Chapter 2

Ideals

A subgroup I of $(R,+)$ is called *left (right) ideal* if

$R{\cdot}I = \{ri | r \in R, i \in I\} \subseteq I$ (resp. $I{\cdot}R \subseteq I$) and *ideal* if it is left and right ideal. We denote by $(X) = \cap\{I$ (left or right) ideal in $R | X \subseteq I\}$, if $X \subseteq R$, called the *(left or right) ideal generated by* X. In an arbitrary ring $(X) = \{\sum_{i=1}^{n} r_i x_i + \sum_{k=1}^{m} x'_k r'_k + \sum_{s=1}^{l} r''_s x''_s r'''_s + \sum_{j=1}^{t} n_j y_j\}$

with $r_i, r'_k, r''_s, r'''_s \in R, x_i, x'_k, x''_s, y_j \in X, n_j \in \mathbb{Z}$ and the reader can provide the simplier forms if the ring has identity or is commutative (or both).

An ideal I is called *finitely generated (resp. principal)* if for a finite (resp. one element) subset F one has $I = (F)$. A domain R is called a *principal ideal domain* if all its ideals are principal.

An left ideal I of R is called

maximal if $I \neq R$ and it is not properly contained in any left ideal $\neq R$;

minimal if $I \neq (0)$ and it contains not properly any nonzero left ideal of R. Maximal, minimal right ideals and maximal, minimal ideals are defined in a similar way.

A subset $X \subseteq R$ is called

nil if each element in X is nilpotent;

nilpotent if there is a $n \in \mathbb{N} : X^n = X \circ X \circ .. \circ X = (0)$. The smallest n with this property is called *the nilpotency index* of X. Nil and nilpotent left (right) ideals or ideals are defined in a similar way.

In a ring with identity each proper (left,right) ideal is contained in a maximal (left,right) ideal.

A ring R is called *regular (von Neumann)* if for each element $a \in R$ there is an element $b \in R : a = aba$.

Noether Isomorphism Theorems. (1) *If $f : R \to R'$ is a ring homomorphism then*

$$f(R) \cong R/\ker(f).$$

(2) *If I, J are ideals in a ring R then $(I+J)/I \cong J/(I \cap J)$.*

(3) *If $I \subseteq J$ are ideals in a ring R then J/I is an ideal in R/I and $(R/I)/(J/I) \cong R/J$.*

A right ideal I of a ring R is called *modular* if there is an element $e \in R$ such that for all $r \in R$ the element $r - er \in I$ (R has a left identity element modulo I).

The subset $l(X) = Ann.l.(X) = \{r \in R | rx = 0, \forall x \in X\}$ for a subset X of a ring R is called the *left annihilator of X in R*. Similarly,

$r(X) = Ann.r.(X) = \{r \in R | xr = 0, \forall x \in X\}$ is called the *right annihilator of X in R.*

Ex. 2.1 Let $F = \{f | f : [-1, 1] \to \mathbb{R}\}$ the commutative ring together with the usual addition and multiplication. Which of the following subsets are subrings and which are ideals:

(i) P the polynomial functions;
(ii) P_n the polynomial functions of degree at most n $(n \in \mathbb{N}^*)$;
(iii) Q_n the polynomial functions of degree n;
(iv) $A = \{f \in F | f(0) = 0\}$;
(v) $B = \{f \in F | f(0) = 1\}$.

Ex. 2.2 Let $R = \mathcal{P}(T)$, where $T = [0, 1] \subseteq \mathbb{R}$, the ring together with the usual ring laws of symmetric difference and intersection (see 1.6). For $A = \left[0, \frac{1}{2}\right]$ and $B = \left\{\frac{1}{4}\right\}$ compute $(A), (B), (A, B)$ and $(A) \circ (B)$, the ideals generated by the corresponding subsets of T.

Ex. 2.3 (a) For a subset X of a ring R show that $l(X)$ is a left ideal and $r(X)$ is a right ideal in R.

(b) If X is a left (right) ideal in R then $l(X)$ (resp. $r(X)$) is an ideal in R.

(c) The following inclusions and equalities hold: (i) $X \subseteq Y \Rightarrow l(Y) \subseteq l(X)$ and $r(Y) \subseteq r(X)$; (ii) $X \subseteq r(l(X))$ and $X \subseteq l(r(X))$; (iii) $l(X) = l(r(l(X)))$ and $r(X) = r(l(r(X)))$.

Ex. 2.4 Show that if $R = I + J$ holds with I, J right modular ideals then $I \cap J$ is a right modular ideal too.

Ex. 2.5 Find all the ideals of $(n\mathbb{Z}, +, \cdot)$ for $n \in \mathbb{N}^*$.

Ex. 2.6 Let I, J, K be ideals in a ring R. Show that $I \subseteq K$ implies $(I + J) \cap K = I + (J \cap K)$.

Ex. 2.7 In $\mathbb{Z}$ show that the set-reunion of two ideals needs not to be an ideal too.

Ex. 2.8 Give an example of ring (without identity) such that not every ideal is included in a maximal ideal.

Ex. 2.9 If M is a maximal right ideal of a ring R with identity and $a \in R \backslash M$ then verify that $a^{-1}M = \{r \in R | ar \in M\}$ is also a maximal right ideal.

Ex. 2.10 Show that in a Boole ring each finitely generated ideal is principal.

Ex. 2.11 For two commutative rings R and S, determine the ideals of the direct product (sum) $R \times S$. Applications: $\mathbb{Z} \times \mathbb{Z}$ and $K \times K$ for a division ring K.

Ex. 2.12 Show that the following implication is false: "if A is a left ideal and B is a right ideal in the same ring R then $A \cap B$ is an ideal of R ".

Ex. 2.13 Let $A : B$ denote $\{r \in R | \forall b \in B : rb \in A\}$ for A and B ideals of a ring R.

(a) Show that $A : B$ is an ideal of R;

(b) $A : B$ is the l.u.b.$\{I \trianglelefteq R | IB \subseteq A\}$.

(c) Verify the following equalities : $A : A = R$; $(A_1 \cap .. \cap A_n) : B = (A_1 : B) \cap .. \cap (A_n : B)$ and $A : (B_1 + .. + B) = (A : B_1) \cap .. \cap (A : B_n)$.

(d) In $\mathbb{Z}$ show that $n\mathbb{Z} : m\mathbb{Z} = \frac{[n;m]}{m}\mathbb{Z}$ (here $[n;m]$ denotes the l.c.m.(n,m)).

Ex. 2.14 In a commutative ring R prove the following properties:

(i) $(A : B) \circ B \subseteq A, (A : (A + B)) = (A : B)$;

(ii) $((A : B) : C) = (A : (B \circ C)) = (A : (B : C))$;

(iii) if R has identity $A : B = R$ iff $B \subseteq A$.

Ex. 2.15 Let I be an ideal in a commutative ring R.

(a) Show that $\sqrt{I} = \{r \in R | \exists n \in \mathbb{N} : r^n \in I\}$ is an ideal too.

(b) Verify the following equalities: $\sqrt{\sqrt{I}} = \sqrt{I}$; $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$and $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

(c) Can this exercise be used in order to show that the set of all the nilpotent elements form in a commutative ring an ideal ? Is commutativity essential ?

Ex. 2.16 Let I be a left ideal and J an ideal of R. If I, J are nil (or nilpotent) ideals show that $I + J$ has the same property.

Ex. 2.17 Verify the following properties: subrings and quotient rings of nil rings are nil; for an ideal I in R, if I and R/I are nil then R is nil.

Ex. 2.18 For a prime number p and $n \in \mathbb{N}, n > 1$ show that each proper ideal of $\mathbb{Z}_{p^n}$ is nilpotent. For each n let I_n be a proper ideal in $\mathbb{Z}_{p^n}$ and

$$I = \left\{ (x_n) \in \prod_{n \in \mathbb{N}^*} \mathbb{Z}_{p^n} | x_n \in I_n \text{ and } (x_n) \text{ has finite support} \right\}.$$

Show that I is a nilideal but is not a nilpotent ideal.

Ex. 2.19 Let U be an ideal of the ring R. We say that in R idempotents can be *lifted* modulo U if for each idempotent element $y \in R/U$ there is an idempotent $x \in R$ such that $x + U = y$. Show that if U is a nilideal of R then the idempotents can be lifted modulo U.

Ex. 2.20 Let R be a ring with identity.

(a) Show that every ideal in the ring of all square matrices $\mathcal{M}_n(R)$ has the form $\mathcal{M}_n(A)$ where A is an ideal of R;

(b) Verify the ring isomorphism $\mathcal{M}_n(R)/\mathcal{M}_n(A) \cong \mathcal{M}_n(R/A)$.

(c) In $\mathcal{M}_2(2\mathbb{Z})$, using the set $S = \{a_{ij} \in \mathcal{M}_2(2\mathbb{Z}) | a_{11} \in 4\mathbb{Z}\}$ show that the existence of the identity of the ring is essential.

(d) The result from (a) holds for left (or right) ideals ?

Ex. 2.21 Show that $\mathcal{M}_2(\mathbb{R})$ has no nontrivial ideals.

Ex. 2.22 Give an example of a non-commutative ring with a proper commutative quotient ring.

Ex. 2.23 Let X be a non-empty set and Y a proper subset of X. Consider $\mathcal{P}(X)$ and $\mathcal{P}(X \backslash Y)$ as (boolean) rings (see 1.6) relative to the symmetric difference and the intersection. Show that:

(a) $\mathcal{P}(X \backslash Y) \cong \mathcal{P}(X)/\mathcal{P}(Y)$;

(b) If X is finite every ideal of $\mathcal{P}(X)$ has the form $\mathcal{P}(Y)$ for a suitable subset Y of X;

(c) If X is infinite (b) fails.

Ex. 2.24 Let I and J be ideals in a ring R. Prove that the canonical ring homomorphism $R \,/\, I \cap J \to R/I \times R/J$ is an isomorphism iff $I + J = R$ (so called *comaximal* ideals).

Ex. 2.25 Let R be a commutative ring and $I = I^2$ a finitely generated ideal of R. Find an idempotent element $e \in R$ such that $I = Re$.

Ex. 2.26 Show that in a commutative regular (von Neumann) ring every finitely generated ideal is principal.

Ex. 2.27 For a prime number p let $\mathbb{Q}^{(p)} = \left\{ \frac{m}{n} \in \mathbb{Q} | (n; p) = 1 \right\}$ (as above $(n; p)$ denotes the g.c.d.(n, p) and all fractions are irreductible). Verify the following properties:

(a) $\mathbb{Q}^{(p)}$ is a subring of $\mathbb{Q}$;

(b) For every $x \in \mathbb{Q}$ either $x \in \mathbb{Q}^{(p)}$ or $x^{-1} \in \mathbb{Q}^{(p)}$;

(c) The only subrings of $\mathbb{Q}$ which contain $\mathbb{Q}^{(p)}$ are $\mathbb{Q}^{(p)}$ and $\mathbb{Q}$;

(d) Every ideal of $\mathbb{Q}^{(p)}$ has the form $(p^n) = p^n.\mathbb{Q}^{(p)}$ for a suitable $n \in \mathbb{N}$;

(e) $\bigcap_{p \in \mathbb{P}} \mathbb{Q}^{(p)} = \mathbb{Z}$ (here $\mathbb{P}$ denotes the set of all the prime numbers).