CORRECTIONS FOR THE ARTICLE "ON THE GENERAL THEORY OF QUOTIENT RINGS" BY V. P. ELIZAROV

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In a letter to the author and in an abstract for Mathematical Reviews [1], Professor G. M. Bergman made a series of remarks about the article indicated above [2]. We give here the corresponding corrections.

1. If A is a ring not containing 1, let A' denote the ring obtained from A by identifying units. In [1] it is noted that, if in Theorem 3 of [2] we omit the condition $R \in \Phi$, then the ring $Q = Q(R, \Phi, I)$ will be the essential completion of the ring $\varphi(R)$ as a right $\varphi(R)'$-module; it is also noted that the mapping ψ in Theorem 4 of (2) is an imbedding without assuming that $R \in \Phi_2$. In [3] it is pointed out that if $R \in \Phi$, then Q is exactly the essential completion of $\varphi(R)$. We now give the corrected forms of Theorems 3 and 4 and their corollaries in [2].

THEOREM 3. If Φ is a right-hand I-system and φ is the canonical mapping φ: $R \to Q$, then Q is exactly the essential completion of $\varphi(R)$ as a right $\varphi(R)'$-module.

Proof. We must show that if $0 \neq q_1 = \theta f_A$ and $q_2 = \theta f_B$ are elements of Q and A, $B \in \Phi$, then there exists $\exists \varphi(r) + \alpha \in \varphi(R)'$, where $\alpha \in Z$, such that $0 \neq q_1(\varphi(r) + \alpha) \in \varphi(R)$ and $q_2(\varphi(r) + \alpha) \in \varphi(R)$.

First take $q_2 \neq 0$. Then in $D = A \cap B \in \Phi$ are found elements $r' = r_1 + \alpha$ and $r'' = r_2 + \beta$, where α, $\beta \in Z$ and such that $f_A(r') \notin I$ and $f_B(r'') \notin I$, since otherwise $\theta f_A = \theta f_B = 0$. If $f_B(r') \notin I$ or $f_A(r'') \notin I$, let $d = r'$ or $d = r''$, respectively. If both $f_B(r') \in I$ and $f_A(r'') \in I$, we define $d = r' + r''$. Here $f_A(r' + r'') \notin I$ and $f_B(r' + r'') \in I$.

If $d = r'$ we consider the compositions $q_1(\varphi(r_1) + \alpha) = \theta f_A(\theta f_{r_1} + \alpha) = \theta f_A \theta f_{r_1} + \alpha\theta f_A$ and $q_2(\varphi(r_1) + \alpha) = \theta f_B \theta f_{r_1} + \alpha\theta f_B$. If $q_1(\varphi(r_1) + \alpha) = 0$, then there exists $\exists C \in \Phi$ such that for all $\forall c \in C$ the relations $(f_A f_{r_1} + \alpha f_A)(c) = f_A(r_1 c + \alpha c) = f_A(r')c \in I$ are valid. But then $f_A(r')C \subset I$ and $f_A(r') \in I$, contrary to the assumption. Therefore $q_1(\varphi(r_1) + \alpha) \neq 0$.

It remains to show that $q_1(\varphi(r_1) + \alpha) \in \varphi(R)$ and $q_2(\varphi(r_1) + \alpha) \in \varphi(R)$, i.e., that there exists $\exists G$, $E \in \Phi$ which for all $\forall g \in G$, $e \in E$ satisfy the relations $(f_A f_{r_1} + \alpha f_A - \alpha r_\beta)(g) \in I$ and $(f_B f_{r_1} + \alpha f_B - f_{r_4})(g) \in I$, where r_3, $r_4 \in R$. The left part of the first of the required relations has the form $f_A(r_1 g + \alpha g) - rg = f_A(r')g - rg$. Therefore, letting $r_3 = f_A(r')$ and $G = R$ we obtain $q_1(\varphi(r_1) + \alpha) \in \varphi(R)$. To satisfy the second relation it is sufficient to let $r_4 = f_B(r')$ and $E = R$ (or R'). Now it is clear how it goes for $q_2 = 0$.

In the cases when $d = r''$ or $d = r' + r''$, we argue similarly via the replacement of $\varphi(r_1) + \alpha$ by $\varphi(r_2) + \beta$ or by $\varphi(r_1 + r_2) + \alpha + \beta$, respectively. The theorem is proved.

THEOREM 4. If Φ_1 and Φ_2 are right-hand I-systems such that $\Phi_1 \supset \Phi_2$ and φ_i is the canonical mapping φ_i: $R \to Q_i = Q(R, \Phi_i, I)$, then there exists an imbedding ψ: $Q_2 \to Q_1$ for which $\psi(\varphi_2(r)) = \varphi_1(r)$ for all $\forall r \in R$.

The proof proceeds as in [2] with the corrections of Theorem 3 used.

COROLLARY 1. If for any right I-systems Φ_1 and Φ_2 and for all $\forall A \in \Phi_1$, $B \in \Phi_2$, $f_B \in \mathrm{Hom}_R(B, R)$, $f_B(I) \subset I$, there exist right I-systems Φ_3 and Φ_4 such that $A \cap B \in \Phi_3$ and $f_A^{-1} B \in \Phi_4$, then the ring R has an I(R)-maximal quotient ring.

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COROLLARY 2. Every ring R for which there is a quotient ring of the form $Q(R, \Phi, I)$ has an $I(R)$-maximal quotient ring.

2. In [1] it is shown that if one considers only right $I(R)$-systems (but not arbitrary I-systems, as shown there), then the maximal quotient ring will be the ring $Q(R, \Phi_u, I)$. Therefore, in formulating Theorem 19 of [2], the following necessary conditions are inserted for the equality and read:

THEOREM 19. If I is an S-prime ideal of the ring R, then we have the relations (under the condition that the ring $Q(R, \Phi_J, I)$ exists): $Q(R, \Phi_E, I) \subset Q(R, \Phi_B, I) \subset Q(R, \Phi_{F-L}, I) = Q(R, \Phi_U, I) = Q(R, \Phi_J, I)$.

3. The beginning of Corollary 3 to Theorem 5 of [2] should read: "Let Φ_i be a right I_j-system (i, j = 1, 2)."

4. In [1] it is shown that the conditions of Theorem 7 of [2] are not satisfied for $n > 1$ or for every ring. Therefore, we give a second statement and proof for the cases when $Q(R, \Phi, I) = \tilde{Q}_U(R, 0)$, as in [4].

R_n denotes the $n \times n$ matrix ring over the ring R, and we let $\Phi_n = \{A \text{ is a right ideal of } R_n | \text{there exists } \exists B \in \Phi,\ B_n \subset A\}$.

LEMMA. The systems Φ_n are right I_n-systems for the rings R_n if R contains 1.

Proof. Conditions α), β), and γ) of such I-systems (1, §2, ¶1) are satisfied in a trivial way. Let A, C be elements of Φ_n and B, D of Φ with $B_n \subset A$, $D_n \subset C$, f_A in $\mathrm{Hom}_{R_n}(A, R_n)$ and $f_A(I_n) \subset I_n$. If e_{kl} is a matrix unit in R_n and $b \in B$, then $be_{kl} \in A$. Let $f_A(be_{kl}) = (r_{ij}(b, k, l))$. Define a mapping f_{ij}^{kl}: $B \to R$ by letting $f_{ij}^{kl}(b) = r_{ij}(b, k, l)$. Because f_A is an R_n homomorphism, it easily follows that f_{ij}^{kl} is an R homomorphism. Since $f_A(I_n) \subset I_n$, $f_{ij}^{kl}(I) \subset I$. But then $(f_{ij}^{kl})^{-1}D \in \Phi$. We let $N_{kl} = \bigcap_{i,j=1}^{n} (f_{ij}^{kl})^{-1} D \in \Phi$ and $N = \bigcap_{k,l=1}^{n} N_{kl} \in \Phi$. Since for all $\forall k, l$, $r \in N_{kl} f_A(re_{kl}) = (r_{ij}(r, k, l)) \in D_n$, therefore $f_A(N_n) \subset D_n$, $f_A^{-1}C \supset N_n$ and $f_A^{-1}C \in \Phi_n$. This proves condition δ).

Now let $A \in \Phi_n$, $B \in \Phi$, $B_n \subset A$, $\bar{r} = (r_{ij}) \in R_n$, and $\bar{r}A \subset I_n$. Since in A are contained all matrices in which a single element belongs to B, but the remaining elements are zero, then for all $\forall r_{ij}$ we have $r_{ij}B \subset I$, $r_{ij} \in I$, and $\bar{r} \in I_n$, i.e., condition ε) is satisfied. The lemma is proven.

THEOREM 7'. If R contains 1 and Φ is a right I-system, there is a quotient ring $Q(R_n, \Phi_n, I_n) \cong Q \cdot (R, \Phi, I)_n$.

Proof. By the lemma, the ring $Q(R_n, \Phi_n, I_n)$ exists. If $A \in \Phi_n$, $f_A \in \mathrm{Hom}_{R_n}(A, R_n)$, $f_A(I_n) \subset I_n$, and $r \in R$, then multiplying re_{kl} by elementary matrices which interchange columns we obtain that $f_A(re_{kl}) = f_A(re_{km})$ for all $\forall l$, m. Therefore there are only n^2 different R-homomorphisms f_{ij}^{kl}, which we denote by f_i^k. The element $\theta_n f_A \in Q(R_n, \Phi_n, I_n)$ corresponds to the matrix $(a_{kl} = \theta f_l^k) \in Q(R, \Phi, I)_n$. It is easy to verify that the correspondence gives the required isomorphism.

5) In [1] it is noted that the second mappings of Theorems 8, 9, 11, and 13 in [2], associated with the maximal quotient rings $Q(R, \Phi_J, I)$, $Q(R, \Phi_U, I)$, $Q(R, \Phi_{F-L}, I)$, and $Q(R, \Phi_B, I)$, are not correct. To make this mapping valid condition δ) for right I-systems must be replaced by the following:

$$\delta') \quad \text{if} \quad A, B \in \Phi \text{ and } f_A \in \mathrm{Hom}_R(A/I, R/I), \text{ then } f_A^{-1}B = \{x \in A | f_A(x+I) \in B/I\} \in \Phi.$$

The construction of the ring $Q(R, \Phi, I)$ follows from letting $M_A = \{f_A \in \mathrm{Hom}_R(A/I, R/I)\}$ for all $\forall A \in \Phi$, but the relation θ on $M = \sum M_A$ must be defined in the following way: $f_A \theta f_B$ if and only if there exists $\exists D \in \Phi$, $D \subset A \cap B$ such that for all $\forall d \in D$, $f_A(d) = f_B(d)$. These changes make all results in [2] correct (with the noted corrections given in 1-4).

6. In [1] it is shown that the constructions of Gabriel, Maranda, and Chew (in [2] erroneously written as "Khyu") are equivalent. As shown in [5] p. 413 the constructions of [2] are not equivalent to theirs. In Section 5 of the survey article [5] one must insert the corrections here indicated in 1-6.

LITERATURE CITED

1. G. M. Bergman, Math. Reviews, 43, No. 3, 3301 (1972).
2. V. P. Elizarov, "On the general theory of quotient rings," Sibirsk. Matem. Zh., 11, No. 3, 526-546 (1970).

3. V. P. Elizarov, "Essential and fully essential quotient rings," 11th All-Union Algebraic Colloquium [in Russian], Kishinev, 131-132 (1971).
4. Y. Utumi, "On quotient rings," Osaka J. Math., 8, 1-18 (1956).
5. V. P. Elizarov, "Quotient rings," Algebra i Logika, 8, No. 4, 381-424 (1969).