Optimal Embeddings in the Hamming Cube Networks *

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Abstract. This paper studies network embeddings in the Hamming cubes, a recently designed interconnection topology for multicomputers. The Hamming cube networks are supergraphs of incomplete hypercubes such that the additional edges form an extra binomial spanning tree. The recursively constructible and unit incremental Hamming cubes have better properties than hypercubes, including half of logarithmic diameter and higher fault-tolerance. They also support simple routing and efficient broadcasting schemes. In this paper, we show that Hamiltonian paths and cycles of all lengths, complete binary trees and their several variants are subgraphs of Hamming cubes. Our embeddings have both dilation and expansion equal to one. Furthermore, taking advantage of the enhanced edges in the Hamming cubes, tree machines can be embedded with dilation of one and expansion of $\frac{7}{6}$. Thus, Hamming cubes provide embeddings at a lower cost than (incomplete) hypercubes of the same size.

Keywords: Network embedding, dilation, interconnection network, Hamming cube, incomplete hypercube, binary tree, hypertree, tree machine.

1 Introduction

The demand for high-performance, reliable computing motivates the study of massively parallel, distributed-memory machines. Many static interconnection network topologies have been proposed for multicomputers [Lei92]. Such a network is usually modeled as an undirected graph $G = (V, E)$, where the node-set V represents the processor-memory modules and the edge-set E represents the communication links among these modules.

Among the existing networks, the binary hypercubes have received significant attention because of such attractive characteristics as node and edge symmetries, logarithmic diameter, high fault-tolerance, scalability, simple communication mechanisms, and embeddability of other networks. An n-dimensional binary hypercube is defined as $Q_n = (V, E)$, where $V = \{v_i = BR(i) \mid 0 \le i \le 2^n - 1\}$ consists of labeled nodes and $BR(i)$ is the binary representation of integer i. An edge $(v_i, v_j) \in E$ exists if and only if the *Hamming distance*, $\rho(v_i, v_j)$, between nodes v_i and v_j is one.

* This research is supported by Texas Advanced Technology Program TATP-003594031.

The hypercube topology grows its order by a power of two. There exist two variants, namely *incomplete* hypercubes [Kat88], IQ_k^n, of $2^n + 2^k$ nodes, where $0 \leq k < n$, and ***generalized incomplete hypercubes*** [TY91], $IQ(N)$, for $N \geq 1$, with incrementabilities of 2^k and 1, respectively. The network IQ_k^n consists of two "complete" hypercubes, Q_n (the *front* cube) and Q_k (the *back* cube), while $IQ(N)$ is composed of several complete hypercubes of different orders. These three networks can be classified as the ***hypercube-family.***

Recent efforts have been made to improve the performance of the hypercube-family of networks with additional links, leading respectively to *folded* hypercubes [EAL91], *enhanced incomplete* hypercubes [CT92], and ***enhanced generalized incomplete*** hypercubes [DM94]. For example, an n-dimensional folded hypercube, FQ_n, has the *complementary* edges $(v_i, v_{\bar{i}})$ for every node v_i in the hypercube Q_n, where $\bar{i}$ is the address with all bits of i complemented. These networks can be categorized into the *enhanced* hypercube-family. Another example of this family is the *incrementally extensible hypercubes* [SS92].

There exist networks which modify the hypercube topology in order to derive new networks according to various design options. For example, a *twisted n-cube* [ENS87] twists a pair of edges in the shortest cycle (consisting of four nodes) of the hypercube Q_n. A *crossed cube* [Efe92], on the other hand, recursively twists pairs of edges. These networks can be regarded as the ***hypercube-like*** family.

We have recently derived ***Hamming cubes*** as another member of the enhanced hypercube-family [DM94a]. Our design is based on a theoretical network model, called the *incremental Hamming group*. These networks are supergraphs of incomplete and folded hypercubes. As shown in [DM94a, DM94b], the Hamming cubes have better topological and performance properties than the hypercubes (complete or incomplete) of the same size, without incurring much additional cost. These properties include recursive scalability, unit incrementability, half of logarithmic diameter, high fault-tolerance, simple routing and broadcasting schemes.

The embedding or mapping of one network architecture into another is an important problem because this way parallel algorithms developed for one architecture can be easily ported to another architecture. The (incomplete) hypercubes can efficiently simulate many other networks with a small factor of slowdown [Lei92, TCC90, OD95]. For example, binary hypercubes have only Hamiltonian cycles of even lengths [SS88, SSB93]. The $(2^n - 1)$-node complete binary tree is a subgraph of Q_{n+1}, and also can be embedded into Q_n with dilation two [BI85].

In this paper, we study the embeddability of the Hamming cubes, showing that several standard topologies including Hamiltonian paths and cycles, complete binary trees and their variants, and tree machines can be optimally embedded. These embeddings are better than those into (incomplete) hypercubes. For example, Hamming cubes are *pancyclic*, i.e. cycles of all lengths can be embedded as subgraphs. The complete binary tree is a subgraph of the same-sized Hamming cube, and also tree machines can be embedded at a lower cost than into hypercubes.

The rest of this paper is organized as follows. Section 2 introduces the Hamming cubes and summarizes their properties. Section 3 deals with embeddings of guest topologies into the Hamming cubes, while Section 4 concludes the paper.

Let us define a few notations to be used throughout this paper. Let $BR(i) = (b_k b_{k-1} \dots b_1)$ be the binary representation of a non-negative integer i, where b_1 is the least significant bit. If there is no confusion, for brevity, $BR(i)$ and i will be used interchangeably. For convenience, the following notations are also defined: $i^{[j]} = (b_k b_{k-1} \ldots \overline{b_j} \ldots b_1)$ in which the jth bit of i is complemented; $\overline{i} = (\overline{b_k b_{k-1} \ldots b_1})$, i.e. all bits of i are complemented; and $i^{\{m\}} = (b_k b_{k-1} \ldots \overline{b_m \ldots b_1})$ such that the rightmost m bits of i are complemented.

2 Hamming Cubes

This section formally introduces the Hamming cube networks originally due to Das and Mao [DM94a]. It also summarizes some of their properties relevant for the subsequent sections.

2.1 Network Definition

A Hamming cube of order $N \geq 2$, denoted as $HC(N) = (V, E)$, is an undirected, connected graph in which $V = \{v_i = BR(i) \mid 0 \leq i \leq N-1\}$ is the set of labeled nodes. (For simplicity, node v_i will also be donted as i.) Let v_i and v_j, for $i < j$, be two nodes in $HC(N)$, each being represented by $\lceil \log(j+1) \rceil$ bits. Then an edge $(v_i, v_j) \in E$ exists iff any one of the following two conditions is satisfied [DM94a]:

(E1): The Hamming distance $\rho(v_i, v_j) = 1$; or
(E2): $\rho(v_i, v_j) = h = \lceil \log(j+1) \rceil$ for $j \geq 1$.

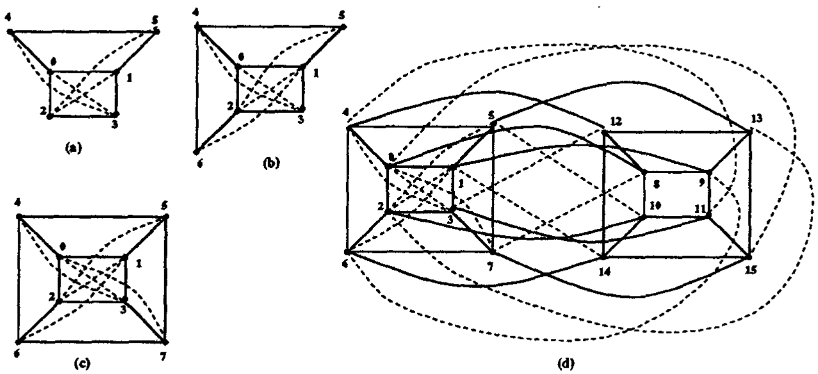

Fig. 1. Hamming cubes $HC(N)$ for N = 6, 7, 8 and 16.

The edges defined by Conditions ($E1$) and ($E2$) are designated as *$E1$-edges* and *$E2$-edges*, respectively. Clearly, the $E1$-edges define the underlying incomplete hypercube topology of $HC(N)$. An $E2$-edge v_i and v_j is said to be *n_h-dimensional* (or in dimension n_h), if $\rho(v_i, v_j) = h = \lceil \log(j+1) \rceil$ and $j \geq 1$. Note that (0, 1) is an $E1$-edge as well as an $E2$-edge. Figures 1(a)-(d) depict Hamming cubes $HC(N)$ for N = 6, 7, 8 and 16, where the $E2$-edges are distinguished by the broken lines. For example, since $\rho(v_3, v_{12}) = 4 = \lceil \log(12+1) \rceil$, there exists an n_4-dimensional $E2$-edge between the nodes v_3 and v_{12} in $HC(16)$. Similarly, $\rho(v_1, v_6) = 3$ implies that v_6 is linked to v_1 through an n_3-dimensional $E2$-edge. For conformity, $HC(2^n)$ will be called the *n-dimensional Hamming cube*, denoted as HC_n.

A *binomial spanning tree* is a binomial tree which spans all nodes in a network. Such a tree of height n has the characteristic that the number of nodes at

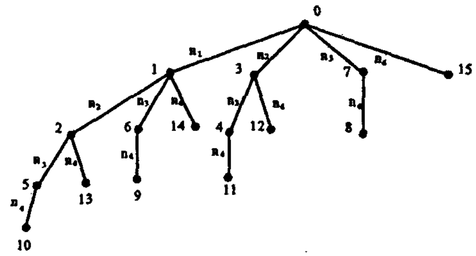

Fig. 2. A binommial spanning tree of HC_4, rooted at node v_0, using the $E2$-edges.

level i is $\binom{n}{i}$, for $0 \leq i \leq n$. A binomial spanning tree rooted at node v_0 can be constructed in HC_n with the help of the $E2$-edges (see Figure 2 for an example), which precisely gives a physical interpretation of these extra edges. By definition, there exists n_n-dimensional $E2$-edges in HC_n between all node-pairs v_i and v_j such that $\rho(v_i, v_j) = n$. Such edges correspond to the *complementary* edges in the folded hypercube [EAL91], which is thus a spanning subgraph of HC_n.

2.2 Topological Properties

Several important topological properties of the Hamming cube networks, including the edge complexity, node degree, and diameter are derived in [DM94a]. The Hamming cube $HC(N)$, where $N = 2^n$ and $n \geq 1$, has $E(N) = \frac{N}{2}\log N + N - 2$ edges, and its diameter is at most $\lceil \frac{\log N}{2} \rceil$. Thus, using only $2^n - 2$ extra edges compared to Q_n, the diameter of the n-dimensional Hamming cube (HC_n) reduces to $\lceil \frac{n}{2} \rceil$.

For an arbitrary order $N \geq 2$, the diameter of $HC(N)$ is given by $\lceil \frac{\lfloor \log N \rfloor}{2} \rceil + 1$. Also the Hamming cubes have been shown to be optimally fault-tolerant since the node-connectivity is equal to the minimum degree [DM94a]. We have shown that the minimum and maximum node-degrees of HC_n are $n+1$ and $2n-1$, respectively.

Table 1 compares the topological properties of several hypercube-like networks. Note that $Q(N)$, $TQ(N)$, $FQ(N)$, and $CC(N)$ have incrementability of $N = 2^n$ and $EIQ(N)$ has incrmentability of 2^k for $0 \leq k \leq n$. Whereas the rest of the networks in Table 1 have unit incrementability.

Clearly, the diameter of the Hamming cube $HC(N)$ of an arbitrary order N is the smallest among all the unit-incremental networks mentioned here. The fact that Hamming cubes are recursive in nature (that is, a smaller order HC is an induced subgraph of a larger order HC) implies that they do not require reconfiguration while expanding, as opposed to the incrementally extensible hypercubes. Also, the diameter of HC_n is almost the same as the n-dimensional crossed cube and folded hypercube at the cost of $2^n - 2$ and $2^{n-1} - 2$ extra edges, respectively. However, folded hypercubes are not recursive in nature.

2.3 Recursive Decomposition

Due to the definition of the Hamming cubes, HC_n can be recursively decomposed into $(n-1)$ induced and disjoint subgraphs, denoted as $HC_n = \{HC_2, Q_2, \ldots, Q_{n-1}\}$. Note that HC_2 consisting of the node-set $V^1 = \{v_0, v_1, v_2, v_3\}$ forms a complete graph of four nodes. The node-sets of other subgraphs Q_i are given by $V^i = \{v_\alpha \mid 2^i \leq \alpha < 2^{i+1}\}$, for $2 \leq i \leq n-1$. The nodes in each induced subgraph of this decomposition have the same degree and satisfy the node-symmetry [DM94a].

Table 1. Topological comparison of several hypercube-like networks.

Networks of N nodes	# Edges $E(N)$ for $N = 2^n$	Degree (ϕ) for $N > 2$	Regular?	Diameter for $N > 2$	Reconfiguration required?
Binary Hypercube $Q(N)$	$\frac{N}{2}\log N$	$\log N$	yes	$\log N$	no
Twisted hypercube $TQ(N)$	$\frac{N}{2}\log N$	$\log N$	yes	$\log N - 1$	yes
Folded hypercube $FQ(N)$	$\frac{N}{2}(\log N + 1)$	$\log N + 1$	yes	$\lceil \frac{\log N}{2} \rceil$	yes
Crossed cube $CC(N)$	$\frac{N}{2}\log N$	$\log N$	yes	$\lceil \frac{\log N + 1}{2} \rceil$	no
Enhanced Incomplete Hypercube $EIQ(N)$ $N = 2^n + 2^k$ for $0 \leq k \leq n$	$\frac{N}{2}(\log N + 1)$	$2 \leq \phi \leq \log N + 1$	no	$\lfloor \frac{\lceil \log N \rceil}{2} + 1 \rfloor$	yes
Incomplete hypercube $IQ(N)$	$\frac{N}{2}\log N$	$1 \leq \phi \leq \log N$	no	$\log N$	no
Enhanced Generalized Incomplete hypercube $EGIQ(N)$	$\frac{N}{2}(\log N + 1)$	$2 \leq \phi \leq \log N + 1$	no	$\lceil \frac{\log N}{2} \rceil$	no
Incrementally Extensible Hypercubes $IEQ(N)$	$\frac{N}{2}\log N$	$\log N \leq \phi \leq \log N + 1$	no	$\lfloor \log N \rfloor + 1$	yes
Hamming cube $HC(N)$	$\frac{N}{2}\log N + N - 2$	$2 \leq \phi \leq 2\log N - 1$	no	$\lceil \frac{\lceil \log N \rceil}{2} \rceil + 1$	no

In $HC(N)$, where $2^{k-1} < N < 2^k$ and $k > 1$, we can partition the node-set into several subsets. Let $N = \sum_{i=1}^{l} 2^{p_i}$, where $1 \leq l \leq k$ and $p_{i+1} > p_i$. The l number of node-subsets are $V^{p_l} = \{v_\alpha \mid 0 \leq \alpha < 2^{p_l}\}$ and $V^{p_{(l-i)}} = \{v_\alpha \mid \sum_{j=l-i+1}^{l} 2^{p_j} \leq \alpha < \sum_{j=l-i}^{l} 2^{p_j}\}$ for $1 \leq i \leq l-1$. The subgraph induced by the node-subset V^{p_l} forms a p_l-dimensional Hamming cube, HC_{p_l}, and the other subgraphs induced by $V^{p_{(l-i)}}$ form the binary hypercubes $Q_{p_{(l-i)}}$. Such a decomposition will be denoted as $HC(N) = \{HC_{p_l}, IQ(N - 2^{p_l})\} = \{HC_{p_l}, Q_{p_{l-1}}, \ldots, Q_{p_1}\}$. For example, $HC(15) = \{HC_3, Q_2, Q_1, Q_0\}$, where HC_3 consists of the nodes $\{v_0, v_1, \ldots, v_7\}$, Q_2 of $\{v_8, \ldots, v_{11}\}$, Q_1 consists of $\{v_{12}, v_{13}\}$, and Q_0 is v_{14}.

3 Embeddings in Hamming Cubes

The *embedding* of a *guest* graph $G = (V_G, E_G)$ into a *host* graph $H = (V_H, E_H)$ is to find two functions, Φ and Ψ, such that $\Phi : V_G \longrightarrow V_H$ is a mapping of their vertices while $\Psi : E_G \longrightarrow$ { paths in H} is a mapping from edges in E_G to paths in H.

There are four metrics to measure the cost of an embedding. The *dilation* of an edge e in G is the length of the path $\Psi(e)$ in H. The dilation of G in an embedding is the maximum dilation over all edges. The *expansion* of an embedding is the ratio $\frac{|V_H|}{|V_G|}$. The *edge-congestion* is the maximum number of edges in G which are mapped by function Ψ to a single edge in H. The *load* is the maximum number of nodes in G mapped by Φ to a single node in H. In our study, the node-mapping function Φ is considered as one-to-one, thereby the maximum load is 1.

If each of dilation and edge-congestion is equal to 1, the guest network is a subgraph of the host. Since there is a trade-off between the dilation and expansion, by *optimal embedding* we mean one with unit dilation and minimum expansion.

In this section, we present subgraph and/or optimal embeddings of various networks into the Hamming cubes.

3.1 Hamiltonian Cycles

Binary hypercubes are Hamiltonian and, in fact, all cycles of even lengths can be embedded in Q_n. It is easy to see that the sequence of nodes traversed along the *binary reflected Gray Codes* [SS88] forms an embedded Hamiltonian cycle. Since there are $n!$ different Gray code sequences of length n, each of which corresponds to a permutation of the set $D = \{0, 1, \ldots, n-1\}$ of dimensions of edges in Q_n, hence Q_n can have $n!$ different Hamiltonian cycles. (Two embedded Hamiltonian cycles are said to be *different* if they differ in at least one edge.) However, it can be shown that Q_n has $2^{n-3}n!$ different Hamiltonian cycles [SSB93]. For the sake of completeness, let us sketch this scheme. A pair of nodes in the hypercube is connected by an i-dimensional edge if and only if their binary labels differ at the $(i+1)$th bit, where the least significant bit corresponds to $i = 0$. A sequence of dimensions, S, determining the traversal of edges in an embedded Hamiltonian cycle is obtained from D as follows [SSB93].

Algorithm Sequencing /* Construct the sequence S of dimensions */
begin
1. Arbitrarily choose a dimension $d_1 \in D$ and let $D = D - \{d_1\}$.
2. Let $S_1 = d_1$.
3. For each i, $2 \leq i \leq n$, choose a dimension $d_i \in D$ and let $D = D - \{d_i\}$.
   Let $S_i = S_{i-1} \bullet d_i \bullet S_{i-1}$, where $\bullet$ indicates the concatenation operation.
4. Let $S = S_n \bullet d_n$.

end

Given a node v_i and a sequence S_π of dimensions, where $\pi = d_1 d_2 \ldots d_n$, a Hamiltonian cycle $C(v_i, S_\pi)$ is traversed which starts at v_i and follows the cycle-edges determined by S_π. Since there are 2^n possible choices of the node v_i and $n!$ possible choices of permutation π, there are at most $2^n n!$ embedded Hamiltonian cycles in Q_n. However, only $2^{n-3}n!$ of them are different, as constructed by Algorithm Sequencing.

In the following, we show that the Hamming cubes are *pancyclic* networks, i.e. cycles of all lengths are embeddable, which is an advantage over binary hypercubes.

Hamiltonian Cycles in HC_n. By definition, the Hamming cube HC_n has two kinds of edges: the $E1$-edges (hypercube edges) and $E2$-edges (enhanced edges). There exist an n_n-dimensional $E2$-edge $(v_i, v_{\bar{i}})$ for every node v_i. Therefore, the size of the dimension-set $D = \{0, 1, \ldots, n-1, n_n\}$ in HC_n is $n+1$, which implies that we have $(n+1)!$ potential choices for the permutation π. Therefore, Algorithm Sequencing applied to HC_n leads to the following theorem.

Theorem 1. *The network HC_n has $2^{n-3}(n+1)!$ different Hamiltonian cycles.*

Example 1: Consider an embedding of a Hamiltonian cycle $C(v_0, S_\pi)$ in HC_4, where the permutation of dimensions is $\pi = 20n_41$. The sequence of dimensions is $S_{20n_41} = 202n_42021202n_42021$ and $C(v_0, S_{20n_41}) = (0,4,5,1,14,10,11,15,13,9,8,12,3,7,6,2,0)$. Note that $(1, 14)$ and $(12, 3)$ are n_4-dimensional edges in this cycle.

Hamiltonian Cycles in $HC(N)$, where $2^{k-1} < N < 2^k$. We consider two cases depending on the value of N.

Case 1: For $2^{k-1} < N \leq 2^{k-1} + 2^{k-2}$ and $k \geq 3$.

Let $N = 2^{k-1}+m$, where $1 \leq m \leq 2^{k-2}$. By Section 2.3, the decomposition yields $HC(N) = \{HC_{k-1}, IQ(m)\}$ in which the subgraphs HC_{k-1} and the incomplete hypercube $IQ(m)$ are induced by the node-sets $V(HC_{k-1}) = \{v_\alpha \mid 0 \leq \alpha < 2^{k-1}\}$ and $V(IQ) = \{v_\alpha \mid 2^{k-1} \leq \alpha < 2^{k-1}+m\}$, respectively. By the definition of Hamming cubes, each node v_i in $IQ(m)$ is linked to two different nodes $v_{i^{[k]}}$ and $v_{i^{\{k\}}}$ in HC_{k-1} through the k-dimensional $E1$-edge and n_k-dimensional $E2$-edge, respectively. Recall here that $i^{[k]}$ is obtained by complementing the kth bit of i, and $i^{\{k\}}$ is obtained by complementing the rightmost k bits of i. Since the Hamming distance $\rho(v_{i^{[k]}}, v_{i^{\{k\}}}) = k-1$, there exists an (n_{k-1})-dimensional $E2$-edge between the nodes $v_{i^{[k]}}$ and $v_{i^{\{k\}}}$. Thus, we have:

Property 1. *Let v_α and v_β be two nodes in HC_{k-1} for $k \geq 3$, such that v_α is linked to v_β through an (n_{k-1})-dimensional $E2$-edge, i.e. $\alpha = \beta^{\{k-1\}}$. Then in HC_k, there exists a path $P = (v_\alpha, v_\gamma, v_\beta)$ of length 2, which goes through node v_γ such that $\alpha = \gamma^{[k-1]}$ and $\beta = \gamma^{\{k\}}$.*

By Theorem 1 and Property 1, Hamiltonian cycles can be embedded into the Hamming cubes of orders satisfying Case 1. We first apply Algorithm Sequencing to construct the Hamiltonian cycle $C(v_i, S_\pi)$ in the subgraph HC_{k-1}. Then in the permutation $\pi = (p_1, p_2, \ldots, p_{k-1})$, we choose the element $p_1 = n_{k-1}$ while the other elements are arbitrarily chosen from the set $D = \{0, 1, \ldots, k-2\}$. Note that the starting node v_i can be any node in HC_{k-1}. The resulting sequence of dimensions corresponding to π has the form $S_{n_{k-1}p_2\cdots p_{k-1}} = n_{k-1}p_2n_{k-1}p_3n_{k-1}p_2n_{k-1}\cdots$. Let the required Hamiltonian cycle be $C(v_i, S_{n_{k-1}p_2\cdots p_{k-1}}) = ia_2a_3\ldots, a_{2^{k-1}-1}i$, in which we search those pairs of nodes that are linked via the (n_{k-1})-dimensional $E2$-edges. Then, between those node-pairs. we appropriately insert the nodes of the subgraph $IQ(m)$ following Property 1.

Since there are 2^{k-1} possible choices of the starting node v_i and $(k-1)!$ possible choices of π, the subgraph HC_{k-1} has $2^{k-4}(k-1)!$ different Hamiltonian cycles, for $k \geq 4$. For each of these cycles, the edges that are (n_{k-1})-dimensional $E2$-edges, are expanded as paths of length two to include the nodes of the subgraph $IQ(m)$. Furthermore, by Property 1, the edges used in the expanded paths are different from those used in the Hamiltonian cycle in HC_{k-1}. Therefore, we have the following lemma.

Lemma 1. *The Hamming cube $HC(N)$, where $2^{k-1} < N < 2^{k-1}+2^{k-2}$ and $k \geq 4$, has $2^{k-4}(k-1)!$ different Hamiltonian cycles.*

Example 2: Consider $HC(11) = \{HC_3, IQ(3)\}$, where $V(HC_3) = \{v_\alpha \mid 0 \leq \alpha \leq 7\}$ and $V(IQ) = \{v_8, v_9, v_{10}\}$. Let $\pi = n_320$ and v_6 be the starting node. Then $S_{n_320} = n_32n_30n_32n_30$. The Hamiltonian cycle in HC_3 is $C(v_6, S_{n_320}) = (\underbrace{6,1},\underbrace{5,2},3,4,\underbrace{0,7},6)$. Now inserting v_8, v_9, and v_{10} into the node-pairs $(0,7)$, $(6,1)$, and $(5,2)$, respectively, the Hamiltonian cycle in $HC(11)$ is obtained as $(6,9,1,5,10,2,3,4,0,8,7,6)$.

Case 2: For $2^{k-1}+2^{k-2} < N < 2^k$ and $k \geq 3$.

Let $N = 2^{k-1}+m$, where $2^{k-2} < m < 2^{k-1}$, and $HC(N) = \{HC_{k-1}, IQ(m)\}$. This case has been proved rigorously by dividing it into two subcases depending on whether m is even or odd. The basic idea involves how to combine the embedded

Hamiltonian cycles in HC_{k-1} and $IQ(m)$ in order to construct the required cycle in $HC(N)$. We summarize the results here. For details, refer to [DM95].

Lemma 2. *The Hamming cube $HC(N)$, where $2^{k-1} + 2^{k-2} < N < 2^k$ for $k \geq 3$, has Δ $(\lceil \log(N - 2^{k-1}) \rceil - 2)!$ different Hamiltonian cycles, where $\Delta = (k-1)!$ for N even and $\Delta = (k-2)!$ for N odd.*

From Lemmas 1 and 2 we obtain

Theorem 2. *The Hamming cube of order N, where $2^{k-1} < N < 2^k$ for $k \geq 3$, has $(k-2)!(\lceil \log(N - 2^{k-1}) \rceil - 2)!$ different Hamiltonian cycles.*

The above results also prove that $HC(N)$ is a *pancyclic* network, for $N \geq 3$.

3.2 Complete Binary Trees and Related Variants

Both the complete binary tree and binary hypercube are bipartite graphs. A node of the hypercube Q_n is said to have *even parity* if its binary representation has an even number of one bits; otherwise, it has an *odd* parity. Also, Q_n has 2^{n-1} even parity nodes and 2^{n-1} odd parity nodes. In the bipartite partition of the binary tree, the nodes at the even (or odd) levels are put together. Therefore, it can be shown that the complete binary tree $CBT(n-1)$ of height $n-1$ and consisting of $2^n - 1$ nodes is not a subgraph of Q_n. However, the 2^n-node *two-rooted* complete binary tree is a subgraph of Q_n [BI85, Lei92]. Thus $CBT(n-1)$ can be embedded into Q_n with dilation two, while it is a subgraph of Q_{n+1}.

In this section, we will show that $CBT(n-1)$ is a subgraph of the n-dimensional Hamming cube, HC_n, consisting of 2^n nodes. This result clearly shows that Hamming cubes have better performance (in terms of tree embeddings) than binary hypercubes of the same size.

The decomposition yields $HC_n = \{HC_{n-1}, Q_{n-1}\}$, induced by the vertex-subsets $V' = \{v_\alpha \mid 0 \leq \alpha < 2^{n-1}\}$ and $V'' = \{v_\alpha \mid 2^{n-1} \leq \alpha < 2^n\}$, respectively. Due to the recursive structure, $HC_{n-1} = \{HC_{n-2}, Q_{n-2}\}$ and $Q_{n-1} = \{Q^1_{n-2}, Q^2_{n-2}\}$. Therefore, $HC_n = \{HC_{n-2}, Q_{n-2}, Q^1_{n-2}, Q^2_{n-2}\}$ such that the nodes in these four subgraphs have the labels $(00*)$, $(01*)$, $(10*)$, and $(11*)$, respectively, where $* \in \{0,1\}^{n-2}$. The following property can be stated for a node $v_i \in HC_{n-2}$.

Property 2. *Let $HC_n = \{HC_{n-2}, Q_{n-2}, Q^1_{n-2}, Q^2_{n-2}\}$ for $n \geq 2$, in which the subgraphs are induced by the vertex-subsets $V^1 = \{v_\alpha \mid 0 \leq \alpha < 2^{n-2}\}$, $V^2 = \{v_\alpha \mid 2^{n-2} \leq \alpha < 2^{n-1}\}$, $V^3 = \{v_\alpha \mid 2^{n-1} \leq \alpha < 2^{n-1} + 2^{n-2}\}$, and $V^4 = \{v_\alpha \mid 2^{n-1} + 2^{n-2} \leq \alpha < 2^n\}$. A node $v_i \in HC_{n-2}$ is linked to $v_{i[n]} \in Q^1_{n-2}$ and $v_{i\{n\}} \in Q^2_{n-2}$, through the $(n-1)$-dimensional $E1$-edge and the n_n-dimensional $E2$-edge, respectively.*

Let us now construct, by induction, the embedded $CBT(n-1)$ in HC_n. Let v_0 be the root of the single node tree. Let $CBT(n-2)$ be the embedded complete binary tree in HC_{n-1}, having the leaves in the set V^2. The embedded $CBT(n-1)$ in HC_n grows from $CBT(n-2)$ by making the nodes in the set $V'' = V^3 \cup V^4$ as the children of the leaves of $CBT(n-2)$ through the $(n-1)$-dimensional $E1$-edges and n_n-dimensional $E2$-edges.

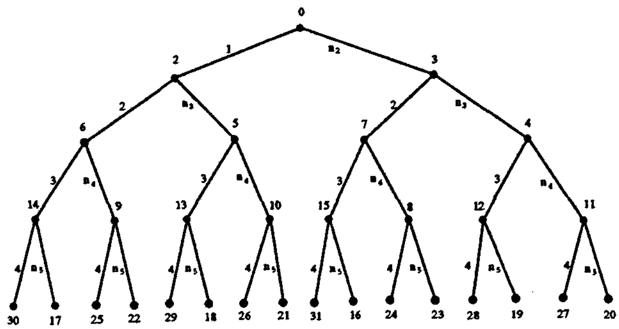

Fig. 3. The embedding of the complete binary tree $CBT(4)$ of height four in HC_5.

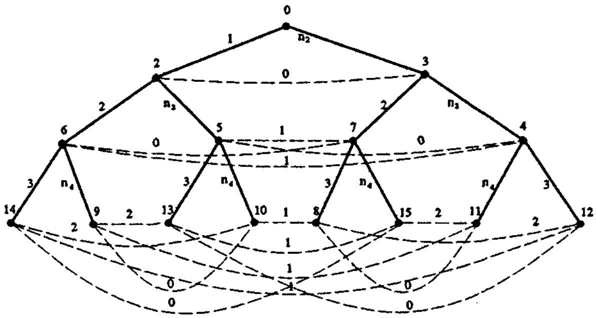

Fig. 4. The embedding of tree-cube $TC(3)$ rooted at node v_0 in HC_4.

Theorem 3. *The $(2^n - 1)$-node complete binary tree $CBT(n-1)$ is a subgraph of the n-dimensional Hamming cube HC_n, having 2^n nodes.*

Figure 3 is an embedding of $CBT(4)$ in HC_5. An edge label indicates its dimension.

Since the subgraph of HC_n induced by the set $V'' = V^3 \cup V^4$ forms the hypercube Q_{n-1}, the nodes at each level j, for $0 \leq j < n$, of the embedded $CBT(n-1)$ in HC_n are connected as a j-dimensional binary hypercube, Q_j, consisting of the nodes $V(Q_j) = \{v_i \mid 2^j \leq i < 2^{j+1}\}$. Let us call such a network architecture a *tree-cube*, $TC(n-1)$, of height $n-1$. Figure 4 shows the embedding of the tree-cube $TC(3)$ in the Hamming cube HC_4. The hypercube edges are shown as broken lines. Due to this structure, several variants of complete binary trees with additional links between the nodes at the same level can be embedded in the n-dimensional Hamming cube.

For example, a *hypertree* structure $HT(n-1)$ of height $n-1$ is a complete binary tree such that the additional links at each level are chosen to be a subset of a hypercube [GS81]. So $HT(n-1)$ is a subgraph of the tree-cube $TC(n-1)$, and hence a subgraph of HC_n. With the help of the embedded tree-cubes and using the fact that the binary hypercubes are Hamiltonian, the full-ringed (hence half-ringed) binary tree of height $n-1$ can also be embedded into HC_n.

3.3 Tree Machines

A *tree machine*, $TM(n)$, of dimension n consists of two $CBT(n)$'s – called the *upper* and *lower* trees – which are connected back to back along the common leaves. Thus, $TM(n)$ has $(3 \cdot 2^n - 2)$ nodes and $(2^{n+2} - 4)$ edges. It can be embedded in the hypercube Q_{n+2} with expansion approximately equal to $\frac{4}{3}$ and dilation one [Efe91].

It is also shown [OD95] that $TM(n)$ can be embedded in the incomplete hypercube $IQ(3 \cdot 2^n)$ with both dilation and edge congestion equal to 2.

We will show that the $TM(n)$ is a subgraph of the Hamming cube $HC(3 \cdot 2^n + 2^{n-1})$, implying that dilation is 1 and expansionis approximately $\frac{7}{6}$. Note that with the same expansion of $\frac{7}{6}$, the tree machine $TM(n)$ cannot be embedded as a subgraph into the incomplete hypercube $IQ(3 \cdot 2^n + 2^{n-1})$. Again, this provides an advantage of the Hamming cubes over the same-sized hypercubes.

Let us view the structure of the tree machine as follows. In $TM(n)$, the 2^n common leaves and their 2^n parents (half of them in the upper tree and the other half in the lower tree) form 2^{n-1} *building blocks*, each being a hypercube Q_2. These building blocks are then connected by the upper and lower complete binary trees of height $n-1$, one less height than the original trees in $TM(n)$. Note that the leaves of these two new trees are now the parents of the cornerwise nodes in the building blocks. When the dimension of the tree machine increases, say from n to $n+1$, the number of building blocks is doubled, from 2^{n-1} to 2^n. Thus, we need 2^n new leaves for each upper and lower tree to connect the new set of 2^n building blocks.

In the Hamming cube $HC(3 \cdot 2^n + 2^{n-1})$ for $n \geq 3$, each node label has length $n+2$. According to the first and second lowest bits of these labels, we can decompose $HC(3{\cdot}2^n{+}2^{n-1})$ into $3{\cdot}2^{n-2}{+}2^{n-3}$ building blocks, Q_2^i, for $0 \leq i \leq 3{\cdot}2^{n-2}{+}2^{n-3}{-}1$. Since the upper and lower trees in $TM(n)$ are symmetric along their common leaves, without loss of generality, we can concentrate on only one tree.

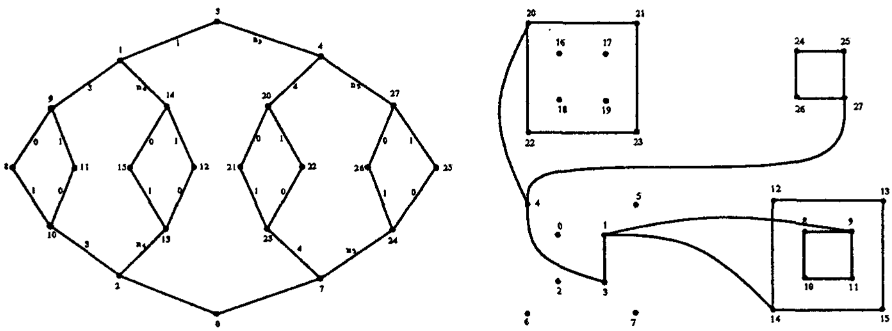

Fig. 5. a) The embedding of $TM(3)$ in $HC(28)$. b) The upper tree of $TM(3)$.

Let nodes v_3 and v_0 in Q_2^0 be respectively the roots of the upper and lower trees in $TM(n)$. The root v_3 has the children v_1 and v_4 through the 1-dimensional $E1$-edge and the n_3-dimensional $E2$-edge. SImilarly, v_0 has the children v_2 and v_7,

In the embedded $TM(n)$ for $n \geq 3$, an internal node v_i at level j, for $1 \leq j \leq n{-}3$, of the upper tree has the left child $v_{i[2j+2]}$ and the right child $v_{i\{2j+2\}}$ linked through a $(2j+1)$-dimensional $E1$-edge and (n_{2j+2})-dimensional $E2$-edge, respectively. By this way, we can construct the top $n-1$ levels of the upper tree. The remaining step is to construct the leaves of the upper tree which are those parents of the cornerwise nodes in the building blocks.

We divide the nodes at level $n-2$ into two subsets, V' and V'' such that V' consists of the first 2^{n-3} nodes from the left, while V'' consists of the remaining 2^{n-3} nodes on that level. A node $v_i \in V'$ has two leaves $v_{i[n+1]}$ and $v_{i\{n+1\}}$ linked through the n-dimensional $E1$-edge and (n_{n+1})-dimensional $E2$-edge, respectively, in the Hamming cube. While a node $v_i \in V''$ has two leaves $v_{i[n+2]}$ and $v_{i\{n+2\}}$. Thus, the entire upper tree is constructed.

By the same method, the lower tree rooted at node v_0 can also be constructed. The common leaves for both the trees are then determined by the parent nodes of the building blocks, which are the leaves of upper and lower trees, through the $E1$-edges of dimensions 0 and 1.

Theorem 4. *The tree machine $TM(n-1)$ can be embedded as a subgraph into the Hamming cube $HC(3 \cdot 2^{n-1} + 2^{n-2})$ with an asymptotic expansion of $\frac{7}{6}$.*

Example 3: Figure 5a) shows the embedding of the tree machine $TM(3)$ in $HC(28)$ and Figure 5b) shows the upper tree. This figure includes all nodes of $HC(28)$, but omits the edges which are not used in the tree. There are four building blocks (i.e., Q_2's) formed by the 0- and 1-dimensional $E1$-edges, and one can clearly see the geometric relation of these building blocks in $HC(28)$.

Table 2. Comparison of embedding results.

Guest Networks	Binary Hypercube Q_n (# of nodes $N = 2^n$)	Incomplete Hypercube $IQ(N)$, $N > 2$	Hamming cube $HC(N)$
Ring $R(m)$	For $0 \leq m \leq 2^n$, • Subgraph, when m is even; • $\mathcal{D}=2$, $\mathcal{EC}=1$, when m is odd.	For $0 \leq m \leq N$, • Subgraph, when m is even; • $\mathcal{D} = \mathcal{EC} = 2$, when m is odd.	pancyclic
Complete Binary Tree $CBT(m)$	• Subgraph $CBT(n-2)$ • $CBT(n-1)$ with $\mathcal{D} = \mathcal{EC} = 2$	• $CBT(n-1)$ in $IQ(2^n - 1)$ with $\mathcal{D} = \mathcal{EC} = 2$	• Subgraph $CBT(n-1)$
X-tree $X(m)$	• $X(n-1)$ with $\mathcal{D} = \mathcal{EC} = 2$	• $X(n-1)$ in $IQ(2^n - 1)$ with $\mathcal{D} = \mathcal{EC} = 2$	• Subgraph $X(n-1)$
Hypertree $HT(m)$	• $HT(n-1)$ with $\mathcal{D}=2$	• $HT(n-1)$ in $IQ(2^n)$ with $\mathcal{D}=2$	• Subgraph $HT(n-1)$
Tree Machine $TM(m)$	• Subgraph $TM(n-2)$	• $TM(n-1)$ in $IQ(3 \cdot 2^{n-1})$ with $\mathcal{D} = \mathcal{EC} = 2$	• Subgraph $TM(n-1)$ in $HC(3 \cdot 2^{n-1} + 2^{n-2})$

4 Conclusions

We have studied the embeddability of the recently proposed Hamming cube networks [DM94a]. Several topologies including Hamiltonian paths and cycles, complete binary trees and their variants, and tree machines are optimally embedded into the Hamming cubes with unit dilation ($\mathcal{D}$) and edge-congestion ($\mathcal{E}$), and minimum expansion. Table 2 compares our embedding results with (incomplete) hypercubes.

Due to the bipartiteness of incomplete hypercubes, a Hamiltonian cycle of odd length cannot be embedded with dilation of one. Using the additional enhanced edges in the Hamming cubes, Hamiltonian cycles of all lengths can be embedded as subgraphs, implying that Hamming cubes are *pancyclic* networks.

Although a complete binary tree is not a subgraph of the same-sized binary hypercube, it is a subgraph of the same-sized Hamming cube. Additionally, X-trees, hypertrees, full-ringed and half-ringed binary trees are all subgraphs of Hamming cubes with unit expansion.

Tree machines can also be embedded into the Hamming cubes with dilation of one and expansion of $\frac{7}{6}$. Whereas, tree machines can be embedded into the incomplete hypercubes with expansion approximately equal to one, and both dilation and edge congestion being equal to two. With the same expansion of $\frac{7}{6}$, the embedding of tree machines into the incomplete hypercubes still have dilation and edge congestion of two. This provides another advantage of the Hamming cubes.

Our future research will aim at the fault-tolerant embedding of guest networks into the Hamming cubes.

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