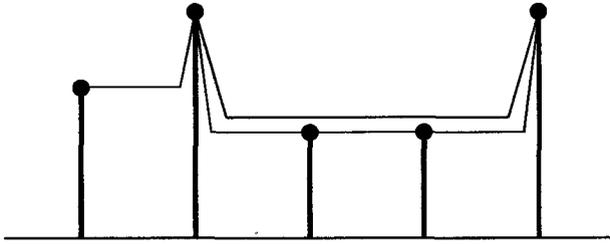
Fig. 3. If $RI(G) = 1 + 1/2$ then G is outerhamiltonian.

We map the vertex v_i to the bar $[0, d_i] \times \{a_i\}$, where $d_i - 1$ for $2 \leq i \leq n - 1$ is the total number of diagonals minus the number of diagonals $\{v_l, v_k\}$ with $l < i < k$, and $d_1 = d_n$ is the total number of diagonals plus 2.

In the second case, if G is not 2-connected, in each block there exist, at most, two cutpoints, place them the first and the last in that block, and we sort the blocks, obtaining, in that way, an ordering of all vertices of the graph. Now, we represent each block as in the first case, but giving the same length (the biggest one) to all bars representing cut-points. □
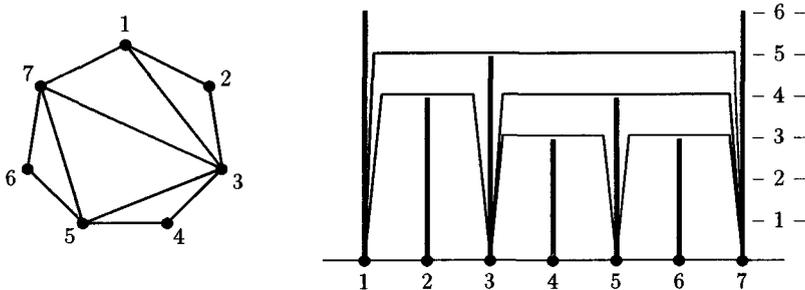
Fig. 4. Construction of a bar-representation in $\mathbf{R}^{1+1/2}$

Finally Bose et al.[3] proved that if G is a planar graph then $RI(G) \leq 3$. Thus, it remains to determine which planar graphs have representation index $2 + 1/2$, this question is still open.

4 Complete Graphs

As it was said before $RI(K_3) = 1 + 1/2$, $RI(K_4) = 2$ and Bose et al. [3] proved that if $n \leq 20$ then $RI(K_n) \leq 3$. In this section, we are going to prove that $RI(K_{10}) = 2 + 1/2$ and that $RI(K_{2n}) \leq RI(K_n) + 1/2$. Observe that this

implies Bose et al. [3] result as a particular case. Moreover, we give lowerbounds to the representation index of complete graphs.

Lemma 5. $RI(K_{10}) = 2 + 1/2$.
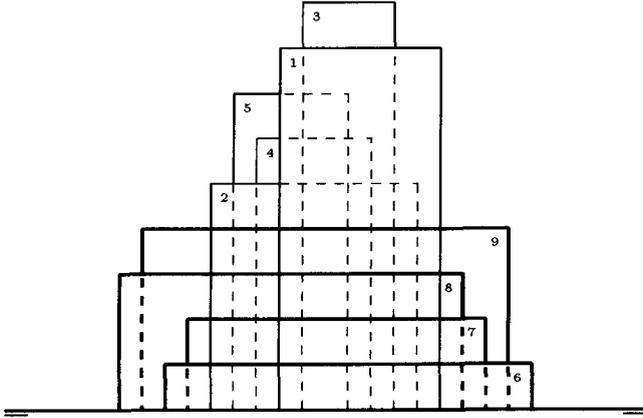
Fig. 5. K_{10} in $\mathbf{R}^{2+1/2}$

Proof. We can give the following configuration (Fig. 5) of 9 rectangles, such that all them are seen from the front. □

Theorem 6. $RI(K_{2n}) \leq RI(K_n) + 1/2$.

Proof. (Outline) We consider two cases. In the first one, we assume that $RI(K_n)$ is not an integer number. If we have a collection of n hyper-rectangles in a half of $\mathbf{R}^m$ representing K_n and such that the last interval of each one is $[0, a_i]$, then we consider a new configuration sustituing those intervals by $[-i, a_i]$, we get a representation of K_{2n} enfrenting to that configuration a new copy of that configuration but flipper over and rotated 90° (see Fig.6).

In the second case, we suppose that $RI(K_n)$ is an integer number. If we have a collection of n hyper-rectangles in $\mathbf{R}^m$ representing K_n, then it must exist a hyper-plane ortogonal to all hyper-rectangles and intersecting all them. Thus, we can suppose that $x_{m-1} = 0$ is such a hyperplane. That means that the i-th hyper-rectangle ends with the product $[a_i, b_i] \times \{a_i\}$ with $a_i < 0 < b_i$, then we sustitute those products by $[-i, b_i] \times [0, -a_i] \times \{a_i\}$. In this way, we get n hyper-rectangles in a half of $\mathbf{R}^{m+1}$ representing K_n such that the intervals before the last intervals in the product defining those hyper-rectangle constitute a sort of staircase, we can face in front of them the same configuration, but now changing $[a_i, b_i] \times \{a_i\}$ by $[a_i, b_i] \times [0, i] \times \{-i\}$ in order to get a representation of K_{2n} in a half of $\mathbf{R}^{m+1}$ (see Fig.7). □
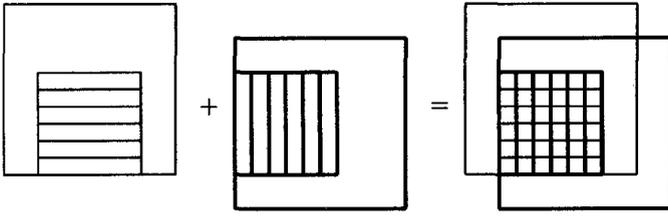
Fig. 6. The first case of Theorem 6.
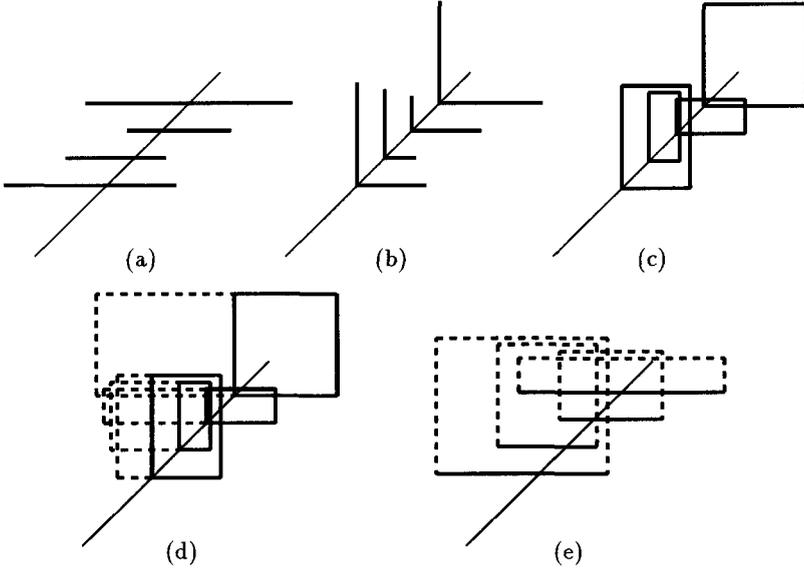
Fig. 7. The second case of Theorem 6.

As an inmediate consequence of Lemma 5 and Theorem 6 we get.

Corollary 7. *For all $n > 2$ if $m \leq 5 \cdot 4^{n-2}$ then $RI(K_m) \leq n$.*

Corollary 7 provides an upperbound to the representation index of the complete graphs. Now, we are going to try to get a lowerbound. As Fekete, Houle and Whitesides do in their paper [6], we will use the following lemma.

Lemma 8. [Attributed by F.R.K. Chung [4] to V. Chvátal and J.M. Steele, among others.] *For all $m > 1$, in every sequence of $\binom{m}{2} + 1$ distinct integers, there exists at last one strongly unimaximal subsequence (with only one local maximum) of length m). On the other hand, there exists a sequence of $\binom{m}{2}$ distinct integers that has no strongly unimaximal subsequence of length m.*

We will use the following easy-to-prove lemma.

Lemma 9. *If the hyper-rectangles $R_i = [a_1^i, b_1^i] \times [a_2^i, b_2^i] \times \cdots \times [a_{t-1}^i, b_{t-1}^i] \times \{a_i\}$ with $1 \le i \le n$ ($b_{t-1}^i \ge 0$) represent K_n, and $\{(a_{t-1}^i) \ 1 \le i \le n\}$ is a strongly unimaximal sequence, then the hyper-rectangles $R_i' = [a_1^i, b_1^i] \times [a_2^i, b_2^i] \times \cdots \times [0, b_{t-1}^i] \times \{a_i\}$ represent also K_n.*

Bose el al. prove in their paper [3] that $RI(K_{103}) \ge 3$ and Fekete, Houle & Whitesides prove in [6] by using of Lemma 8 that $RI(K_{56}) \ge 3$. It is possible to get this same lowerbound as a consequence of Lemma 9 and Lemma 8, in fact, a more general result can be achieved.

Theorem 10. $RI(K_{\binom{m}{2}+1}) > RI(K_m)$.

Proof. (Outline) If the hyper-rectangles $R_i = [a_1^i, b_1^i] \times [a_2^i, b_2^i] \times \cdots \times [a_{t-1}^i, b_{t-1}^i] \times \{a_i\}$ ($b_{t-1}^i \ge 0$) with $1 \le i \le \binom{m}{2} + 1$ represent $K_{\binom{m}{2}+1}$, by Lemma 8, there exists a strongly unimaximal subsequence of (a_{t-1}^i) of length m. Then Lemma 9 assures that K_n has representation index strictly lower than $K_{\binom{m}{2}+1}$. $\square$

5 Complete Bipartite Graphs

In this section we characterize the representation index of all complete bipartite graphs. Firstly, it is easy to observe that $RI(K_{n,m}) \le 3$. But from the results in Sect. 3, we get that $RI(K_{1,2}) = RI(K_{2,2}) = 1 + 1/2$, and that $RI(K_{2,n}) = RI(K_{2,n}) = 2$ for all $n > 2$.

Now, we are going to prove that $RI(K_{3,n}) = 2 + 1/2$, for $n \ge 3$, and that $RI(K_{n,m}) = 3$ when $n, m \ge 4$.

Lemma 11. $RI(K_{3,n}) = 2 + 1/2$, *for $n \ge 3$.*
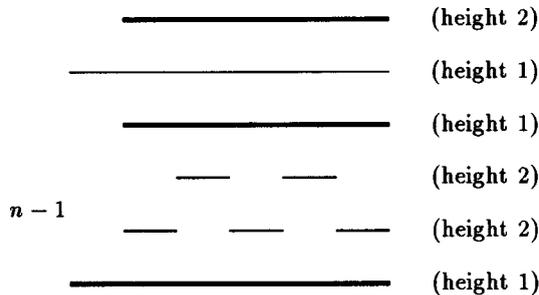
Fig. 8. $K_{3,n}$ in $\mathbf{R}^{2+1/2}$

Proof. As $K_{3,n}$ is not planar, we get that $RI(K_{3,n}) \geq 2 + 1/2$, and we can give the configuration of Fig. 8 to prove that $RI(K_{3,n}) = 2 + 1/2$. $\qquad\Box$

And studying exhaustly all cases it is possible to prove the following lemma.

Lemma 12. $RI(K_{4,4}) = 3$.

Finally, it is trivial to check that $RI(K_{m,n}) = 3$ for all $m, n \in \mathbf{N}$ such that $4 \leq m \leq n$.
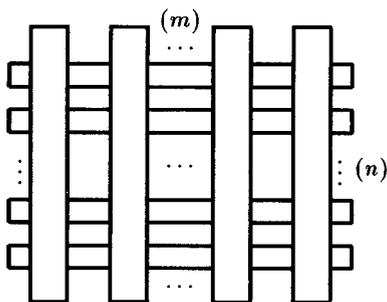
Fig. 9. $K_{n,m}$ in $\mathbf{R}^3$

6   Conclusions

We can sumarize our results in the following table.

Type of graph G	$RI(G)$	Reference
Outerhamiltonian	$1 + 1/2$	Theorem 4
Cutpoint-outerplanar	≤ 2	Theorem 3 [14]
Planar	≤ 3	Bose et al. [3]
$K_n \quad n \leq 3$	$1 + 1/2$	
K_4	2	
K_{10}	$2 + 1/2$	Lemma 5
K_{11}	3	Theorem 10
$K_n \quad n \geq 56$	> 3	Fekete et al. [6]
$K_{5 \cdot 4^{n-2}} \quad n > 2$	$\leq n$	Corollary 7
$K_n \quad n \geq \binom{m}{2} + 1$	$> RI(K_m)$	Theorem 10

$K_{1,2}$ or $K_{2,2}$	$1+1/2$	
$K_{1,n}$ or $K_{2,n}$ $n \geq 3$	2	
$K_{3,n}$ $n \geq 3$	$2+1/2$	Lemma 11
$K_{4,4}$	3	Lemma 12
$K_{m,n}$ $4 \leq m \leq n$	3	Bose et al. [3]

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