Abstract. - We give here an existence result of minimizers for a class of one dimensional integrals of the Calculus of Variations with non convex, non coercive integrands.

1. - Introduction and main result.

Let us consider the functional

\[ F(u) = \int_0^1 f(x, u(x), u'(x)) \, dx \]

defined in the class \( \mathcal{W}_p = \{ u \in W^{1,p}(0, 1): u(0) = 0, u(1) = \lambda, u' \geq 0 \, \text{a.e.} \} \) with \( \lambda \in \mathbb{R}_+ \) and \( p \geq 1 \). The integrand \( f = f(x, s, \xi) \) is not assumed to be neither coercive nor convex with respect to \( \xi \). The closure of \( \mathcal{W}_p \) in the (either strong or weak) topology of \( W^{1,p}_{\text{loc}}(0, 1) \) is given by

\[ \overline{\mathcal{W}}_p = \{ u \in W^{1,p}_{\text{loc}}(0, 1): u(0) \geq 0, u(1) \leq \lambda, u' \geq 0 \, \text{a.e.} \}, \]

where the values \( u(0) \) and \( u(1) \) are defined by

\[ u(0) = \inf_{x \in (0, 1)} u(x), \quad u(1) = \sup_{x \in (0, 1)} u(x). \]

The extension of \( F \) «by lower semicontinuity» from \( \mathcal{W}_p \) to \( \overline{\mathcal{W}}_p \) is the functional \( \overline{F} \) defined for \( u \in \overline{\mathcal{W}}_p \) by

\[ \overline{F}(u) = \inf_{\{u_k\}} \left\{ \liminf_{k \to \infty} F(u_k): \{u_k\} \subset \mathcal{W}_p, \ u_k \rightharpoonup^w u \right\}. \]

Let us precise the hypotheses on the integrand function \( f \):

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Indirizzo degli AA.: Dipartimento di Matematica Pura e Applicata, Facoltà di Scienze, Università dell'Aquila, 67100 L'Aquila, Italia.
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Authors are members of G.N.A.F.A., C.N.R.
A1) \( f \) is a Carathéodory function on \([0, 1] \times R \times R\);
A2) there exist \( K \geq 0 \), a convex function \( h = h(\xi) \) and continuous functions \( a = a(x, s) \) and \( b = b(x, s) \) such that for every \( x \in [0, 1], s \in R, \xi \in R \)
   i) \( a(x, s) h(\xi) - K \leq f(x, s, \xi) \leq a(x, s) h(\xi) + b(x, s) \),
   ii) \( |\xi| \leq h(\xi) \leq L(1 + |\xi|^p), L \in R_+ \),
   iii) \( a(x, s) \geq 0 \).

Then the function \( f^{**} \) (which is the greatest function convex with respect to \( \xi \) and less than or equal to \( f \)) satisfies the same assumptions and the lower semicontinuous extension of

\[
G(u) = \int_0^1 f^{**}(x, u(x), u'(x)) \, dx, \quad u \in \overline{\mathcal{V}_p}
\]

to \( \overline{\mathcal{V}_p} \), can be represented as

\[
G(u) = \int_0^1 f^{**}(x, u(x), u'(x)) \, dx + \int_0^1 a(0, s) \, ds + \int_0^1 a(1, s) \, ds,
\]

where, for simplicity, we set \( \bar{h} = h_+ = h_- \),

\[
\bar{h}_\pm = \lim_{\xi \to \pm} \frac{h(\xi)}{\xi},
\]

(see [B.-M., Theorem 2.4]).

We are interested in the existence of solutions, for the following problem:

\[
\min \left\{ \int_0^1 f(x, u(x), u'(x)) \, dx + \int_0^1 a(0, s) \, ds + \int_0^1 a(1, s) \, ds \right\}, \quad u \in \overline{\mathcal{V}_p}
\]

where \( \overline{\mathcal{V}_p} \) is defined by (1.2).

Usually existence for a non convex problem is achieved in two steps: find a minimizer \( u_0 \in \overline{\mathcal{V}_p} \) of the relaxed functional (here (1.3)) and then prove that, for such \( u_0 \), \( f(x, u_0(x), u_0'(x)) = f^{**}(x, u_0(x), u_0'(x)) \) a.e. in \( \Omega \).

Therefore we need assumptions on \( f^{**} \) in order to prove existence of minima of the functional (1.3) in the class (1.2):

B1) \( f^{**} \) admits continuous partial derivatives

\[
f_{\xi \xi}^{**}, f_{\xi \xi}^{**}, f_{\xi \xi \xi}^{**}, f_{\xi \xi \xi \xi}^{**};
\]

B2) there exist an exponent \( p \geq 1 \) and a function \( M: R_+ \times R_+ \to R_+ \) such that for \( \varepsilon, r > 0 \),

\[
|f_{\xi \xi}^{**}(x, s, \xi)| \leq M(\varepsilon, r)(1 + |\xi|^p), \quad \forall (x, s, \xi) \in [\varepsilon, 1 - \varepsilon] \times [-r, r] \times R.
\]
REMARK 1. - The assumptions $B_2$ implies that the functions defined by

\[(1.5) \quad \varphi = \varphi(x, s, \xi) = f^{**} - f_{s}^{**} - f_{\xi}^{**},\]

\[(1.6) \quad \psi = \psi(x, s, \xi) = \varphi_x + \varphi_s \xi,\]

are continuous.

Existence results for problem (1.4), when the integrand $f$ is convex, are proved in [B.-M.]. Here the authors consider both the cases where $\varphi = f_s - f_{s\xi} - f_{\xi\xi}$ has a definite sign or it changes its sign.

The non convex case is considered in [M.2] under the assumption that the function $\varphi$ given by (1.5) has a definite sign.

Our aim in this paper is to prove (see the theorem below) an existence result for (1.4) in the non convex case when $\varphi$ changes its sign.

This framework could be a general approach to prove existence of a minimizer for the following non convex functional related to the problem of cavitation in non linear elasticity:

\[(1.7) \quad \text{Min} \left\{ \int_0^1 r^{n-1} \Phi \left[ \frac{v}{r}, v' \right] dr + \bar{h} \frac{[v(0)]^n}{n} : v \in W_{\text{loc}}^{1,p}(0, 1); v \geq 0, v(1) = 0, v' \geq 0 \text{ a.e.} \right\},\]

where the energy $\Phi = \Phi(\gamma, \xi)$, satisfies assumptions of type $A_1, A_2, B_2$ and $\bar{h}$ is defined as above. The functional to minimize in problem (1.7) has first been considered by P. MARCELLINI in [M.1].

Up to now, no existence result seems to be applicable to problem (1.7), when $\Phi$ is not convex with respect to $\xi$.

The problem of cavitation has been first studied by J. BALL in [B.1] and [B.2].

More exhaustive references on the subject can be found in [M.2].

We now state our main theorem.

**THEOREM.** - Assume that $f(x, s, \xi)$ satisfies $A_1, A_2, B_1$ and $B_2$ and that the functions $\varphi$ and $\psi$ defined in (1.5) and (1.6) satisfy the following assumption

\[C) \quad \begin{align*}
&\text{i)} \quad \varphi(x, s, 0) \equiv 0 \quad \forall x, s, \\
&\text{ii)} \quad \varphi(x, s, \xi) = 0, \xi \neq 0 \Rightarrow \xi \varphi(x, s, \xi) > 0.
\end{align*}\]

Then the variational problem (1.4) has a solution $u_0$ which belongs to $W_{\text{loc}}^{1,\infty}(0, 1)$ and satisfies the following estimate

\[(1.7) \quad |u_0(x)| \leq 4 \frac{A}{\varepsilon} \quad \forall x \in [\delta, 1 - \delta], \quad \varepsilon \in \left[0, \frac{1}{2}\right].\]
REMARK 2. - Let us point out that, if $h = + \infty$ and $a(0, s), a(1, s)$ are almost everywhere positive, the minimum $u_0$ satisfies $u_0(0) = 0, u_0(1) = \lambda$ and therefore $u_0$ is also a minimum of the functional $F$ defined by (1.1) in the class $W_p$.

Moreover our theorem also looks at the case where $f = f(x, s, \xi)$ grows at most linearly when $|\xi| \to \infty$.

REMARK 3. - The existence result in [B.-M.] is related to a convex integrand $f$ such that $\varphi = f_\epsilon - f_{\varphi\xi} - f_{\varphi\xi} \xi$ changes its sign according to the following assumption:

$\forall \xi \in [0, 1/2]$ and $r > 0$ there exists $k_0 = k_0(\varepsilon, r) > 0$ such that for every $(x, s, \xi)$ belonging to $[\varepsilon, 1 - \varepsilon] \times [-r, r] \times R$ with $|\xi| > k_0$, if $\varphi(x, s, \xi) = 0$ then $\xi \varphi(x, s, \xi) > 0$.

The plan of the paper is the following: in Section 2 we define approximating problems which are convex and coercive and we prove some properties of their solutions.

In Section 3 we prove some geometrical properties (concavity-convexity properties) of the approximating solutions defined in Section 2 and a priori estimates.

Finally, in Section 4, we prove the main theorem.

2. - Approximating solutions, monotonicity properties.

In this section a double approximating scheme is introduced in order to obtain smooth convex and coercive integrand functions. We consider

\begin{equation}
\tag{2.1}
g_{\varepsilon k}(x, s, \xi) = \alpha_k s \ast f_{**}(x, s, \xi) + \varepsilon (1 + |\xi|^2)^{q/2} + k(\xi^-)^q
\end{equation}

where $q \geq \max \{p, 4\}, \alpha = \alpha(\varepsilon)$ is a positive mollifier with compact support in $[-1, 1]$, $\alpha_k(\varepsilon) = k\alpha(k\varepsilon)$ and $\xi^- = -\min \{\xi, 0\}$.

The variational problem

\begin{equation}
\tag{2.2}
\min \left\{ G_{\varepsilon k}(u) = \frac{1}{0 \leq} g_{\varepsilon k}(x, u(x), u'(x)) \right\} \text{\text{d}}x: u \in W^{1, q}(0, 1), u(0) = 0, u(1) = \lambda
\end{equation}

related to the convex and coercive integral $G_{\varepsilon k}(u)$ admits a solution $u_{\varepsilon k}(x)$ which satisfies the properties stated in the following lemma.

**Lemma 2.1.** - For $\varepsilon \in [0, 1]$ and $k > 0$, $u_{\varepsilon k} \in C^2[0, 1]$ and satisfies

\begin{equation}
\tag{2.3}
\frac{d}{dx} [g_{\varepsilon k}(x, u_{\varepsilon k}, u_{\varepsilon k}')] = g_{\varepsilon k}(x, u_{\varepsilon k}, u_{\varepsilon k}').
\end{equation}

Moreover, for fixed $\varepsilon \in [0, 1], \|u_{\varepsilon k}\|_{L^\infty(0, 1)}$ is bounded uniformly with respect to $k$. 
PROOF. - A classical argument due to Morrey (see Th. 1.10.1 in [Mo.]) provides solutions $u_{ek} \in C^3[0, 1]$ of the Euler's equation (2.3). The uniform $C^1$ bound of $u_{ek}$ is obtained following the outline of the proof of Lemma 5.7 in [M.2].

**LEMMA 2.2.** The sequence $\{u_{ek}\}_{k \in \mathbb{N}}$, for fixed $\varepsilon$, is relatively compact in the weak topology of $W^{1, q}(0, 1)$: up to a subsequence, $\{u_{ek}\}_{k \in \mathbb{N}}$ weakly converges to a solution $u_\varepsilon$ of the following minimum problem:

$$
\text{Min} \ \left\{ \int_0^1 g^\varepsilon(x, u(x), u'(x)) \, dx : u \in W^{1, q}(0, 1), u(0) = 0, u(1) = \lambda, u' > 0 \ \text{a.e.} \right\}
$$

where $g^\varepsilon(x, s, \xi) = f^{**}(x, s, \xi) + \varepsilon(1 + |\xi|^2)^{q/2}$.

**PROOF.** Let us begin by proving that the sequence $\{u_{ek}\}_{k \in \mathbb{N}}$ is bounded in the $W^{1, q}(0, 1)$-norm uniformly with respect to $k$.

Since $u_{ek}$ solves the problem (2.2), for $v = \varepsilon x$, $v' \in (0, 1)$, we get

$$
G_{ek}(u_{ek}) < - G_{ek}(v) < C_1
$$

where $C_1$ is a positive constant independent of $\varepsilon \in [0, 1]$ and $k \in \mathbb{N}$.

By the growth condition on $f$ (see i) in $A_2$) and the definition of $f^{**}$, we get

$$
\varepsilon \|u_{ek}'\|_{L^q(0, 1)} + k \|u_{ek}'\|_{L^q(0, 1)} - K \leq G_{ek}(u_{ek}) \leq C_1.
$$

This proves the boundedness of the sequence $\{u_{ek}\}_{k \in \mathbb{N}}$ in $W^{1, q}(0, 1)$. Then there exists $u_\varepsilon \in W^{1, q}(0, 1)$ which is the weak limit in $W^{1, q}(0, 1)$ of $\{u_{ek}\}_{k \in \mathbb{N}}$ (up to a subsequence).

By (2.5), since $k \|u_{ek}'\|_{L^q(0, 1)}$ is bounded for each $k \in \mathbb{N}$, then the negative part of $u_{ek}'$ converges strongly to zero in $L^q(0, 1)$ and thus $u_\varepsilon' \geq 0$ a.e. in $[0, 1]$.

We show now that $u_\varepsilon$ solves problem (2.4).

Indeed, by Lemma 2.1, $u_{ek}$ is bounded in $L^{\infty}(0, 1)$ uniformly with respect to $k$ and since $\varepsilon k f^{**}$ converges uniformly on bounded sets of $[0, 1] \times \mathbb{R} \times \mathbb{R}$, then we have, for $\varepsilon \in [0, 1/2]$,

$$
\lim_{k \to +\infty} \int \left\{ \varepsilon k f^{**}(x, u_{ek}, u_{ek}') - f^{**}(x, u_{ek}, u_{ek}') \right\} \, dx = 0.
$$

Therefore, using lower semicontinuity arguments, for $v \in W^{1, q}(0, 1)$ such that
v(0) = 0, v(1) = λ, v' ≥ 0 a.e. in [0, 1], we get

\[
\int_0^1 g^\varepsilon(x, u_\varepsilon, u'_\varepsilon) \, dx \leq \liminf_{k \to +\infty} \int_0^1 g^\varepsilon(x, u_{\varepsilon k}, u'_{\varepsilon k}) \, dx =
\]

\[
= \liminf_{k \to +\infty} \left\{ \varepsilon k + f^{**}(x, u_{\varepsilon k}, u'_{\varepsilon k}) + \varepsilon(1 + |u'_{\varepsilon k}|^2)^{\alpha/2} \right\} \, dx \leq
\]

\[
\liminf_{k \to +\infty} G^{\varepsilon k}(u_{\varepsilon k}) \leq \liminf_{k \to +\infty} G^{\varepsilon k}(v) = \int_0^1 g^\varepsilon(x, v, v') \, dx.
\]

By the monotone convergence theorem, as ε → 0 we get the result.

A strict monotonicity property of u_ε is stated in the following lemma.

**Lemma 2.3.** For fixed ε, the functions u_ε are strictly increasing in (0, 1).

**Proof.** First of all, let us prove that there not exists any interval I ⊆ [0, 1] such that u''_ε(x) = 0 ∀x ∈ I, where u_ε is defined in the previous lemma.

Indeed, if such an interval I exists, set I = (x_1, x_2) ⊆ [0, 1], u_ε solves Euler's equation in weak form and also in the form

\[
g^\varepsilon(x, u_\varepsilon(x), u'_\varepsilon(x)) = \text{const} + \int_{x_1}^{x_2} g^\varepsilon(t, u_\varepsilon(t), u'_\varepsilon(t)) \, dt, \quad \forall x \in (x_1, x_2)
\]

Differentiation with respect to x, taking into account that u'_ε(x) = 0 ∀x ∈ (x_1, x_2), gives

\[
f^{**}_\varepsilon(x, u_\varepsilon(x), 0) = f^{**}_\varepsilon(x, u_\varepsilon(x), 0) \quad \forall x \in (x_1, x_2),
\]

which contradicts the assumption i) in C).

Since, by Lemma 2.2, we know that u'_ε ≥ 0 a.e. in [0, 1], u_ε is an increasing function in [0, 1]. Indeed the first part of the proof implies that it is strictly increasing.

As a consequence, we get

\[
0 = u_\varepsilon(0) \leq u_\varepsilon(x) \leq u_\varepsilon(1) = \lambda \quad \forall x \in [0, 1].
\]

**3. Geometrical properties and a priori estimates for approximating solutions.**

This section is devoted to the study of concavity-convexity properties of the approximating solutions u_{εk} and to the related a priori estimates. Both of them will hold true for the limit function u_ε.
Let $\varepsilon \in [0, 1/2]$ and $\varepsilon \in [0, 1]$ be fixed. For $k \in N$, define the following subsets of $]2, 1 - \varepsilon[$

(3.1) $Y_k = \{x \in ]2, 1 - \varepsilon[ : u_{\varepsilon k}(x) \neq 0\}$,

(3.2) $Z_k = \{x \in Y_k : u_{\varepsilon k}(x) = 0\}$.

By Lemma 2.3, $\{k \in N : Y_k \neq \emptyset\}$ is infinite.
In order to prove the stated properties of $u_{\varepsilon k}$, we need the following lemma.

**Lemma 3.1.** - If the set $\{k \in N : Z_k \neq \emptyset\}$ is infinite, up to a subsequence, the functions $u_{\varepsilon k}$ have a unique global minimum point $x_k$ with $u_{\varepsilon k}(x_k) > 0$.

**Proof.** - Since $u_{\varepsilon k} \in C^3$, the Euler's equation (2.3) can be differentiated obtaining:

(3.3) $g_{\varepsilon k} u_{\varepsilon k} = \alpha_k f_{\varepsilon x}^* - \{\alpha_k f_{\varepsilon x}^{**} + u_{\varepsilon k} \cdot \alpha_k f_{\varepsilon x}^{**}\}$

and in the set $Z_k$:

(3.4) $g_{\varepsilon k} u_{\varepsilon k} = \alpha_k (f_{\varepsilon x}^{**} + f_{\varepsilon x}^{**} u_{\varepsilon k}') - \alpha_k f_{\varepsilon x}^{**} -$

$- u_{\varepsilon k}'(\alpha_k f_{\varepsilon x}^{**}) - \alpha_k (f_{\varepsilon x}^{**} u_{\varepsilon k}') - u_{k}(\alpha_k f_{\varepsilon x}^{**} u_{\varepsilon k}')$.

If we set

(3.5) $L_1(r) = \sup \{ |f_{\varepsilon x}^{**}(x, s, \xi)| : x \in [0, 1], |s| \leq r, |\xi| \leq r \}$,

(3.6) $L_2(r) = \sup \{ |f_{\varepsilon x}^{**}(x, s, \xi)| : x \in [0, 1], |s| \leq r, |\xi| \leq r \}$,

then for such values $x, s, \xi$, we have

(3.7) $|\xi f_{\varepsilon x}^{**} - \alpha_k f_{\varepsilon x}^{**}| =$

$= \left| \xi \int_R \alpha_k(t) f_{\varepsilon x}^{**}(x, s, \xi - t) dt - \int_R \alpha_k(t)(\xi - t) f_{\varepsilon x}^{**}(x, s, \xi - t) dt \right| \leq$

$\leq L_1(r + 1) \int_R |\alpha_k(t)| dt \leq \frac{L_1(r + 1)}{k} \int_R x(t) |t| dt \leq$

$\leq \frac{L_1(r + 1)}{k} \int_R x(t) dt = \frac{L_1(r + 1)}{k}$,

(3.8) $|\xi f_{\varepsilon x}^{**}(\xi) - \alpha_k f_{\varepsilon x}^{**} \xi^2| =$

$= \left| \xi \int_R \alpha_k(t) f_{\varepsilon x}^{**}(x, s, \xi - t)(\xi - t) dt - \int_R \alpha_k(t) f_{\varepsilon x}^{**}(x, s, \xi - t)(\xi - t)^2 dt \right| \leq$
\( L_2(r + 1) \int \left[ |\xi_k(t)(\xi - t) - \alpha_k(t)(\xi - t)^2| \right] dt = \)

\( = L_2(r + 1) \int |\alpha_k(t)\xi - \alpha_k(t)t^2| dt \leq L_2(r + 1) \int |\alpha_k(t)| \xi - |t| |t| dt \leq \)

\( \leq (r + 1)L_2(r + 1) \int |t| \alpha_k(t) dt \leq \frac{L_2(r + 1)}{k} (r + 1). \)

By (3.4)-(3.8), for \( r \geq \sup_{k \to 0} \|u_{\bar{k}}\|_{L^\infty([0, 1])} \), taking into account the definition (1.6), we have, for \( x \in Z_k \),

\( |g_{k,2}^\xi u_{\bar{k}}'' - \alpha_k^* \varphi| \leq \frac{L_1(r + 1)}{k} + \frac{L_2(r + 1)}{k} (r + 1). \)

In a similar way, we can prove (see also (5.18) in [M.2])

\( |g_{k,2}^\xi u_{\bar{k}}'' - \alpha_k^* \varphi| \leq \frac{L(r + 1)}{k}, \)

where \( \varphi \) is defined in (1.5) and \( L(r) \) is defined by

\( L(r) = \sup \{ |f_{k,2}^\xi(x, s, \xi)| : x \in [0, 1], |s| \leq r, |\xi| \leq r \}. \)

Consider now the infinite set \( \{ k \in N : Z_k \neq \phi \} \). We can assume, possibly extracting a subsequence, that for each \( k, Z_k \neq \phi \). Let be \( x_k \in Z_k \), then \( u_{\bar{k}}(x_k) \neq 0, u_{\bar{k}}'(x_k) = 0 \) and \( \{ (x_k, u_{\bar{k}}(x_k), u_{\bar{k}}'(x_k)) \}_{k \in N} \) converges to some point \( (x, s, \xi) \in [0, 1] \times [-r, r] \times \times [-r, r] \).

On the other hand, by the continuity of \( \varphi \) and (3.10) used for \( x = x_k \), \( \lim_{k \to \infty} \varphi(x_k, u_{\bar{k}}(x_k), u_{\bar{k}}'(x_k)) = \varphi(x, s, \xi) = 0 \)

therefore, by assumption i) in C), \( \xi \) must be different from zero, and by ii) in C), \( \xi \varphi(x, s, \xi) > 0 \) which implies definitively that \( u_{\bar{k}}'(x_k) \cdot \varphi(x_k, u_{\bar{k}}(x_k), u_{\bar{k}}'(x_k)) > 0 \). Now we use (3.9) and, taking into account that \( g_{k,2}^\xi \) and \( \alpha_k \) are positive, we conclude that definitively \( u_{\bar{k}}'(x_k) \) and \( u_{\bar{k}}''(x_k) \) have the same sign.

It follows that definitively \( x_k \) is a local minimum for \( u_{\bar{k}}(x) \) with \( u_{\bar{k}}'(x_k) > 0 \) if \( \xi > 0 \) or, definitively, \( x_k \) is a local maximum with \( u_{\bar{k}}'(x_k) < 0 \) if \( \xi < 0 \).

Indeed \( x_k \) is a strict global minimum for the function \( |u_{\bar{k}}| \), because if it was strict local but not global, it would imply the existence elsewhere of a local positive maximum, which is excluded by the previous argument. For the same reason it is unique. The lemma follows now from the strong \( L^\infty \)-convergence of \( u_{\bar{k}} \) to \( u \) and Lemma 2.3.

**Remark 4.** From the above proof it follows also that \( u_{\bar{k}} \) cannot have a positive local maximum.

Now we can state the lemma which exhibits the mentioned geometrical properties of the approximating solutions \( u_{\bar{k}} \).
**Lemma 3.2.** Let be \( \varepsilon \in ]0, 1/2[ \). There exists a subsequence of \( \{u_{nk}\}_{k \in N} \), still denoted by \( \{u_{nk}\}_{k \in N} \) and two sequences \( \{x^1_k\} \) and \( \{x^2_k\} \), \( \varepsilon \leq x^1_k \leq x^2_k \leq 1 - \varepsilon \) such that

i) \( u'_{nk}(x) = 0 \ \forall x \in ]x^1_k, x^2_k[ \);

ii) if \( u'_{nk}(x) > 0 \) (resp. \( u'_{nk}(x) < 0 \)) in \( ]\varepsilon, x^1_k[ \), then \( u_{nk} \) is concave (resp. convex) in \( ]\varepsilon, x^1_k[ \);

\( \) if \( u'_{nk}(x) > 0 \) (resp. \( u'_{nk}(x) < 0 \)) in \( ]x^2_k, 1 - \varepsilon[ \), then \( u_{nk} \) is convex (resp. concave) in \( ]x^2_k, 1 - \varepsilon[ \).

**Proof.** Assume first that the set \( \{k \in N: Y_k = ]\varepsilon, 1 - \varepsilon[ \} \) is infinite (the set \( Y_k \) is defined by (3.1)); up to a subsequence, we can assume that \( Y_k = ]\varepsilon, 1 - \varepsilon[ \ \forall k \in N \).

If the set \( \{k \in N: Z_k \neq 0\} \) is finite then definitively \( Z_k = 0 \) and \( u_{nk} \) are convex in \( ]\varepsilon, 1 - \varepsilon[ \) or concave in \( ]\varepsilon, 1 - \varepsilon[ \) and the lemma is proved by choosing \( x^1_k = x^2_k = \varepsilon \) or \( x^1_k = x^2_k = 1 - \varepsilon \).

If the set \( \{k \in N: Z_k \neq 0\} \) is infinite, by Lemma 3.1, up to subsequence, \( u'_{nk}(x) \) is decreasing for \( x < x_k \) and increasing for \( x > x_k \). We can conclude also in this case that the lemma is true, by choosing \( x^1_k = x^2_k = x_k \).

Assume now that the set \( \{k \in N: Y_k = ]\varepsilon, 1 - \varepsilon[ \} \) is finite. Therefore, definitively \( Y_k = ]\varepsilon, 1 - \varepsilon[ \), i.e. there exists \( \tilde{k} \in N \) such that, for \( k > \tilde{k} \), there exists at least one point \( x_k \in ]\varepsilon, 1 - \varepsilon[ \) satisfying \( u'_{nk}(x_k) = 0 \). Moreover for large values of \( k \), \( Z_k = 0 \) because if not, the set \( \{k \in N: Z_k \neq 0\} \) would be infinite and, by Lemma 3.1, it would exists a positive local minimum for \( u'_{nk} \) in \( x_k \in Y_k \). On the other hand we have that \( u'_{nk}(x_k) = 0 \), which implies the presence of a local maximum point for \( u_{nk} \) in the interval with end points \( x_k \) and \( \overline{x}_k \) and this contradicts Remark 4.

Now we prove that, for \( k \) large enough, the set \( \{x \in ]\varepsilon, 1 - \varepsilon[: u'_{nk}(x) = 0\} \) is an interval. In fact, let be \( x, y \) such that \( u'_{nk}(x) = u'_{nk}(y) = 0 \); if \( u'_{nk}(x) \) is different from zero in some point \( \overline{x} \) between \( x \) and \( y \), the function \( u_{nk} \) must have an extremum between \( x \) and \( y \) in contradiction with the fact that \( Z_k = 0 \) definitively.

Setting

\[ x^1_k = \inf \{x \in ]\varepsilon, 1 - \varepsilon[: u'_{nk}(x) = 0\} , \]

\[ x^2_k = \sup \{x \in ]\varepsilon, 1 - \varepsilon[: u'_{nk}(x) = 0\} , \]

then assertion i) in the statement of the lemma is proved.

Since \( Z_k = 0 \) for \( k \) large enough, \( |u'_{nk}| \) is decreasing in \( ]\varepsilon, x^1_k[ \) and increasing in \( ]x^2_k, 1 - \varepsilon[ \) which proves assertion ii).

Finally we are able to prove the a priori local estimate on \( u_{nk} \).
LEMMA 3.3. - Let \( \{u_{nk}\}_{k \in \mathbb{N}} \) be the subsequence satisfying the statement in the Lemma 3.2, then the following estimate holds:

\[
\forall \delta \in \left[0, \frac{1}{2}\right] \quad \left\| u_{nk} \right\|_{L^\infty((\delta, 1 - \delta))} \leq \frac{4}{\delta} \left\| u_{nk} \right\|_{L^\infty((0, 1))}.
\]

PROOF. - Let us apply Lemma 3.2 with \( \delta \) replaced by \( \delta/2 \). Different situations are possible, but in any case we get the following estimate:

\[
|u_{nk}(x)| \leq \frac{|u_{nk}(x) - u_{nk}(\delta/2)|}{|x - \delta/2|} \quad \forall x \in [\delta, x_k^1[.
\]

Then, \( \forall x \in [\delta, x_k^1[ \)

\[
|u_{nk}(x)| \leq \frac{4}{\delta} \left\| u_{nk} \right\|_{L^\infty((0, 1))}.
\]

In a similar way we proceed to prove estimate (3.12) for \( x \in ]x_k^2, 1 - \delta[ \). By (3.12) and i) in Lemma 3.2 we get the estimate (3.11).

Let us observe that estimate (3.11) holds true passing to the limit for \( k \to \infty \). In fact the boundedness in \( L^\infty((\delta, 1 - \delta)) \) of \( \{u_{nk}\} \) implies that this sequence converges in the weak* topology to \( u' \) and by lower semicontinuity of the norm, we get

\[
\|u'\|_{L^\infty((\delta, 1 - \delta))} \leq \liminf_{k \to \infty} \|u_{nk}\|_{L^\infty((\delta, 1 - \delta))} \leq \frac{4\lambda}{\delta}.
\]

4. - Proof of the main theorem.

Here we follow the outline of the proof of Theorem 5.4 of [M.2]. Let us consider for each \( \varepsilon \) the function \( u_\varepsilon(x) \) obtained as limit, for \( k \to \infty \), of \( u_{nk} \). By inequality (3.13), \( \{u_\varepsilon\} \) is relatively compact in the weak* topology of \( W^{1, \infty}_{loc}(0, 1) \) and there exists a function \( u_0 \in W^{1, \infty}_{loc}(0, 1) \) such that, up to a subsequence,

\[
u_\varepsilon \rightharpoonup u_0 \quad \text{in} \quad W^{1, \infty}_{loc}(0, 1) \quad \text{for} \quad \varepsilon \to 0.
\]

By the definition of \( \overline{G} \) (see 1.3), recalling that \( u_\varepsilon \) is a solution of problem (2.4) (see Lemma 2.2), \( \forall \varphi \in \mathcal{W}_0^1 = \mathcal{W}_0^1 \cap W^{1, q}(0, 1) \), we get

\[
\overline{G}(u_0) \leq \liminf_{\varepsilon \to 0} \int_0^1 f^{**}(x, u_\varepsilon, u'_\varepsilon) \, dx \leq \liminf_{\varepsilon \to 0} \int_0^1 g^*(x, u_\varepsilon, u'_\varepsilon) \, dx \leq
\]

\[
\leq \liminf_{\varepsilon \to 0} \int_0^1 g^*(x, v, v') \, dx = \int_0^1 f^{**}(x, v, v') \, dx = G(v)
\]
then
\begin{equation}
\overline{G}(u_0) \leq G(v) \quad \forall v \in \mathcal{W}_q.
\end{equation}

Let now be \( w \in \mathcal{W}_p \), because of the density of \( W^{1,q} \) in \( W^{1,p} \), there exists a sequence \( \{v_k\} \subset \mathcal{W}_q \) such that \( v_k \rightharpoonup w \) in \( W^{1,p}(0,1) \). Moreover since, by \( A_2 \), \( G \) is strongly continuous in \( W^{1,p} \), inequality (4.2) applied to \( v = v_k \), to the limit, gives
\begin{equation}
\overline{G}(u_0) \leq G(w) \quad \forall w \in \mathcal{W}_p.
\end{equation}

Finally let \( v \in \mathcal{W}_p \), by the definition of \( \overline{G} \), for a sequence \( \{v_k\} \subset \mathcal{W}_q \) such that \( v_k \rightharpoonup v \) in the weak topology of \( W^{1,p}_{bc}(0,1) \), \( \lim G(v_k) = \overline{G}(v) \). By replacing \( w \) with \( v_k \) in the previous inequality (4.3) and passing to the limit, we see that
\begin{equation}
\overline{G}(u_0) \leq \overline{G}(v) \quad \forall v \in \mathcal{W}_p
\end{equation}
and \( u_0 \) solves the minimum problem related to the functional (1.3) in \( \mathcal{W}_p \).

To conclude our proof we must only prove that
\begin{equation}
f(x, u_0(x), u'_0(x)) = f^{**}(x, u_0(x), u'_0(x)) \quad \text{a.e. in } (0,1)
\end{equation}

since from (3.13) immediately follows the analogous estimate for \( u'_0 \), by semicontinuity arguments.

Let us point out that \( u'_0 \) is a piecewise monotone function because of the geometrical properties of \( u_0 \) stated in the Lemma 3.2. Then \( u'_0 \) is almost everywhere continuous. Let be \( A = \{ x \in (0,1) : u'_0 \text{ is continuous in } x \} \) and choose \( x \in A \) such that \( f(x, u_0(x), u'_0(x)) \neq f^{**}(x, u_0(x), u'_0(x)) \). We recall that \( f^{**} \) is a linear function with respect to \( \xi = u'_0(x) \) and therefore, taking the derivative at \( x \) of the Euler's equation in the weak form,
\[ f^{**}_x(x, u_0(x), u'_0(x)) = c + \int_0^x f^{**}_t(t, u_0(t), u'_0(t)) \, dt, \]
we get
\[ \varphi(x, u_0(x), u'_0(x)) = f^{**} - f^{**}_x - f^{**}_y u'_0(x) = 0. \]

By i) in the assumption C), it follows that \( u'_0(x) \neq 0 \).

On the other hand \( \bar{\varphi}(x) = \varphi(x, u_0(x), u'_0(x)) \) is strictly increasing in this point \( x \) because of assumption ii) in C). Then there exists a neighbourhood \( I(x) \) such that, for each \( y \in I(x) \) \( \bar{\varphi}(y) = 0 \). It follows then that, for each \( y \in I(x) \) \( \bar{\varphi}(y) = 0 \), either \( y \in A \) or \( f(y, u_0(y), u'_0(y)) = f^{**}(y, u_0(y), u'_0(y)) \) otherwise, by the previous arguments, \( \bar{\varphi}(y) \) would be equal to zero.

Since \( u'_0 \) is almost everywhere continuous, then \( (f - f^{**})(y, u_0(y), u'_0(y)) = 0 \) a.e. in \( I(x) - \{x\} \). This contradicts the fact that \( (f - f^{**})(x, u_0(x), u'_0(x)) \) is different from zero in \( x \) which is a continuity point for \( u'_0 \). We conclude that (4.4) holds true.
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