Three Variables Suffice for Real-Time Logic

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Abstract. A natural framework for real-time specification is monadic first-order logic over the structure $(\mathbb{R}, <, +1)$—the ordered real line with unary $+1$ function. Our main result is that $(\mathbb{R}, <, +1)$ has the 3-variable property: every monadic first-order formula with at most 3 free variables is equivalent over this structure to one that uses 3 variables in total. As a corollary we obtain also the 3-variable property for the structure $(\mathbb{R}, <, f)$ for any fixed linear function $f : \mathbb{R} \to \mathbb{R}$. On the other hand, we exhibit a countable dense linear order $(E, <)$ and a bijection $f : E \to E$ such that $(E, <, f)$ does not have the k-variable property for any k.

1 Introduction

Monadic first-order logic is an expansion of first-order logic by infinitely many unary predicate variables. In this setting a class of structures $\mathcal{C}$ is said to have the *k-variable property* if every formula with at most k free first-order variables is equivalent over $\mathcal{C}$ to a formula with at most k first-order variables in total (allowing multiple binding occurrences of the same variable). The k-variable property for monadic first-order logic over linearly ordered structures has been studied in [1,2,4,8,12,14,15], among others. In finite model theory the k-variable property plays an important role in descriptive complexity. Over infinite models it is closely connected with expressive completeness of temporal logics.

It is well known that Linear Temporal Logic (LTL) with Stavi modalities is expressively complete for monadic first-order logic over the class of linear orders [3,13]. More precisely, LTL is expressively complete for the class of monadic first-order formulas with one free variable (corresponding to the fact that LTL formulas are evaluated at a single point of a linear order). The translation from LTL to first-order logic is a straightforward inductive construction that maps into the 3-variable fragment of first-order logic. It follows that every monadic first-order formula with at most one free variable is equivalent to a 3-variable formula over linear orders. However this is a strictly weaker condition than the 3-variable property in general: Hodkinson and Simon [8] give a class of partial orders over which every monadic first-order formula with at most one free variable is equivalent to a 3-variable formula, but which does not have the k-variable property for any k. Nevertheless the 3-variable property *does* hold over linear orders, as shown by Poizat [14] and Immerman and Kozen [12], using Ehrenfeucht-Fraïssé games.

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Going beyond pure linear orders, Venema [16] gives a dense linear order with a single equivalence relation over which monadic first-order logic does not have the k-variable property for any k. A more powerful result by Rossman [15] shows that the class of finite linearly ordered graphs does not have the k-variable property for any k, resolving a longstanding conjecture of Immerman [11].

In this paper we are concerned with monadic first-order logic over the ordered reals with unary $+1$ function $(\mathbb{R}, <, +1)$. This logic has been extensively studied in the context of real-time verification. An expansion of $(\mathbb{R}, <, +1)$ with interpretations of the unary predicate variables can be seen as a real-time signal, with the unary predicates denoting propositions that may or may not hold at any given time. First-order logic over signals can express both metric and order-theoretic temporal properties and is an expressive meta-language into which many different real-time logics can directly be translated [5,6]. In particular, first-order logic over signals is expressively equivalent with Metric Temporal Logic (MTL) [9,10].

Our main result is that $(\mathbb{R}, <, +1)$ has the 3-variable property. For example, the property

$$\forall x_1 \exists x_2 \exists x_3 \exists x_4 \left(x_4 < x_1 + 1 \wedge \bigwedge_{1 \leq i \leq 3} x_i < x_{i+1} \wedge \bigwedge_{2 \leq i \leq 4} P(x_i) \right)$$

that P is true at least 3 times in every unit interval can equivalently be written

$$\forall x \exists y \big(x < y \wedge P(y) \wedge \exists z (y < z \wedge P(z) \wedge \exists y (z < y < x + 1 \wedge P(y)))\big) .$$

From the expressive completeness of MTL it follows that every monadic first-order formula with at most one free variable is equivalent to a 3-variable formula over $(\mathbb{R}, <, +1)$. However, as remarked above, this condition is weaker than the 3-variable property in general. Moreover the proof of expressive completeness of MTL combines intricate syntactic manipulations of MTL formulas together with technically involved results of [3] for LTL. On the other hand, the model-theoretic argument given here, using Ehrenfeucht-Fraïssé games, is self-contained and exposes a novel two-level compositional technique that can potentially be applied in more general settings and to other ends (see the Conclusion).

As a corollary of our main result we straightforwardly derive the 3-variable property for each structure $(\mathbb{R}, <, f)$ with $f : \mathbb{R} \to \mathbb{R}$ a linear function $f(x) = ax + b$. We believe that the result can be generalised to other linear orders and suitably well-behaved functions. However, unsurprisingly, the property fails for sufficiently 'wild' functions. Adapting Venema's construction [16], we give an example of a countable dense linear order E and a (far from monotone) bijection $f : E \to E$ such that $(E, <, f)$ does not have the k-variable property for any k.

The paper naturally divides into two parts. Sections 3 to 4 are exclusively concerned with the structure $(\mathbb{R}, <, +1)$, while Sections 5 and 6 consider other unary functions in place of $+1$.

2 Background

2.1 Ehrenfeucht-Fraïssé Games

Throughout the paper we work with a first-order signature σ with a binary relation symbol $<$ and a unary function symbol f. The monadic first-order language over σ is defined as follows:

- There is an infinite collection of monadic predicate variables $P_1, P_2, \ldots$.
- The atomic formulas are $x = y$, $x < y$, $P_n(x)$, and $x = f(y)$ for first-order variables x and y and $n \in \mathbb{N}$.
- If φ_1 and φ_2 are formulas and x is a variable then $\neg\varphi_1$, $\varphi_1 \wedge \varphi_2$ and $\exists x\, \varphi_1$ are also formulas.

Referring to the restricted use of the function symbol f in atomic formulas, we say that the formulas above are *unnested*. The unnesting assumption essentially amounts to treating the function symbol f as a binary relation symbol. We make this assumption as an alternative to restricting to a purely relational signature. The unnesting assumption does not affect expressiveness since we can translate an arbitrary formula to an equivalent unnested formula by successively replacing atomic formulas $f^m(x) = f^n(y)$ with $m > 0$ by $\exists z\, (z = f(x) \wedge f^{m-1}(z) = f^n(y))$, and similarly for $f^m(x) < f^n(y)$. While this transformation may increase the quantifier depth, it preserves the subclass of 3-variable formulas.

Let $\mathbf{A} = (A, <^{\mathbf{A}}, f^{\mathbf{A}}, \overline{P}^{\mathbf{A}})$ denote a σ-structure expanded with interpretations of the monadic predicate variables $P_1, P_2, \ldots$. We call $\mathbf{A}$ a *labelled σ-structure*. Given first-order variables $x_1, \ldots, x_k$, an *assignment* in $\mathbf{A}$ with domain $\{x_1, \ldots, x_k\}$ is a tuple $\overline{u} = u_1 \ldots u_k$ in A^k. Given another assignment $\overline{v}$ with the same domain in a labelled σ-structure $\mathbf{B}$, we say that $(\overline{u}, \overline{v})$ is a *partial isomorphism* between $\mathbf{A}$ and $\mathbf{B}$ if $\mathbf{A} \models \varphi[\overline{u}]$ iff $\mathbf{B} \models \varphi[\overline{v}]$ for all atomic formulas $\varphi(x_1, \ldots, x_k)$.

The *Ehrenfeucht-Fraïssé* (EF) game on structures $\mathbf{A}$ and $\mathbf{B}$ is played by two players—*Spoiler* and *Duplicator*.[1] Each player has a collection of pebbles, respectively labelled $x_1, x_2, \ldots$. The game is played over a fixed number of rounds. In each round Spoiler chooses a structure and places a pebble on an element of the structure (either an unused pebble or one that has already been placed); Duplicator responds by placing a pebble with the same label on some element of the other structure. A placement of k pebbles on each structure naturally determines a pair of assignments $(\overline{u}, \overline{v})$, called a *$k$-configuration*. (Our notation for k-configurations leaves the structures $\mathbf{A}$ and $\mathbf{B}$ implicit.) If the configuration after each round is a partial isomorphism then Duplicator wins, otherwise Spoiler wins. For each configuration $(\overline{u}, \overline{v})$ and number of rounds n, exactly one of the players has a winning strategy in the n-round game starting from $(\overline{u}, \overline{v})$ (see [12] for more details).

A natural restriction on Ehrenfeucht-Fraïssé games is to limit each player to a fixed number of pebbles. In the *k-pebble game* both Spoiler and Duplicator

[1] By convention, Spoiler is male and Duplicator is female.

possess only k pebbles, respectively labelled $x_1, \ldots, x_k$. The following theorem shows how Ehrenfeucht-Fraïssé games can be used to characterise the expressiveness of first-order logic according to the number of variables.

Theorem 2.1 ([12]). *Let $\mathcal{C}$ be a class of σ-structures such that for all n there exists m such that if Spoiler wins the n-round Ehrenfeucht-Fraïssé game on a pair of labelled structures from $\mathcal{C}$ starting in a k-configuration $(\overline{u}, \overline{v})$, then he also wins the m-round k-pebble game starting in $(\overline{u}, \overline{v})$. Then $\mathcal{C}$ has the k-variable property.*

In the remainder of this section we specialise our attention to the σ-structure $(\mathbb{R}, <, +1)$. In this case we call a labelled σ-structure a *signal*.

In addition to k-pebble games, on signals we introduce another restriction of Ehrenfeucht-Fraïssé games. Given an assignment $\overline{u} \in \mathbb{R}^k$ with domain $\{x_1, \ldots, x_k\}$, the *diameter* of $\overline{u}$ is $diam(\overline{u}) = \max\{|u_i - u_j| : 1 \leq i, j \leq k\}$. Given $D \in \mathbb{R}$, the D*-local game* on a pair of signals is such that Spoiler and Duplicator must maintain the invariant that all assignments have diameter at most D.

We will always explicitly indicate any restrictions on the number of pebbles or the diameter of configurations in games: thus the default notion of Ehrenfeucht-Fraïssé game is without restriction on the number of pebbles or the diameter.

Recall that our main result is that $(\mathbb{R}, <, +1)$ has the 3-variable property. The main conceptual insight underlying the proof is that one should first prove the 3-variable property for "local" formulas. We treat locality semantically through the notion of local EF games, as defined above, but intuitively a local formula is one that asserts properties of elements at a bounded distance from one another. For example, $\exists x \exists y\, (P(x) \land Q(y) \land x, < y < x+1)$ is local but $\exists x \exists y\, (P(x) \land Q(y))$ is not local.

We prove the 3-variable property for local formulas by a compositional argument based on the fractional-part preorder on $\mathbb{R}$. We then extend the 3-variable property to all formulas by adapting the well-known composition lemma for sums of linear orders to the structure $(\mathbb{R}, <, +1)$. Roughly speaking, this second compositional lemma shows that Duplicator strategies on summands can be composed provided that there is sufficient distance between pebbles in different summands. However this precondition is not always met and here it is crucial that we have already established the 3-variable property for local formulas.

2.2 Interpretations

In this section we briefly deviate from the setting of linear orders and unary functions to recall from [7, Chapter 4.3] the notion of an interpretation of one first-order structure in another.

Let σ_1 and σ_2 be signatures, $\mathbf{A}$ a σ_1-structure with domain A, $\mathbf{B}$ a σ_2-structure with domain B, and n a positive integer. An n-dimensional *interpretation* Γ of $\mathbf{B}$ in $\mathbf{A}$ consists of three items:

- a σ_1-formula $\partial_\Gamma(x_1, \ldots, x_n)$ denoting the *domain* of the interpretation, which is the set $\partial_\Gamma(A^n) := \{\overline{a} \in A^n : \mathbf{A} \models \partial_\Gamma[\overline{a}]\}$.

– for each unnested atomic σ_2-formula $\varphi(x_1, \ldots, x_m)$, a σ_1-formula $\varphi_\Gamma(\overline{x}_1, \ldots, \overline{x}_m)$ in which the $\overline{x}_i$ are disjoint n-tuples of distinct variables,
– a surjective *coding map* $f_\Gamma : \partial_\Gamma(A^n) \to B$ such that for all unnested atomic σ_2-formulas φ and all $\overline{a}_i \in \partial_\Gamma(A^n)$,

$$\mathbf{B} \models \varphi[f_\Gamma\overline{a}_1, \ldots, f_\Gamma\overline{a}_m] \text{ iff } \mathbf{A} \models \varphi_\Gamma[\overline{a}_1, \ldots, \overline{a}_m]\,.$$

3 From Local Games to 3-Pebble Games

In this section we consider an Ehrenfeucht-Fraïssé game on two signals **A** and **B**. Here $\overline{u} = u_1 \ldots u_s$ will always denote an assignment in **A** and $\overline{v} = v_1 \ldots v_s$ will always denote an assignment in **B**.

Write $\overline{u} \equiv \overline{v}$ if $\overline{u}$ and $\overline{v}$ are indistinguishable by difference constraints, that is, $u_i - u_j < c \Leftrightarrow v_i - v_j < c$ and $u_i - u_j = c \Leftrightarrow v_i - v_j = c$ for all constants $c \in \mathbb{Z}$ and indices $1 \leq i, j \leq s$. Equivalently, $\overline{u} \equiv \overline{v}$ if and only if $\lfloor u_i - u_j \rfloor = \lfloor v_i - v_j \rfloor$ for all indices $1 \leq i, j \leq s$.[2] Assignments that are indistinguishable by difference constraints are, in particular, ordered the same way.

Define the *fractional part* of $u \in \mathbb{R}$ by $\text{frac}(u) = u - \lfloor u \rfloor$. The proof of the following proposition can be found in the Appendix.

Proposition 3.1. *Let $\overline{u} = u_1 \ldots u_s$ and $\overline{v} = v_1 \ldots v_s$ be two assignments with $\overline{u} \equiv \overline{v}$. Then*

$$\text{frac}(u_i - u_k) < \text{frac}(u_j - u_k) \;\Leftrightarrow\; \text{frac}(v_i - v_k) < \text{frac}(v_j - v_k)$$

for all indices $i, j, k \in \{1, \ldots, s\}$.

We say that $u_1 \ldots u_s$ is in *increasing order* if $\text{frac}(u_i - u_1) \leq \text{frac}(u_{i+1} - u_1)$ for $i = 1, \ldots, s-1$. Intuitively $u_1 \ldots u_s$ is in increasing order if it is listed in increasing order of fractional parts relative to u_1. Note that if $u_1 \ldots u_s$ is in increasing order then any cyclic permutation is also in increasing order. By Proposition 3.1, if $\overline{u} \equiv \overline{v}$ then $\overline{u}$ and $\overline{v}$ can both be brought into increasing order by a common permutation.

The following proposition can be seen as a compositional lemma for $\equiv$. The proof can be found in the Appendix.

Proposition 3.2. *Suppose that $u_1 \ldots u_s$ and $v_1 \ldots v_s$ are both increasing and that $u_1 \ldots u_m \equiv v_1 \ldots v_m$ and $u_m \ldots u_s \equiv v_m \ldots v_s$ for some m, $1 \leq m \leq s$. Then $u_1 \ldots u_s \equiv v_1 \ldots v_s$.*

Proposition 3.3 and Corollary 3.4 show that three pebbles suffice to determine equivalence of configurations under the relation $\equiv$.

Proposition 3.3. *Let $n \in \mathbb{N}$. Consider a 2-configuration (u_1u_2, v_1v_2) such that either (i) $u_1 - u_2 < c$ and $v_1 - v_2 \not< c$ for some non-negative integer $c < 2^n$ or (ii) $u_1 - u_2 = c$ and $v_1 - v_2 \neq c$ for some non-negative integer $c \leq 2^n$. Then Spoiler wins the n-round 3-pebble game from (u_1u_2, v_1v_2).*

[2] Note that $\lceil u_i - u_j \rceil = \lceil v_i - v_j \rceil$ if and only if $\lfloor u_j - u_i \rfloor = \lfloor v_j - v_i \rfloor$, so there is no need to add a separate clause for ceiling in the characterisation of $\equiv$.

Proof. The proof is by induction on n.

Base case ($n = 0$). Under either assumption (i) or (ii) the configuration (u_1u_2, v_1v_2) is not a partial isomorphism, and is therefore immediately winning for Spoiler in the 3-pebble game.

Induction step ($n \geq 1$). Suppose $u_1 - u_2 < c$ but $v_1 - v_2 \not< c$, where $c < 2^n$. Write $c' = \lfloor c/2 \rfloor$, so that $c' < 2^{n-1}$ and $c - c' \leq 2^{n-1}$. Suppose that Spoiler places a pebble on u_3 such that $u_1 - u_3 = c - c'$ and $u_3 - u_2 < c'$. Since $v_1 - v_2 \not< c$, for any response v_3 of Duplicator we either have $v_1 - v_3 \neq c - c'$ or $v_3 - v_2 \not< c'$. In the first case, by the induction hypothesis, (u_1u_3, v_1v_3) is winning in $n-1$ rounds for Spoiler; likewise in the second case (u_2u_3, v_2v_3) is winning in $n-1$ rounds for Spoiler. Thus in either case $(u_1u_2u_3, v_1v_2v_3)$ is winning in $n-1$ rounds for Spoiler. We conclude that (u_1u_2, v_1v_2) is winning in n rounds for Spoiler. This handles (i); Case (ii) is almost identical. □

Corollary 3.4. *Let $(\overline{u}, \overline{v})$ be a 3-configuration such that $\overline{u} \not\equiv \overline{v}$ and at least one of $\overline{u}$ and $\overline{v}$ has diameter at most 2^m. Then Spoiler wins the m-round 3-pebble game from $(\overline{u}, \overline{v})$.*

Proof. Since $\overline{u} \not\equiv \overline{v}$, there are indices i, j such that $u_i - u_j \sim c$ and $v_i - v_j \not\sim c$ for some non-negative integer constant c and comparison operator $\sim \in \{<, =\}$. Moreover, since at least one of $\overline{u}$ and $\overline{v}$ has diameter at most 2^m, we can assume that $c \leq 2^m$. But then Spoiler wins the m-round 3-pebble game from $(\overline{u}, \overline{v})$ by Proposition 3.3. □

One can think of following proposition as showing the 3-variable property for local formulas. The proof uses the compositional principle in Proposition 3.2.

Proposition 3.5. *Let $(\overline{u}, \overline{v})$ be a 3-configuration of diameter at most 2^m. If Spoiler wins the n-round 2^m-local game from $(\overline{u}, \overline{v})$ then he wins the $(m+n)$-round 3-pebble game from $(\overline{u}, \overline{v})$.*

Proof. If $\overline{u} \not\equiv \overline{v}$ then the result follows from Corollary 3.4. Thus it suffices to prove the proposition under the assumption $\overline{u} \equiv \overline{v}$.

Without loss of generality assume that $\overline{u}$ and $\overline{v}$ are both increasing. The proof is by induction on n, with the following induction hypothesis.

Induction Hypothesis: Let assignments $u_1 \ldots u_s \equiv v_1 \ldots v_s$ be increasing and have diameter at most 2^m. If Spoiler wins the n-round 2^m-local game from $(\overline{u}, \overline{v})$, then he wins the $(m+n)$-round 3-pebble game from a 2-configuration of the form (u_iu_{i+1}, v_iv_{i+1}), $1 \leq i \leq s-1$, or (u_su_1, v_sv_1).

Base case ($n = 0$). By assumption $(\overline{u}, \overline{v})$ is immediately winning for Spoiler in the local game. Since $\overline{u} \equiv \overline{v}$, u_i and v_i must disagree on a unary predicate for some index i. Then (u_i, v_i) is immediately winning for Spoiler in the 3-pebble game. Clearly the position remains immediately winning for Spoiler if we add an extra pebble to each assignment. Thus the base case of the induction is established.

Induction step ($n \geq 1$). Pick a Spoiler move according to his winning strategy in the local game in configuration $(\overline{u}, \overline{v})$. Without loss of generality, assume that this move, say u', is in structure **A**. Since any cyclic permutation of an increasing configuration is also increasing, we may assume without loss of generality that $u_1 \dots u_s u'$ is increasing.

If $(u_1 u_s, v_1 v_s)$ is winning for Spoiler in the $(m+n)$-round 3-pebble game then we are done, so suppose that this is not the case. Then there exists a Duplicator move v' such that $(u_1 u_s u', v_1 v_s v')$ is winning for Duplicator in the $(m+n-1)$-round 3-pebble game. Since $\mathrm{diam}(u_1 u_s u') \leq 2^m$, by Corollary 3.4 we must have $u_1 u_s u' \equiv v_1 v_s v'$. It follows that $v_1 \dots v_s v'$ is increasing.

Since $u_1 \dots u_s u'$ and $v_1 \dots v_s v'$ are increasing, $u_1 \dots u_s \equiv v_1 \dots v_s$, and $u_s u' \equiv v_s v'$, by Proposition 3.2 we have

$$u_1 \dots u_s u' \equiv v_1 \dots v_s v' . \tag{1}$$

Since the pair of assignments in (1) is winning for Spoiler in the $(n-1)$-round local game, by the induction hypothesis there exists a sub-configuration (comprising two consecutive pebbles in each assignment) from which Spoiler wins the $(m+n-1)$-round 3-pebble game. This 2-configuration cannot be $(u_s u', v_s v')$ nor $(u_1 u', v_1 v')$, since $(u_1 u_s u', v_1 v_s v')$ is winning for Duplicator in the $(m+n-1)$-round 3-pebble game. Thus Spoiler must win the $(m+n-1)$-round 3-pebble game from a 2-configuration $(u_i u_{i+1}, v_i v_{i+1})$ for some $i \in \{1, \dots, s-1\}$. *A fortiori* Spoiler also wins the $(m+n)$-round 3-pebble game from this configuration. □

4 Main Results

4.1 Composition Lemma

In this section we consider an Ehrenfeucht-Fraïssé game on two signals **A** and **B**. We will prove a Composition Lemma that allows us to compose winning Duplicator strategies under certain assumptions. From this we obtain our main result, that monadic first-order logic over signals has the 3-variable property.

Assume assignments $\overline{u} = u_1 \dots u_s$ in **A** and $\overline{v} = v_1 \dots v_s$ in **B** with $u_1 < \dots < u_s$ and $v_1 < \dots < v_s$. The Composition Lemma is predicated on a decomposition of $\overline{u}$ into a *left part* $\overline{u}_\triangleleft = u_1 \dots u_l$, *middle part* $\overline{u}_\diamond = u_l \dots u_r$, and *right part* $\overline{u}_\triangleright = u_r \dots u_s$, where $1 \leq l \leq r \leq s$. We call u_l the *left boundary* and u_r the *right boundary*. The *left margin* is defined to be $\mathit{margin}(\overline{u}_\triangleleft) = u_l - u_{l-1}$, where $u_0 = -\infty$ by convention. Likewise the *right margin* is defined to be $\mathit{margin}(\overline{u}_\triangleright) = u_{r+1} - u_r$, where $u_{s+1} = \infty$ by convention. We consider a corresponding decomposition of $\overline{v}$ into $\overline{v}_\triangleleft = v_1 \dots v_l$, $\overline{v}_\diamond = v_l \dots v_r$, and $\overline{v}_\triangleright = v_r \dots v_s$, for the same values of l and r.

The Composition Lemma gives conditions under which we can obtain a winning strategy for Duplicator in a configuration $(\overline{u}, \overline{v})$ by composing winning Duplicator strategies in the *left configuration* $(\overline{u}_\triangleleft, \overline{v}_\triangleleft)$, the *middle configuration* $(\overline{u}_\diamond, \overline{v}_\diamond)$, and *right configuration* $(\overline{u}_\triangleright, \overline{v}_\triangleright)$, see Figure 1. The main idea behind the proof is to maintain adequate separation between pebbles played by the left and middle Duplicator strategies, and likewise between pebbles played by the middle and right strategies. We do this by maintaining the left and right margins

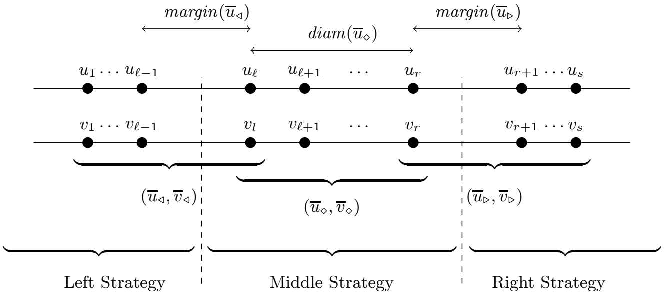

Fig. 1. Situation of the Composition Lemma

appropriately. Importantly for later use, we need only assume that Duplicator has a local winning strategy in the middle configuration.

Lemma 4.1 (Composition Lemma). *Suppose that Duplicator wins the n-round games from configurations $(\overline{u}_\triangleleft, \overline{v}_\triangleleft)$ and $(\overline{u}_\triangleright, \overline{v}_\triangleright)$ respectively, and let D be such that Duplicator wins the $3n$-round D-local game from configuration $(\overline{u}_\diamond, \overline{v}_\diamond)$. If $margin(\overline{u}_\triangleleft) > 2^n$, $margin(\overline{u}_\triangleright) > 2^n$, $D \geq diam(\overline{u}_\diamond) + 2^{n+1}$, and the corresponding three conditions also hold for $\overline{v}$, then Duplicator wins the n-round game from configuration $(\overline{u}, \overline{v})$.*

Proof. We show that configuration $(\overline{u}, \overline{v})$ is winning for Duplicator in the n-round game. The proof is by induction on n.

Base case ($n = 0$). Note that $(\overline{u}, \overline{v})$ is a partial isomorphism since $(\overline{u}_\triangleleft, \overline{v}_\triangleleft)$, $(\overline{u}_\diamond, \overline{v}_\diamond)$, and $(\overline{u}_\triangleright, \overline{v}_\triangleright)$ are all partial isomorphisms, $margin(\overline{u}_\triangleleft)$ and $margin(\overline{v}_\triangleleft)$ are both greater than one, and likewise for $margin(\overline{u}_\triangleright)$ and $margin(\overline{v}_\triangleright)$.

Induction step ($n > 0$). Without loss of generality assume that Spoiler plays a move u' in structure $\mathbf{A}$. We consider three cases.

Case (i). Suppose that $u' < u_l - 2^{n-1}$. Then Duplicator's winning strategy in configuration $(\overline{u}_\triangleleft, \overline{v}_\triangleleft)$ yields a response v' such that $(\overline{u}_\triangleleft u', \overline{v}_\triangleleft v')$ is winning for Duplicator in the $(n-1)$-round game. In particular, applying Proposition 3.3, we have $v' < v_l - 2^{n-1}$. Applying the induction hypothesis to $(\overline{u}_\triangleleft u', \overline{v}_\triangleleft v')$, $(\overline{u}_\diamond, \overline{v}_\diamond)$, and $(\overline{u}_\triangleright, \overline{v}_\triangleright)$ we get that $(\overline{u}u', \overline{v}v')$ is winning for Duplicator in the $(n-1)$-round game.

Case (ii). Suppose that $u' > u_r + 2^{n-1}$. This case is entirely analogous to Case (i), except that Duplicator's response to u' is generated from her winning strategy in configuration $(\overline{u}_\triangleright, \overline{v}_\triangleright)$.

Case (iii). Suppose that $u_l - 2^{n-1} \leq u' \leq u_r + 2^{n-1}$. Then Duplicator's winning strategy in configuration $(\overline{u}_\diamond, \overline{v}_\diamond)$ yields a response v' such that Duplicator wins the $(3n-1)$-round D-local game from $(\overline{u}_\diamond u', \overline{v}_\diamond v')$. By Proposition 3.3 we must have $v_l - 2^{n-1} \leq v' \leq v_r + 2^{n-1}$.

To apply the induction hypothesis, the idea is to "expand the middle configuration" by adding new left and right boundary pebbles u'_l, u'_r and v'_l, v'_r respectively. Formally, Spoiler moves $u'_l := u_l - 2^{n-1}$ and $u'_r := u_r + 2^{n-1}$ in the D-local game in position $(\overline{u}_\diamond u', \overline{v}_\diamond v')$ force Duplicator responses $v'_l := v_l - 2^{n-1}$ and $v'_r := v_r + 2^{n-1}$ such that $(u'_l \overline{u}_\diamond u' u'_r, v'_l \overline{v}_\diamond v' v'_r)$ is winning for Duplicator in the $3(n-1)$-round D-local game. By the same reasoning, $(\overline{u}_\triangleleft u'_l, \overline{v}_\triangleleft v'_l)$ and $(u'_r \overline{u}_\triangleright, v'_r \overline{v}_\triangleright)$ are both winning positions for Duplicator in the $(n-1)$-round game. *A fortiori* $(u_1 \dots u_{l-1} u'_l, v_1 \dots v_{l-1} v'_l)$ and $(u'_r u_{r+1} \dots u_s, v'_r v_{r+1} \dots v_s)$ are also both winning for Duplicator in the $(n-1)$-round game. Finally, applying the induction hypothesis with left configuration $(u_1 \dots u_{l-1} u'_l, v_1 \dots v_{l-1} v'_l)$, middle configuration $(u'_l \overline{u}_\diamond u' u'_r, v'_l \overline{v}_\diamond v' v'_r)$, and right configuration $(u'_r u_{r+1} \dots u_s, v'_r v_{r+1} \dots v_s)$, we conclude that $(\overline{u}u', \overline{v}v')$ is winning for Duplicator in $n-1$ rounds. □

4.2 3-Variable Theorem

Proposition 4.2. *Suppose that Duplicator wins the* $(4n+2)$*-round 3-pebble game from a configuration* $(\overline{u}, \overline{v})$ *with* $|\overline{u}| = |\overline{v}| \leq 3$*. Then she also wins the* n*-round (unrestricted-pebble) game from configuration* $(\overline{u}, \overline{v})$*.*

Proof. The proof is by induction on n. The base case $(n = 0)$ is immediate, and the induction step $(n > 0)$ is as follows. Suppose that $|\overline{u}| = |\overline{v}| < 3$. Then for any Spoiler move, Duplicator replies using her 3-pebble strategy, leading to a 3-configuration $(\overline{u}', \overline{v}')$. Duplicator now has a winning strategy for the $(4n+1)$-round 3-pebble game starting from the configuration $(\overline{u}', \overline{v}')$, and therefore she also has a winning strategy for the $(4(n-1)+2)$-round 3-pebble game from $(\overline{u}', \overline{v}')$. By the induction hypothesis, she has a winning strategy for the $(n-1)$-round unrestricted game from $(\overline{u}', \overline{v}')$, and therefore a winning strategy for the n-round game from $(\overline{u}, \overline{v})$.

Now suppose that $|\overline{u}| = |\overline{v}| = 3$. We claim that given any 3-configuration $(\overline{u}, \overline{v})$, we can decompose it into a left part $(\overline{u}_\triangleleft, \overline{v}_\triangleleft)$, a middle part $(\overline{u}_\diamond, \overline{v}_\diamond)$ and a right part $(\overline{u}_\triangleright, \overline{v}_\triangleright)$, satisfying the following desiderata:

1. $diam(\overline{u}_\diamond) \leq 2^{n+1}$,
2. $margin(\overline{u}_\triangleleft) > 2^n$ and $margin(\overline{u}_\triangleright) > 2^n$,
3. $|\overline{u}_\triangleleft| \leq 2$ and $|\overline{u}_\triangleright| \leq 2$,
4. Conditions 1–3 hold for $\overline{v}_\triangleleft$, $\overline{v}_\diamond$, and $\overline{v}_\triangleright$.

By Proposition 3.5, if the above four conditions hold, we obtain that Duplicator has a winning strategy for the $3n$-round 2^{n+2}-local game from the configuration $(\overline{u}_\diamond, \overline{v}_\diamond)$. Furthermore, by (3) and the case described above for configurations of size strictly less than 3, it follows that Duplicator has a winning strategy for the n-round games from the configurations $(\overline{u}_\triangleleft, \overline{v}_\triangleleft)$ and $(\overline{u}_\triangleright, \overline{v}_\triangleright)$. Thus, by applying the Composition Lemma 4.1, Duplicator has a winning strategy for the n-round game from the configuration $(\overline{u}, \overline{v})$.

It remains to show that given any 3-configuration $(\overline{u}, \overline{v})$, we can always find a decomposition that satisfies the above conditions. We show this by the following case analysis. Without loss of generality, assume that $u_1 \leq u_2 \leq u_3$ and $v_1 \leq v_2 \leq v_3$.

Case(i). Suppose that $u_2 - u_1 \leq 2^n$ and $u_3 - u_2 \leq 2^n$. Then it is also the case that $v_2 - v_1 \leq 2^n$ and $v_3 - v_2 \leq 2^n$, since otherwise Spoiler would have a n-round 3-pebble winning strategy by the contraposition of Corollary 3.4. Then let

$\overline{u}_\lhd = u_1$, $\overline{u}_\rhd = u_3$ and $\overline{u}_\diamond = u_1u_2u_3$, and assume a corresponding decomposition of $\overline{v}$.

Case(ii). Suppose that $u_3 - u_2 > 2^n$ and $u_2 - u_1 > 2^n$. Then it is also the case that $v_3 - v_2 > 2^n$ and $v_2 - v_1 > 2^n$ by Corollary 3.4. Let then $\overline{u}_\diamond = u_2$, $\overline{u}_\lhd = u_1u_2$, $\overline{u}_\rhd = u_2u_3$, and consider the corresponding decomposition for $\overline{v}$.

Case(iii). Suppose finally that $u_3 - u_2 > 2^n$ and $u_2 - u_1 \leq 2^n$. By Corollary 3.4, we also have that $v_3 - v_2 > 2^n$ and $v_2 - v_1 \leq 2^n$. Let $\overline{u}_\lhd = u_1$, $\overline{u}_\diamond = u_1u_2$, $\overline{u}_\rhd = u_2u_3$ and consider the corresponding decomposition of $\overline{v}$.

The case where $u_3 - u_2 \leq 2^n$ and $u_2 - u_1 > 2^n$ is symmetric. □

From Proposition 4.2 and Theorem 2.1 we immediately obtain our main result:

Theorem 4.3. $(\mathbb{R}, <, +1)$ *has the 3-variable property.*

5 Linear Functions

In this section we show the 3-variable property for the σ-structure $(\mathbb{R}, <, f)$ with $f : \mathbb{R} \to \mathbb{R}$ a linear function $f(x) = ax + b$. This follows fairly straightforwardly from our main result, Theorem 4.3, using the classical compositional method for sums of ordered structures.

5.1 Monotone Linear Functions

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$, where $a, b \in \mathbb{R}$ and $a > 0$. We prove that $(\mathbb{R}, <, f)$ has the 3-variable property.

Suppose that $a = 1$, that is, $f(x) = x + b$. If $b > 0$ then $(\mathbb{R}, <, f)$ is isomorphic to $(\mathbb{R}, <, +1)$. If $b < 0$ then $(\mathbb{R}, <, f)$ is isomorphic to $(\mathbb{R}, <^{\mathrm{op}}, +1)$, where $<^{\mathrm{op}}$ is the opposite order on $\mathbb{R}$. In either case $(\mathbb{R}, <, f)$ inherits the 3-variable property from $(\mathbb{R}, <, +1)$.

Assume now that $a \neq 1$. Notice that f has a unique fixed point $x^* = \frac{b}{1-a}$. Moreover, considering the intervals $I_0 = (-\infty, x^*)$ and $I_1 = (x^*, \infty)$, f restricts to bijections $f_i : I_i \to I_i$ for $i = 0, 1$. Now the map $\Phi_0(x) = -\log(x^* - x)$ defines an isomorphism of σ-structures from $(I_0, <, f_0)$ to $(\mathbb{R}, <, +a)$. Likewise the map $\Phi_1(x) = \log(x - x^*)$ defines an isomorphism from $(I_1, <, f_1)$ to $(\mathbb{R}, <, +a)$. It follows that $(I_0, <, f_0)$ and $(I_1, <, f_1)$ both have the 3-variable property.

We argue that $(\mathbb{R}, <, f)$ has the 3-variable property as follows. Let $\mathbf{A}$ and $\mathbf{B}$ be expansions of $(\mathbb{R}, <, f)$ with interpretations of the monadic predicate variables. Let $\mathbf{A}_0$ be the sub-structure of $\mathbf{A}$ with domain I_0 and let $\mathbf{A}_1$ be the sub-structure of $\mathbf{A}$ with domain I_1. Define $\mathbf{B}_0$ and $\mathbf{B}_1$ likewise. Then if Spoiler wins the n-round EF game on $\mathbf{A}$ and $\mathbf{B}$ he also wins the n-round game on the substructures $\mathbf{A}_0$ and $\mathbf{B}_0$ and the n-round game on $\mathbf{A}_1$ and $\mathbf{B}_1$. Thus there exists m, depending only on n, such that Spoiler wins the m-round 3-pebble EF games on $\mathbf{A}_0$ and $\mathbf{B}_0$ and on $\mathbf{A}_1$ and $\mathbf{B}_1$. Then by the usual composition argument on sums of ordered structures [12], we can show that Spoiler wins the m-round 3-pebble game on $\mathbf{A}$ and $\mathbf{B}$.

5.2 Antitone Linear Functions

Consider a linear function $f(x) = ax + b$, where $a < 0$. Note that the map $f^2 := f \circ f : \mathbb{R} \to \mathbb{R}$ is monotone and linear. The idea is to exploit the fact that $(\mathbb{R}, <, f^2)$ has the 3-variable property to rewrite a given monadic first-order σ-sentence φ to a 3-variable sentence φ'' that is equivalent to φ over $(\mathbb{R}, <, f)$. In this rewriting it is convenient to use x^* as an additional constant symbol in intermediate forms, where x^* is the unique fixed point of f. We also allow nested applications of f in intermediate formulas.

We obtain φ'' as follows. Motivated by the fact that f maps the open interval (x^*, ∞) onto $(-\infty, x^*)$ and *vice versa*, working bottom-up, replace each subformula $\exists x\, \psi$ by

$$\exists x\, (x > x^* \wedge (\psi \vee \psi[f(x)/x] \vee \psi[x^*/x]))\,.$$

Now simplify the atomic subformulas as follows, bearing in mind that all variables range over (x^*, ∞). Replace every term $f^n(x^*)$ with x^*. Replace $f^n(x) = x^*$ with **false**. Replace $f^n(x) = f^m(y)$ with $f^{n-1}(x) = f^{m-1}(y)$ if n and m are both odd, and with **false** if n and m have different parity. If n is odd then replace $x^* < f^n(x)$ with **false** and $f^n(x) < x^*$ with **true**. Replace $f^n(x) < f^m(y)$ by $f^{n-1}(x) < f^{m-1}(y)$ if n and m are both odd, by **true** if n is odd and m is even, and by **false** if n is even and m is odd. Finally eliminate the constant symbol x^* using the fact that it is definable in terms of f^2, e.g., replace each subformula $P(x^*)$ with $\exists y\, (y = f^2(y) \wedge P(y))$.

Let φ' denote the sentence arising from the above transformation. Treating the atomic formulas $P(f(x))$ as unary predicate variables, we can interpret φ' as a monadic first-order sentence over the structure $(\mathbb{R}, <, f^2)$. Since f^2 is monotone we can use the result of Section 5.1 to transform φ' to an equivalent 3-variable sentence φ'' over $(\mathbb{R}, <, f^2)$. Then φ'' is equivalent to φ considered as a formula over the structure $(\mathbb{R}, <, f)$.

6 Counterexample

In this section we exhibit a countable dense linear order E and function $g : E \to E$ such that $(E, <, g)$ does not have the k-variable property for any k.

Let $(S, <)$ be the set of non-empty finite sequences of integers under the lexicographic order, and let E be the equivalence relation on S that relates any two such sequences that end with the same element. Since the integers have no greatest or least element, any non-empty interval in S contains an element of each E-equivalence class. Venema [16] has shown that the structure $(S, <, E)$ does not have the k-variable property for any k. For example, one can express the property *"predicate P holds on at least $k+1$ E-inequivalent elements"* with $k+1$ variables but not k variables. Indeed it is not hard to see that in the k-pebble EF game (over any number of rounds) Spoiler cannot distinguish the cases that predicate P is a union of k E-equivalence classes and that P is a union of $k+1$ E-equivalence classes.

We next translate this example to the setting of linear orders with unary functions. Consider the equivalence relation E above as an ordered set under the lexicographic order on $S \times S$. Define $g : E \to E$ by $g(s,t) = (t,s)$ and consider the σ-structure $\mathbf{E} = (E, <, g)$ (where σ is the signature for linear orders and unary functions, defined in Section 2.1). Note that g is very far from being monotone.

To each labelled expansion $\mathbf{S}$ of $(S, <, E)$ we associate a labelled expansion $\mathbf{E}$ of $(E, <, g)$, where $P^{\mathbf{E}} = \{(s,s) : s \in P^{\mathbf{S}}\}$ for each monadic predicate symbol P. There is moreover a one-dimensional interpretation Γ (cf. Section 2.2) of $\mathbf{S}$ in $\mathbf{E}$. The domain formula $\partial_\Gamma(x)$ of Γ is $x = g(x)$ so that $\partial_\Gamma(E) = \{(s,t) \in E : s = t\}$. The coding map $f_\Gamma : \partial_\Gamma(E) \to S$ is given by $f_\Gamma(s,s) = s$. The interpretation also specifies for each atomic formula $\varphi(x_1, \ldots, x_m)$ over $\mathbf{S}$ a corresponding formula $\varphi_\Gamma(x_1, \ldots, x_m)$ over $\mathbf{E}$, with $\mathbf{S} \models \varphi[s_1, \ldots, s_m]$ if and only if $\mathbf{E} \models \varphi_\Gamma[(s_1,s_1), \ldots, (s_m,s_m)]$ for all $s_1, \ldots, s_m \in S$. This correspondence sends $x < y$ and $P(x)$ to themselves and $E(x,y)$ to the formula $\psi(x,y) \vee \psi(y,x)$, where

$$\begin{aligned}\psi(x,y) :=\, & \exists u\, (x < u < g(u) < y \,\wedge \\ & \forall v (x < v < u \vee g(u) < v < y \to g(v) \neq v))\,.\end{aligned}$$

Conversely there is a natural two-dimensional first-order interpretation Γ of $\mathbf{E}$ in $\mathbf{S}$. The domain formula is $\partial_\Gamma(x,y) = E(x,y)$, and thus $\partial_\Gamma(S^2) = \{(s,t) \in S \times S : (s,t) \in E\}$. The coding map $f_\Gamma : \partial_\Gamma(S^2) \to E$ is given by $f_\Gamma(s,t) = (s,t)$. The translation of atomic formulas over $\mathbf{E}$ to corresponding formulas over $\mathbf{S}$ is similarly straightforward, e.g., $x < y$ is mapped to $x_1 < y_1 \vee (x_1 = y_1 \wedge x_2 < y_2)$.

As observed in Dawar [1, Section 3] in a similar context, the existence of such a two-way interpretation entails that if $(E, <, g)$ has the k-variable property for some k then $(S, <, E)$ has the k'-variable property for some k'. It follows that $(E, <, g)$ does not have the k-variable property for any k.

7 Conclusion and Future Work

We have shown that the structure $(\mathbb{R}, <, f)$ has the 3-variable property for linear functions $f : \mathbb{R} \to \mathbb{R}$. In future work it would be natural to consider whether the k-variable property holds, for some k, for richer classes of functions, e.g., classes of polynomials.

Moving beyond the reals, we would like to explore whether the results in this paper generalise to arbitrary linear orders and families of monotone functions thereon. More generally, there is the problem, raised by Immerman and Kozen in the conclusion of [12], of finding a model-theoretic characterisation of those classes of structures possessing the k-variable property for some k.

In those settings in which the k-variable property holds, following [4], it is natural to consider how the number of variables affects the succinctness of formulas and, in view of [2], also to seek expressively complete temporal logics.

References

1. Dawar, A.: How many first-order variables are needed on finite ordered structures? In: We Will Show Them! Essays in Honour of Dov Gabbay, vol. 1, pp. 489–520. College Publications (2005)
2. Gabbay, D.M.: Expressive functional completeness in tense logic. In: Mönnich, U. (ed.) Aspects of Philosophical Logic, pp. 91–117. Reidel, Dordrecht (1981)
3. Gabbay, D.M., Pnueli, A., Shelah, S., Stavi, J.: On the temporal basis of fairness. In: POPL, pp. 163–173. ACM Press (1980)
4. Grohe, M., Schweikardt, N.: The succinctness of first-order logic on linear orders. Logical Methods in Computer Science 1(1) (2005)
5. Hirshfeld, Y., Rabinovich, A.: Continuous time temporal logic with counting. Inf. Comput. 214, 1–9 (2012)
6. Hirshfeld, Y., Rabinovich, A.M.: Timer formulas and decidable metric temporal logic. Inf. Comput. 198(2), 148–178 (2005)
7. Hodges, W.: A Shorter Model Theory. Cambridge University Press, New York (1997)
8. Hodkinson, I., Simon, A.: The k-variable property is stronger than H-dimension k. Journal of Philosophical Logic 26(1), 81–101 (1997)
9. Hunter, P.: When is metric temporal logic expressively complete? In: CSL. LIPIcs, vol. 23, pp. 380–394. Schloss Dagstuhl (2013)
10. Hunter, P., Ouaknine, J., Worrell, J.: Expressive completeness for metric temporal logic. In: LICS, pp. 349–357. IEEE Computer Society Press (2013)
11. Immerman, N.: Upper and lower bounds for first order expressibility. J. Comput. Syst. Sci. 25(1), 76–98 (1982)
12. Immerman, N., Kozen, D.: Definability with bounded number of bound variables. Inf. Comput. 83(2), 121–139 (1989)
13. Kamp, H.: Tense Logic and the Theory of Linear Order. PhD thesis, University of California (1968)
14. Poizat, B.: Deux ou trois choses que je sais de L_n. J. Symb. Log. 47(3), 641–658 (1982)
15. Rossman, B.: On the constant-depth complexity of k-clique. In: STOC, pp. 721–730. ACM (2008)
16. Venema, Y.: Expressiveness and completeness of an interval tense logic. Notre Dame Journal of Formal Logic 31(4), 529–547 (1990)

A Appendix

A.1 Missing Proofs from Section 3

Proposition 3.1. *Let $u_1 \dots u_s \equiv v_1 \dots v_s$ be two assignments. Then*

$$\mathrm{frac}(u_i - u_k) < \mathrm{frac}(u_j - u_k) \;\Leftrightarrow\; \mathrm{frac}(v_i - v_k) < \mathrm{frac}(v_j - v_k)$$

for all indices $i, j, k \in \{1, \dots, s\}$.

Proof. Fix $i, j, k \in \{1, \dots, s\}$. From the assumption $u_1 \dots u_s \equiv v_1 \dots v_s$ we have the following chain of equivalences:

$$\begin{aligned}
\mathrm{frac}(u_i - u_k) < \mathrm{frac}(u_j - u_k) &\Leftrightarrow u_i - u_k - \lfloor u_i - u_k \rfloor < u_j - u_k - \lfloor u_j - u_k \rfloor \\
&\Leftrightarrow u_i - u_j < \lfloor u_i - u_k \rfloor - \lfloor u_j - u_k \rfloor \\
&\Leftrightarrow v_i - v_j < \lfloor u_i - u_k \rfloor - \lfloor u_j - u_k \rfloor \\
&\Leftrightarrow v_i - v_j < \lfloor v_i - v_k \rfloor - \lfloor v_j - v_k \rfloor \\
&\Leftrightarrow v_i - v_k - \lfloor v_i - v_k \rfloor < v_j - v_k - \lfloor v_j - v_k \rfloor \\
&\Leftrightarrow \mathrm{frac}(v_i - v_k) < \mathrm{frac}(v_j - v_k)\,.
\end{aligned}$$

□

Proposition 3.2. *Suppose that $u_1 \dots u_s$ and $v_1 \dots v_s$ are both increasing and that $u_1 \dots u_m \equiv v_1 \dots v_m$ and $u_m \dots u_s \equiv v_m \dots v_s$ for some m, $1 \le m \le s$. Then $u_1 \dots u_s \equiv v_1 \dots v_s$.*

Proof. We must show that $\lfloor u_j - u_i \rfloor = \lfloor v_j - v_i \rfloor$ for all $i \le m < j$. To this end, we observe that since $u_1 \dots u_s$ is increasing,

$$\begin{aligned}
\mathrm{frac}(u_j - u_i) &= \mathrm{frac}(u_j - u_m + (u_m - u_i)) \\
&= \mathrm{frac}(u_j - u_m) + \mathrm{frac}(u_m - u_i)\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
\lfloor u_j - u_i \rfloor &= u_j - u_i - \mathrm{frac}(u_j - u_i) \\
&= (u_j - u_m) + (u_m - u_i) - (\mathrm{frac}(u_j - u_m) + \mathrm{frac}(u_m - u_i)) \\
&= (u_j - u_m) - \mathrm{frac}(u_j - u_m) + (u_m - u_i) - \mathrm{frac}(u_m - u_i) \\
&= \lfloor u_j - u_m \rfloor + \lfloor u_m - u_i \rfloor\,.
\end{aligned}$$

We can similarly show that

$$\lfloor v_j - v_i \rfloor = \lfloor v_j - v_m \rfloor + \lfloor v_m - v_i \rfloor\,.$$

But $\lfloor u_j - u_m \rfloor = \lfloor v_j - v_m \rfloor$ since $u_m \dots u_s \equiv v_m \dots v_s$. Likewise $\lfloor u_m - u_i \rfloor = \lfloor v_m - v_i \rfloor$ since $u_1 \dots u_m \equiv v_1 \dots v_m$. We conclude that $\lfloor u_j - u_i \rfloor = \lfloor v_j - v_i \rfloor$. □