Introduction and Some Problems Encountered in the Construction of a Relativistic Quantum Theory

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1.1 States in Relativistic Quantum and Classical Mechanics

One of the deepest and most difficult problems of theoretical physics in the past century has been the construction of a simple, well-defined one-particle theory which unites the ideas of quantum mechanics and relativity. Early attempts, such as the construction of the Klein-Gordon equation and the Dirac equation were inadequate to provide such a theory since, as shown by Newton and Wigner (1949), they are intrinsically non-local, in the sense that the solutions of these equations cannot provide a well-defined local probability distribution. This result will be discussed in detail below. Relativistic quantum field theories, such as quantum electrodynamics, provide a manifestly covariant framework for important questions such as the Lamb shift and other level shifts, the anomalous moment of the electron and scattering theory, but the discussion of quantum mechanical interference phenomena and associated local manifestations of the quantum theory are not within their scope; the one particle sector of such theories display the same problem pointed out by Newton and Wigner since they satisfy the same one-particle field equations.

On the other hand, the nonrelativistic quantum theory carries a completely local interpretation of probability density; it can be used as a rigorous basis for the development of nonrelativistic quantum field theory, starting with the construction of tensor product spaces to build the Fock space, and on that space to define annihilation and creation operators (e.g., Baym 1969). The development of a manifestly covariant single particle quantum theory, with local probability interpretation, could be used in the same way to develop a rigorous basis for a relativistic quantum field theory which carries such a local interpretation. A central problem in formulating such a theory is posed by the requirement of constructing a description of the quantum state of an elementary system (e.g., a "particle") as a manifestly covariant function on a manifold of observable coordinates which belongs to a Hilbert space. The essential properties of the quantum theory, such as the notions of probability, transition amplitudes, linear superposition, observables and their expectation values, are realized in terms of the structure of a Hilbert space.

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Nonrelativistic quantum mechanics, making explicit use of the Newtonian notion of a universal, absolute time, provides such a description in terms of a square integrable function over spatial variables at a given moment of this Newtonian time. This function is supposed to develop dynamically, from one moment of time to another, according to Schrödinger's equation, with some model Hamiltonian operator for the system. The theory furthermore satisfies the property of manifest covariance under the Galilean group.

This non-relativistic description of a state is, however, inconsistent with special relativity from both mathematical and physical points of view. The wave function $\psi_t(\mathbf{x})$, as a function of spatial variables, and parametrized by the Newtonian time, described in a frame in inertial motion with respect to another and related to it by a Lorentz transformation, undergoes a transformation which makes its interpretation in the new frame very difficult. In particular, if an event is predicted by this function with a certain probability to take place at the point $\mathbf{x}$ at the time t in the original frame, that event should occur with the same probability, as seen in the new frame, at the point $\mathbf{x}'$ at the time t'. According to the structure of the Lorentz transformation, the time t' depends on the location of the point $\mathbf{x}$ in the original frame as well as t, so that it is inconsistent to label the wave functions in the new frame according to t', now no longer a parameter, but partly dependent on the variable $\mathbf{x}$, with a value associated with the probability distribution defined by the original wave function. Since the Hilbert spaces associated with different times are distinct, the transformed function therefore loses its interpretation as the description of a state.

The situation for classical nonrelativistic mechanics is quite analogous; the state of a system is described by a set of canonical coordinates and momenta (the variables of the phase space) at a given time. These canonical variables develop in time according to the first order Hamilton equations of motion. The variables of the phase space, under the transformations of special relativity, are mapped into a new set in which the time parameter for each of them depends on the spatial location of the points; in addition, there is a structural lack of covariance of the phase space variables themselves (as for the quantum wave function, they become mixed with the time parameter).

On the other hand, observed interference phenomena, such as the Davisson-Germer experiment (Davisson 1927), showing the interference pattern due to the coherence of the wave function over the spatial variables at a given time, clearly should remain when observed from a moving frame. In this case, the parts of the wave function that interfere appear to pass the scattering centers (or slits, in a double slit experiment) at different times,and would not be coherent in the framework of the nonrelativistic theory. Hence one would expect that there is a more general, covariant, description of the state of a system, with Hilbert space based on a scalar product of the form, for example, for scalar wave functions, $\int d^4x \psi^*{}_\tau(\mathbf{x}, t)\psi_\tau(\mathbf{x}, t)$ (for which the time t is considered as an observable), instead of the nonrelativistic form $\int d^3\mathbf{x}\psi^*{}_\tau(\mathbf{x})\psi_\tau(\mathbf{x})$, where τ is a parameter that we shall discuss below, which would predict such an interference pattern, modified only by the laws of special relativity when observed from a moving frame. In the succeeding chapters, I shall discuss such a theory based on the original work of Stueckelberg (1941) and Horwitz and Piron (1973), and describe some important results that have been achieved in this framework.

1.2 The Problem of Localization for the Solutions of Relativistic Wave Equations

Attempts to take into account the required relativistic covariance of the quantum theory by means of relativistic wave equations such as the Klein-Gordon equation (Schrödinger 1926) for spin zero particles, and the Dirac equation (Dirac 1930) for spin $1/2$ particles, have not succeeded in resolving the difficulties associated with the definition and evolution of quantum states. These equations are of manifestly covariant form, with the potential interpretation of providing a description of a quantum state, with spatial properties, in each frame, evolving according to the time parameter associated with that frame. The well-known problem posed by the lack of a positive definite probability density for the Klein-Gordon equation (Schweber 1964) is formally managed by passing to the second quantized formalism (Pauli 1934); the Dirac equation admits a positive definite density, but the problem of localization remains. In both cases, in the second quantized formalism, the vacuum to one particle matrix element of the field operator, which should have a quantum mechanical interpretation (the one-particle sector), poses the same problem of localization. Predictions of particle detection which follow from the formation of interference patterns remain ambiguous in this framework.

Foldy-Wouthuysen type (Foldy 1950) transformations (for both spin zero and spin $1/2$ cases) restore the local property of the wave functions, but in this representation, manifest Lorentz covariance is lost. It is clear that the problem of localization is a fundamental difficulty in realizing a covariant quantum theory by means of the usual wave equations confining the energy momentum to a definite value of mass m. I describe the problem of localization in the following.

Newton and Wigner (1949), showed that the solution $\phi(x)$, for example, of the Klein-Gordon equation, cannot have the interpretation of an amplitude for a *local* probability density. The function $\phi_0(x)$, corresponding to a particle localized at $\mathbf{x} = 0$, at $t = 0$, has support in a range of $\mathbf{x}$ of order $1/m$, where m is the mass of the particle. The argument of Newton and Wigner is as follows. The Klein-Gordon equation (we use indices $\mu = 0, 1, 2, 3$ for time and space, with Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$; $x \equiv t, \mathbf{x}$), and $\partial_\mu = \partial/\partial x^\mu$, with $\hbar = c = 1$)

$$(-\partial_\mu \partial^\mu + m^2)\phi(x) = (\partial_t^2 - \nabla^2 + m^2)\phi(x) = 0 \tag{1.1}$$

has the conserved current

$$J_\mu(x) = \frac{i}{2}(\phi^*(x)\partial_\mu \phi(x) - (\partial_\mu \phi^*(x))\phi(x)). \tag{1.2}$$

The scalar function $\phi(x)$ has the Fourier representation

$$\begin{aligned}\phi(x) &= \int d^4p\delta(-p_\mu p^\mu - m^2)e^{ip^\mu x_\mu}\phi(p) \\ &= \int d^4p\delta\left(\left(E-\sqrt{\mathbf{p}^2+m^2}\right)\left(E+\sqrt{\mathbf{p}^2+m^2}\right)\right)e^{i\mathbf{p}\cdot\mathbf{x}-iEt}\phi(\mathbf{p},E) \quad (1.3)\\ &= \int \frac{d^3\mathbf{p}}{2E}\{e^{i\mathbf{p}\cdot\mathbf{x}-iEt}\phi(\mathbf{p},E)+e^{i\mathbf{p}\cdot\mathbf{x}+iEt}\phi(\mathbf{p},-E)\},\end{aligned}$$

where in the last equality, $E \equiv +\sqrt{\mathbf{p}^2+m^2}$, and, with the δ-function in the first term, we have confined the integration to the "mass shell" m. The two terms in the first equality correspond to the contributions from the positive and negative values of energy in the integration. Assuming that the wave function has contributions only from positive energy, the scalar product may be derived from the positive definite norm obtained by integrating the fourth component of the current (1.2) over all space (a Lorentz invariant construction), i.e., using just the first term of (1.3) (containing the positive energy part)

$$\int d^3\mathbf{x}J_0(x) = \int \frac{d^3\mathbf{p}}{2E}|\phi(\mathbf{p},E)|^2, \quad (1.4)$$

This norm is associated with a scalar product

$$(\phi_1,\phi_2) = \int \frac{d^3\mathbf{p}}{2E}\phi_1^*(\mathbf{p})\phi_2(\mathbf{p}). \quad (1.5)$$

Newton and Wigner then assume that $\phi_1 \equiv \phi_0$ corresponds to the wave function in momentum space describing a particle known to be with certainty at the point $\mathbf{x}=0$, and $\phi_2 = e^{i\mathbf{p}\cdot\mathbf{a}}\phi_0$, i.e., translated by $\mathbf{a}$. Since the two points are separated for $\mathbf{a}\neq 0$, the scalar product must be zero. It is a basic theorem in quantum mechanics that two macroscopically separated systems are in orthogonal quantum states. It then follows from (1.5) that

$$\int \frac{d^3\mathbf{p}}{2E}|\phi_0(\mathbf{p})|^2 e^{i\mathbf{p}\cdot\mathbf{a}} = 0. \quad (1.6)$$

This result has the form of a Fourier transform of the function $|\phi(\mathbf{p})|^2/2E$ which must vanish for all $\mathbf{a}\neq 0$, and therefore it must be a constant. Newton and Wigner argue that since it must be a representation of the Poincaré group, up to an overall constant phase,

$$\phi_0(\mathbf{p}) = C\sqrt{2E},$$

where C is some constant, and therefore the state of a particle known to be precisely at the point $\hat{\mathbf{x}}$ is

$$\phi_{\hat{\mathbf{x}}}(\mathbf{p}) = C\sqrt{2E}e^{-i\mathbf{p}\cdot\hat{\mathbf{x}}}. \quad (1.7)$$

This implies that the wave function in space, the inverse Fourier transform (with weight factor $1/2E$ is given by

$$\phi_{\hat{\mathbf{x}}}(\mathbf{x}) = C \int \frac{d^3\mathbf{p}}{\sqrt{2E}} e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{1.8}$$

This function, due to the momentum dependent denominator, is not localized, but rather spread out in a somewhat oscillatory way (a form of Bessel function), with a width for the central peak of the order of $1/m$. One learns from this that a particle with very small mass is very nonlocalized (this result gives rise to the common statement that the photon is not a localized particle). The operator for which the wave function (1.7) has eigenvalue $\mathbf{x}$ is

$$\mathbf{x}_{NW} = i\left(\frac{\partial}{\partial \mathbf{p}} - \frac{\mathbf{p}}{2E^2}\right), \tag{1.9}$$

known as the Newton-Wigner operator. It is a Hermitian operator in the scalar product (1.5), the second term compensating for the derivative of the weight factor in the process of integrating by parts. We remark the for the scalar function discussed above, the Foldy Wouthuysen transformation corresponds to a map on the vector space by the factor $1/\sqrt{2E}$, which returns the scalar product to the usual form, and the representation of $\mathbf{x}$ as $i\partial/\partial\mathbf{p}$ as well as the locality of the theory, but, as in the case of the Dirac spinor theory, destroys its covariance. We remark that in the limit $c \to \infty$, i.e., the nonrelativistic limit, the momentum dependence in the denominator of (1.8) becomes negligible, and the wave function goes over to the local Schrödinger form.

One concludes from this discussion that the Klein Gordon wave function cannot represent a proper quantum theory, since the square of the wave function, which should correspond to a probability distribution, does not vanish in regions where the particle is known with certainty not to be present.

A similar conclusion was found for the solutions of the spin 1/2 Dirac equation (Newton 1949).

In this chapter we have discussed some of the fundamental issues involved in developing a relativistic quantum theory which have been encountered historically. We shall see in the next chapter that these difficult conceptual problems have a simple and natural resolution in the framework of a consistent manifestly covariant quantum theory. We furthermore discuss a relation closely related to the Newton-Wigner problem derived by Landau and Peierls (1931) that further illustrates the utility and content of the relativistic theory.

In Chap. 3 we treat the induced representation for the spin of a relativistic particle in the framework of the relativistic quantum theory, and discuss the associated quantum field theory for identical particles. It is shown that there is necessarily a universality of the orbit parameter on the whole set of identical particles, and that the nonrelativistic Clebsch-Gordan coefficients may be applied to compute angular momentum states independently of the state if motion of the particles.

In Chap. 4, we discuss the 5D gauge fields associated with the Stueckelberg theory. Along with the current of charged events, the field equations of the 5D theory reduce to the standard Maxwell form under integration over the invariant world time parameter. The Lorentz force, however, is not of linear form, and therefore integration over τ introduces a convolution, indicating that the particle does not stay on "mass shell" during the interaction.

The generalization of the classical radiation reaction problem for the relativistic charged particle is then formulated, and it is shown that the resulting Abraham-Dirac-Lorentz (Abraham 1903) equation is highly nonlinear, and the solution has chaotic behavior. Although it is highly unstable, as is the solution of the standard Abraham-Dirac-Lorentz equation (which has the so-called runaway solutions), the attractor that exists in this very non-linear equation appears to stabilize the macroscopic behavior of the classical solutions, as well as to provide a mechanism, under certain conditions, for the stability of the observed mass of a charged particle.

We also show in Chap. 4 how a simple description of flavor oscillations for neutrinos can emerge from a simple extension of the basic Stueckelberg semi-classical picture. The Lorentz force for both abelian and non-abelian gauge fields are treated.[1]

In Chap. 5, we shall show that the two-body bound state in an invariant phenomenological action-at-a-distance potential has a solution with spectrum in agreement with the corresponding non-relativistic two body problems, up to relativistic corrections, showing that the theory is a proper generalization of the non-relativistic Schrödinger quantum theory. The two body scattering amplitude is discussed in Chap. 7, providing further insights into how the relativistic theory can provide results consistent with the usual nonrelativistic structure.

In Chap. 6, we describe the experiment of Lindner et al. (2005) which demonstrates the existence of coherence in time, a fundamental property of the covariant relativistic theory. Calculating the effect in the framework of the covariant quantum theory, using the conditions of the experiment, one finds very good agreement with experimental results. We discuss in some depth as well why this result is not consistent with the nonrelativistic quantum theory. The formulas were actually obtained many years earlier by Horwitz and Rabin (1976) in their early investigations of the consequences of the relativistic theory, but at that time the necessary experimental tools for confirming the predictions were not available. A similar, but somewhat more complex problem occurs in the proposed experiment of Palacios et al. (2009) where spin correlations presumed to be maintained between particles at different times. The application of Wigner's induced representation theory (Wigner 1939), discussed on Chap. 3, to the spin of a many body system, accounting for correlations between spins of particle at different times, may be applied to discuss this experiment in much the same way as the description of the Lindner et al. experiment; the point is that the wave functions, carrying information on the particle spin, are extended in time as well as space, and therefore entanglement can occur between particles located at different times t.

[1] Shnerb and Horwitz (1993) have carried out the full canonical second quantization of the $U(1)$ gauge theory following the methods discussed by Henneaux and Teitelboim (1992) and Haller (1972).

In Chap. 6, I also discuss the consequences of the construction of a *spacetime lattice*, which one might imagine as the picture of an electromagnetic standing wave in a cavity, periodic in both space and time; the corresponding Stueckelberg wave function, like the Bloch waves in a crystal, has forbidden bands which could, in principle, be seen experimentally (Engelberg 2009).

Chapter 7 discusses scattering theory. Since the structure of the Stueckelberg theory is based on the existence of a Hamiltonian, the scattering theory has a very strong parallel to the nonrelativistic scattering theory, and in the same way makes accessible the use of rigorous mathematical techniques. We show that the partial wave expansion for scattering theory for potential models can be achieved in a form close to that of the non-relativistic theory. The problem of describing resonances in scattering theory for which a semigroup decay law can be achieved is described in the framework of the relatively recently developed theory of Lax and Phillips (1967), Strauss (2000), is here extended to systems of relativistic particles. A relativistic Lee-Friedrichs model (Horwitz 1995) is worked out (Strauss 2000a) as an illustration of this very powerful technique.

Since the Stueckelberg quantum theory is covariant, there is an open and important question of how the theory can be applied to problems previously only accessible to quantum field theory.

In Chap. 8, we show that the anomalous moment of the electron can be computed in this framework without resort to quantum yield theory (Bennett 2012), and that it therefore carries some of the information usually attributed to the effects of vacuum polarization. Some further results of this type are also discussed. In this chapter, we discuss also the existence of Berry (1984), Bahar (2014) phases for the perturbed relativistic oscillator problem.

Chapter 9 discusses the existence of a conformal map in the framework of general relativity that results in a description of Milgrom's approach (Milgrom 1983) to the modification of Newton's law to account for the radiation curves of galaxies as an alternative to dark matter; the TeVeS theory of Bekenstein and Sanders (1994), Bekenstein (2004) emerges from a nonabelian gauge construction in the Stueckelberg theory (Horwitz 2010).

In Chap. 10, the statistical mechanics of the N-body problem is worked out, discussing both the Gibbs ensembles and the non-equilibrium generalization to the Boltzmann equation (Horwitz 1981). The general H-theorem that follows from this equation shows that there is an entropy increase monotonically in τ; an increase in entropy in the Einstein t variable follows, in general, only if there is no pair formation or annihilation. All of the standard thermodynamic relations are obtained in this framework, with some new features. In particular, there may be a high temperature Bose-Einstein condensation (Burakovsky 1996) to a state with a sharp (average) mass determined by a chemical potential.

In Chap. 11, there is a review of the main ideas underlying the theory and their phenomenological basis, and some discussion pointing to possible future developments.