On Classical Nonassociative Lambek Calculus

Wojciech Buszkowski(✉)

Faculty of Mathematics and Computer Science,
Adam Mickiewicz University, Poznań, Poland
buszko@amu.edu.pl

Abstract. CNL, intoduced by de Groote and Lamarche [11], is a conservative extension of Nonassociative Lambek Calculus (NL) by a De Morgan negation $^\sim$, satisfying $A^\sim/B \Leftrightarrow A\backslash B^\sim$. [11] provides a fine theory of proof nets for CNL and shows cut elimination and polynomial decidability. Here the purely proof-theoretic approach of [11] is enriched with algebras and phase spaces for CNL. We prove that CNL is a strongly conservative extension of NL, CNL has the strong finite model property, the grammars based on CNL (also with assumptions) generate the context-free languages, and the finitary consequence relation for CNL is decidable in polynomial time.

Keywords: Lambek calculus · Phase space · Sequent system · Type grammar

1 Introduction

NL, due to Lambek [13], admits formulas built from variables and the connectives $\otimes, \backslash, /$. The axioms and the rules are as follows.

$$\text{(NL-id) } A \Rightarrow A$$

$$(\otimes\Rightarrow)\ \frac{\Gamma[(A,B)]\Rightarrow C}{\Gamma[A\otimes B]\Rightarrow C} \quad (\Rightarrow\otimes)\ \frac{\Gamma\Rightarrow A \quad \Delta\Rightarrow B}{(\Gamma,\Delta)\Rightarrow A\otimes B}$$

$$(\backslash\Rightarrow)\ \frac{\Gamma[B]\Rightarrow C \quad \Delta\Rightarrow A}{\Gamma[(\Delta,A\backslash B)]\Rightarrow C} \quad (\Rightarrow\backslash)\ \frac{(A,\Gamma)\Rightarrow B}{\Gamma\Rightarrow A\backslash B}$$

$$(/\Rightarrow)\ \frac{\Gamma[A]\Rightarrow C \quad \Delta\Rightarrow B}{\Gamma[(A/B,\Delta)]\Rightarrow C} \quad (\Rightarrow/)\ \frac{(\Gamma,B)\Rightarrow A}{\Gamma\Rightarrow A/B}$$

$$\text{(NL-cut) } \frac{\Gamma[A]\Rightarrow B \quad \Delta\Rightarrow A}{\Gamma[\Delta]\Rightarrow B}$$

This is a sequent system for NL. Sequents are of the form $\Gamma \Rightarrow A$, where A is a formula and Γ is a formula structure. Formula structures are defined recursively: (i) all formulas are formula structures, (ii) if Γ and Δ are formula structures, then (Γ, Δ) is a formula structure. Formula structures represent the elements of the free groupoid generated by formulas. A context $\Gamma[\,]$ is a formula structure

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containing one special formula x. $\Gamma[\Delta]$ denotes the substitution of Δ for x in $\Gamma[\,]$. We reserve A, B, C, D for formulas and Γ, Δ, Θ for formula structures.

NL is strongly complete with respect to residuated groupoids (see Sect. 2 for the definition). Recall that a logic (in the form of a sequent system) is strongly complete with respect to a class of (ordered) algebras $\mathcal{C}$, if the following equivalence holds: $\Gamma \Rightarrow A$ is provable in this logic from the set of sequents Φ if and only if, for any algebra from $\mathcal{C}$ and any valuation μ, $\Gamma \Rightarrow A$ is true for μ whenever all sequents from Φ are true for μ. The right-hand side of this equivalence expresses the semantic entailment: $\Gamma \Rightarrow A$ follows from Φ in $\mathcal{C}$. For systems considered here, $\Gamma \Rightarrow A$ is *true* for μ, if $\mu(\Gamma) \leq \mu(A)$.

NL1 is NL admitting empty antecedents of sequents and containing the constant 1, the axiom (a-1) $\Rightarrow 1$ and the rules:

$$(1\Rightarrow)\ \frac{\Gamma[\Delta] \Rightarrow A}{\Gamma[(1,\Delta)] \Rightarrow A}\,, \quad \frac{\Gamma[\Delta] \Rightarrow A}{\Gamma[(\Delta,1)] \Rightarrow A}\,.$$

NL1 is strongly complete with respect to residuated unital groupoids.

Classical Nonassociative Lambek Calculus (CNL) can be presented as an extension of NL with negation $^{\sim}$, admitting the axioms $A^{\sim\sim} \Leftrightarrow A$, $A^{\sim}/B \Leftrightarrow A\backslash B^{\sim}$ and the transposition rule:

$$\frac{A \Rightarrow B}{B^{\sim} \Rightarrow A^{\sim}}\,.$$

Here $A \Leftrightarrow B$ replaces two sequents: $A \Rightarrow B$ and $B \Rightarrow A$. In [11], CNL is presented as a Schütte style (i.e. one-sided) sequent system in language $\otimes, \oplus, ^{\sim}$, where $A \oplus B$ is equivalent to $(B^{\sim} \otimes A^{\sim})^{\sim}$. So $\oplus$ corresponds to the operation 'par' in linear logics. We do not follow the popular notation of Girard [10], but replace it with a notation used in substructural logics [9]. CNL is a nonassociative variant of Cyclic Noncommutative MALL [15], but it lacks the multiplicative units.

In Sect. 2 we define CNL-algebras, i.e. the ordered algebras corresponding to CNL. We also define phase spaces, appropriate for nonassociative logics without units. We show that CNL-algebras arise from symmetric phase spaces, satisfying a compatibility condition.

In Sect. 3 we present CNL as a dual Schütte style system, which seems closer to the syntax of NL and the framework of type grammars. We discuss the strong completeness of CNL with respect to CNL-algebras and phase spaces. In particular, we outline a model-theoretic proof of cut elimination, similar to those for different substructural logics (see [9] for a discussion). Theorem 2 states that CNL is a strongly conservative extension of NL; we give a model-theoretic proof. At the end we briefly discuss analogous results for related logics: CNL1, i.e. CNL with constants 1 and 0, CNL and CNL1 with $\vee, \wedge$, and others.

In Sect. 4 we prove an interpolation lemma for CNL (with assumptions), analogous to the interpolation lemma for NL [4,8]. Using this lemma, we prove the strong finite model property (SFMP) for CNL (see [9] for the definition), the context-freeness of the languages generated by CNL-grammars and the polynomial time decidability of the consequence relation for CNL. These results remain true for CNL1. At the end, we discuss their status for other logics.

The size limits do not allow us to study CNL^-, i.e. the variant of CNL with two negations $^\sim, ^-$, satisfying $A^{\sim -} \Leftrightarrow A$, $A^{-\sim} \Leftrightarrow A$, $A^\sim/B \Leftrightarrow A\backslash B^-$ and the transposition rules. CNL^- is a nonassociative variant of Noncommutative MALL [1], also called Classical Bilinear Logic in [14]; again it lacks units. The corresponding algebras are briefly discussed in Sect. 2. We only note here that CNL^- does not have SFMP. $A^\sim \Rightarrow A^-$ entails $A^- \Rightarrow A^\sim$ in finite CNL^--algebras, since $a^\sim < a^-$ enforces the infinite chain $a < a^{\sim\sim} < a^{\sim\sim\sim\sim} < \ldots$; there exist infinite CNL^--algebras such that $a^\sim < a^-$, for some element a.

2 Algebras and Phase Spaces

The algebraic models of NL are residuated groupoids $\mathbf{M} = (M, \otimes, \backslash, /, \leq)$ such that $(M, \leq)$ is a nonempty poset and $\otimes, \backslash, /$ are binary operations on M, satisfying:

$$\text{(RES) } a \otimes b \leq c \text{ iff } b \leq a\backslash c \text{ iff } a \leq c/b\,,$$

for all $a, b, c \in M$. The models of NL1 are residuated unital groupoids, i.e. residuated groupoids containing the unit element for $\otimes$ (denoted by 1). It follows that $1\backslash a = a$, $a/1 = a$.

A pair $^\sim, ^-$ of unary operations on a poset $(P, \leq)$ is called *an involutive pair of negations*, if for all $a, b \in P$ the following conditions are satisfied:

(TR) if $a \leq b$ then $b^\sim \leq a^\sim$ and $b^- \leq a^-$,
(DN) $a^{-\sim} = a$, $a^{\sim -} = a$;

if $^\sim$ equals $^-$, then $^\sim$ is called *a De Morgan negation* (then $a^{\sim\sim} = a$).

The models of CNL are residuated groupoids $\mathbf{M}$ with a De Morgan negation $^\sim$ satisfying the compatibility condition:

(COM) for all $a, b, c \in M$, if $a \otimes b \leq c$ then $c^\sim \otimes a \leq b^\sim$.

We refer to these algebras as *CNL-algebras*. Unital CNL-algebras (i.e. with the unit for $\otimes$) are called *CNL1-algebras*.

In any CNL-algebra the following conditions are equivalent: $a \otimes b \leq c$, $c^\sim \otimes a \leq b^\sim$, $b \otimes c^\sim \leq a^\sim$. On the basis of other axioms, (COM) is equivalent to:

(TR') $a\backslash b^\sim = a^\sim/b$ for all $a, b \in M$,

and either of the following transposition laws: $a\backslash b = a^\sim/b^\sim$, $a/b = a^\sim\backslash b^\sim$.

In any CNL-algebra one defines *the dual product*: $a \oplus b = (b^\sim \otimes a^\sim)^\sim$. The following equations hold:

$$a\backslash b = a^\sim \oplus b,\ a/b = a \oplus b^\sim.$$

Consequently, $\oplus, \backslash, /$ are definable in terms of $\otimes, ^\sim$.

In any CNL1-algebra one defines: $0 = 1^\sim$. Then, $1 = 0^\sim$, 0 is the unit for $\oplus$, and $a^\sim = a\backslash 0 = 0/a$.

CNL^--algebras are residuated groupoids $\mathbf{M}$ with an involutive pair of negations, satisfying:

(COM$^-$) for all $a, b, c \in M$, if $a \otimes b \leq c$ then $c^- \otimes a \leq b^-$ and $b \otimes c^\sim \leq a^\sim$.

Unital CNL$^-$-algebras are referred to as *CNL1$^-$-algebras*. In any CNL$^-$-algebra the three conditions in (COM$^-$) are equivalent. (COM$^-$) is equivalent to:

(TR") $a^\sim/b = a\backslash b^-$, for all $a, b \in M$.

Hence in any CNL1$^-$-algebra, $1^\sim = 1^-$. One defines $0 = 1^\sim$ and obtains: $a^\sim = a\backslash 0$, $a^- = 0/a$.

CNL$^-$-algebras (resp. CNL-algebras) are term equivalent to (resp. cyclic) involutive p.o. groupoids [9].

The equation $(a^- \otimes b^-)^\sim = (a^\sim \otimes b^\sim)^-$ is valid in CNL$^-$-algebras. One defines $a \oplus b = (b^- \otimes a^-)^\sim$ and obtains:

$$a\backslash b = a^\sim \oplus b,\ a/b = a \oplus b^-.$$

Consequently, $\oplus, \backslash, /$ are definable in terms of $\otimes, ^\sim, ^-$.

These algebras can be constructed from *phase spaces*, i.e. structures $(M, \cdot, R)$ such that $(M, \cdot)$ is a groupoid and $R \subseteq M^2$. We focus on *symmetric* phase spaces (R is symmetric).

A closure operation on a poset $(P, \leq)$ is a map $C : P \mapsto P$, satisfying: (C1) $x \leq C(x)$, (C2) if $x \leq y$ then $C(x) \leq C(y)$, (C3) $C(C(x)) \leq C(x)$, for all $x, y \in P$. *A nucleus* on a p.o. groupoid $(M, \cdot, \leq)$ is a closure operation C on $(M, \leq)$, satisfying: (C4) $C(x) \cdot C(y) \leq C(x \cdot y)$. If $\mathbf{M}$ is a residuated groupoid, then C is a nucleus on $(M, \cdot, \leq)$ iff C is a closure operation on $(M, \leq)$ and satisfies: (C4') $x\backslash y$ and y/x are C-closed for any $x \in M$ and any C-closed $y \in M$. Recall that x is *C-closed*, if $C(x) = x$.

Let $R \subseteq M^2$. For $X \subseteq M$, one defines:

$$X^\sim = \{a \in M : \forall_{b \in X} R(b, a)\},\ X^- = \{a \in M : \forall_{b \in X} R(a, b)\}.$$

The maps $^\sim, ^-$ are a Galois connection on $\mathcal{P}(M)$: $X \subseteq Y^\sim$ iff $Y \subseteq X^-$. Consequently, $X \subseteq Y$ entails $Y^\sim \subseteq X^\sim$ and $Y^- \subseteq X^-$. The maps $\phi_R(X) = X^{-\sim}$ and $\psi_R(X) = X^{\sim-}$ are closure operations on $(\mathcal{P}(M), \subseteq)$. It follows that X is ϕ_R-closed (resp. ψ_R-closed) iff $X = Y^\sim$ (resp. $X = Y^-$) for some Y.

Proposition 1. *The following conditions are equivalent. (i) $X^\sim = X^-$ for all $X \subseteq M$, (ii) R is symmetric: $R(a, b)$ entails $R(b, a)$, for all $a, b \in M$.*

Let $(M, \cdot, R)$ be a symmetric phase space. Then, $\phi_R = \psi_R$. By M_R we denote the family of ϕ_R-closed subsets of M. Clearly $^\sim$ is a De Morgan negation on $(M_R, \subseteq)$.

Let $(M, \cdot, R)$ be a phase space. For $X, Y \subseteq M$, one defines: $X \cdot Y = \{a \cdot b : a \in X, b \in Y\}$, $X\backslash Y = \{y \in M : X \cdot \{y\} \subseteq Y\}$, $X/Y = \{x \in M : \{x\} \cdot Y \subseteq X\}$. $\mathcal{P}(\mathcal{M})$ with $\cdot, \backslash, /, \subseteq$ is a residuated groupoid. Let C be a nucleus on $(\mathcal{P}(M), \cdot, \subseteq)$. Then, $(M_C, \otimes^C, \backslash^C, /^C, \subseteq)$ is a residuated groupoid, where M_C is the family of $C-$closed subsets of M, $X \otimes^C Y = C(X \cdot Y)$, and $\backslash, /$ are the operations defined on $\mathcal{P}(M)$, restricted to M_C. If $\cdot$ is associative (resp. commutative), then $\otimes^C$ is associative (resp. commutative). If 1 is the unit for $\cdot$ in M, then $C(\{1\})$ is the unit for $\otimes^C$ (see e.g. [9]).

By *a phase space for CNL* we mean a symmetric phase space $(M, \cdot, R)$, satisfying the compatibility condition:

(COM-R) for all $a, b, c \in M$, $R(a \cdot b, c)$ iff $R(a, b \cdot c)$.

Phase spaces for CNL$^-$ are defined in a similar way except that the symmetry of R is replaced with $\phi_R = \psi_R$.

Proposition 2. *For any phase space, (COM-R) holds if and only if, for all $X, Y, Z \subseteq M$, $X \cdot Y \subseteq Z^\sim$ iff $Z \cdot X \subseteq Y^-$.*

Proof. We show ($\Rightarrow$). $X \cdot Y \subseteq Z^\sim$ is equivalent to $\forall_{x \in X} \forall_{y \in Y} \forall_{z \in Z} R(z, x \cdot y)$, and $Z \cdot X \subseteq Y^-$ to iff $\forall_{z \in Z} \forall_{x \in X} \forall_{y \in Y} R(z \cdot x, y)$. Both statements are equivalent, by (COM-R). For ($\Leftarrow$), take $X = \{b\}$, $Y = \{c\}$, $Z = \{a\}$. Now $\{b\} \cdot \{c\} \subseteq \{a\}^\sim$ iff $R(a, b \cdot c)$, and $\{a\} \cdot \{b\} \subseteq \{c\}^-$ iff $R(a \cdot b, c)$. □

Corollary 1. *For any phase space, (COM-R) holds if and only if, for all $Y, Z \subseteq M$, $Z^\sim / Y = Z \backslash Y^-$.*

Theorem 1. *Let $(M, \cdot, R)$ be a phase space for CNL. Then M_R, ordered by $\subseteq$, with operations $\otimes^{\phi_R}$ and $\backslash, /, ^\sim$, restricted to M_R, is a CNL-algebra.*

Proof. First, we show that ϕ_R satisfies (C4'). Using (COM-R), we show that $\{a\} \backslash \{b\}^\sim = \{b \cdot a\}^\sim$ and $\{a\}^\sim / \{b\} = \{b \cdot a\}^\sim$ for all $a, b \in M$. We have: $c \in \{a\} \backslash \{b\}^\sim$ iff $a \cdot c \in \{b\}^\sim$ iff $R(b, a \cdot c)$ iff $R(b \cdot a, c)$ iff $c \in \{b \cdot a\}^\sim$. The second equation is proved similarly (use the symmetry of R). This yields $X \backslash Y^\sim = (Y \cdot X)^\sim$ and $X^\sim / Y = (Y \cdot X)^\sim$, for all $X, Y \subseteq M$, by the well-known distribution laws: $\cdot$ distributes over infinite joins in both arguments, $\backslash$ (resp. $/$) distributes over infinite meets in the second (resp. first) argument and converts joins into meets in the first (resp. second) argument, and $^\sim$ converts joins into meets. So for $X = \{a_i\}_{i \in I}$, $Y = \{b_j\}_{j \in J}$ we have:

$$X \backslash Y^\sim = \bigcap_{i \in I} \bigcap_{j \in J} \{a_i\} \backslash \{b_j\}^\sim = \bigcap_{i \in I} \bigcap_{j \in J} \{b_j \cdot a_i\}^\sim = (Y \cdot X)^\sim.$$

Let $X \subseteq M$, $Z \in M_R$. Then $Z = Y^\sim$ for some Y. Hence $X \backslash Z = (Y \cdot X)^\sim$ belongs to M_R, and similarly for Z/X.

Since ϕ_R is a nucleus on $(\mathcal{P}(M), \cdot, \subseteq)$, then M_R with $\otimes^{\phi_R}, \backslash, /, \subseteq$ is a residuated groupoid. Since R is symmetric, $^\sim$ is a De Morgan negation on M_R. (TR') $X \backslash Y^\sim = X^\sim / Y$, for $X, Y \in M_R$, has been shown in the preceding paragraph; (TR') also follows from Corollary 1. □

If $(M, \cdot, R)$ is a phase space for CNL, then the CNL-algebra constructed above is referred to as *the complex algebra* of the phase space. Worthy of noting, every CNL-algebra **M** is isomorphic to a subalgebra of the complex algebra of the phase space $(M, \otimes, R)$, where R is defined by: $R(a, b)$ iff $a \leq b^\sim$. Let $[a]^\downarrow$ denote the principal downset in $(M, \leq)$ generated by a, i.e. $[a]^{\downarrow} = \{x \in M : x \leq a\}$. Then, $[a]^{\downarrow} = \{a^{\sim}\}^{\sim}$. (Here $^{\sim}$ is used in two meanings: the inner one as an operation in $\mathbf{M}$, the outer one as an operation on $\mathcal{P}(M)$.) The so-defined R is symmetric and $[a]^{\downarrow} \in M_R$. The map $h(a) = [a]^{\downarrow}$ is the required isomorphism. We omit the proof.

Remark 1. In fact, for any symmetric phase space $(M, \cdot, R)$, (COM-R) holds if and only if ϕ_R is a nucleus and (TR') (equivalently (COM)) holds in the complex algebra.

A unital phase space is a structure $(M, \cdot, 1, R)$ such that $(M, \cdot, 1)$ is a unital groupoid and $R \subseteq M^2$. *A phase space for* CNL1 is a unital phase space $(M, \cdot, 1, R)$ such that $(M, \cdot, R)$ is a phase space for CNL. The analogue of Theorem 1 remains true. Now $\phi_R(\{1\})$ is the unit for $\otimes^{\phi_R}$ in the complex algebra.

For unital phase spaces, (COM-R) implies:

$$\text{(Eq-R) } R(a, b) \text{ iff } R(1, a \cdot b) \text{ iff } R(a \cdot b, 1).$$

R can be represented by a set $O \subseteq M$, satisfying:

(COM-O) for all $a, b, c \in M$, $a \cdot (b \cdot c) \in O$ iff $(a \cdot b) \cdot c \in O$.

For $R \subseteq M^2$, we define $O_R = \{a \in M : R(1, a)\}$, and for $O \subseteq M$, we define $R_O = \{(a, b) \in M^2 : a \cdot b \in O\}$. By (Eq-R), $R_{O_R} = R$ and $O_{R_O} = O$. Furthermore, R satisfies (COM-R) iff O_R satisfies (COM-O). So there is a one-one correspondence between relations $R \subseteq M^2$ satisfying (COM-R) and sets $O \subseteq M$ satisfying (COM-O). Therefore, unital phase spaces, satisfying (COM-R), can also be defined as structures $(M, \cdot, 1, O)$ such that $(M, \cdot, 1)$ is a unital groupoid and $O \subseteq M$ satisfies (COM-O). This resembles the standard definitions of phase spaces for linear logics [1,10,15].

Remark 2. If 1 is not present, then we can define $O_R = \{a \cdot b : R(a, b)\}$ and R_O as above, but this only yields the inclusions: $R \subseteq R_{O_R}$ and $O_{R_O} \subseteq O$. O satisfies (COM-O) iff R_O satisfies (COM-R). On the other hand, if O_R satisfies (COM-O), then R satisfies (COM-R), but the converse implication fails. If, however, $(M, \cdot)$ is a free groupoid, then there is a one-one correspondence between relations $R \subseteq M^2$ and sets $O \subseteq M$ such that each element of O is of the form $x \cdot y$, for some $x, y \in M$. Also R satisfies (COM-R) iff O_R satisfies (COM-O).

Let $(M, \cdot, 1, O)$ be a unital phase space. The symmetry of R_O is equivalent to *the cyclic law* for O:

(Cy) for all $a, b \in M$, if $a \cdot b \in O$ then $b \cdot a \in O$.

Accordingly, a phase space for CNL1 can be defined as a unital phase space $(M, \cdot, 1, O)$, satisfying (COM-O) and (Cy). Observe that $X^{\sim} = X \backslash O = O / X$, for any $X \subseteq M$. We denote $\phi_O = \phi_{R_O}$, and similarly for ψ_O, M_O. O is ϕ_O-closed, since $O = \{1\}^{\sim}$. So $O \in M_O$; also $O^{\sim}$ is the unit for $\otimes^{\phi_O}$ and O is the unit for the dual product. If M does not contain 1, then O, even satisfying (COM-O) and (Cy), need not belong to M_O.

Example 1. Consider the phase space $(M, \cdot, O)$ such that $M = \Sigma^+$, $\cdot$ is the concatenation of strings, and O is the set of all strings of length 1. Clearly O satisfies (COM-O) and (Cy). So the complex algebra of $(M, \cdot, R_O)$ is a CNL-algebra. We have $\emptyset^\sim = \Sigma^+$ and $X^\sim = \emptyset$ for $X \neq \emptyset$. Therefore $M_O = \{\emptyset, \Sigma^+\}$ and $O \notin M_O$.

Example 2. We construct a phase space $(M, +, R)$ such that $(M, +)$ is a commutative semigroup, $R \subseteq M^2$ is symmetric and satisfies (COM-R), but $R \neq R_O$, for any $O \subseteq M$. Let M consist of all pairs of positive integers. For $a, b \in M$, $a = (a_1, a_2)$, $b = (b_1, b_2)$, we set $a + b = (a_1 + b_1, a_2 + b_2)$. Let R consist of all $(a, b) \in M^2$ such that neither a, nor b is of the form $x + y$, for any $x, y \in M$. Clearly R is symmetric and satisfies (COM-R). Assume $R = R_O$ for some $O \subseteq M$. Since $R((1,2),(2,1))$, then $(3,3) \in O$. We have $(3,3) = (1,1) + (2,2)$, which yields $R((1,1),(2,2))$. This contradicts the definition of R, since $(2,2) = (1,1) + (1,1)$. □

This example shows that the notion of a phase space with a relation R is essentially wider than that with a set O for the non-unital spaces, even based on (commutative) semigroups. Therefore the former may also be useful in the theory of associative linear logics with no multiplicative units (not only in language, but in the corresponding algebras). Clearly (COM-O) (resp. (Cy)) holds for any $O \subseteq M$, if $\cdot$ is associative (resp. commutative).

3 Logics

We present a dual Schütte style system for CNL. Formulas are built from variables $p, q, \ldots$, negated variables $p^\sim, q^\sim, \ldots$, and connectives $\otimes, \oplus$. A, B, C, D range over formulas. By $\mathcal{S}$ we denote the free groupoid generated by all formulas. Γ, Δ, Θ range over elements of $\mathcal{S}$. These elements are represented as formula structures. The groupoid operation is: $\Gamma \cdot \Delta = (\Gamma, \Delta)$.

In CNL, sequents are formula-structures, containing at least two formulas; the set of all sequents is denoted by $\mathcal{S}^{(2)}$. So the distinction between quasi-sequents and sequents in [11] corresponds to our distinction between formula-structures and sequents. In axioms and rules of our systems (and after the provability symbol $\vdash$) we omit outer parentheses, e.g. we write $\vdash \Gamma, \Delta$ for $\vdash (\Gamma, \Delta)$. The axioms and the rules of CNL are as follows.

$$\text{(id) } p, p^\sim$$

$$\text{(r-}\otimes\text{) } \frac{(A,B),\Gamma}{A\otimes B,\Gamma} \quad \text{(r-}\oplus\text{) } \frac{A,\Gamma \quad B,\Delta}{A\oplus B,(\Delta,\Gamma)}$$

$$\text{(r-sym) } \frac{\Gamma,\Delta}{\Delta,\Gamma} \quad \text{(r-com) } \frac{(\Gamma,\Delta),\Theta}{\Gamma,(\Delta,\Theta)}$$

(r-$\otimes$), (r-$\oplus$) are the introduction rules for connectives, and (r-sym), (r-com) are the structural rules (expressing the symmetry of R and the condition (COM-R) in phase spaces for CNL).

We write $\Gamma \sim \Delta$, if Δ can be derived from Γ by finitely many applications of (r-sym), (r-com). Clearly $\sim$ is an equivalence relation (but not a congruence in $\mathcal{S}$).

Proposition 3. *For any sequent $\Gamma' \in \mathcal{S}^{(2)}$, containing one marked formula $\underline{A}$, there exists a unique $\Delta' \in \mathcal{S}$ such that $\Gamma' \sim (\underline{A}, \Delta')$.*

Proof. We describe an algorithm which reduces Γ' to some sequent $(\underline{A}, \Delta')$. We underline the substructure containing $\underline{A}$. The reduction rules are as follows.

$$\begin{array}{l}(\mathrm{R1})\ (\Gamma, \underline{\Delta}) \to (\underline{\Delta}, \Gamma)\\ (\mathrm{R2})\ ((\underline{\Gamma}, \Delta), \Theta) \to (\underline{\Gamma}, (\Delta, \Theta))\\ (\mathrm{R3})\ ((\Gamma, \underline{\Delta}), \Theta) \to (\underline{\Delta}, (\Theta, \Gamma))\end{array}$$

Each reduction step can be executed by applying at most three instances of (r-sym), (r-com). This procedure is deterministic. If we run it on a sequent $\Gamma' \in \mathcal{S}^{(2)}$, then the algorithm terminates in finitely many steps and yields $(\underline{A}, \Delta')$.

The uniqueness of Δ', satisfying $\Gamma' \sim (\underline{A}, \Delta')$, follows from the fact:

(F1) if Γ' reduces to $(\underline{A}, \Delta')$ and $\Theta' \sim \Gamma'$, then Θ' reduces to $(\underline{A}, \Delta')$.

The proof of (F1) has two parts: (I) one proves it for Θ' resulting from Γ' by one application of (r-sym) or (r-com), (II) one proves (F1) by induction on the number of applications of (r-sym), (r-com) leading from Γ' to Θ'. We skip details.

Now assume that $\Gamma' \sim (\underline{A}, \Delta)$ and $\Gamma' \sim (\underline{A}, \Delta')$. Then $(\underline{A}, \Delta) \sim (\underline{A}, \Delta')$. By (F1), $(\underline{A}, \Delta)$ reduces to $(\underline{A}, \Delta')$. Since the algorithm stops on sequents of this form, then $\Delta = \Delta'$. □

Example 3. Take $\Gamma' = ((B, (C', \underline{A})), (C, D))$. The reduction looks as follows:

$$\Gamma' \to_{R3} ((C', \underline{A}), ((C, D), B)) \to_{R3} (\underline{A}, (((C, D), B), C')).$$

Due to Proposition 3, the introduction rules can be restricted to the left-most occurrences of formulas in sequents, as above.

We say that a reduction of Γ' to $(\underline{A}, \Delta')$ *preserves* a substructure Θ of Γ', if Θ can be replaced by a variable in the whole reduction. The reduction in Example 3 preserves (C, D).

Lemma 1. *Assume that Γ' reduces to $(\underline{A}, \Delta')$ and Θ is a substructure of Γ', which does not contain $\underline{A}$. Then, the reduction preserves Θ.*

Proof. Let Γ_1 result from Γ' after one has replaced Θ by a new variable p. By Proposition 3, Γ_1 reduces to a sequent $(\underline{A}, \Delta_1)$. Now we substitute Θ for p in the whole reduction, which yields the reduction of Γ' to a sequent $(\underline{A}, \Delta)$. We have $\Delta = \Delta'$, since the algorithm is deterministic. Consequently, the reduction of Γ' to $(\underline{A}, \Delta')$ preserves Θ. □

Let $\mathbf{M}$ be a CNL-algebra. *A valuation in* $\mathbf{M}$ is a homomorphism of the free algebra of CNL-formulas into $\mathbf{M}$ such that $\mu(p^{\sim}) = \mu(p)^{\sim}$, for any (non-negated) variable p. The valuation μ is extended for sequents, by setting: $\mu((\Gamma, \Delta)) = \mu(\Gamma) \otimes \mu(\Delta)$. The sequent (Γ, Δ) is *true* for μ in $\mathbf{M}$, if $\mu(\Gamma) \leq \mu(\Delta)^{\sim}$. A sequent is *valid* in $\mathbf{M}$, if it is true for all valuations in $\mathbf{M}$.

The above system of CNL is *weakly complete*: the provable sequents are precisely the sequents valid in all CNL-algebras. Since the system is cut-free, its weak completeness entails the cut-elimination theorem (see below). Soundness is easy. The proof of completeness is a routine modification of similar proofs for different substructural logics, tracing back to Lafont [12]; see [9] for a wider discussion. Since for CNL and its variants no proof can be found in the literature, we give some details. We write $\vdash \Gamma$ if Γ is provable in CNL.

In metalanguage, one defines $A^{\sim}$ for any formula A:

$$(p^{\sim})^{\sim} = p$$

$$(A \otimes B)^{\sim} = B^{\sim} \oplus A^{\sim} \quad (A \oplus B)^{\sim} = B^{\sim} \otimes A^{\sim}$$

By formula induction, one proves $A^{\sim\sim} = A$ and $\mu(A^{\sim}) = \mu(A)^{\sim}$, for any formula A and any valuation μ in $\mathbf{M}$. Also $\vdash A, A^{\sim}$, for any A.

It is convenient to write $\Gamma \Rightarrow A$ for the sequent $(\Gamma, A^{\sim})$; due to (r-sym), it is deductively equivalent to $(A^{\sim}, \Gamma)$. Clearly $\Gamma \Rightarrow A$ is true for μ in $\mathbf{M}$, if $\mu(\Gamma) \leq \mu(A)$. We define $[A] = \{\Gamma \in \mathcal{S} : \vdash \Gamma \Rightarrow A\}$.

We consider the phase space $(M, \cdot, R)$ such that $(M, \cdot) = (\mathcal{S}, \cdot)$ and $R = \{(\Gamma, \Delta) \in \mathcal{S}^2 : \vdash \Gamma, \Delta\}$. Since $(M, \cdot)$ is a free groupoid, R can be replaced by the set $O_R = \{(\Gamma, \Delta) \in \mathcal{S} : R(\Gamma, \Delta)\}$ (see Remark 2 in Sect. 2). Due to (r-com), (r-sym), R is symmetric and satisfies (COM-R). By Theorem 1, M_R with inclusion and $\otimes^{\phi_R}, \backslash, /, ^{\sim}$ is a CNL-algebra. For any formula A, we have: $[A] = \{A^{\sim}\}^{\sim}$. So $[A]$ is ϕ_R-closed for any formula A.

We define a valuation μ in M_R:

$$\mu(p) = [p] = \{p^{\sim}\}^{\sim}, \ \mu(p^{\sim}) = \mu(p)^{\sim} \, . \tag{1}$$

By formula induction, one proves:

$$A \in \mu(A) \subseteq [A], \text{ for any formula } A. \tag{2}$$

We only consider the case: $A \otimes B$. Since $A \in \mu(A)$, $B \in \mu(B)$, then $(A, B) \in \mu(A) \cdot \mu(B) \subseteq \mu(A \otimes B)$. We use the fact:

(F2) if $(A, B) \in X$ and X is ϕ_R-closed then $A \otimes B \in X$.

Let $X = Y^{\sim}$, $(A, B) \in X$. Then, for all $\Gamma \in Y$, $\vdash (A, B), \Gamma$, hence $\vdash A \otimes B, \Gamma$, by (r-$\otimes$). So $A \otimes B \in X$. Consequently $A \otimes B \in \mu(A \otimes B)$.

We show $\mu(A \otimes B) \subseteq [A \otimes B]$. Since $[A \otimes B]$ is ϕ_R-closed, it suffices to show $\mu(A) \cdot \mu(B) \subseteq [A \otimes B]$. Let $\Gamma \in \mu(A)$, $\Delta \in \mu(B)$. Then, $\Gamma \in [A]$, $\Delta \in [B]$, hence $\vdash A^{\sim}, \Gamma$, $\vdash B^{\sim}, \Delta$. By (r-$\oplus$), $\vdash (A \otimes B)^{\sim}, (\Gamma, \Delta)$, which yields $(\Gamma, \Delta) \in [A \otimes B]$.

Now assume $\not\vdash \Gamma, \Delta$. By Proposition 3, there exists a sequent $(A, \Theta) \sim (\Gamma, \Delta)$. Then $\not\vdash A, \Theta$, hence $\Theta \notin [A^{\sim}]$. By (2), $\Theta \notin \mu(A^{\sim})$ and $\Theta \in \mu(\Theta)$. Consequently (A, Θ) is not true for μ in the complex algebra of $(M, \cdot, R)$. It follows that (Γ, Δ) is not true, since the set of true sequents is invariant under $\sim$. This finishes the proof of weak completeness.

The sequents valid in CNL-algebras are closed under the cut rule:

$$(\text{cut})\frac{\Gamma[A] \quad A^{\sim}, \Delta}{\Gamma[\Delta]} .$$

Therefore (cut) is admissible in the cut-free system of CNL. By Proposition 3, this rule can also be formulated in the form:

$$(\text{cut'})\ \frac{A, \Gamma \quad A^{\sim}, \Delta}{\Delta, \Gamma} .$$

The system of CNL with (cut') is *strongly complete* with respect to CNL-algebras: the sequents provable from a set of assumptions Φ are precisely those which follow from Φ in CNL-algebras.

Let $f(\Gamma)$ be the formula arising from Γ after one has replaced each comma by $\otimes$. Every sequent (Γ, Δ) is deductively equivalent to $(f(\Gamma), f(\Delta))$. This is easy to prove with applying (cut); for the cut-free system one can use the reversibility of (r-$\otimes$). Therefore, without lost of generality, we assume that all sequents in Φ are of the form (A, B).

In the proof of strong completeness, one constructs the complex algebra of $(\mathcal{S}, \cdot, R)$, where $R = \{(\Gamma, \Delta) \in \mathcal{S} : \Phi \vdash \Gamma, \Delta\}$. Now $[A] = \{\Gamma \in \mathcal{S} : \Phi \vdash \Gamma \Rightarrow A\}$, and μ is defined by (1).

In the presence of (cut'), the inclusion in (2) can be replaced by $\mu(A) = [A]$; so $A \in \mu(A)$ may be omitted. We use the fact:

(F3) if X is ϕ_R-closed, $A \in X$ and $\Phi \vdash \Gamma \Rightarrow A$, then $\Gamma \in X$.

This is needed to prove that all sequents from Φ are true for μ in the complex algebra. Let $(A, B) \in \Phi$. Then, $A \in [B^{\sim}]$, hence $[A] \subseteq [B^{\sim}]$, by (F3). Consequently $\mu(A) \subseteq \mu(B^{\sim}) = \mu(B)^{\sim}$.

Remark 3. We have shown in Sect. 2 that not every phase space $(M, \cdot, R)$ can be replaced by $(M, \cdot, O)$. The above proof shows that CNL is strongly complete with respect to phase spaces of the latter form, satisfying (COM-O) and (Cy) (even based on free groupoids). It follows that every CNL-algebra is isomorphic to a subalgebra of the complex algebra of some space $(M, \cdot, O)$ such that $(M, \cdot)$ is a free groupoid.

The connectives $\backslash, /$ can be defined by: $A \backslash B = A^{\sim} \oplus B$, $A / B = A \oplus B^{\sim}$. Each NL-sequent $\Gamma \Rightarrow A$ can be treated as a CNL-sequent $\Gamma \Rightarrow A$, i.e. $(A^{\sim}, \Gamma)$. We prove that CNL with (cut') is a *strongly conservative extension* of NL with (NL-cut). The weak conservativeness was proved in [11] by proof-theoretic methods.

Theorem 2. *Let Φ be a set of NL-sequents (of the form $C \Rightarrow D$), and let $\Gamma \Rightarrow A$ be an NL-sequent. Then, $\Phi \vdash_{NL} \Gamma \Rightarrow A$ iff $\Phi \vdash_{CNL} \Gamma \Rightarrow A$.*

Proof. The only-if part is easy. The easiest proof uses the strong completeness, hence soundness, of NL with respect to residuated groupoids and the strong completeness of CNL with respect to CNL-algebras. We prove the if-part.

We consider the free groupoid $(M, \cdot)$ generated by all NL-formulas and formally negated NL-formulas $A^\sim$, i.e. A with superscript $^\sim$. The elements of M are represented as formula-structures, as above. We define $O \subseteq M$ as the smallest set which contains all $(A^\sim, \Gamma)$ such that $\Phi \vdash_{NL} \Gamma \Rightarrow A$ and is closed under (r-sym), (r-com). Clearly each element of O contains at least two formulas and exactly one negated formula.

We consider the complex algebra M_O, i.e. M_R for $R = R_O$. Since O satisfies (COM-O) and (Cy), M_O is a CNL-algebra, by Theorem 1.

For any NL-formula A, we define $[A] = \{\Gamma : \Phi \vdash_{NL} \Gamma \Rightarrow A\}$. We show $[A] = \{A^\sim\}^\sim$. Clearly $[A] \subseteq \{A^\sim\}^\sim$, by the definition of O. We prove $\{A^\sim\}^\sim \subseteq [A]$. Let $\Gamma \in \{A^\sim\}^\sim$. Then $(A^\sim, \Gamma) \in O$. By the definition of O, there exists a NL-sequent $\Delta \Rightarrow A$ such that $\Phi \vdash_{NL} \Delta \Rightarrow A$ and $(A^\sim, \Gamma) \sim (A^\sim, \Delta)$. By Proposition 3, $\Gamma = \Delta$ (take $A^\sim$ as the marked formula). Consequently $\Gamma \in [A]$.

So all sets $[A]$ are ϕ_O-closed. We define μ by (1). By formula induction we show $\mu(A) = [A]$ for any NL-formula A. This is obvious for p.

The cases $A\backslash B$, A/B are treated in the same way as in analogous proofs for NL. Let us consider $A\backslash B$. Assume $\Gamma \in \mu(A\backslash B)$. Since $A \in \mu(A)$, then $(A, \Gamma) \in \mu(B)$. So $(A, \Gamma) \in [B]$, which yields $\Gamma \in [A\backslash B]$, by $(\Rightarrow \backslash)$. Assume $\Gamma \in [A\backslash B]$. By the reversibility of $(\Rightarrow \backslash)$ in NL, $(A, \Gamma) \in [B]$. Let $\Delta \in \mu(A)$. Then $\Delta \in [A]$, which yields $(\Delta, \Gamma) \in [B]$, by (NL-cut). So $(\Delta, \Gamma) \in \mu(B)$ for any $\Delta \in \mu(A)$, and consequently $\Gamma \in \mu(A\backslash B)$.

The case $A \otimes B$ needs (F2), (F3), which remain true for NL-formulas. We prove (F2). Let $X = Y^\sim$, $(A, B) \in X$. Then, $(\Gamma, (A, B)) \in O$ for any $\Gamma \in Y$. We fix $\Gamma \in Y$. Let $C^\sim$ be the only negated formula in Γ; we treat $C^\sim$ as the marked formula. By Proposition 3, there is a unique Δ such that $(\Gamma, (A, B)) \sim (C^\sim, \Delta)$. By the construction of O, $\Phi \vdash_{NL} \Delta \Rightarrow C$. By Lemma 1, the reduction of $(\Gamma, (A, B))$ to $(C^\sim, \Delta)$ preserves (A, B), hence $\Delta = \Theta[(A, B)]$. Accordingly $\Phi \vdash_{NL} \Theta[A \otimes B] \Rightarrow C$, by $(\otimes \Rightarrow)$, hence $(C^\sim, \Theta[A \otimes B]) \in O$. Clearly $(\Gamma, A \otimes B) \sim (C^\sim, \Theta[A \otimes B])$. Consequently $(\Gamma, A \otimes B) \in O$. This yields $A \otimes B \in X$. (F3) can be proved in a similar way ($\vdash$ in (F3) means $\vdash_{NL}$).

We prove $[A \otimes B] \subseteq \mu(A \otimes B)$. Since $A \in \mu(A)$, $B \in \mu(B)$, then $(A, B) \in \mu(A) \cdot \mu(B) \subseteq \mu(A \otimes B)$. By (F2), $A \otimes B \in \mu(A \otimes B)$. Hence $[A \otimes B] \subseteq \mu(A \otimes B)$, by (F3). We prove $\mu(A \otimes B) \subseteq [A \otimes B]$. Since $[A \otimes B]$ is ϕ_O-closed, it suffices to show $\mu(A) \cdot \mu(B) \subseteq [A \otimes B]$, which amounts to $[A] \cdot [B] \subseteq [A \otimes B]$. This holds, by $(\Rightarrow \otimes)$.

Now assume $\Phi \not\vdash_{NL} \Gamma \Rightarrow A$. Then $\Gamma \in \mu(\Gamma)$, $\Gamma \notin [A] = \mu(A)$, and consequently $\Gamma \Rightarrow A$ is not true for μ. Let $C \Rightarrow D \in \Phi$. $\mu(C) \subseteq \mu(D)$ follows from $[C] \subseteq [D]$. Therefore $\Gamma \Rightarrow A$ does not follow from Φ in CNL-algebras. Consequently $\Phi \not\vdash_{CNL} \Gamma \Rightarrow A$. □

The results of this section can be extended for several richer logics. Proofs are similar, and we omit them.

First, we consider CNL with $^\sim$ in the language. So formulas are built from variables and $\otimes, \oplus, ^\sim$. One adds the rules:

$$(\text{r-}^{\sim\sim})\,\frac{A,\Gamma}{A^{\sim\sim},\Gamma}$$

$$(\text{r-}\otimes^\sim)\;\frac{A^\sim,\Gamma\quad B^\sim,\Delta}{(A\otimes B)^\sim,(\Gamma,\Delta)}\quad(\text{r-}\oplus^\sim)\;\frac{(B^\sim,A^\sim),\Gamma}{(A\oplus B)^\sim,\Gamma}.$$

This system is equivalent to the former one in a strong sense. Every formula with $^\sim$ can be translated into a formula without $^\sim$ (except its occurrences at variables), using the metalanguage definition of $^\sim$, given above. The translation can be extended for sequents and sets of sequents. Γ is provable from Φ in CNL with $^\sim$ if and only if the translation of Γ is provable from the translation of Φ in CNL without $^\sim$.

CNL1 is obtained by adding the constants $1, 0$, treated as atomic formulas, and:

$$(\text{a-0})\ 0,\quad (\text{r-1})\ \frac{\Gamma}{1,\Gamma}\,.$$

The new axiom (a-0) introduces a sequent containing only one formula. We define sequents as all elements of $\mathcal{S}$. The set of formula-structures is defined as the free unital groupoid $\mathcal{S}_1 = \mathcal{S}\cup\{\lambda\}$, where λ satisfies $\Gamma\cdot\lambda = \Gamma = \lambda\cdot\Gamma$. One may imagine λ as the 'empty structure'. Γ and Δ may be empty in (r-$\otimes$), (r-$\oplus$).

For CNL1 without $^\sim$, the metalanguage negation is defined as above, with: $1^\sim = 0$, $0^\sim = 1$. Given a CNL1-algebra and a valuation μ, one sets $\mu(\lambda) = 1$. A sequent $\Gamma \in \mathcal{S}$ is said to be *true* for μ, if $\mu(\Gamma) \leq 0$. For sequents (Γ, Δ) this amounts to the former definition of a true sequent.

CNL1 (in both versions) admits cut elimination, since the cut-free system is weakly complete with respect to CNL1-algebras. With (cut') it is strongly complete. CNL1 is a strongly conservative extension of NL1.

CNL* is obtained from CNL1 by dropping 1 and 0. Since CNL* is strongly complete with respect to CNL1-algebras, then CNL1 is a strongly conservative extension of CNL*. Notice that CNL* is stronger than CNL; $p\otimes p^\sim$ is provable in CNL*, by (id) and (r-$\otimes$), but not in CNL. In CNL1-algebras this law expresses $a\otimes a^\sim \leq 0$, which lacks sense in CNL-algebras without 0. The axiom (id) expresses $a \leq a$, which holds in all ordered algebras.

In the completeness proofs, the underlying unital groupoid is $(\mathcal{S}_1, \cdot, \lambda)$ and O consists of all provable sequents. Then O satisfies (COM-O) and (Cy), hence the complex algebra is a CNL1-algebra. (1) is extended by: $\mu(0) = O$, $\mu(1) = \phi_O(\{\lambda\})$.

If C is a closure operation on a complete lattice, then the C-closed sets are closed under infinite meets. So they form a complete lattice. The results of this section can be extended to CNL and CNL1 with lattice connectives $\vee, \wedge$, satisfying the lattice laws. These logics may be called Full CNL and Full CNL1 (FCNL and FCNL1) by analogy with FNL, i.e. NL with $\vee, \wedge$. FCNL (resp. FCNL1) is a strongly conservative extension of FNL (resp. FNL1).

For FCNL, the connectives are $\otimes, \oplus, \vee, \wedge$. One adds three new rules.

$$\text{(r-}\wedge\text{)}\ \frac{A, \Gamma}{B \wedge A, \Gamma}\quad \frac{A, \Gamma}{A \wedge B, \Gamma}\quad \text{(r-}\vee\text{)}\ \frac{A, \Gamma \quad B, \Gamma}{A \vee B, \Gamma}$$

In the complex algebra of $(M, \cdot, R)$ (an arbitrary phase space) one defines: $X \wedge Y = X \cap Y$, $X \vee Y = \phi_R(X \cup Y)$. With these operations the complex algebra of a phase space for CNL is a lattice-ordered CNL-algebra. We refer to these algebras as FCNL-algebras. FCNL1-algebras are defined in a similar way.

These results remain true for associative and/or commutative CNL-algebras and CNL1-algebras. The associative FCNL1-algebras are the algebras of Cyclic Noncommutative MALL [15]; the commutative and associative FCNL1-algebras are the algebras of MALL [10]. The completeness results were proved in these papers. The fact that Cyclic Noncommutative MALL is a (weakly) conservative extension of FL1 was proved in Abrusci [2] by a tedious proof-theoretic argument. This can be proved like Theorem 2, which yields the strong conservativeness.

4 Main Results

We need an extended subformula property for $\Phi \vdash_{CNL} \Gamma$. Let T be a set of formulas. $\mathcal{S}_T$ consists of all $\Gamma \in \mathcal{S}$ such that every formula in Γ belongs to T. *A T-sequent* is a sequent $\Gamma \in \mathcal{S}^2 \cap \mathcal{S}_T$. A T-proof is a formal proof from Φ in CNL which consists of T-sequents only. We write $\Phi \vdash^T_{CNL} \Gamma$, if there exists a T-proof of Γ from Φ in CNL. We write $\vdash$ for $\vdash_{CNL}$ and $\vdash^T$ for $\vdash^T_{CNL}$. We define $[A]^T = \{\Gamma \in \mathcal{S}_T : \Phi \vdash^T \Gamma \Rightarrow A\}$.

Lemma 2. *Let T be a set of formulas, closed under subformulas and $^\sim$. Let Φ be a set of T-sequents of the form (A, B). For any T-sequent Γ_0, $\Phi \vdash \Gamma_0$ if and only if $\Phi \vdash^T \Gamma_0$.*

Proof. The if-part is obvious. For the only-if part, we consider the phase space $(M, \cdot, R)$ such that $M = \mathcal{S}_T$, $\cdot$ is defined as in Sect. 3, and $R = \{(\Gamma, \Delta) \in (\mathcal{S}_T)^2 : \Phi \vdash^T \Gamma, \Delta\}$. Clearly R is symmetric and satisfies (COM-R). So the complex algebra of $(M, \cdot, R)$ is a CNL-algebra. We define: $\mu(p) = [p]^T = \{p^\sim\}^\sim$ for $p \in T$; the values of μ for $p \notin T$ may be arbitrary. One proves: $\mu(A) = [A]^T$ for any $A \in T$, by the same argument as in Sect. 3. Consequently, if $\Phi \vdash^T \Gamma_0$ does not hold, then Γ_0 is not true for μ, but all sequents in Φ are true for μ. Therefore $\Phi \vdash \Gamma_0$ does not hold. □

Corollary 2. *Let T be the smallest set of formulas, containing all formulas occurring in Φ or Γ and being closed under subformulas and $^\sim$. If $\Phi \vdash \Gamma$, then $\Phi \vdash^T \Gamma$.*

We prove an interpolation lemma for CNL: every proper substructure Δ of a provable sequent Γ can be replaced by a formula (*an interpolant*) from a finite set.

Lemma 3. *Let T, Φ, Γ_0 be as in Lemma 2, and let Δ_0 be a substructure of Γ_0, $\Delta_0 \neq \Gamma_0$. We write $\Gamma_0 = \Theta_0[\Delta_0]$. If $\Phi \vdash^T \Gamma_0$, then there exists $D \in T$ such that $\Phi \vdash^T D^\sim, \Delta_0$ and $\Phi \vdash^T \Theta_0[D]$.*

Proof. Assume $\Phi \vdash^T \Gamma_0$. We proceed by induction on T-proofs from Φ. If Δ_0 is a formula, then $D = \Delta_0$. So the thesis holds, if Γ_0 is an axiom (id) or belongs to Φ. We assume that Δ_0 is not a formula.

Case: (r-$\otimes$). Then D is the same as in the premise.

Case: (r-$\oplus$). 1°. $\Delta_0 = (\Delta, \Gamma)$. Then $D = (A \oplus B)^\sim$. 2°. Δ_0 is a substructure of Γ or Δ. Then D is as in the appropriate premise.

Cases: (r-sym). D is as in the premise.

Case: (r-com) downwards. If $\Delta_0 = (\Delta, \Theta)$, then $D = D_1^\sim$, where D_1 is the interpolant of Γ in the premise. Otherwise D is the interpolant of Δ_0 in the premise. (r-com) upwards is treated in a similar way.

Case: (cut'). D is as in the appropriate premise. □

There are two important consequences of Lemma 3.

Theorem 3. *CNL has the strong finite model property (SFMP).*

Proof. Let Φ be a finite set of sequents of the form (A, B). We show that for any sequent Γ, if $\Phi \vdash_{CNL} \Gamma$ does not hold, then there exist a finite CNL-algebra $\mathbf{M}$ and a valuation μ in $\mathbf{M}$ such that all sequents from Φ are true for μ, but Γ is not true for μ.

Assume $\Phi \not\vdash \Gamma$. Let T be defined as in Corollary 2. Clearly T is finite and $\Phi \not\vdash^T \Gamma$. Let $\mathbf{M}$ be the complex algebra constructed in the proof of Lemma 2, and let μ be defined as there. It suffices to show that $\mathbf{M}$ is finite, this means: there are only finitely many ϕ_R-closed sets, i.e. sets of the form $X^\sim$, for $X \subseteq \mathcal{S}_T$. We have $X^\sim = \bigcap_{\Gamma \in X} \{\Gamma\}^\sim$. So it suffices to show that there are only finitely many sets of the form $\{\Gamma\}^\sim$.

Let $\Delta \in \{\Gamma\}^\sim$. Then, $\Phi \vdash^T \Gamma, \Delta$. By Lemma 3, $\Phi \vdash^T \Gamma, D$, for some $D \in T$ such that $\Phi \vdash^T D^\sim, \Delta$. We have: $D \in \{\Gamma\}^\sim$ and $\Delta \in [D]^T$. By (F3) (precisely: its version for T-sequents and T-proofs), $[D]^T \subseteq \{\Gamma\}^\sim$. Consequently, $\{\Gamma\}^\sim$ is the union of some family of sets $[D]^T$, for $D \in T$. There are only finitely many sets $[D]^T$ such that $D \in T$, which yields our claim. □

By *a CNL-grammar* we mean a triple $G = (\Sigma, I, A_0)$ such that Σ is a nonempty, finite alphabet, I is a map from Σ to the family of finite sets of CNL-formulas, and A_0 is a CNL-formula. For any $\Gamma \in \mathcal{S}$, we define a sequence of formulas $s(\Gamma)$: $s(A) = A$, $s((\Gamma, \Delta)) = s(\Gamma)s(\Delta)$, i.e. the concatenation of $s(\Gamma)$ and $s(\Delta)$. We say that G assigns A to the string $a_1 \ldots a_n$ ($a_i \in \Sigma$), if there exists $\Gamma \in \mathcal{S}$ such that $(A^\sim, \Gamma)$ is provable, $s(\Gamma) = A_1 \ldots A_n$ and $A_i \in I(a_i)$ for $i = 1, \ldots, n$. Here 'provable' means 'provable in CNL'. We also consider grammars based on CNL augmented with finitely many assumptions; then 'provable' means 'provable from Φ in CNL', where Φ is the set of assumptions. *The language* of G is the set of all $x \in \Sigma^+$ such that G assigns A_0 to x.

Theorem 4. *Let Φ be a finite set of sequents. Let G be a CNL-grammar based on CNL augmented with the assumptions from Φ. Then, the language of G is a context-free language.*

Proof. Fix a grammar $G = (\Sigma, I, A_0)$. Let T be the smallest set of formulas which contains A_0 and all formulas appearing in Φ, I and is closed under subformulas and $^\sim$. Clearly T is finite. Let $(A_0^\sim, \Gamma)$ be provable, $\Gamma \in \mathcal{S}_T$. Let (A, B) be a substructure of Γ; so $\Gamma = \Theta[(A, B)]$. By Lemma 3, there exists $D \in T$ such that $(D^\sim, (A, B))$ and $(A_0^\sim, \Theta[D])$ are provable. Accordingly, every $\Gamma \in \mathcal{S}_T$ such that $(A_0^\sim, \Gamma)$ is provable can be derived (as a derivation tree) from A_0 by means of context-free rules: $A \mapsto B$ (resp. $A \mapsto B, C$) such that $A, B, C \in T$ and $B \Rightarrow A$ (resp. $(B, C) \Rightarrow A$) is provable. The language of G is generated by the context-free grammar with the terminal alphabet Σ, the nonterminal alphabet T, the start symbol A_0, and the production rules as above plus $A \mapsto a$ for $A \in I(a)$. □

Conversely, every ϵ-free context-free language is generated by some CNL-grammar (without assumptions). This follows from Theorem 2 and the fact that every ϵ-free context-free language is generated by an NL-grammar [3].

Theorem 3 implies the decidability of the finitary consequence relation for CNL. We prove that it is decidable in polynomial time. [11] shows the polynomial time decidability of CNL.

Theorem 5. *The relation $\Phi \vdash \Gamma$, for finite sets Φ and $\Gamma \in \mathcal{S}^{(2)}$, is decidable in polynomial time.*

Proof. A sequent $\Gamma \in \mathcal{S}^{(2)}$ is said to be *restricted*, if it is of the form (A, B), $(A, (B, C))$ or $((A, B), C)$. So (id) and all sequents from Φ are restricted. Fix a finite set Φ and $\Gamma_0 \in \mathcal{S}^{(2)}$. Let T be defined as in Corollary 2 (for $\Gamma = \Gamma_0$).

By CNL_r^T we denote the system whose axioms and rules are those of CNL with (cut'), limited to restricted T-sequents. Clearly there are finitely many restricted T-sequents. All sequents provable in CNL_r^T from Φ can be determined in polynomial time (in the size of $\Phi \cup \{\Gamma_0\}$).

By CNL_Φ^T we denote the system whose axioms are all sequents provable in CNL_r^T from Φ and the only inference rule is (cut) (now admitting unrestricted T-sequents). Notice that (cut) is not the same as (cut'). Observe that every restricted $T-$sequent provable in CNL_Φ^T must be provable in CNL_r^T from Φ (if the conclusion of (cut) is restricted, then the premises are restricted; also (cut) limited to restricted T-sequents is derivable in CNL_r^T). We prove:

$$\Phi \vdash_{CNL}^T \Gamma \text{ iff } \Gamma \text{ is provable in } \text{CNL}_\Phi^T.$$

($\Leftarrow$) is obvious. For ($\Rightarrow$), we observe that CNL_Φ^T has the interpolation property: if $\Theta_0[\Delta_0]$ is provable and $\Delta_0 \neq \Theta_0[\Delta_0]$, then there exists $D \in T$ such that $(D^\sim, \Delta_0)$ and $\Theta_0[D]$ are provable.

First, one proves this property for CNL_r^T with the assumptions from Φ in the same way as Lemma 3. For rules (r-$\oplus$), (r-com) one uses the fact that $(A, A^\sim)$, for $A \in T$, is provable in CNL_r^T.

Second, one shows this property for CNL_{Φ}^{T} by induction on derivations based on (cut), which is easy. The only interesting case is the following: $\Theta_0[\Theta_1[\Delta]]$ arises by (cut) from $A^{\sim}, \Delta$ and $\Theta_0[\Theta_1[A]]$, and $\Delta_0 = \Theta_1[\Delta]$. Then, the interpolant of Δ_0 equals the interpolant of $\Theta_1[A]$ in $\Theta_0[\Theta_1[A]]$.

Third, one shows that all rules of CNL, restricted to T-sequents, are admissible in CNL_{Φ}^{T}. We only consider (r-sym). Let (Γ, Δ) be provable in CNL_{Φ}^{T}. By interpolation, there exist $C \in T$, $D \in T$ such that (C, D), $(C^{\sim}, \Gamma)$, $(D^{\sim}, \Delta)$ are provable in CNL_{Φ}^{T}. Since (C, D) is provable in CNL_{r}^{T} from Φ, then (D, C) is provable in CNL_{r}^{T} from Φ, and consequently, (Δ, Γ) is provable in CNL_{Φ}^{T}, by two applications of (cut). This yields ($\Rightarrow$).

By Lemma 2, $\Phi \vdash_{CNL} \Gamma_0$ if and only if Γ_0 is provable in CNL_{Φ}^{T}. In particular, for a restricted Γ_0, Γ_0 is provable in CNL from Φ if and only if Γ_0 is provable in CNL_{r}^{T}. □

We have noted in Sect. 1 that CNL^{-} does not have SFMP. The status of Theorems 4 and 5 for CNL^{-} remains an open problem. They are true for the pure CNL^{-} (i.e. $\Phi = \emptyset$); the proof will be given in another paper.

Chvalovsky [7] proves that the consequence relation for FNL is undecidable. Since FCNL is a strongly conservative extension of FNL, then the consequence relation for FCNL is undecidable (hence SFMP fails). On the other hand, the analogues of Theorems 3 and 4 hold for DFCNL, i.e. FCNL admitting the distributive laws for $\vee, \wedge$, like for DFNL and its variants [5,6].

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