The Invariance of Limit Sets for Retarded Differential Equations (*).

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Résumé. – *On démontre la semi-invariance ou la quasi-invariance des ensembles limites pour une équation différentielle à retard, sans supposer l'unicité ni la prolongeabilité des solutions, et sans supposer non plus que la solution engendrant l'ensemble limite soit contenue dans un ensemble compact ou fermé.*

1. – Introduction.

By way of introduction, let us consider the positive limit set $\Lambda^+(y)$ of a solution $y(t)$ of an autonomous differential equation $y' = f(y)$, whose second member is defined and continuous on some open subset Ω of R^n. The invariance of such a limit set has often been studied in a setting where one assumes the following:

(1) uniqueness of the solution through every initial point;

(2) continuability of every solution up to $-\infty$ to the left and $+\infty$ to the right;

(3) the solution $y(t)$ generating the limit set either is bounded, or remains in some compact, or at least closed, subset of Ω (Amongst many others, see e.g. J.P. LASALLE [1] and [2].

But it is also well known that the invariance property is true irrespective of these hypotheses. Of course, when $\Lambda^+(y)$ is considered as a subset no more of Ω but of its closure $\bar{\Omega}$ (i.e. when part of $\Lambda^+(y)$ may belong to the boundary of Ω), the invariant set is no more $\Lambda^+(y)$ but $\Lambda^+(y) \cap \Omega$ (cf. P. HARTMAN [3]). This observation is important because for many equations arising from various physical or technical fields, conditions (2) and (3) at least are not verified. For a conspicuous example of this fact, see J. L. CORNE and N. ROUCHE [4].

When the invariance, or some kind of pseudo-invariance of limit sets has been studied for situations more general than the one associated with autonomous ordinary differential equations, *part* or *all* of hypotheses (1) to (3) above have been

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retained, *mutatis mutandis*, by most authors, for instance J. P. LASALLE [5] (periodic ordinary differential equations), T. YOSHIZAWA [6] (asymptotically autonomous ordinary differential equations), R. K. MILLER [7] (asymptotically almost periodic ordinary differential equations), F. KAPPEL [8] (autonomous functional differential equations), F. KAPPEL [8] (autonomous functional differential equations), R. K. MILLER and G. R. SELL [9] (Volterra integral equations), M. A. CRUZ and J. K. HALE [10] (functional equations of neutral type), J. K. HALE [11] and M. R. HILDEBRANDO [12] (retarded differential equations). As an exception, i.e. a paper where none of conditions (1) to (3) has been used, cf. A. STRAUSS and J. A. YORKE [13] (asymptotically autonomous ordinary differential equations).

The same invariance problem has been studied for *processes*, a process being a kind of non autonomous generalization of a dynamical system: see C. M. DAFERMOS [14] as well as J. K. HALE, J. P. LASALLE and M. SLEMROD [15]. But (1) and (2) above are contained in the axioms of a process, and when a hypothesis like (3) has been discarded (C. M. Dafermos, loc. cit.), only positive invariance has been proved.

The only objective of this paper is to give some answer to the following question: for a retarded ordinary differential equation, what properties of pseudo-invariance is it possible to prove without assuming uniqueness or continuability, and with or without a boundedness condition? In particular, can we generalize to retarded differential equations the most elegant and seemingly fundamental property quoted above, saying that $\Lambda^+(y) \cap \Omega$ is invariant? It seems that the answer is no *in general*: to prove more than some kind of positive invariance for the unbounded case, we shall need a supplementary hypothesis, namely uniform continuity of the solution $y(t)$.

We think that, as a by-product of our study, some proofs have been simplified with respect to previous works in this field. As is expected, most of them rely heavily on Ascoli's theorem. Following the model given by P. HARTMAN, loc. cit., we have tried to reduce them to some straightforward application of a preliminary theorem on the regularity of solutions. On a study like this one, but regarding non autonomous Carathéodory differential equations, see N. ROUCHE [16].

2. – Notations and general hypotheses.

Most of our notations for retarded equations are standard (J. K. HALE [11]). Let us recall them rapidly. Let r be some real number, $r > 0$, and let $C = C([-r, 0], R^n)$ be the real linear vector space of continuous functions on $[-r, 0]$ into R^n. A norm in R^n will be chosen arbitrarily and symbolized by $\|\cdot\|$. There will be no possible confusion if we use the same notation for the norm of uniform convergence on C. Thus, if $\varphi \in C$ and θ designates the argument of φ, we have

$$\|\varphi\| = \sup_{\theta \in [-r,0]} \|\varphi(\theta)\| .$$

If, for a given $b > 0$, $x: [\sigma - r, \sigma + b] \to R^n$, $t \mapsto x(t)$ is a continuous function, then for every $t \in [\sigma, \sigma + b]$ we define the function $x_t \in C$ by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. If x is defined only on the half open interval $[\sigma - r, \sigma + b[$, the x_t will be defined alike for $t \in [g, \sigma + b[$.

Let $D = R \times \Omega$ where Ω is an open set of C, and let $\mathcal{F}$ be the class of continuous functions $f: D \to R^n$. For some $(t_0, \varphi_0) \in D$, we consider hereafter the initial value problem

$$\dot{x} = f(t, x_t)\,, \tag{1}$$

$$x_{t_0} = \varphi_0\,, \tag{2}$$

where the dot represents a derivative with respect to t. We adopt the usual definition for the solution of this problem. Further, we shall deal with a sequence of similar problems. If $(t_{0k}, \varphi_{0k}) \in D$ for $k = 1, 2, \ldots$, these problems read

$$\dot{x}^k = f_k(t, x_t^k)\,, \tag{3}$$

$$x_{t_0}^k = \varphi_{0k}\,. \tag{4}$$

We say that the functions f_k, $k = 1, 2, \ldots$, *take closed bounded subsets of D into bounded sets of R^n* uniformly with respect to k, if for every closed bounded subset $F \subset D$, there exists an $m > 0$ such that, for $k = 1, 2, \ldots$, and every $(t, \varphi) \in F$, one has $\|f_k(t, \varphi)\| < m$.

3. – Invariance of limit sets of bounded solutions.

The lemma to follow deals with a sequence of solutions corresponding to the sequence of problems (3), (4). All these solutions will be defined on one and the same compact time interval, all will remain in a single closed bounded subset of Ω.

LEMMA 1. – *In the general hypotheses above, assume that:*

(i) $\varphi_{0k} \to \varphi_0$ *as* $k \to \infty$; *for every* $(t, \varphi) \in D$ *and every sequence* $\{\varphi_k\}$ *such that* $\varphi_k \to \varphi$ *as* $k \to \infty$: $f_k(t, \varphi_k) \to f(t, \varphi)$ *as* $k \to \infty$;

(ii) *the functions f and f_k take closed bounded sets into bounded sets, uniformly with respect to k.*

For some $a \geqslant 0$ and $b > 0$, let $\{x^k\}$ be a sequence of solutions of problems (3), (4), *all defined on* $[t_0 - r - a, t_0 + b]$; *assume that*

(iii) *there exists a closed bounded set $M \subset \Omega$ such that for every k and every* $t \in [t_0 - a, t_0 + b]$: $x_t^k \in M$;

(iv) *the sequence* $\{x_{t_0-a}^k\}$ *is equi-continuous.*

Then

(*a*) *there exists a subsequence* $\{x^{k(i)}|: i = 1, 2, \dots\}$ *and a function* $x: [t_0 - r - a, t_0 + b] \to R^n$ *such that* $x^{k(i)}(t) \to x(t)$ *as* $i \to \infty$, *uniformly on* $[t_0 - r - a, t_0 + b]$;

(*b*) x *is a solution of problem* (1), (2);

(*c*) *if there exists no other solution of problem* (1), (2) *on* $[t_0 - r - a, t_0 + b]$, *then* $x^k(t) \to x(t)$ *as* $k \to \infty$, *uniformly on this interval.*

PROOF. – Due to hypothesis (iii), the x^k are uniformly bounded. Since $[t_0 - a, t_0 + b] \times M$ is closed and bounded, we know by (ii) and (3) that there exists an $m > 0$ such that $\|\dot{x}^k(t)\| \leq m$ for every k and $t \in [t_0 - a, t_0 + b]$. This shows the equicontinuity of the x^k restricted to $[t_0 - a, t_0 + b]$ and at last, because of (iv), the equicontinuity of the x^k themselves. Thesis (*a*) results therefore from Ascoli's theorem.

For ease of notation, let us now write $\{x^k\}$ for the subsequence of thesis (*a*), and x for its limit. One has for every k and $t \in [t_0 - a, t_0 + b]$ that

$$x^k(t) = \varphi_{0k}(0) + \int_{t_0}^{t} f_k(\tau, x_\tau^k)\, d\tau \,.$$

By hypotheses (ii) and (iii), the integrand is bounded, uniformly with respect to k. Therefore we may pass to the limit for $k \to \infty$, using the dominated convergence theorem. The equation obtained in this way, i.e.

$$x(t) = \varphi_0(0) + \int_{t_0}^{t} f(\tau, x_\tau)\, d\tau \,,$$

along with the fact that $x_{t_0} = \varphi_0$, proves thesis (*b*). At last, thesis (*c*) is obvious.

Some further preliminaries are needed before we can introduce the main definitions of semi- and quasi-invariance. First, the *translate* by a given amount $a > 0$ of a function $f \in \mathcal{F}$ is the function $f_a \in \mathcal{F}$ defined for every $(t, \varphi) \in D$ by $f_a(t, \varphi) = f(t + a, \varphi)$. The following hypotheses on f will be called upon repeatedly:

(*A*) *There exists an* $f^* \in \mathcal{F}$ *such that for every* $(t, \varphi) \in D$

$$f_a(t, \psi) \to f^*(t, \varphi)$$

as $a \to \infty$ *and* $\psi \to \varphi$.

(*B*) *For every sequence* $\{a_k\}$ *tending to* ∞, *there exists a function* $f^* \in \mathcal{F}$ *and a subsequence* $\{a_{k(i)}\}$ *such that for every* $(t, \varphi) \in D$ *and every sequence* $\{\varphi_k\}$ *tending to* φ:

$$f_{a_{k(i)}}(t, \varphi_{k(i)}) \to f^*(t, \varphi) \qquad \text{as } i \to \infty \,.$$

A function of the type of f^* in hypotheses (A) and (B) will be called a *limit function*. The corresponding equation $\dot{x} = f^*(t, x_t)$ will be called a *limit equation*. The following observations are important:

1) If f possesses property (A), then for any $\tau > 0$ and every $(t, \varphi) \in D$

$$f_{a+\tau}(t, \psi) \to f^*(t, \varphi)$$

but also

$$f_a(t+\tau, \psi) \to f^*(t+\tau, \varphi)$$

as $a \to \infty$ and $\psi \to \varphi$. Therefore $f^*(t+\varphi) = f^*(t+\tau, \varphi)$ and f^* doesn't actually vary with t. In this sense, property (A) characterizes *asymptotically autonomous equations.*

2) Suppose $f(t, \varphi)$ possesses property (B). One might wonder what kind of function of t it is for fixed φ. But property (B) implies that for every sequence $\{a_k\}$ there exists a function $f^* \in \mathcal{F}$ and a subsequence $\{a_{k(i)}\}$ such that for every $(t, \varphi) \in D$

$$f_{a_{k(i)}}(t, \varphi) \to f^*(t, \varphi) \qquad \text{as } i \to \infty .$$

Consider the case where this convergence of the $f_{a_{k(i)}}$ towards f^* is, for any $\tau \in R$, uniform on $[\tau, \infty[$. One knows then (cf. M. FRÉCHET [17]) that $f(t, \varphi)$ is the sum of two functions $g(t, \varphi)$ and $h(t, \varphi)$ such that, for fixed φ, g is almost periodic in the sense of Bohr and $h(t, \varphi) \to 0$ as $t \to \infty$. In this case, a function possessing property (B) is *asymptotically almost periodic*. It would probably be interesting to characterize further the class of functions with property (B).

(3) (A) implies (B).

(4) $\mathcal{F}$ being a linear vector space for the usual sum of functions and product of a function by a real scalar, the subset of functions of $\mathcal{F}$ verifying hypothesis (A) is a vector subspace of $\mathcal{F}$. A similar property holds true for (B).

A subset $F \subset \Omega$ will be said to be *semi-invariant* with respect to equation (1), whose second member is supposed to possess property (A) if, for every $(t_0, \varphi_0) \in R \times F$, there exist $\alpha < t_0 - r$, $\omega > t_0$ and a non continuable solution $x\colon]\alpha, \omega[\to R^n$ of the Cauchy problem $\dot{x} = f^*(t, x_t)$, $x_{t_0} = \varphi_0$, such that, for every $t \in]\alpha + r, \omega[: x_t \in F$.

A subset $F \subset \Omega$ will be said to be *quasi-invariant* with respect to equation (1), whose second member is supposed to possess property (B) if, for every $(t_0, \varphi_0) \in R \times F$, there exist a limit function of the type mentioned in hypothesis (B), two quantities $\alpha < t_0 - r$, $\omega > t_0$ and a noncontinuable solution $x\colon]\alpha, \omega[\to R^n$ of the Cauchy problem $\dot{x} = f^*(t, x)$, $x_{t_0} = \varphi_0$, such that, for every $t \in]\alpha + r, \omega[: x_t \in F$.

In this section, where we study bounded solutions only, we shall always have $\alpha = -\infty$ and $\omega = \infty$. The definition of a semi-invariant set becomes that of an *invariant* one in the well known sense, in case equation (1) is autonomous and it is specified further that *all* solutions of the Cauchy problem $\dot{x} = f^*(t, x_t)$, $x_{t_0} = \varphi_0$

remain in F. This restriction is of course superfluous if uniqueness of solutions is assumed throughout D.

Let us at last recall that if $x: [t_0 - r, \infty[$ is a solution of problem (1), (2), its *positive limit set* (or, in this context, its *limit set*), written $\Lambda^+(x)$, is the set of points $\psi \in \bar{\Omega}$ for each of which there is a sequence $\{t_k\} \subset [t_0, \infty[$, such that $t_k \to \infty$ and $x_{t_k} \to \psi$ as $k \to \infty$.

THEOREM 1. – *Assume that*

(i) *f verifies hypothesis* (A);

(ii) *for every sequence* $\{a_k\} \subset [0, \infty[$, $a_k \to \infty$, *the functions f and f_{a_k} take closed bounded sets into bounded sets, uniformly with respect to k; let $x: [t_0 - r, \infty[\to R^n$ be a solution of problem* (1), (2) *and assume further that*

(iii) *for some closed bounded set $M \subset \Omega$ and all $t \in [t_0, \infty[: x_t \in M$; then $\Lambda^+(x)$ is semi-invariant.*

PROOF. – Consider any $\varphi_0^* \in \Lambda^+(x)$ and $t_0^* \in R$. Let $\{t_i\} \subset [t_0, \infty[$ be a sequence such that $t_i \to \infty$ and $x_{t_i} \to \varphi_0^*$ as $i \to \infty$. We shall write $x_{t_i} = \varphi_{0i}$ and suppose, without loss of generality, that $t_i - t_0^* > 0$ for every i. Let f^* be the limit function mentioned in hypothesis (A). Thus, for every $(t, \varphi) \in D$ and every sequence $\{\varphi_i\}$ such that $\varphi_i \to \varphi$,

$$f_{t_i - t_0^*}(t, \varphi_i) \to f^*(\varphi) \qquad \text{as } i \to \infty .$$

Putting $x^i(t) = x(t + t_i - t_0^*)$, we observe that $x^i(t)$ is a solution of the Cauchy problem

$$\dot{x}^i = f_{t_i - t_0^*}(t, x_i^t) , \tag{5}$$

$$x_{t_0^*}^i = x_{t_i} . \tag{6}$$

Let a and b be two arbitrary numbers, $a > 0$, $b > 0$. For i sufficiently large, the solution just mentioned of problem (5), (6) is defined on $[t_0^* - r - a, t_0^* + b]$. The hypotheses of lemma 1 are verified for the sequence of second members $f_{t_i - t_0^*}(t, x_t)$, the sequence of initial points $x_{t_i} = \varphi_{0i}$ and the sequence of solutions $x^i(t)$. In particular, as it can be supposed that the solutions $x^i(t)$ are defined far enough in the left direction, hypothesis (iv) of lemma 1 is but a consequence of hypothesis (ii) above. Of course, the final sequence $x^i(t)$ has to begin with some i large enough. We conclude that there exists on $[t_0^* - r - a, t_0^* + b]$ a solution $x^*(t)$ of the problem $\dot{x}^* = f^*(t, x_t^*)$, $x_{t_0}^* = \varphi_0^*$. And since the $x^i(t)$ converge uniformly towards $x^*(t)$ on $[t_t^* - r - a, t_0^* + b]$, every x_0^* belongs to $\Lambda^+(x)$ for $t \in [t_0^* - a, t_0^* + b]$. The thesis of the theorem results from the fact that a and b have been chosen arbitrarily.

THEOREM 2. – *If one replaces, in theorem* 1, *property* (A) *by property* (B), *then $\Lambda^+(x)$ is quasi-invariant.*

The proof is similar to that of theorem 1.

4. – A regularity theorem.

The regularity theorem to follow will prove helpful in our attempt, postponed until next section, to get rid of hypothesis (iii) of theorem 1. Let us first introduce some preliminaries. Consider, for $n = 1, 2, \ldots$, the sets

$$\Psi_n = \left[\Omega \setminus B\left(\partial\Omega, \frac{1}{n}\right)\right] \cap \overline{B(0, n)} ,$$

where $\partial\Omega$ is the boundary of Ω, $B(\partial\Omega, 1/n) = \{\varphi \in C : d(\varphi, \partial\Omega) < 1/n\}$ representing the distance from a point to a set in C. Further, consider the sets $\Theta_n = [-n, +n] \times \times \Psi_n \subset D$. Their union is equal to D.

LEMMA 2. – *For every $n = 1, 2, \ldots$, there exist $b > 0$ and $\varrho > 0$ such that, for every $(t_0, \varphi_0) \in \Theta_n$:*

(*a*) *the cylinder $T = [t_0, t_0 + b] \times \{\varphi : \|\varphi - \varphi_0\| \leqslant \varrho\}$ is contained in Θ_{n+1}; if*

(i) $\{\varphi_{0k}\} \subset \Theta_n$, $\varphi_{0k} \to \varphi_0$ as $k \to \infty$;

(ii) *the f_k take closed bounded sets into bounded sets, uniformly with respect to k; then*

(*b*) *for k large enough, all solutions of problems* (3), (4) *exist on $[t_0 - r, t_0 + b]$ and $(t, x_t^k) \in T$ for $t \in [t_0, t_0 + b]$.*

PROOF. – Thesis (*a*) is obvious. In order to prove that (t, x_t^k) remains in T, we shall need the following expressions for $x_t^k(\tau) = x^k(t + \tau)$:

$$x^k(t + \tau) = \varphi_{0k}(0) + \int_{t_0}^{t+\tau} f_k(\sigma, x_\sigma^k)\, d\sigma \qquad t_0 - t \leqslant \tau \leqslant 0 ,$$

$$= \varphi_{0k}(\tau + t - t_0) \qquad -r \leqslant \tau \leqslant t_0 - t .$$

Let m be a bound on the f_k, associated with Θ_{n+1}. If necessary, reduce the value of b in order that

$$b < \min\left(r, \frac{\varrho}{3m}\right). \tag{7}$$

Also choose b small enough in order that

$$(|\tau_1 - \tau_2| < b) \Rightarrow (\|\varphi_0(\tau_1) - \varphi_0(\tau_2)\|) < \frac{\varrho}{3} , \tag{8}$$

which is possible, as φ_0 in uniformly continuous. Then choose k large enough to get

$$\|\varphi_0 - \varphi_{0k}\| < \frac{\varrho}{3} .$$

As long as (t, x_t^k) is in T, one has

$$\sup_{t_0-t\leqslant\tau\leqslant 0}\|\varphi_0(\tau)-x_t^k(\tau)\|\leqslant\sup_{-b\leqslant\tau\leqslant 0}\|\varphi_0(\tau)-\varphi_0(0)\|+\|\varphi_0(0)-\varphi_{0k}(0)\|+\int_{t_0}^{t_0+b}\|f_k(\sigma, x_\sigma^k)\|\,d\sigma<\varrho\,.$$

Further

$$\sup_{-r\leqslant\tau\leqslant t_0-t}\|\varphi_0(\tau)-x_t^k(\tau)\|\leqslant$$
$$\leqslant\sup_{-r\leqslant\tau\leqslant t_0-t}\{\|\varphi_0(\tau)-\varphi_0(\tau+t-t_0)\|+\|\varphi_0(\tau+t-t_0)-\varphi_{0k}(\tau+t-t_0)\|\}<\varrho\,.$$

Therefore (t, x_t^k) cannot come out of T for $t\in[t_0, t_0+b]$.

THEOREM 3. – *In the general hypotheses of section 2, assume that:*

(i) $\varphi_{0k}\to\varphi_0$ *as* $k\to\infty$; *for every* $(t,\varphi)\in D$ *and every sequence* $\{\varphi_k\}$ *such that* $\varphi_k\to\varphi$ *as* $k\to\infty$: $f_k(t,\varphi_k)\to f(t,\varphi)$ *as* $k\to\infty$;

(ii) *the functions* f *and* f_k *take closed bounded sets into bounded sets, uniformly with respect to* k.

Let $x^k\colon [t_0-r, \infty[\;\to R^n$, $k=1,2,\dots$ *be a sequence of solutions of problems* (3), (4). *Then there exists an* $\omega>t_0$, *a non-continuable solution* $x\colon[t_0-r,\omega[\;\to R^n$ *of problem* (1), (2) *and a subsequence* $x^{k(i)}$, $i=1,2,\dots$, *such that, for every time-value* t_1 *with* $t_0<t_1<\omega$, $x^{k(i)}(t)\to x(t)$ *as* $i\to\infty$, *uniformly on* $[t_0-r, t_1]$.

PROOF. – Let $\{\Theta_n\}$ be the sequence of sets considered in the lemma. As $(t_0,\varphi_0)\in D$, there is an integer n_0 such that $(t_0,\varphi_0)\in\Theta_{n_0}$. Let b_1 and ϱ_1 be the length and radius of the cylinder associated with Θ_{n_0+1} in thesis (a) of lemma 2. This lemma, along with lemma 1 proves the existence of a solution $x(t)$ of (1), (2) on $[t_0-r, t_0+b_1]$ and of a subsequence of the x^k, again written $\{x^k\}$, such that $x^k(t)\to x(t)$ as $k\to\infty$, uniformly on $[t_0-r, t_0+b_1]$. Either $(t_0+b_1, x(t_0+b_1))$ belongs to Θ_{n_0}, or it does not. If it does, we start from this point as a new initial point to prove, by the same argument, the existence of a new subsequence, again written $\{x^k\}$, with the same convergence property, but this time on $[t_0-r, t_0+2b_1]$. Repeating this process proves either the existence of a subsequence $\{x^k\}$ converging to $x(t)$ uniformly on every finite subinterval of $[t_0-r,\infty[$, or allows one to reach a point outside Θ_{n_0}. But this point will be in Θ_{n_0+2}, and we can repeat in this set what we have done previously, choosing of course new dimensions b_2 and ϱ_2 for the cylinder. The rest of the proof is obvious.

REMARK. – It is possible, without substantial modification, to prove a more general theorem: for instance D would be an arbitrary open set of $R\times C$, variations of the initial time t_0 would be considered (a sequence $t_{0k}\to t_0$ as $k\to\infty$) and the x^k would no more necessarily be continuable up to $+\infty$. We refrain here from considering such generalizations, because they are not needed to study the invariance of limit sets, and further, up to small variations, similar theorems exist, for instance the one given by J. K. HALE [11]. Here, we could not content ourselves with merely quoting Hale's theorem, because it assumes uniqueness of solutions. Further, the proof presented here is simpler than the usual one: it is a straightforward application of Ascoli's theorem and requires no use of Schauder's fixed point theorem.

5. – Invariance of limit sets of unbounded solutions.

We can now easily prove some positive invariance properties of the limit set for a solution which is no more confined in the future to some closed bounded subset of Ω.

A subset $F \subset \Omega$ will be said to be *positively semi-invariant* with respect to equation (1), whose second member is supposed to possess property (A), if, for every $(t_0, \varphi_0) \in R \times F$, there exists a solution $x: [t_0 - r, \omega[\to R^n$ of the Cauchy problem $\dot{x} = f^*(t, x_t)$, $x_{t_0} = \varphi_0$ such that it is not continuable to the right and for every $t \in [t_0, \omega[: x_t \in F$.

A subset $F \subset \Omega$ will be said to be *positively quasi-invariant* with respect to equation (1), whose second member is supposed to possess property (B) if, for every $(t_0, \varphi_0) \in R \times F$, there exist a limit function f^* of the type mentioned in hypothesis (B) and a solution $x: [t_0 - r, \omega[\to R^n$ of the Cauchy problem $\dot{x} = f^*(t, x_t)$, $x_{t_0} = \varphi_0$ such that, for every $t \in R: x_t \in F$.

THEOREM 4. – *Assume that*

(i) *f verifies hypothesis* (A);

(ii) *for every sequence* $\{a_k\} \subset [0, \infty[$, $a_k \to \infty$, *the functions f and f_{a_k} take closed bounded sets into bounded sets, uniformly with respect to k if*; $x: [t_0 - r, \infty[\to R^n$ *is a solution of problem* (1), (2), $\Lambda^+(x) \cap \Omega$ *is positively semi-invariant.*

PROOF. – The proof runs like that of theorem 1 up to equations (5) and (6). Then one observes that the hypotheses of theorem 3 are verified for the sequence of second members $f_{t_i - t_0^*}(t, x_t)$, the sequence of initial points $x_{t_i} = \varphi_{0i}$ and the sequence of solutions $x^i(t)$. The conclusion follows from theorem 3 in the same way that of theorem 1 followed from lemma 1.

THEOREM 5. – *If one replaces, in theorem* 4, *property* (A) *by property* (B), *then* $\Lambda^+(x) \cap \Omega$ *is positively quasi-invariant.*

Let us now try to answer a last question: can we find some further hypothesis enabling us to convert the conclusions of theorems 4 and 5 from positive semi- or quasi-invariance into semi- or quasi-invariance? To achieve this, we have to go back to our treatment of the regularity in section 4 and introduce some substantial changes.

LEMMA 3. – *If the sets Θ_n are defined as in section 4, then for every $n=1,2,\ldots$, there exist $b>0$ and $\varrho>0$ such that, for every $(t_0,\varphi_0)\in\Theta_n$:*

(*a*) *the cylinder $T\colon [t_0-b, t_0+b]\times\{\varphi\colon \|\varphi-\varphi_0\|\leqslant\varrho\}$ is contained in Θ_{n+1}; further, if*

(i) $\{\varphi_{0k}\}\subset\Theta_n$, $\varphi_{0k}\to\varphi_0$ as $k\to\infty$;

(ii) *for some $a>0$ and for k large enough, the solutions of problems* (3), (4) *exist and are uniformly equi-continuous on $[t_0-r-a, t_0+a]$; then*

(*b*) *for k large enough, $(t,x_t^k)\in T$ for $t\in[t_0-b,t_0+b]$.*

PROOF. – Thesis (*a*) is obvious. Let us suppose, without loss of generality, that all the x^k exist on $[t_0-r-b, t_0+b]$. Choose b small enough, in order that for any k and $t_1,t_2\in[t_0-r-b,t_0+b]$

$$(|t_1-t_2|<2b)\Rightarrow\left(\|x^k(t_1)-x^k(t_2)\|<\frac{\varrho}{2}\right),$$

which is possible using hypothesis (ii). Then of course

$$\|x^k_{t_0-b}(\tau)-x^k_t(\tau)\|<\frac{\varrho}{2}$$

for $t_0-b\leqslant t\leqslant t_0+b$ and $-r\leqslant\tau\leqslant 0$, and therefore

$$\|x^k_{t_0-b}-x^k_t\|<\frac{\varrho}{2}\qquad \text{for } t_0-b\leqslant t\leqslant t_0+b\ .$$

If one chooses k large enough to get $\|\varphi_0-\varphi_{0k}\|<\varrho/2$ then $(t,x_t)\in T$ for $t\in[t_0-b,t_0+b]$

THEOREM 6. – *In the general hypotheses of section 2, assume that:*

(i) *$\varphi_{0k}\to\varphi_0$ as $k\to\infty$; for every $(t,\varphi)\in D$ and every sequence $\{\varphi_k\}$ such that $\varphi_k\to\varphi$ as $k\to\infty$: $f_k(t,\varphi_k)\to f(t,\varphi)$ as $k\to\infty$;*

(ii) *the functions f and f_k take closed bounded sets into bounded sets uniformly with respect to k.*

Let $x^k\colon [t_0-r-a_k, \infty[\,\to R^n, k=1,2,\ldots$, be a sequence of solutions of problems (3), (4) *such that $a_k\to\infty$ as $k\to\infty$ and the x^k are uniformly equi-continuous. Then there exist $\alpha<t_0-r$, $\omega>t_0$, a non continuable solution $x\colon\,]\alpha,\omega[\,\to R^n$ of problem* (1), (2) *and a subsequence $x^{k(i)}$, $i=1,2,\ldots$, such that, for every time values t_1, t_2 with $\alpha<t_1<$ $<t_2<\omega$, $x^{k(i)}(t)\to x(t)$ as $i\to\infty$, uniformly on $[t_1,t_2]$.*

PROOF. – The proof is similar to that of theorem 3, but here continuation is possible to the left, as well as to the right, using lemmas 1 and 3.

Now we get the following theorems on semi- and quasi-invariance of limit sets for unbounded solutions.

THEOREM 7. – *If one adds to the hypotheses of theorem* 4 *that* x *is uniformly continuous, then* $\Lambda^+(x) \cap \Omega$ *is semi-invariant.*

PROOF. – This is due to the fact that, if x is uniformly continuous, the sequence $x^i(t) = x(t - t_i - t_0^*)$ is uniformly equi-continuous.

If f is bounded, then of course every solution of problem (1), (2) is uniformly continuous. Naturally, one also gets the following theorem on quasi-invariance.

THEOREM 8. – *If one adds to the hypotheses of theorem* 5 *that* x *is uniformly continuous, then* $\Lambda^+(x) \cap \Omega$ *is quasi-invariant.*

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BIBLIOGRAPHY

[1] J.-P. LASALLE, *Some extensions of Liapunov's second method*, IRE Trans. Circuit Theory, CT-7 (1960), pp. 520-527.
[2] J.-P. LASALLE, *Stability theory for ordinary differential equations*, J. Diff. Equ., **4** (1968), pp. 57-65.
[3] P. HARTMAN, *Ordinary differential equations*, J. Wiley and Sons, New York (1964).
[4] J. L. CORNE - N. ROUCHE, *Attractivity of closed sets proved by using a family of Liapunov functions*, J. Diff. Equ., **13** (1973), pp. 231-246.
[5] J.-P. LASALLE, *Asymptotic stability criteria*, Proc. Symp. Appl. Math., **13**, pp. 299-307, Amer. Math. Soc., Providence, R. I. (1962).
[6] T. YOSHIZAWA, *Asymptotic behavior of solutions of nonlinear differential equations*, Contr. Diff. Equ., **1** (1963), pp. 371-387.
[7] R. K. MILLER, *Asymptotic behavior of solutions of nonlinear differential equations*, Trans. Amer. Math. Soc., **115** (1965), pp. 400-416.
[8] F. KAPPEL, *The invariance of limit sets for autonomous functional differential equations*, SIAM J. Appl. Math., **19** (1970), pp. 408-419.
[9] R. K. MILLER - G. R. SELL, *Volterra integral equations and topological dynamics*, Memoir of Amer. Math. Soc., **102**, Providence, R. I. (1970).
[10] M. A. CRUZ - J. K. HALE, *Stability of functional differential equations of neutral type*, J. Diff. Equ., **7** (1970), pp. 334-355.
[11] J. K. HALE, *Functional differential equations*, Springer-Verlag, New York (1971).
[12] M. R. HILDEBRANDO, *Invariança para sistemas autônomos de equaçoes differenciais com retardamento e applicaçoes*, Master's Thesis, São Carlos (1970).
[13] A. STRAUSS - J. A. YORKE, *On asymptotically autonomous differential equations*, Math. Systems Theor., **1** (1967), pp. 175-182.
[14] C. M. DAFERMOS, *An invariance principle for compact processes*, J. Diff. Equ., **9** (1971), pp. 239-252.
[15] J. K. HALE - J. P. LASALLE - M. SLEMROD, *Theory of a general class of dissipative processes*, J. Math. Analysis Appl., **39** (1972), pp. 177-191.
[16] N. ROUCHE, *The invariance principle applied to non-compact limit sets*, to appear in Boll. Un. Mat. Ital.
[17] M. FRÉCHET, *Les fonctions asymptotiquement presque périodiques*, Rev. Sci., **79** (1941), pp. 341-357.