Biased Anisotropic Diffusion —A Unified Regularization and Diffusion Approach to Edge Detection

Niklas Nordström
Department of Electrical Engineering and Computer Sciences
University of California, Berkeley, CA 94720

Abstract

We present a global edge detection algorithm based on variational regularization. The algorithm can also be viewed as an anisotropic diffusion method. We thereby unify these two from the original outlook quite different methods. The algorithm to be presented moreover has the following attractive properties: 1) It only requires the solution of a *single* boundary value problem over the *entire* image domain—almost always a very simple (rectangular) region. 2) It converges to a solution of interest.

1 Introduction

Edge detection can in short be described as the process of finding the discontinuities of the partial derivatives up to some order of an *image function* defined on an open bounded connected *image domain* $B \subseteq \mathbf{R}^2$. The image function can represent various kinds of data collected from the visible surfaces in the scene. We will be concerned with brightness data. In this case the image function is real-valued, and the discontinuities of interest appear in the zeroth order derivative, that is the image function itself. If the *true image function* one would obtain by pure projection of the brightness in the scene onto the image domain were known, the edge detection problem would be easy. However, because of imperfections in the image formation process, the *original image function* one is given, is distorted so that the discontinuities in the true image function are disguised into large gradients. Edge detection therefore essentially boils down to numerical differentiation—a problem well-known to be ill-posed (in the sense of Hadamard) due to its instability with respect to the initial data. Since measurement noise and other undesirable disturbances cannot be avoided, the edge detection problem thus has to be stabilized in order to have a meaningful solution. In more practical terms this means that the undesirable disturbances must be suppressed without the disappearance or dislocation of any of the edges. Over the last six years or so many attempts along these lines have appeared in the literature. One can distinguish between two seemingly quite different approaches, viz. regularization and anisotropic diffusion.

Regularization can be achieved in different ways. In probabilistic regularization the problem is reformulated as Bayesian estimation. In variational regularization it is posed as a cost (or energy) functional minimization problem leading to a variational principle. In spite of the different outlooks of these approaches they essentially end up with the same mathematical and computational problem; given the original image function $\zeta : B \to \mathbf{R}$, minimize a cost functional $\mathcal{C}_\zeta(w, z)$ where w is some function representing the edges, and $z : B \to \mathbf{R}$ is the *estimated* (or reconstructed) *image function.* In each case the total cost can furthermore be divided into three components according to $\mathcal{C}_\zeta(w, z) = \mathcal{E}(w) + \mathcal{D}(z, \zeta) + \mathcal{S}(w, z)$ where the *edge cost* $\mathcal{E}$ measures the extent of the edges, the *deviation cost* $\mathcal{D}$ measures the discrepancy between the estimated and the original image functions, and the *stabilizing cost* measures the unsmoothness or the a priori "unlikeliness" of the estimated image function. The "edge function" w can be defined in a variety of ways. Here we will only be concerned with edge functions of the form $w : B \to \mathbf{R}$.

Given a specific edge function w it is generally the case that there exists a unique optimal estimated image function $\check{z}_w$, which can be found by solving a linear partial differential equation. While most of the regularization approaches do take advantage of this condition, none of them is capable of solving for the optimal edges in a similar way. The optimality conditions for the edges do either not exist, or else they consist of unsolvable equations. For the minimization of $\mathcal{C}_\zeta(w, z)$ with respect to w all of the regularization approaches referred to above therefore resort to some kind of stochastic or deterministic search method such as the "Metropolis algorithm" or "steepest descent". Because of the tremendous size of the solution space any such search method is by itself quite expensive. In addition the general nonconvexity of the cost function causes any converging search algorithm to get stuck at local minima. The common response to this unfortunate situation has been to solve whole sequences of minimization problems, as a mechanism for "gravitating" towards a good local possibly a global minimum. The GNC-algorithm introduced in [1, 2] and simulated annealing [3] are both examples thereof. In summary most global edge detection methods up to date involve some form of repeated iterative minimization process, and because of the high computational cost that this implies, the optimality of the solution is often compromised.

In contrast to the regularization-based methods, the anisotropic diffusion method presented in the literature does not seek an optimal solution of any kind. Instead it operates by repeatedly filtering the image function with a smoothing kernel of small support, thereby producing a sequence of *diffused image functions* of successively lower resolution. In order to retain the strength and correct location of the edges, the "smoothing power" of the filter kernel is made to depend (inversely) on the magnitude of the image function gradient in a heuristic fashion. At some stage in the iterated filtering process remarkably impressive edges can be extracted by postprocessing the diffused image function with a rudimentary local edge detector. In the limit, however, all edges disappear, and the diffused image function converges to a constant. Needless to say, any solution of interest therefore has to be selected from the sequence of diffused image functions way before convergence. This selection has so far been a matter of manual inspection. If automation is necessary, one can of course, in the absence of more sophisticated rules, simply prespecify a number of filter iterations. A more serious problem due to the necessity to select a solution prior to convergence, may arise in an analog circuit implementation where the diffusion process must be "latched" or halted in order to retrieve the diffused image function of interest.

In this paper we show how the variational regularization approach by Terzopoulos [4, 5] can be modified so that the calculus of variations yields useful optimality conditions, not only for the estimated image function, but for the edges as well. The result is a global edge detection algorithm, which as it turns out, also can be viewed as a (new) *biased anisotropic diffusion* method. This unification of the from the original outlook quite different regularization and diffusion approaches is in itself quite interesting. It also brings the anisotropic diffusion approach an appealing sense of optimality. Anisotropic diffusion is thus a method for solving a well-defined mathematical problem, not just an image processing technique, by which one image can be transformed into another more pleasing looking one. Even more exciting than the unification just mentioned, is the strong indications that the new algorithm shares the better properties of both the regularization-based methods and the anisotropic diffusion method. First of all, it does only require the solution of a *single* boundary value problem on the *entire* image domain—almost always a very simple region. In particular, no explicit search method is necessary. Neither is the solution of a sequence of minimization problems. Secondly, the algorithm converges to a solution of interest. The problem of manual selection of which one in the sequence of diffused image functions to be postprocessed with the local edge detector, is thereby removed from the anisotropic diffusion method.

Before we continue, some notation needs to be settled: $\mathbf{R}_-$ and $\mathbf{R}_+$ will denote the sets $]-\infty, 0[$ and $]0, \infty[$ respectively. The norm $\|\cdot\|$ will always refer to the Euclidean norm in $\mathbf{R}^n$. Finally $\vee$, $\wedge$ and $\circ$ will denote the binary maximum, minimum and function composition operators respectively.

2 Controlled-Continuity Stabilization

The "classical" stabilizers that first appeared in early vision problems did not allow estimation or reconstruction of image functions with discontinuities. In order to improve on this for vision problems far too restrictive model Terzopoulos [4, 5] introduced a more general class of stabilizing functionals referred to as *controlled-continuity stabilizers*. These are of the form $\mathcal{S}(w, z) \doteq \int_{\mathbf{R}^K} \sum_{i=1}^{I} w_i \sum_{k_1=1}^{K} \cdots \sum_{k_i=1}^{K} (\partial^i z / \partial x_{k_1} \cdots \partial x_{k_i})^2 \, dx$ where $w \doteq [w_1 \cdots w_I]^T$, and the weighting functions $w_1, \ldots, w_I : \mathbf{R}^K \to [0, 1]$, referred to as *continuity control functions* are in general discontinuous. In particular they are able to make jumps to zero, and edges, where the partial derivatives of z of order $\geq j$ are allowed to be discontinuous, are represented by the sets $\bigcap_{i=j+1}^{I} w_i^{-1}(\{0\})$, $j = 0, \ldots, I-1$. For the edge cost Terzopoulos proposes the functional $\mathcal{E}(w) \doteq \int_{\mathbf{R}^K} \sum_{i=1}^{I} \lambda_i (1 - w_i) \, dx$ where the constants $\lambda_1, \ldots, \lambda_I \in \overline{\mathbf{R}_+}$ satisfy $\sum_{i=1}^{I} \lambda_i > 0$. This paradigm apparently fails to support a genuinely variational technique for minimizing the total cost with respect to the continuity control function vector w. First of all, all the control functions that represent nonempty sets of edges, belong to the boundary of the continuity control function space. Secondly, the total cost functional is affine in w, whence it does not have any critical points. Terzopoulos resolves this problem by first discretizing the entire space of continuity control functions; w is defined on a finite subset D—a dual pixel grid—and only allowed to take the values 0 or 1. The edge cost is modified accordingly to $\mathcal{E}(w) \doteq \sum_{x \in D} \sum_{i=1}^{I} \lambda_i [1 - w_i(x)]$ For a solution he then applies a descent method in the space of all possible continuity control function vectors.

3 Genuinely Variational Edge Detection

For our problem of detecting discontinuities of a bivariate image function, the appropriate deviation and stabilizing costs in the paradigm above are given by $\mathcal{D}(z,\zeta) \doteq \int_B (z-\zeta)^2\,dx$ and $\mathcal{S}(w,z) \doteq \int_B w\|\nabla z^T\|^2\,dx$. In order to remedy the difficulties with Terzopoulos' method, we propose the use of a smooth continuity control function $w : B \to \overline{\mathbf{R}_+}$. To avoid the problem with non-critical optimal continuity control functions, which are impossible to find by means of variational calculus, we will choose the edge cost so that for each estimated image function z the total cost $\mathcal{C}_\zeta(w,z)$ attains its minimum for exactly one optimal continuity control function $\check{w}_z$ whose range is confined to lie in $]0,1]$. This idea is similar to the use of barrier functions in finite dimensional optimaization [6]. The uniqueness of $\check{w}_z$ for a given z also allows us to solve for $\check{w}_z$ in terms of z in a way similar to Blake and Zisserman's elimination of their "line process" [1, 2]. The edge costs we propose for this purpose are of the form $\mathcal{E}(w) \doteq \int_B \lambda f \circ w\,dx$ where the *edge cost coefficient* $\lambda > 0$ is constant, and the *edge cost density function* $f : \mathbf{R}_+ \to \mathbf{R}$ is twice differentiable. Our total cost functional is thus given by

$$\mathcal{C}_\zeta(w,z) \doteq \int_B [\lambda f \circ w + (z-\zeta)^2 + w\|\nabla z^T\|^2]\,dx \tag{1}$$

Setting the first variation of $\mathcal{C}_\zeta(w,z)$ to zero yields the Euler equations

$$z(x) - \zeta(x) - \nabla\cdot(w\nabla z)(x) = 0 \qquad \forall x \in B \tag{2a}$$

$$\lambda f'(w(x)) + \|\nabla z(x)^T\|^2 = 0 \qquad \forall x \in B \tag{2b}$$

$$w(x)\frac{\partial z}{\partial e_n}(x) = 0 \qquad \forall x \in \partial B \tag{2c}$$

where $\nabla\cdot$ denotes the divergence operator, and $\partial/\partial e_n$ denotes the directional derivative in the direction of the outward normal. The second variation of $\mathcal{C}_\zeta$ with respect to w is also easily found to be

$$\delta^2_{ww}\mathcal{C}_\zeta(w,z) = \int_B \frac{\lambda}{2}(f''\circ w)(\delta w)^2\,dx \tag{3}$$

Together with the desired existence of a unique optimal continuity control function $\check{w}_z$ for each possible estimated image function z these equations put some restrictions on the edge cost density f. In fact from (2b) it follows that $f'|]0,1] \to \overline{\mathbf{R}_-}$ must be bijective, and that $f'(]1,\infty[) \subseteq \mathbf{R}_+$. Likewise from (3) we see that f'' must be strictly positive on $]0,1[$, and that $f''(1) \geq 0$. Two of the simplest functions satisfying these requirements are given by $f(\omega) \doteq \omega - \ln\omega$ and $f(\omega) \doteq \omega\ln\omega - \omega$, but there are of course many other possibilities [7].

Given that f satisfies the conditions above, $f'|]0,1]$ is invertible. Since moreover w is strictly positive, we end up with the equations

$$z(x) = \zeta(x) + \nabla\cdot(w\nabla z)(x) \qquad \forall x \in B \tag{4a}$$

$$w(x) = g(\|\nabla z(x)^T\|) \qquad \forall x \in B \tag{4b}$$

$$\frac{\partial z}{\partial e_n}(x) = 0 \qquad \forall x \in \partial B \tag{4c}$$

where the function $g : \overline{\mathbf{R}_+} \rightarrow]0,1]$, referred to as the *diffusivity anomaly*, is defined by

$$g(\gamma) \doteq (f'|]0,1])^{-1}\left(-\frac{\gamma^2}{\lambda}\right) \qquad \gamma \geq 0 \tag{5}$$

The properties of the edge cost density f clearly imply that g is a strictly positive strictly decreasing differentiable bijection. For the two edge cost densities proposed above the diffusivity anomaly takes the form $g(\gamma) \doteq (1+\gamma^2/\lambda)^{-1}$ and $g(\gamma) \doteq e^{-\gamma^2/\lambda}$ respectively.

Since our method necessarily yields continuity control functions for which $w^{-1}(\{0\}) = \emptyset$, Terzopoulos' edge representation is inadequate. The simplest and most reasonable modification is to consider the edges to consist of the set $w^{-1}(]0,\theta])$ where θ is a fixed threshold. Since the diffusivity anomaly g is strictly decreasing, we have $w^{-1}(]0,\theta]) = \|\nabla z^T\|^{-1}([g^{-1}(\theta),\infty[)$ whence the edges are obtained by thresholding the magnitude of the gradient of the estimated image function.

4 Biased Anisotropic Diffusion

Perona and Malik [8, 9] have introduced anisotropic diffusion as a method of suppressing finer details without weakening or dislocating the larger scale edges. The initial value problem governing their method is given by

$$\frac{\partial z}{\partial t}(x,t) = \nabla \cdot (w \nabla z)(x,t) \qquad \forall x \in B \qquad \forall t > 0 \tag{6a}$$

$$w(x,t) = g(\|\nabla z(x,t)^T\|) \qquad \forall x \in B \qquad \forall t > 0 \tag{6b}$$

$$\frac{\partial z}{\partial e_n}(x,t) = 0 \qquad \forall x \in \partial B \qquad \forall t > 0 \tag{6c}$$

$$z(x,0) = \zeta(x) \qquad \forall x \in B \tag{6d}$$

where the diffused image function z and the *diffusivity* w are functions of both position $x \in B$ and time $t \geq 0$, $\nabla\cdot$ and ∇ denote the divergence and the gradient respectively with respect to x, and the diffusivity anomaly $g : \overline{\mathbf{R}_+} \rightarrow \overline{\mathbf{R}_+}$ is a decreasing function.

The Euler equations we derived in the previous section are very similar to the initial value problem (6). In fact, a solution of (4) is given by the steady state of the initial value problem

$$\frac{\partial z}{\partial t}(x,t) = \zeta(x,t) - z(x,t) + \nabla \cdot (w \nabla z)(x,t) \qquad \forall x \in B \qquad \forall t > 0 \tag{7a}$$

$$w(x,t) = g(\|\nabla z(x,t)^T\|) \qquad \forall x \in B \qquad \forall t > 0 \tag{7b}$$

$$\frac{\partial z}{\partial e_n}(x,t) = 0 \qquad \forall x \in \partial B \qquad \forall t > 0 \tag{7c}$$

$$z(x,0) = \chi(x) \qquad \forall x \in B \tag{7d}$$

which is obtained from (6) by replacing the anisotropic diffusion equation (6a) by the closely related *"biased" anisotropic diffusion* equation (7a). Since our interest is in the steady state solution, the initial condition (6d) can also be replaced by an arbitrary initial condition (7d). The continuity control function w thus plays the role of the diffusivity, and will be referred to as such whenever the context so suggests. The bias term $\zeta - z$

intuitively has the effect of locally moderating the diffusion as the diffused image function z diffuses further away from the original image function ζ. It is therefore reasonable to believe that a steady state solution does exist. This belief is further substantiated by our experimental results.

The possibility of suppressing finer details while the more significant edges remain intact, or are even strengthened, is a consequence of the anisotropy, which in both the diffusions described above in turn is caused by the non-constancy of the diffusivity anomaly g. For our variational method governed by the boundary value problem (4) the choice of g was based on optimality considerations. Perona and Malik select their function g by demanding that the resulting ***unbiased*** anisotropic diffusion enhances the already pronounced edges while the less significant edges are weakened. Based on an analysis including only blurred linear step edges they vouch for diffusivity anomalies of the form $g(\gamma) \doteq c/[1+(\gamma^2/\lambda)^a]$ where $c, \lambda > 0$ and $a > 1/2$ are constants. It is easy to check that, if these functions were substituted in the Euler equation (4b), the corresponding edge cost densities would satisfy the requirements of our variational method. Incidentally, for their experimental results Perona and Malik use exactly the diffusivity anomalies we proposed in section 3.

5 The Extremum Principle

The extremum principle is a common tool for proving uniqueness and stability with respect to boundary data for linear elliptic and linear parabolic problems [10]. For quasi-linear equations, such as the Euler equation (4a) and the biased anisotropic diffusion equation (7a), it is not quite as conclusive. Nevertheless it provides bounds on the solution and useful insight for convergence analysis of the numerical methods employed to find such a solution.

Theorem 5.1 *Let $z : \overline{B} \times \overline{\mathbf{R}_+} \to \mathbf{R}$ be a solution of the biased anisotropic diffusion problem (7) where it is assumed that $g : \overline{\mathbf{R}_+} \to \overline{\mathbf{R}_+}$ is continuously differentiable and $\zeta : B \to \mathbf{R}$ is uniformly continuous. Assume further that z and its first and second order partial derivatives with respect to x are continuous (on $\overline{B} \times \overline{\mathbf{R}_+}$). Then the following claims are true:*

(i) *If $\pm y_\tau : \overline{B} \to \mathbf{R} : x \mapsto \pm z(x,\tau)$ has a local maximum at $\xi \in \overline{B}$ for some fixed $\tau > 0$, then $\pm(\partial z/\partial t)(\xi,\tau) \leq \pm\zeta(\xi) \mp z(\xi,\tau)$.*

(ii) *If $\pm z$ has a local maximum at $(\xi,\tau) \in \overline{B} \times \mathbf{R}_+$, then $\pm z(\xi,\tau) \leq \pm\zeta(\xi)$.*

(iii) $\inf_{\xi\in B}[\zeta(\xi) \wedge \chi(\xi)] \leq z(x,t) \leq \sup_{\xi\in B}[\zeta(\xi) \vee \chi(\xi)] \qquad \forall x \in \overline{B} \qquad \forall t \geq 0$

Theorem 5.2 *Let $z : \overline{B} \to \mathbf{R}$ be a solution of the boundary value problem (4) where it is assumed that $g : \overline{\mathbf{R}_+} \to \overline{\mathbf{R}_+}$ is continuously differentiable. Assume further that z and its first and second order partial derivatives are continuous (on $\overline{B}$). Then*

$$\inf_{\xi\in B} \zeta(\xi) \leq z(x) \leq \sup_{\xi\in B} \zeta(\xi) \qquad \forall x \in \overline{B}$$

According to the two theorems above the solutions of the biased anisotropic diffusion problem are well-behaved, in that they do not stray too far away from the original image function ζ unless forced to by the initial condition, and even if so, they eventually approach the range of ζ as $t \to \infty$. In particular, our variational edge detection method produces an estimated image function whose range is contained inside that of the original image function.

6 Edge Enhancement

It was mentioned earlier that the biased anisotropic diffusion (7) in similarity with its unbiased counterpart (6) has the important property of suppressing finer details while strengthening the more significant edges. In order to see this we define the edges to consist of the points in the image domain B at which the magnitude of the gradient of the diffused image function z has a strict local maximum along the direction perpendicular to the edge, that is the direction of ∇z. Letting $\partial/\partial e_\nu$ denote the directional derivative in this direction, Δ denote the Laplacian operator, and $\sigma \doteq \|\nabla z^T\|$ represent the strength of the edge, we furthermore observe that $\partial^2\sigma/\partial e_\nu^2 \approx \Delta\sigma < 0$ at typical edge points of interest. At such edge points it can then be shown [7] that

$$\frac{\partial}{\partial t}(\sigma - \sigma_\zeta) = -(\sigma - \sigma_\zeta) + (g' \circ \sigma)\frac{\partial^2\sigma}{\partial e_\nu^2}\sigma + (g \circ \sigma)\Delta\sigma \approx -(\sigma - \sigma_\zeta) + (\varphi' \circ \sigma)\Delta\sigma \quad (8)$$

where $\sigma_\zeta \doteq \partial\zeta/\partial e_\nu$, and $\varphi(\gamma) \doteq g(\gamma)\gamma,\ \gamma \geq 0$. From this equation it is clear that the bias term $-(\sigma - \sigma_\zeta)$ merely has a moderating effect on the enhancement/blurring of the edge while the decision between enhancement vs. blurring depends on the sign of the "driving" term $(\varphi' \circ \sigma)\Delta\sigma$ associated with the unbiased anisotropic diffusion. Since $\Delta\sigma < 0$, the desired performance of weakening the weak edges while strengthening the strong ones in a consistent manner, therefore requires that there exists an *edge enhancement threshold* $\gamma_0 \in \mathbf{R}_+$ such that $\varphi'^{-1}(\mathbf{R}_-) =]\gamma_0, \infty[$, $\varphi'^{-1}(\{0\}) = \{\gamma_0\}$ and $\varphi'^{-1}(\mathbf{R}_+) = [0, \gamma_0[$. If that is the case, the threshold γ_0 clearly determines the sensitivity of the edge detector, and one would hence expect it to be closely related to the intuitively similarly acting edge cost coefficient λ. Indeed, from (5) and the definition of φ it follows that γ_0 is proportional to $\sqrt{\lambda}$. It is also easy to verify that the two diffusivity anomalies proposed in section 3 satisfy the threshold condition above with $\gamma_0 = \sqrt{\lambda}$ and $\gamma_0 = \sqrt{\lambda/2}$ respectively.

If the diffused image function converges to a steady state solution, that is a solution of the boundary value problem (4), according to (8) we obtain the steady state edge enhancement $\sigma - \sigma_\zeta = (\varphi' \circ \sigma)\Delta\sigma$. Since the range of the steady state solution by the extremum principle is confined to lie within the range of the original image function, an amply enhanced edge strength σ can only be maintained along a very short distance across the edge. Such edges are therefore sharpened.

Besides being of vital importance for the edge enhancement mechanism, the existence of the edge enhancement threshold γ_0 also provides a natural choice for the threshold to be used in the postprocessing whereby the edges are finally extracted from the estimated image function. For our edge representation to be consistent with the edge enhancement mechanism, the edge representation threshold in section 3 should thus be given by $\theta \doteq g(\gamma_0)$.

7 Experimental Results

The experiments presented here were conducted with the edge cost density $f(\omega) \doteq \omega - \ln\omega$ corresponding to the diffusivity anomaly $g(\gamma) \doteq 1/(1+\gamma^2/\lambda)$. The images involved were obtained by solving a discrete approximation of the boundary value problem (4) with a Gauss-Seidel-like iteration method.

As mentioned earlier, the iteration method converges to a solution of interest. This condition is illustrated by figure 1, which shows an original image and the estimated image obtained after convergence in the "sense of insignificant perceptible changes". With the initial image function $z^{(0)}$ set equal to ζ this convergence required about 100 iterations. For the purpose of edge detection, however, reasonably good results were obtained already before 50 iterations.

The variational edge detection method itself as well as the iteration method we employed to solve it appear to be remarkably robust with respect to changes in the initial image function. To demonstrate this behavior we tried the algorithm on the same original image function as in figure 1, but with the for convergence particularly unfavorable initial image function $z^{(0)} = 0$. Once again the algorithm converged if yet at a slower rate. The two solutions obtained with the two different initial image functions were not identical, but very similar. The significant differences were indeed limited to affect only small blobs of high contrast relative to the local background.

In order to extract a set of edges from the estimated image function z we followed the strategy outlined in section 3, and simply thresholded the gradient magnitude. Figure 2 shows the edges extracted from the estimated image function in figure 1 (b) with a gradient magnitude threshold ϑ 50% higher than the edge enhancement threshold γ_0. The edges obtained with ϑ 50% lower than γ_0 were almost identical. Since the edge enhancement/blurring mechanism discussed earlier depletes the set of points at which the gradient magnitude takes values close to γ_0, this ought to be expected.

8 Acknowledgements

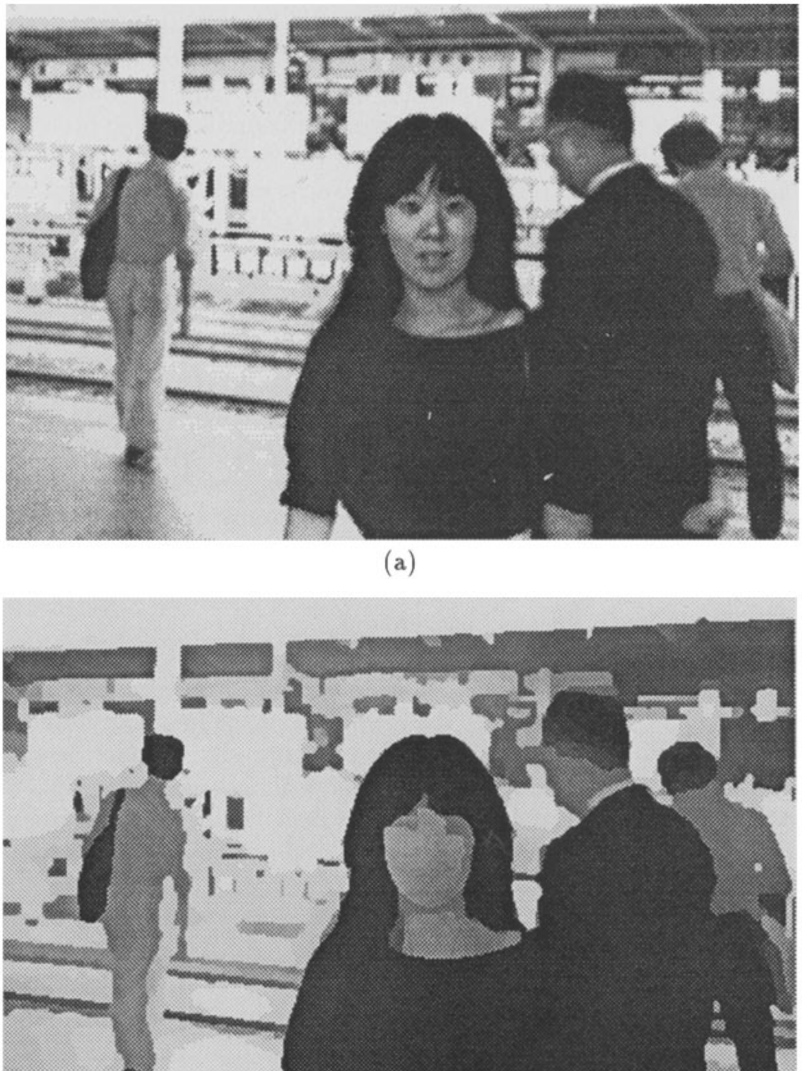
The author is grateful to M. Singer for writing most of the user interface for the software implementation of the algorithm and to K. Pister for generous help with generating the images.

References

[1] A. Blake and A. Zisserman, "Some properties of weak continuity constraints and the GNC algorithm," in *Conference on Computer Vision and Pattern Recognition*, pp. 656–660, IEEE, 1986.

[2] A. Blake and A. Zisserman, *Visual Reconstruction*. The MIT Press, 1987.

[3] S. Geman and D. Geman, "Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 6, pp. 721–741, Nov. 1984.



(a)

(b)

Figure 1: Estimated image after i iterations, when $z^{(0)} = \zeta$. (a) $i = 0$ (original image). (b) $i = 100$.

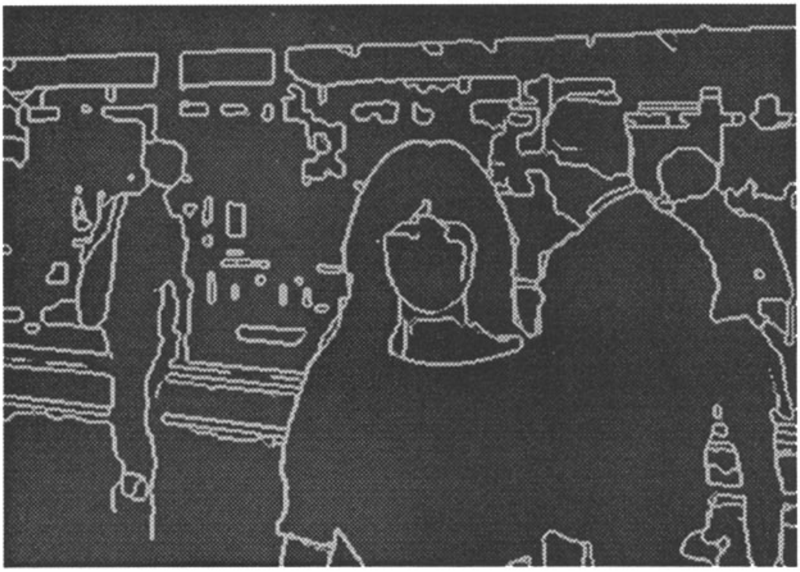


Figure 2: Extracted edges from estimated image in figure 1 (b).

[4] D. Terzopoulos, "Computing visible-surface representations," A.I. Memo 800, Massachusetts Institute of Technology Artificial Intelligence Laboratory, Cambridge, MA, Mar. 1985.

[5] D. Terzopoulos, "Regularization of inverse visual problems involving discontinuities," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 8, pp. 413–424, July 1986.

[6] D. G. Luenberger, *Linear and Nonlinear Programming.* Reading, MA: Addison-Wesley, second ed., 1984.

[7] K. N. Nordström, "Biased anisotropic diffusion—a unified regularization and diffusion approach to edge detection," Report UCB/CSD 89/514, Computer Science Division University of California, Berkeley, CA, June 1989.

[8] P. Perona and J. Malik, "Scale space and edge detection using anisotropic diffusion," in *Workshop on Computer Vision—Miami*, pp. 16–22, IEEE Computer Society, June 1987.

[9] P. Perona and J. Malik, "Scale-space and edge detection using anisotropic diffusion," Report UCB/CSD 88/483, Computer Science Division University of California, Berkeley, CA, Dec. 1988.

[10] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. II. John Wiley & Sons, 1962.