Liapunov Direct Method in Approaching Bifurcation Problems (*) (**).

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Summary. – *A concept of total stability for continuous or discrete dynamical systems and a generalized definition of bifurcation are given: it is possible to show the link between an abrupt change of the asymptotic behaviour of a family of flows and the arising of new invariant sets, with determined asymptotic properties. The theoretical results are a contribution to the study of the behaviour of flows near an invariant compact set. They are obtained by means of an extension of Liapunov's direct method.*

Introduction.

Let us consider a flow, possibly induced by a differential equation, which is supposed to describe the evolution of a physical system. One may ask which regions in the phase space are relevant for a significant picture of the analyzed phenomenon. First of all it is clear that we may restrict our attention to those regions which are left invariant by the flow. In many cases we are actually concerned with compact invariant sets only. In this respect one has to remember that, in a concrete case, the initial data will not be known with arbitrary precision, so that it is essential that a small perturbation will not result in a drastic change in the behaviour of the flow near the compact invariant set M of interest; that is that Liapunov stability be guaranteed. Further, if all trajectories starting in a neighbourhood of M will asymptotically reach it, then the set M will dominate the behaviour of the system in a whole neighbourhood. Such sets are asymptotically stable and are those of special physical significance.

We emphasize this interest in view of the following consideration: we must take into account not only perturbations in the initial data but of the whole system (e.g. of the right hand side in a differential equation), since a determined flow arises

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from a specific schema, neglecting a number of side effects which are judged not essential, but that are in fact equivalent to a perturbation of the whole system. For such a procedure to be reasonable it is fundamental that any « small perturbation » of the system will not drastically alter the local behaviour of the flow: that is that, roughly speaking, total stability be guaranteed. It is remarkable that for asymptotically stable sets total stability is assured, under not restrictive smoothness conditions. Here is another significant reason for the physical relevance of asymptotically stable sets.

In this paper we shall give a concept of total stability for continuous or discrete dynamical systems defined on a locally compact metric space E: this kind of total stability implies the classical one according to the definition due to DUBOSIN [1], in the case of dynamical systems induced by ordinary differential equations with lipschitzian right hand side. It is possible to show that asymptotic stability of a compact subset M of E implies its total stability in our sense. Note that, if we consider a system with a uniformly continuous dependence on a parameter, a small change in the parameter will induce a « small perturbation » (following our definition) of the system. We next prove that if for a fixed flow a certain compact invariant set is asymptotically stable, then under a « small perturbation » a compact asymptotically stable set will appear in a neighbourhood of the original one. What about the first set? If it changes its behaviour to a « complete instability » we can add that new asymptotically stable sets disjoint from it arise in a neighbourhood of it; this situation will be better described in the following.

We can compare these general facts with models such as those proposed by LANDAU for turbulence [2]. Starting from the Navier-Stokes equation for a certain value of a suitable parameter (e.g. the Reynolds number) we first have a stationary solution asymptotically stable. For values of the parameter greater than a certain constant, the stationary solution becomes completely unstable but a periodic asymptotically stable solution appears in its neighbourhood. Now it can be assumed that further increases of the parameter will cause analogous phenomena with higher and higher dimensional tori appearing, carrying quasi-periodic motions. An alternative model is that of D. RUELLE and F. TAKENS [3] which is not based on quasi-periodic motions for the description of turbulence, but relies all the same upon this kind of phenomenon. This successive « branching » of compact invariant sets is usually called bifurcation. Particular and well-known cases are branchings of stationary or periodic solutions of differential equations from stationary ones [4, 5] and analogously for diffeomorphisms [3] (*). All of these papers rely upon an inspection of the spectrum of the linear approximation of the right hand side of the differential equation or of the diffeomorphism. A common feature is that a switching of the asymptotic behaviour induces a bifurcation, under suitable hypotheses on the spectrum.

(*) For a wide list of references concerning this kind of problems see [6].

Our purpose is to show that the qualitative aspect of this phenomenon depends only on the switching in the asymptotic behaviour, regardless of any particular structure of the linear approximation. Moreover this analysis goes over, without changes, to more general cases (e.g. when no linear approximation exists at all). The core of our approach is an extension of the so-called Liapunov direct method of qualitative analysis to continuous dynamical systems not necessarily induced by differential equations [7]: it can easily be fitted to cover discrete dynamical systems too (these systems arise in a natural way, e.g. when considering Poincaré maps of continuous flows; transversal mappings are often used in the study of bifurcation phenomena). A remarkable result of this theory is that asymptotic stability is equivalent to the existence of a continuous function strictly decreasing along the trajectories. It is through the use of such a function that we can prove that, if each dynamical system of a one-parameter family admits an invariant compact set $M_\mu \subset E$, asymptotically stable for $\mu = \mu_0$ and completely unstable for $\mu > \mu_0$, then, for $\mu > \mu_0$, a new invariant compact set M'_μ exists, encircling the corresponding M_μ, disjoint from it and asymptotically stable. In this framework the case of $\boldsymbol{R}^2$ is investigated as an application of general results and some examples are given (*).

I. – Preliminaries.

1. – Let I be either the set $\boldsymbol{R}$ of real numbers or the set Z of integers. Let E be a locally compact metric space and ϱ the distance in E. Let U be a non-empty subset of E. For $\varepsilon > 0$ we set $S(U, \varepsilon) = \{x \in E \colon \varrho(x, U) < \varepsilon\}$ and $S[U, \varepsilon] = \{x \in E \colon \varrho(x, U) \leqslant \varepsilon\}$. For a map $\Gamma \colon E \to 2^E$, we set $\Gamma(U) = \bigcup \{\Gamma(x) \colon x \in U\}$. Consider the set S of continuous mappings from $I \times E$ into E such that for each $\pi \in \mathrm{S}$: (i) $\pi(0, x) = x$ for all $x \in E$; (ii) $\pi(t_1, \pi(t_2, x)) = \pi(t_1 + t_2, x)$ for all $t_1, t_2 \in I$ and $x \in E$. The triplet $\{I, E, \pi\}$ defines a dynamical system, continuous or discrete, according to wether $I = \boldsymbol{R}$ or $I = Z$.

For each $x \in E$ we denote by: (i) $\gamma^+_\pi(x)[\gamma^-_\pi(x)]$ the positive [negative] π-semitrajectory through x; (ii) $\Lambda^+_\pi(x)[\Lambda^-_\pi(x)]$ the positive [negative] π-limit set of x; (iii) $J^+_\pi(x)[J^-_\pi(x)]$ the positive [negative] π-first prolongational limit set of x; that is

$$\gamma^+_\pi(x) = \{\pi(t, x) \colon t \in I^+\} ;$$

$$\Lambda^+_\pi(x) = \{y \in E \colon \text{there exists a sequence } \{t_n\} \subset I^+, \text{ such that } t_n \to +\infty,\ \pi(t_n, x) \to y\} .$$

$$J^+_\pi(x) = \{y \in E \colon \text{there exist sequences } \{t_n\} \subset I^+,\ \{x_n\} \subset E, \text{ such that } t_n \to +\infty,\ x_n \to x,\ \pi(t_n, x_n) \to y\} .$$

Analogously for $\gamma^-_\pi(x)$, $\Lambda^-_\pi(x)$, $J^-_\pi(x)$.

(*) Most of these results have been furnished, in a preliminary version, as a communication to the « II Congresso Nazionale AIMETA » [8].

Let us recall some well-known definitions and theorems concerning a non-empty compact subset M of E. The set

$$A_\pi(M) = \{x \in E \colon \varrho(\pi(t, x), M) \to 0 \text{ as } t \to +\infty\},$$

is said to be the region of attraction of M with respect to π.

1.1 DEFINITIONS. – M is said to be:

1.1.1 a π-attractor, if $A_\pi(M)$ is a neighbourhood of M;

1.1.2 a π-uniform attractor, if M is a π-attractor and for any $x \in A_\pi(M)$ and for any neighbourhood V of M there exist a neighbourhood U of x and $T \in I^+$ such that $\pi(t, U) \subset V$ for $t \geqslant T$;

1.1.3 π-stable, if $(\forall \varepsilon > 0)(\exists \delta > 0) \colon \gamma_\pi^+(S(M, \delta)) \subset S(M, \varepsilon)$;

1.1.4 π-asymptotically stable, if 1.1.1 and 1.1.3 are both satisfied;

1.1.5 π-unstable, if it is not π-stable.

1.2 THEOREMS.

1.2.1 *If M is a π-attractor, then $A_\pi(M)$ is open.*

1.2.2 *If M is π-stable, then it is π-positively invariant.*

1.2.3 *If M is a π-uniform attractor, then the set*

$$\{x \in E \colon J_\pi^+(x) \neq \emptyset,\ J_\pi^+(x) \subset M\}$$

is a neighbourhood of M (and coincides with $A_\pi(M)$); the inverse holds if the dynamical system is continuous, or if a compact neighbourhood of M exists that is π-positively invariant.

1.2.4 *If M is π-positively invariant and a π-uniform attractor, then M is π-asymptotically stable.*

1.2.5 *If M is π-asymptotically stable, then M is a π-uniform attractor.*

1.3 REMARKS. – Dual concepts are trivially defined with respect to the asymptotic behaviour in the past: in this case we shall use the word «negative» (e.g. we say «negative attractor», «negative stability», and so on). Thus $A_\pi^-(M)$ will be the region of the π-negative attraction of M. The negative asymptotic stability will also be called «complete instability».

2. – We conclude these preliminaries by recalling some concepts and theorems about relative stability and attraction for a non-empty compact set $M \subset E$.

2.1 DEFINITIONS. – Let U be a non-empty subset of E. M is said to be:

2.1.1 a π-attractor relative to U, if $U \subset A_\pi(M)$;

2.1.2 π-stable relative to U, if $(\forall \varepsilon > 0)(\exists \delta > 0): \gamma_\pi^+(S(M, \delta) \cap U) \subset S(M, \varepsilon)$.

2.2 THEOREMS.

2.2.1 *If $U \subset E$ is a π-positively invariant set and M is a π-stable attractor relative to U, then M is a π-uniform attractor relative to U.*

2.2.2 *Suppose that M is π-positively invariant and denote by $\tilde{M}$ the largest π-invariant set contained in M. Then $\tilde{M}$ is a π-stable attractor relative to M; therefore (by 2.2.1) it is a π-uniform attractor relative to M.*

The proofs of Theorems 1.2 and 2.2 are performed in the same way as for continuous dynamical systems (see e.g. [7]).

II. – Total stability for continuous or discrete dynamical systems.

1. – The concept of total stability for ordinary differential equations is well-known as well as the interest of such a concept as a test of evolution schemes constructed in order to describe physical dynamics. Now we shall give a definition of total stability, that allows to consider dynamical systems (not necessarily generated by autonomous differential equations) in terms of a suitable «measure» of the perturbations in the phase space. Let M be a non-empty compact subset of E. For π, $p \in \mathcal{S}$ and $\varepsilon > 0$, $\bar{t} \in I^+ \setminus \{0\}$ we put

$$D(p, \pi; \varepsilon, \bar{t}) = \sup\{\varrho(p(t, x), \pi(t, x)) : 0 \leqslant t \leqslant \bar{t},\ x \in S(M, \varepsilon)\}.$$

1.1 DEFINITION. – Let $\pi \in \mathcal{S}$. We say that M is π-totally stable if:

1.1.1 $(\forall \varepsilon > 0)(\forall \bar{t} \in I^+ \setminus \{0\})(\exists \delta_1, \delta_2 > 0)(\forall p \in \mathcal{S} : D(p, \pi; \varepsilon, \bar{t}) < \delta_2):$

$$\gamma_p^+(S(M, \delta_1)) \subset S(M, \varepsilon)\ (*).$$

1.2 REMARKS. – We easily have:

1.2.1 if M is π-totally stable, then M is π-stable;

(*) Another concept of total stability, for generalized continuous dynamical systems, has been previously given by P. SEIBERT in [9]. The extension of Def. 1.1 to these dynamical systems and its connection with the definition due to P. SEIBERT, have been recently discussed by P. BONDI and V. MOAURO in a forthcoming paper.

1.2.2 if for given values of ε, $\bar{t}$ there exist positive numbers δ_1, δ_2 which fulfill condition 1.1.1, then this condition is still satisfied by the same numbers δ_1, δ_2, for any $\varepsilon' \geqslant \varepsilon$, $\bar{t}' \geqslant \bar{t}$;

1.2.3 by virtue of 1.2.2 if $I = Z$ and M is π-totally stable, the numbers δ_1, δ_2 in Def. 1.1 can be chosen independent of $\bar{t}$.

1.3 DEFINITION. – The compact M is said to be π-totally stable with respect to a subset $\mathcal{S}'$ of $\mathcal{S}$ if condition 1.1.1 holds with $\mathcal{S}'$ substituted to $\mathcal{S}$.

2. – We want now to give a connection between Def. 1.1 and the concept of total stability due to Dubosin, when we restrict ourselves to dynamical systems generated by autonomous ordinary differential equations in $\boldsymbol{R}^n$.

2.1 Let $\mathcal{A}$ be the set of continuous mappings from $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$ such that, whatever be $g \in \mathcal{A}$, for the differential equation

2.1.1 $$\dot{x} = g(x)$$

uniqueness and global existence in $\boldsymbol{R}^n$ of solutions through every point $x \in \boldsymbol{R}^n$ be guaranteed. For $x \in \boldsymbol{R}^n$ we denote by $p_g(\cdot, x)$ the solution such that $p_g(0, x) = x$. Then the triplet $\{\boldsymbol{R}, \boldsymbol{R}^n, p_g\}$ defines a continuous dynamical system and we obtain a subset $\mathcal{S}^*$ of $\mathcal{S}$ as g varies in $\mathcal{A}$. Let $\|\cdot\|$ be any norm in $\boldsymbol{R}^n$ and ϱ the induced distance. Let M a non-empty compact subset of $\boldsymbol{R}^n$. For $f, g \in \mathcal{A}$ and $\varepsilon' > 0$ we put

$$\Delta(f, g; \varepsilon') = \sup\{\varrho(f(x), g(x)) : x \in S(M, \varepsilon')\}.$$

2.2 DEFINITION. Let $f \in \mathcal{A}$ and $\{\boldsymbol{R}, \boldsymbol{R}^n, p_g\}$ be the dynamical system generated by eq. 2.1.1, $g \in \mathcal{A}$. M is said to be f-totally stable with respect to $\mathcal{A}$, in the sense of Dubosin, if

2.2.1 $$(\forall \varepsilon' > 0)(\exists \delta_1', \delta_2' > 0)(\forall g \in \mathcal{A}: \Delta(f, g; \varepsilon') < \delta_2') : \gamma^+_{p_g}(S(M, \delta_1')) \subset S(M, \varepsilon').$$

We have the following

2.3 THEOREM. – *Let $f \in \mathcal{A}$ be a locally lipschitzian function. Let $\{\boldsymbol{R}, \boldsymbol{R}^n, \pi\}$ be the dynamical system defined by the equation $\dot{x} = f(x)$. If M is π-totally stable with respect to $\mathcal{S}^*$ (in the sense of Def.* 1.3), *then M is f-totally stable with respect to $\mathcal{A}$ (in the sense of Def.* 2.2).

PROOF. – From Def. 1.3 it follows that

2.3.1 $$(\forall \varepsilon > 0)(\forall \bar{t} > 0)(\exists \delta_1, \delta_2 > 0)(\forall p_g \in \mathcal{S}^*: D(p_g, \pi; \varepsilon, \bar{t}) < \delta_2) : \gamma^+_{p_g}(S(M, \delta_1)) \subset S(M, \varepsilon).$$

We have to show that 2.3.1 implies 2.2.1. Let $\varepsilon' > 0$ and let $k > 0$ be a Lipschitz constant of f in $S[M, \varepsilon']$. Take $L = \max\{\|f(x)\| : x \in S[M, \varepsilon']\}$ and $\lambda > 0$. Set $U = \{\varphi \in \mathcal{A}: \|\varphi(x)\| < L + \lambda \text{ in } S[M, \varepsilon']\}$. It is easy to prove the existence of a $\bar{t} = \bar{t}(\varepsilon', \lambda)$ so that for any $g \in U$ eq. 2.1.1 defines a dynamical system $\{\boldsymbol{R}, \boldsymbol{R}^n, p_g\}$ with the property

2.3.2 $$p_g(t, z) \in S(M, \varepsilon') \qquad \forall (t, z) \in [0, \bar{t}] \times S(M, \varepsilon'/2)\,.$$

For instance if the norm in $\boldsymbol{R}^n$ is Euclidian, one can assume $\bar{t} = \varepsilon'[2\sqrt{n}(L + \lambda)]^{-1}$. Let δ_1', δ_2' be such that

$$\delta_1' = \delta_1\,, \qquad 0 < \delta_2' < \min\{\lambda, \delta_2 \bar{t}^{-1} \exp[-k\bar{t}]\}\,,$$

where δ_1, δ_2 are positive numbers associated through condition 2.3.1 to $\varepsilon = \varepsilon'/2, \bar{t}$. Then, for any $g \in \mathcal{A}$ satisfying

2.3.3 $$\Delta(f, g; \varepsilon') < \delta_2'\,,$$

it turns out $g \in U$, and hence, by virtue of Gronwall's lemma,

$$D(p_g, \pi; \varepsilon, \bar{t}) \leqslant \delta_2' \bar{t} e^{k\bar{t}} < \delta_2\,.$$

Therefore, since $\delta_1' = \delta_1$, condition 2.3.1 implies that, for all $g \in \mathcal{A}$ satisfying 2.3.3, one has $\gamma_{p_g}^+(S(M, \delta_1')) \subset S(M, \varepsilon) \subset S(M, \varepsilon')$.

3. – Consider again any dynamical system $\{I, E, \pi\}$ continuous or discrete and a non-empty compact subset M of E. The following theorem holds:

3.1 THEOREM. – *If M is π-asymptotically stable, then it is π-totally stable.*

PROOF. – (i) By virtue of the π-asymptotic stability of M, there exists a continuous function $V: A_\pi(M) \to \boldsymbol{R}^+$ such that:

3.1.1 $$V(x) = 0 \text{ for all } x \in M; \qquad V(x) > 0 \text{ for all } x \in A_\pi(M) \setminus M\,;$$

3.1.2 $$V(\pi(t, x)) < V(x) \qquad \text{for all } t \in I^+ \setminus \{0\} \text{ and } x \in A_\pi(M) \setminus M\,.$$

This fact is well-known in the case $I = \boldsymbol{R}$ (see [7], V, 2); the extension to the case of discrete dynamical systems is trivial. For $x \in M$ it is $V(\pi(t, x)) = 0 \;\; \forall t \in I^+$, inasmuch as M is π-stable and therefore π-positively invariant. Assume $\lambda > 0$ such that $S[M, \lambda]$ be a compact subset of $A_\pi(M)$: this is possible by virtue of the local compactness of E and because $A_\pi(M)$ is a neighbourhood of M. Let $\bar{t} \in I^+ \setminus \{0\}$; there exist three mappings a, b, c from $\boldsymbol{R}^+$ into $\boldsymbol{R}^+$ of class K (in the sense of W. HAHN, that is continuous, strictly increasing and equal to zero in the origin) such that for every $x \in S[M, \lambda]$:

3.1.3 $$a(\varrho(x, M)) \leqslant V(x) \leqslant b(\varrho(x, M)) ;$$

3.1.4 $$V(\pi(\bar{t}, x)) - V(x) \leqslant -c(\varrho(x, M)) .$$

(ii) Fix $\bar{\lambda} \in]0, \lambda[$ and choose $\varepsilon \in]0, \bar{\lambda}[$ so that

3.1.5 $$\gamma_\pi^+(S(M, \varepsilon)) \subset S(M, \bar{\lambda}) .$$

The existence of such an ε is guaranteed by the π-stability of M. Analogously assume $\varepsilon_1 \in]0, \varepsilon[$, $\varepsilon_2 \in]0, \varepsilon_1[$, $\varepsilon_3 \in]0, b^{-1}(a(\varepsilon_2))[$ in such a way that

3.1.6 $$\gamma_\pi^+(S(M, \varepsilon_1)) \subset S(M, \varepsilon) ,$$

3.1.7 $$\gamma_\pi^+(S(M, \varepsilon_2)) \subset S(M, \varepsilon_1) ,$$

3.1.8 $$\gamma_\pi^+(S(M, \varepsilon_3)) \subset S(M, \varepsilon_2) .$$

By virtue of 3.1.3 it is $b^{-1}(a(\varepsilon_2)) \leqslant \varepsilon_2$. We now give two positive numbers δ_1, δ_2 and we want to show that they fulfill condition 1.1.1 relatively to the couple ε, $\bar{t}$. First we assume $\delta_1 \in]0, \varepsilon_3[$ with the condition

3.1.9 $$\gamma_\pi^+(S[M, \delta_1]) \subset S(M, \varepsilon_3) .$$

Take $L \in]0, +\infty[$ such that

3.1.10 $$|V(y) - V(z)| \leqslant L\varrho(y, z) + \frac{c(\delta_1)}{4} \qquad \text{for all } y, z \in S[M, \lambda] ;$$

this is possible because V is continuous and of the compactness of $S[M, \lambda]$. Then we assume δ_2 in the following way:

3.1.11 $$0 < \delta_2 < \min\left\{\frac{c(\delta_1)}{4L}, \lambda - \bar{\lambda}, \varepsilon - \varepsilon_1, \varepsilon_1 - \varepsilon_2, b^{-1}(a(\varepsilon_2)) - \varepsilon_3\right\} .$$

(iii) Consider a dynamical system $\{I, E, p\}$ satisfying the condition

3.1.12 $$D(\pi, p; \varepsilon, \bar{t}) < \delta_2 ;$$

it follows that

3.1.13 $$\varrho(p(t, z), M) < \varrho(\pi(t, z), M) + \delta_2 \qquad \text{for all } t \in I: 0 \leqslant t \leqslant \bar{t} \text{ and } z \in S(M, \varepsilon) .$$

Let $x \in S(M, \delta_1)$. Suppose now that there exists $\theta \in I^+$ such that $p(\theta, x) \notin S(M, \varepsilon)$. Set

$$t'' = \min \{t \in I: 0 \leqslant t \leqslant \theta,\ p(t, x) \notin S(M, \varepsilon)\},$$

$$t' = \max \{t \in I: 0 \leqslant t \leqslant t'', p(t, x) \in S[M, \delta_1]\}.$$

For every $\tau \in I: 0 \leqslant \tau \leqslant \bar{t}$ one gets, by 3.1.9, 3.1.13, 3.1.11,

$$p(t' + \tau, x) \in S(M, \varepsilon_3 + \delta_2) \subset S(M, \varepsilon_2) \subset S(M, \varepsilon);$$

hence by virtue of 3.1.7, and again 3.1.13, 3.1.11,

$$p(t' + 2\tau, x) = p(\tau, p(t' + \tau, x)) \in S(M, \varepsilon_1 + \delta_2) \subset S(M, \varepsilon).$$

Then $t'' - t' > 2\bar{t}$. Thus we can put $t'' = t' + (N + \alpha)\bar{t}$ where $N \geqslant 2$ is an integer and $\alpha \in I: 0 < \alpha \leqslant 1$. Define

$$x_n = p(t' + n\bar{t}, x) \quad \text{for } n \in \{1, 2, \dots, N\} \text{ and } x^* = p(t' + (N + \alpha)\bar{t}, x).$$

One gets

$$\delta_1 < \varrho(x_n, M) < \varepsilon \quad \text{for } n \in \{1, 2, \dots, N\};$$

further taking into account 3.1.5, 3.1.13 and the fact that $\bar{\lambda} + \delta_2 < \lambda$, we have

$$\varepsilon \leqslant \varrho(x^*, M) < \lambda.$$

Now, by virtue of 3.1.4 one obtains

$$V(x_{n+1}) - V(x_n) = V(p(\bar{t}, x_n)) - V(\pi(\bar{t}, x_n)) + V(\pi(\bar{t}, x_n)) - V(x_n) <$$
$$< V(p(\bar{t}, x_n)) - V(\pi(\bar{t}, x_n)) - c(\delta_1) \quad \text{for } n \in \{1, 2, \dots, N - 1\}.$$

Since $p(\bar{t}, x_n)$ and $\pi(\bar{t}, x_n)$ belong to $S[M, \lambda]$ and the conditions 3.1.10, 3.1.11, 3.1.12 hold, we have

$$V(x_{n+1}) - V(x_n) < L\delta_2 + \frac{c(\delta_1)}{4} - c(\delta_1) < -\frac{c(\delta_1)}{2} \qquad \text{for } n \in \{1, \dots, N - 1\}.$$

Analogously, keeping 3.1.2 in mind in place of 3.1.4, we get

$$V(x^*) - V(x_N) < \frac{c(\delta_1)}{2}.$$

Then

3.1.14 $$V(x^*) - V(x_1) < -\frac{N - 2}{2} c(\delta_1) \leqslant 0.$$

On the other hand it is $V(x^*)\geqslant a(\varepsilon)$, and

$$V(x_1)\leqslant b(\varrho(x_1, M)) < b(\varepsilon_3+\delta_2) < b\big(b^{-1}(a(\varepsilon_2))\big) = a(\varepsilon_2) < a(\varepsilon)\,;$$

that is $V(x^*)>V(x_1)$, in contradiction with 3.1.14. Thus the theorem is proved. Indeed $\bar t\in I^+\setminus\{0\}$ has been chosen arbitrarily; as for ε recall Definition 3.1.5 and Remark 1.2.2.

3.2 REMARK. – The statement of Th. 3.1 still holds if the hypothesis that M is π-asymptotically stable is replaced by the weaker one that M possesses a fundamental system of π-asymptotically stable compact neighbourhoods.

3.3 REMARK. – Let $\{\boldsymbol{R}, \boldsymbol{R}^n, \pi\}$ be a dynamical system generated by a differential equation $\dot x = f(x)$, where $f\in\mathcal{A}$ is locally lipschitzian. Then, by virtue of Th. 2.3, one gets a well-known result which follows as a corollary, in the case of autonomous differential equations, from the classical theorem of GORSIN and MALKIN [10, 11].

III. – Bifurcation and total stability.

1. – Let $\bar\mu>0$ and p be a map from $[0,\bar\mu[$ into $\mathcal{S}$, $\mu\mapsto p_\mu$. Assume that $p\colon [0,\bar\mu[\times I\times E\to E$, $p(\mu,t,x)=p_\mu(t,x)$, is a continuous function. We shall say that p defines a one-parameter continuous family of dynamical systems. In the following we shall denote by $\mathcal{C}$ the set of all proper compact subsets of E other than $\emptyset$.

1.1 DEFINITION. – Consider a map M from $[0,\bar\mu[$ into $\mathcal{C}$, $\mu\to M_\mu$, such that

(1) for every $\mu\in[0,\bar\mu[$ the compact set M_μ is p_μ-invariant;

(2) $\max\{\varrho(x, M_0)\colon x\in M_\mu\}\to 0$ as $\mu\to 0$.

Then $\mu=0$ is said to be a bifurcation point for the mapping M if there exists a $\mu^*\in]0,\bar\mu[$ and another mapping $M'\colon]0,\mu^*[\to\mathcal{C}$, $\mu\mapsto M_\mu$, such that

(α) for every $\mu\in]0,\mu^*[$ the compact set M'_μ is p_μ-invariant and $M'_\mu\cap M_\mu=\emptyset$;

(β) $\max\{\varrho(x, M_0)\colon x\in M'_\mu\}\to 0$ as $\mu\to 0$.

Our aim is now to give a sufficient condition in order to ensure the occurrence of bifurcation and to characterize the asymptotic behaviour of bifurcated sets. First of all we have to premise the following

1.2 THEOREM. – *Let $M_0\subset E$ be a compact set which is p_0-asymptotically stable. Then there exist a $\mu^*\in]0,\bar\mu[$ and a function $v\colon \boldsymbol{R}^+\to\boldsymbol{R}^+$ of class K such that for* *every* $\mu \in]0, \mu^*[$ *the set* $P_\mu = \overline{\gamma^+_{p_\mu}(S(M_0, \nu(\mu)))}$ *satisfies the following conditions:*

(*a*) P_μ *is a* p_μ*-asymptotically stable compact set;*

(*b*) *there exists a neighbourhood* $\mathcal{N}$ *of* M_0 *for which*

$$(\exists T \in I^+)(\forall x \in \mathcal{N})(\forall t \geqslant T): p_\mu(t, x) \in P_\mu .$$

Further we have:

(*c*) $P_\mu \to M_0$ *as* $\mu \to 0$, *in the Hausdorff metric* (*).

PROOF. – (i) M_0 is p_0-totally stable (II, Th. 3.1). Fix $\bar{t} \in I^+ \setminus \{0\}$ and $\lambda > 0$ so that $S[M_0, \lambda]$ be a compact subset of $A_{p_0}(M_0)$. By virtue of condition II, 1.1.1 and the continuity of $p(\mu, t, x)$ it is not difficult to prove the existence of two mappings h, k, from $\boldsymbol{R}^+$ into $\boldsymbol{R}^+$ of class K for which

1.2.1 $$\gamma^+_{p_\mu}(S(M_0, h(\varepsilon))) \subset S(M_0, \varepsilon) \quad \text{for all } \varepsilon \in [0, \lambda] \text{ and } \mu \in [0, k(\varepsilon)] .$$

where $k(\lambda) \leqslant \bar{\mu}$. Let $V: A_{p_0}(M_0) \to \boldsymbol{R}^+$ be a continuous function associated to the p_0-asymptotic stability of M_0 in the way we have specified in the proof of Th. II, 3.1. This function satisfies in $S[M_0, \lambda]$ the conditions 3.1.3, 3.1.4 of Sect. II, with $\pi = p_0$ and $M = M_0$. For each $\varepsilon \in]0, \lambda]$ there exists (cfr. II, 3.1.10) a number $L(\varepsilon) \in]1, +\infty[$ so that

1.2.2 $$|V(y) - V(z)| \leqslant L\varrho(y, z) + \frac{c(h(\varepsilon))}{4} \qquad \text{for all } y, z \in S[M_0, \lambda] .$$

Because of the continuity of the function p, one can prove the existence of another function $\psi: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ of class K for which

1.2.3 $$\varrho(p(\mu, t, x), p(0, t, x)) \leqslant \frac{c(h(\varepsilon))}{4L} \quad \text{for all } \mu \in [0, \psi(\varepsilon)] ,\ (t, x) \in [0, \bar{t}] \times S[M_0, \lambda] .$$

Consider a function $\chi: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ of class K such that $\chi(\varepsilon) \leqslant \min\{k(\varepsilon), \psi(\varepsilon)\}$ for every $\varepsilon \in [0, \lambda]$. Let $\nu = h \circ \chi^{-1}$ and $\mu^* = \chi(h^2(\lambda))$, where $h^2 = h \circ h$. Then

1.2.4 $$\mu^* = \chi(h^2(\lambda)) \leqslant \chi(\lambda) \leqslant k(\lambda) \leqslant \bar{\mu} .$$

(ii) For every $\mu \in]0, \mu^*[$ consider the set $P_\mu = \overline{\gamma^+_{p_\mu}(S(M_0, \nu(\mu)))}$. First of all, we note that $S(M_0, h^2(\lambda))$ is an open neighbourhood of P_μ: this is a consequence of 1.2.1

(*) This theorem extends well known results ([12], Chap. VI, Th. 25.3) to the case of perturbed dynamical systems (continuous or discrete) not necessarily defined by differential equations.

when we assume $\varepsilon = \chi^{-1}(\mu)$, $\mu \in]0, \mu^*[$. We prove that if $x \in S(M_0, h^2(\lambda))$, then there exists a $\tau_x \in I^+$ so that $p_\mu(\tau_x, x) \in S(M_0, \nu(\mu))$. Indeed suppose $\gamma^+_{p_\mu}(x) \cap S(M_0, \nu(\mu)) = \emptyset$ and let N be a positive integer. Setting $x_n = p_\mu(n\bar{t}, x)$ for $n \in \{0, 1, \ldots, N\}$, one has $x_n \in S(M_0, \lambda)$ by virtue of 1.2.1, 1.2.4; the same holds for the points $p_0(\bar{t}, x_n)$. Then, taking into account inequalities 1.2.2, 1.2.3 for the values $\varepsilon = \chi^{-1}(\mu)$, $\mu \in]0, \mu^*[$, we easily obtain (as in the proof of Th. II, 3.1) that $V(x_{n+1}) - V(x_n) \leqslant -[c(\nu(\mu))]/2$, for all $n \in \{0, 1, \ldots, N-1\}$, and therefore $V(p_\mu(N\bar{t}, x)) - V(x) \leqslant -N[c(\nu(\mu))]/2$. Hence $V(p_\mu(N\bar{t}, x)) \to -\infty$ as $N \to +\infty$, which is absurd. The existence of the required number τ_x is then proved. Further, as $S(M_0, \nu(\mu)) \subset P_\mu$, and P_μ is p_μ-positively invariant, one has that $p_\mu(t, x) \in P_\mu$ for all $t \geqslant \tau_x$: We can assume as τ_x a number T independent of $x \in S(M_0, h^2(\lambda))$, e.g. any integer greater than the number $\max\{2V(x)/c(\nu(\mu)) : x \in S[M_0, h^2(\lambda)]\}$. Thus (*b*) is proved. In particular the set P_μ is a compact p_μ-uniform attractor, and therefore by virtue of I, 1.2.4 it is also proved that P_μ is p_μ-asymptotically stable.

Finally we have $M_0 \subset P_\mu \subset \overline{S(M_0, \chi^{-1}(\mu))}$, from which (*c*) immediately follows, since $\chi^{-1} \in K$.

Now we can prove the following

1.3 THEOREM. – *Suppose that E be connected. Let $\bar{\mu} > 0$ and $M: [0, \bar{\mu}[\to \mathcal{C}$ be a mapping satisfying conditions* (1), (2) *of Def.* 1.1. *Suppose that M_0 be p_0-asymptotically stable and that for each $\mu \in]0, \bar{\mu}[$ the compact set M_μ be p_μ-completely unstable. Then, $\mu = 0$ is a bifurcation point for the map M. Further the positive number μ^* and the map M' (of Def.* 1.1) *can be determined so that, for each $\mu \in]0, \mu^*[$:*

(*a*) *M'_μ is p_μ-asymptotically stable;*

(*b*) *$M'_\mu = \tilde{P}_\mu \setminus A^-_{p_\mu}(M_\mu)$, where $\tilde{P}_\mu$ is a compact set containing $A^-_{p_\mu}(M_\mu)$.*

PROOF. – We keep the notations introduced in the proof of Th. 1.2.

(i) Since $\max\{\varrho(x, M_0) : x \in M_\mu\} \to 0$ as $\mu \to 0$, there exists a function $\sigma: \mathbf{R}^+ \to \mathbf{R}^+$ of class K such that

1.3.1 $$M_\mu \subset S(M_0, h(\varepsilon)) \quad \text{for all } \mu \in [0, \sigma(\varepsilon)] \text{ and } \varepsilon \in [0, \lambda],$$

where λ is now chosen so that $S[M_0, \lambda]$ be a proper compact subset of $A_{p_0}(M_0)$. We choose the function $\chi \in K$ (also used in the proof of Th. 1.2) subject to the additional condition

$$\chi(\varepsilon) \leqslant \sigma(\varepsilon) \quad \text{for all } \varepsilon \in [0, \lambda],$$

and again correspondingly define $\nu = h \circ \chi^{-1}$ and $\mu^* = \chi(h^2(\lambda))$. For every $\mu \in]0, \mu^*]$ the compact set $P_\mu = \overline{\gamma^+_{p_\mu}(S(M_0, \nu(\mu)))}$ satisfies conditions (*a*), (*b*), (*c*) of Th. 1.2. Let $\tilde{P}_\mu$ be the largest p_μ-invariant subset of P_μ; $\tilde{P}_\mu$ is a compact p_μ-uniform attractor with respect to P_μ (I, 2.2.2). Then, by virtue of the condition (*b*) of Th. 1.2, $\tilde{P}_\mu$ is a p_μ-uniform attractor, hence a p_μ-asymptotically stable set (I, 1.2.4).

(ii) Now we can prove: $\tilde{P}_\mu \supset A_\mu^-$, where for sake of simplicity we have set $A_\mu^- = A_{p_\mu}^-(M_\mu)$. From 1.3.1 it follows that there exists $r_\mu > 0$ such that $S(M_\mu, r_\mu) \subset \subset S(M_0, \nu(\mu))$. Let $y \in A_\mu^-$. M_μ is p_μ-negatively asymptotically stable; then there exists a $t' \in I^-$ such that $p_\mu(t', y) \in S(M_\mu, r_\mu) \subset S(M_0, \nu(\mu)) \subset P_\mu$. But P_μ is p_μ-positively invariant; then $p_\mu(-t', p_\mu(t', y)) = y \in P_\mu$. Hence $A_\mu^- \subset P_\mu$. By the definition of $\tilde{P}_\mu$ and because of the p_μ-invariance of A_μ^-, we have $\tilde{P}_\mu \supset A_\mu^-$. The open set A_μ^- is a proper subset of $\tilde{P}_\mu$, by virtue of the connectedness of E and because $\tilde{P}_\mu \subset S[M_0, \lambda] \neq E$.

(iii) Consider the p_μ-invariant compact set $M'_\mu = \tilde{P}_\mu \setminus A_\mu^-$. We prove that it is p_μ-asymptotically stable. Because of (i) and I, 1.2.3 the proof is obviously achieved if we show that $x \in A_\mu^- \setminus M_\mu \Rightarrow J_{p_\mu}^+(x) \neq \emptyset$ and $J_{p_\mu}^+(x) \subset \partial A_\mu^-$. Since A_μ^- is a p_μ-invariant compact set, it is $J_{p_\mu}^+(x) \neq \emptyset$ and $J_{p_\mu}^+(x) \subset \bar{A}_\mu^-$. Thus we have to prove that $y \in J_{p_\mu}^+(x)$ implies $y \notin A_\mu^-$. Indeed, if $y \in A_\mu^-$ we should have $J_{p_\mu}^-(y) \neq \emptyset$ and $J_{p_\mu}^-(y) \subset M_\mu$: On the other hand, from the definition of $J_{p_\mu}^+(x)$ it follows $x \in J_{p_\mu}^-(y)$. Then $x \in M_\mu$, which is absurd.

Lastly we have $\max\{\varrho(x, M_0): x \in M'_\mu\} \to 0$ as $\mu \to 0$. This, taking into account that $M'_\mu \subset \tilde{P}_\mu \subset P_\mu$, follows immediately from the condition (*c*) of Th. 1.2.

2. – We may now apply Th. 1.3 to the study of the bifurcation of a critical point with respect to a family of dynamical systems induced by ordinary differential equations in $\boldsymbol{R}^2$. The analysis in the neighbourhood of this critical point will rely upon hypotheses merely concerning the asymptotic behaviour of solutions near this rest point (which will be chosen as the origin in $\boldsymbol{R}^2$). The specific results which will be obtained are a generalization of a classical theorem due to E. HOPF [4] (a proof of this theorem can also be found in [3]).

2.1. Let $\bar{\mu} > 0$ and $f: [0, \bar{\mu}[\times \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be a continuous map satisfying the following conditions:

(*a*) $f(\mu, 0) \equiv 0$;

(*b*) for each $\mu \in [0, \bar{\mu}[$ the differential equation

2.1.1 $$\dot{x} = f_\mu(x),$$

where $f_\mu(x) \equiv f(\mu, x)$, defines a dynamical system $\{\boldsymbol{R}, \boldsymbol{R}^2, p_\mu\}$ on $\boldsymbol{R}^2$;

(*c*) a neighbourhood Ω of the origin $0 \in \boldsymbol{R}^2$ exists such that for every $\mu \in [0, \bar{\mu}[$ equation 2.1.1 does not admit any critical point other than 0.

Specialize now the map $M: [0, \bar{\mu}[\to \mathcal{C}$ defined in Def. 1.1 assuming $M_\mu = \{0\}$ for all $\mu \in [0, \bar{\mu}[$. The bifurcation of this map (with respect to the family of dynamical systems $\{\boldsymbol{R}, \boldsymbol{R}^2, p_\mu\}$) will be called bifurcation of the origin as well as bifurcation of the static solution $x \equiv 0$ of each equation defined by 2.1.1. One has the following

2.2 THEOREM. – *Let $\{0\}$ be p_0-asymptotically stable and p_μ-completely unstable for every $\mu \in]0, \bar{\mu}[$. Then $\mu = 0$ is a bifurcation point of the origin. Further the number $\mu^* > 0$ and the map M': $]0, \mu^*[\to \mathcal{C}$ of Def. 1.1 can be determined in such a way that for $\mu \in]0, \mu^*[$ these properties hold:*

(*a*) M'_μ *is p_μ-asymptotically stable;*

(*b*) M'_μ *is the compact annular region included between two cycles C_μ, C'_μ of $\{\boldsymbol{R}, \boldsymbol{R}^2, p_\mu\}$ both containing 0 in their interior, the inside one, C'_μ, being equal to $\partial A^-_{p_\mu}(0)$.*

PROOF. – (i) By virtue of Theorem 1.3, $\mu = 0$ is a bifurcation point of the origin; besides a number $\mu^* > 0$ and a map M'': $]0, \mu^*[\to \mathcal{C}$ exist (with $M''_\mu \to \{0\}$ as $\mu \to 0$) such that, for each $\mu \in]0, \mu^*[$, it results: (1) M''_μ is p_μ-asymptotically stable; (2) $M''_\mu = \tilde{P}_\mu \setminus A^-_\mu$, where A^-_μ is now the set $A^-_{p_\mu}(0)$ and $\tilde{P}_\mu \supset A^-_\mu$ is a compact subset of Ω, p_μ-invariant and p_μ-asymptotically stable. Choose $\mu \in]0, \mu^*[$ and put $r_\mu = \max\{\varrho(0, x): x \in \tilde{P}_\mu\}$. Choose $\xi \in A_{p_\mu}(\tilde{P}_\mu)$ such that $\varrho(0, \xi) > r_\mu$. Now the limit set $\Lambda^+_{p_\mu}(\xi)$ is not empty and is contained in $\tilde{P}_\mu$, since $\xi \in A_{p_\mu}(\tilde{P}_\mu)$. Moreover, ξ being in the complement of the compact invariant set $\tilde{P}_\mu$, we may state that $\Lambda^+_{p_\mu}(\xi) \subset \partial\tilde{P}_\mu$; therefore $\Lambda^+_{p_\mu}(\xi)$ is bounded and does not contain any critical point (since $\partial\tilde{P}_\mu \subset \Omega\setminus\{0\}$). Then $\Lambda^+_{p_\mu}(\xi)$ is a cycle, whose interior contains the origin (by virtue of some well-known theorems of Bendixon). We put $C_\mu = \Lambda^+_{p_\mu}(\xi)$ and denote by D_μ the compact region bounded by C_μ.

(ii) We now prove that $\tilde{P}_\mu \subset D_\mu$. Since $0 \in D_\mu$, choose $z \in \tilde{P}_\mu \setminus \{0\}$ and suppose that $z \notin D_\mu$: we have to show that this is absurd. The set $\Lambda^+_{p_\mu}(z)$ is not empty and $\Lambda^+_{p_\mu}(z) \subset \tilde{P}_\mu \setminus \{0\} \subset \Omega$: then $\Lambda^+_{p_\mu}(z)$ is a cycle contained in $\tilde{P}_\mu$ and containing 0 in its interior. By the uniqueness of trajectories, the only cases to consider are: (α) $\Lambda^+_{p_\mu}(z) \subset \mathring{D}_\mu$; ($\beta$) $\Lambda^+_{p_\mu}(z) \subset R^2 \setminus D_\mu$; ($\gamma$) $\Lambda^+_{p_\mu}(z) \equiv C_\mu$. ($\alpha$) is impossible, for the set D_μ is invariant and $z \notin D_\mu$. If (β) holds the point ξ defined in (i) should be interior to the cycle $\Lambda^+_{p_\mu}(z)$ (which is contained in $\tilde{P}_\mu \setminus \{0\}$): this is obviously absurd. Then (γ) only is possible; analogously $\Lambda^-_{p_\mu}(z) = C_\mu$. Therefore C_μ should be both positive and negative limit set for the point $z \notin C_\mu$: this is absurd (see e.g. [13], Chap. IV, Th. 8).

(iii) D_μ is a p_μ-invariant compact set containing $\tilde{P}_\mu$. It is easy now to show that D_μ is p_μ-asymptotically stable. Obviously, there exists $\varepsilon > 0$ such that $S(D_\mu, \varepsilon) \setminus \mathring{D}_\mu \subset A_{p_\mu}(\tilde{P}_\mu)$: then for any $x \in S(D_\mu, \varepsilon) \setminus \mathring{D}_\mu$ it happens that $J^+_{p_\mu}(x) \neq \emptyset$ and $J^+_{p_\mu}(x) \subset \tilde{P}_\mu \subset D_\mu$. If $x \in \mathring{D}_\mu$ one gets trivially, by virtue of the p_μ-invariance of D_μ, $J^+_{p_\mu}(x) \subset D_\mu$.

(iv) Let $\zeta \in A^-_\mu \setminus \{0\}$. Since $A^-_\mu \subset \tilde{P}_\mu \subset D_\mu \subset \Omega$ and A^-_μ is invariant, then the limit set $\Lambda^+_{p_\mu}(\zeta)$ is a cycle contained in D_μ and containing 0 in its interior. We set $C'_\mu = \Lambda^+_{p_\mu}(\zeta)$ and denote by D'_μ the compact region bounded by C'_μ: it is true that $D'_\mu \subset D_\mu$. We prove that $C'_\mu = \partial A^-_\mu$. It is sufficient to show that $\mathring{D}'_\mu = A^-_\mu$. If $y \in A^-_\mu$ then $\Lambda^-_{p_\mu}(y) = \{0\}$ and there exists a $\bar{t} < 0$ such that $p_\mu(\bar{t}, y) \in \mathring{D}'_\mu$. Taking into account that $\mathring{D}'_\mu$ is p_μ-invariant, it will be $y \in \mathring{D}'_\mu$, that is $A^-_\mu \subset \mathring{D}'_\mu$. Conversely, since $0 \in \mathring{A}^-_\mu$, suppose now that $y \in \mathring{D}'_\mu \setminus \{0\}$. The limit set $\Lambda^+_{p_\mu}(y)$ is a cycle C''_μ contained in D'_μ and containing 0 in its interior. It is easy to see that $C''_\mu = C'_\mu$; otherwise either $\gamma^+_{p_\mu}(\zeta)$ or $\gamma^-_{p_\mu}(\zeta)$, the semi-trajectories through ζ, would cross C''_μ in a point, without coinciding with C''_μ. This is absurd. Then in D'_μ there exist a unique cycle, C'_μ, and a unique critical point, 0, with respect to $\{\boldsymbol{R}, \boldsymbol{R}^2, p_\mu\}$; since $\Lambda^+_{p_\mu}(y) = C'_\mu$, necessarily $\Lambda^-_{p_\mu}(y) = \{0\}$. This means that $y \in A^-_\mu$, that is $\mathring{D}'_\mu \subset A^-_\mu$.

(v) We can conclude that the set $M'_\mu = D_\mu \setminus \mathring{D}'_\mu$ satisfies all conditions required by Theorem 2.2: indeed this is the ring between the two cycles C_μ, $C'_\mu = \partial A^-_\mu$ of $\{\boldsymbol{R}, \boldsymbol{R}^2, p_\mu\}$, each having in its interior the origin. Secondly, it is true that $M'_\mu \to \{0\}$ as $\mu \to 0$, since $\tilde{P}_\mu \to \{0\}$ as $\mu \to 0$ $(C_\mu \subset \partial \tilde{P}_\mu)$. Lastly M'_μ is p_μ-invariant and asymptotically stable. To prove the p_μ-asymptotic stability of M'_μ it is sufficient, by virtue of (iii), to observe that $x \in A^-_\mu \setminus \{0\}$ implies $J^+_{p_\mu}(x) \subset \partial A^-_\mu \subset M'_\mu$.

2.3 REMARK. – As a particular case, we note that if in the region Ω the dynamical system admits for each $\mu \in]0, \mu^*[$ a unique cycle, then this cycle is attractive and one obtains the Hopf bifurcation.

2.4 It is now suggestive to provide some examples of bifurcation in $\boldsymbol{R}^2$. Let n be an odd integer and $a_0, a_1, \dots, a_n$ be functions of a parameter $\mu \geqslant 0$: consider the system of two differential equations

2.4.1
$$\begin{aligned} \dot{x} &= a_0 x - y + a_1 x(x^2 + y^2) + a_2 x(x^2 + y^2)^2 + \dots + a_n x(x^2 + y^2)^n \\ \dot{y} &= a_0 y + x + a_1 y(x^2 + y^2) + a_2 y(x^2 + y^2)^2 + \dots + a_n y(x^2 + y^2)^n , \end{aligned}$$

which admits the null solution $x \equiv y \equiv 0$.

(i) Suppose now that for any $\mu \geqslant 0$, $a_0(\mu), \dots, a_n(\mu)$ are such that the following identity is true:

$$a_0 + a_1 z + \dots + a_n z^n = -(z-\mu)^2 (z - 2\mu)^2 \dots (z - k\mu)^2 [z - (k+1)\mu] ,$$

where $n = 2k+1$. One has $a_0(0) = a_1(0) = \dots = a_{n-1}(0) = 0$; $a_n(\mu) \equiv -1$; $a_0(\mu) > 0$ for $\mu > 0$. It is straightforward to check that for every $\mu > 0$ the circumferences $C_h(\mu)$, $h \in \{1, \dots, k+1\}$, with the origin as center and radius $\sqrt{h\mu}$, are cycles of 2.4.1. Introduce now the function $V(x, y) = (x^2 + y^2)/2$ whose derivative along the solution of 2.4.1 is

$$\dot{V}(x, y) = (x^2 + y^2)[a_0 + a_1(x^2 + y^2) + \dots + a_n(x^2 + y^2)^n] .$$

Then it is easy to see that the origin of $\boldsymbol{R}^2$ is asymptotically stable for $\mu = 0$ and completely unstable for $\mu > 0$. Further, one gets

$$\dot{V}(r) \geqslant 0 \qquad \text{if and only if } 0 \leqslant r \leqslant \sqrt{(k+1)\mu} ,$$

$$\dot{V}(r) = 0 \qquad \text{if and only if } r \in \{0, \sqrt{\mu}, \sqrt{2\mu}, \dots, \sqrt{(k+1)\mu}\} ,$$

where $r = \sqrt{(x^2 + y^2)}$. It follows immediately that no other cycles of 2.4.1 exist except the circumferences $C_h(\mu)$ and that for every $\mu > 0$ there exists a unique set M'_μ satisfying conditions (a), (b) of Th. 2.2: the compact ring included between $C_1(\mu)$ and $C_{k+1}(\mu)$.

(ii) Suppose that the hypotheses of (i) hold and $n = 1$. Then the system 2.4.1 admits, for any $\mu > 0$, the unique cycle $C_1(\mu)$ which is attractive.

(iii) Suppose that, for any $\mu \geqslant 0$, $a_0(\mu), \ldots, a_n(\mu)$ are such that:

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = -(z - \mu)(z - 2\mu) \ldots (z - n\mu) .$$

It can be easily shown that the circumferences $C_h(\mu)$, $h \in \{1, 2, \ldots, n\}$, with the origin as center and radius $\sqrt{h\mu}$, are the only cycles of 2.4.1. For any $\mu > 0$ the set M'_μ will be the attractive cycle $C_1(\mu)$ or, if $n \geqslant 3$, any compact ring included between $C_1(\mu)$ and $C_j(\mu)$, $j \in \{3, 5, \ldots, n\}$.

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