Advances on Testing C-Planarity of Embedded Flat Clustered Graphs*

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Abstract. We show a polynomial-time algorithm for testing c-planarity of embedded flat clustered graphs with at most two vertices per cluster on each face.

1 Introduction

A *clustered graph* $C(G,T)$ consists of a graph $G(V,E)$, called *underlying graph*, and of a rooted tree T, called *inclusion tree*, representing a cluster hierarchy on V. The vertices in V are the leaves of T, and the inner nodes of T, except for the root, are called *clusters*. The vertices that are descendants of a cluster α in T *belong to* α or *are in* α. A *c-planar drawing* of C is a planar drawing of G together with a representation of each cluster α as a simple connected region R_α enclosing all and only the vertices that are in α; further, the boundaries of no two such regions R_α and R_β intersect; finally, only the edges connecting vertices in α to vertices not in α cross the boundary of R_α, and each does so only once. A clustered graph is *c-planar* if it admits a c-planar drawing.

Clustered graphs find numerous applications in computer science [22], thus theoretical questions on clustered graphs have been deeply investigated. From the visualization perspective, the most intriguing question is to determine the complexity of testing c-planarity of clustered graphs. Unlike for other planarity variants [21], like *upward planarity* [14] and *partial embedding planarity* [1], the complexity of testing c-planarity remains unknown since the problem was posed nearly two decades ago [13].

Polynomial-time algorithms to test the c-planarity of a clustered graph C are known if C belongs to special classes of clustered graphs [7–11,13,15,16,18,19], including *c-connected clustered graphs*, that are clustered graphs $C(G,T)$ in which, for each cluster α, the subgraph $G[\alpha]$ of G induced by the vertices in α is connected [8,10,13]. Effective ILP formulations and FPT algorithms for testing c-planarity have been presented [5,6]. Generalizations of the c-planarity testing problem have also been considered [2,12].

An important variant of the c-planarity testing problem is the one in which the clustered graph $C(G,T)$ is *flat* and *embedded*. That is, every cluster is a child of the root of T and a planar embedding for G (an order of the edges incident to each vertex) is fixed

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in advance; then, the c-planarity testing problem asks whether a c-planar drawing exists in which G has the prescribed planar embedding. This setting can be highly regarded for several reasons. First, several NP-hard graph drawing problems are polynomial-time solvable in the fixed embedding scenario, e.g., *upward planarity testing* [3, 14] and *bend minimization in orthogonal drawings* [14, 23]. Second, testing c-planarity of embedded flat clustered graphs generalizes testing c-planarity of triconnected flat clustered graphs. Third, testing c-planarity of embedded flat clustered graphs is strongly related to a seemingly different problem, that we call *planar set of spanning trees in topological multigraphs* (PSSTTM): Given a non-planar topological multigraph A with k connected components $A_1, \dots, A_k$, do spanning trees $S_1, \dots, S_k$ of $A_1, \dots, A_k$ exist such that no two edges in $\bigcup_i S_i$ cross? Starting from an embedded flat clustered graph $C(G, T)$, an instance A of the PSSTTM problem can be constructed that admits a solution if and only if $C(G, T)$ is c-planar: A is composed of the edges that can be inserted inside the faces of G between vertices of the same cluster, where each cluster defines a multigraph A_i. The PSSTTM problem is NP-hard, even if $k = 1$ [20].

Testing c-planarity of an embedded flat clustered graph $C(G, T)$ is a polynomial-time solvable problem if G has no face with more than five vertices and, more in general, if C is a *single-conflict* clustered graph [11], i.e., the instance A of the PSSTTM problem associated with C is such that each edge has at most one crossing. A polynomial-time algorithm is also known for testing c-planarity of embedded flat clustered graphs such that the graph induced by each cluster has at most two connected components [17]. Finally, the c-planarity of clustered cycles with at most three clusters [9] or with each cluster containing at most three vertices [19] can be tested in polynomial time.

Our Contribution. In this paper we show how to test c-planarity in polynomial time for embedded flat clustered graphs $C(G, T)$ such that at most two vertices of each cluster are incident to any face of G. While this setting might seem unnatural at a first glance, its study led to a deep (in our opinion) exploration of some combinatorial properties of highly non-planar topological graphs. Namely, every instance A of the PSSTTM problem arising from our setting is such that there exists no sequence $e_1, e_2, \dots, e_h$ of edges in A with e_1 and e_h in the same connected component of A and with e_i crossing e_{i+1}, for every $1 \leq i \leq h - 1$; these instances might contain a quadratic number of crossings, which is not the case for single-conflict clustered graphs [11]. Within our setting, performing all the "trivial local" tests and simplifications results in the rise of nice global structures, called *α-donuts*, whose study was interesting to us.

Refer to the full version of the paper [4] for complete proofs.

2 Saturators, Con-Edges, and Spanning Trees

A natural approach to test c-planarity of a clustered graph $C(G(V, E), T)$ is to search for a *saturator* for C. A set $S \subseteq V \times V$ is a saturator for C if $C'(G'(V, E \cup S), T)$ is a c-connected c-planar clustered graph. Determining the existence of a saturator for C is equivalent to testing the c-planarity of C [13]. Thus, the core of the problem consists of determining S so that $G'[\alpha]$ is connected, for each $\alpha \in T$, and so that G' is planar. For embedded flat clustered graphs (see Fig. 1(a)), the problem of finding saturators

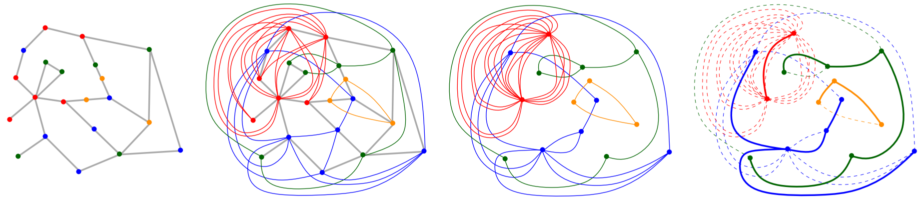

Fig. 1. (a) A clustered graph C. (b) Con-edges in C. (c) Multigraph A. (d) A planar set S of spanning trees for A. Edges in S are thick and solid, while edges in $A \setminus S$ are thin and dashed.

becomes seemingly simpler. Since the embedding of G is fixed and since G' has to be planar, edges in S can only be embedded inside faces of G. This implies that, for any two edges e_1 and e_2 that can be inserted inside a face f of G, it is known *a priori* whether e_1 and e_2 can be both in S, namely only if their end-vertices do not alternate along the boundary of f. Also, S can be assumed to contain only edges between vertices in distinct connected components of $G[\alpha]$, for each cluster α, as other types of edges do not help to connect any cluster.

Consider a face f of G and let $(o_1, \ldots, o_k)$ be the clockwise order of the occurrences of vertices along the boundary of f, where o_i and o_j might be occurrences of the same vertex u (this might happen if u is a cut-vertex of G). A *con-edge* (short for *connectivity-edge*) is a pair of occurrences (o_i, o_j) of distinct vertices both belonging to a cluster α, both incident to f, and belonging to different connected components of $G[\alpha]$ (see Fig. 1(b)). If there are ℓ distinct pairs of occurrences of vertices u and v along a single face f, then there are ℓ con-edges connecting u and v in f, one for each pair of occurrences. A *con-edge for* α is a con-edge connecting vertices in a cluster α. Two con-edges e and e' in f *have a conflict* or *cross* (we write $e \otimes e'$) if the occurrences in e alternate with the occurrences in e' along the boundary of f.

The *multigraph* A *of the con-edges* is an embedded multigraph that is defined as follows. Starting from G, insert all the con-edges inside the faces of G; then, for each cluster α and for each connected component $G_i[\alpha]$ of $G[\alpha]$, contract $G_i[\alpha]$ into a single vertex; finally, remove all the edges of G. See Fig. 1(c). With a slight abuse of notation, we denote by A both the multigraph of the con-edges and the set of its edges. For each cluster α, we denote by $A[\alpha]$ the subgraph of A induced by the con-edges for α. A *planar set of spanning trees for* A is a set $S \subseteq A$ such that: (i) for each cluster α, the subset $S[\alpha]$ of S induced by the con-edges for α is a tree that spans the vertices belonging to α; and (ii) there exist no two edges in S that have a conflict. See Fig. 1(d). The PSSTTM problem asks whether a planar set of spanning trees for A exists.

The following lemma relates the c-planarity problem for embedded flat clustered graphs to the PSSTTM problem.

Lemma 1 ([11]). *An embedded flat clustered graph $C(G, T)$ is c-planar if and only if: (1) G is planar; (2) there exists a face f in G such that when f is chosen as outer face for G no cycle composed of vertices of the same cluster encloses a vertex of a different cluster; and (3) a planar set of spanning trees for A exists.*

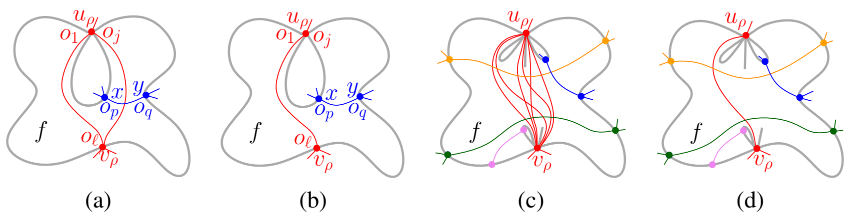

Fig. 2. Illustration for the reduction to a multigraph of the con-edges satisfying Property 1

We now introduce the concept of *conflict graph* K_A, which is defined as follows. Graph K_A has a vertex for each con-edge in A and has an edge (e, e') if $e \otimes e'$. In the remainder of the paper we will show how to decide whether a set of planar spanning trees for A exists by assuming that the following property holds for A.

Property 1. *No two con-edges for the same cluster belong to the same connected component of K_A.*

We now show that A can be assumed to satisfy Property 1, since $C(G, T)$ has at most two vertices per cluster on each face. Consider any face f of G and any cluster ϱ that has two vertices u_ϱ and v_ϱ incident to f. No con-edge for ϱ that connects a pair of vertices different from (u_ϱ, v_ϱ) is in the connected component of K_A containing (u_ϱ, v_ϱ), given that no vertex of ϱ different from u_ϱ and v_ϱ is incident to f. However, it might be the case that several con-edges (u_ϱ, v_ϱ) are in the same connected component of K_A, which happens if u_ϱ, or v_ϱ, or both have several occurrences on the boundary of f. We show a simple reduction that gets rid of these multiple con-edges.

Let $(o_1, \dots, o_k)$ be the clockwise order of the occurrences of vertices along f. Let o_1, o_j, and o_ℓ be occurrences of u_ϱ, u_ϱ, and v_ϱ, respectively, with $1 < j < \ell \leq k$. Suppose that o_p and o_q are occurrences of vertices x and y in a cluster $\tau \neq \varrho$, for some $1 < p < j < q < \ell$, as in Fig. 2(a). Then, all the con-edges (x, y) have a conflict with con-edge $e_\varrho = (o_j, o_\ell)$; moreover, the con-edges (x, y) form a separating set for $A[\tau]$, hence any planar set S of spanning trees for A contains one of them. Thus, $e_\varrho \notin S$ and e_ϱ can be removed from A, as in Fig. 2(b). Similar reductions can be performed if $\ell < q \leq k$ and by exchanging the roles of u_ϱ and v_ϱ. If no two occurrences o_p and o_q as above exist, then all the con-edges (u_ϱ, v_ϱ) left cross the same set of con-edges for clusters different from ϱ (see Fig. 2(c)). Hence, a single edge (u_ϱ, v_ϱ) can be kept in A, and all the other con-edges (u_ϱ, v_ϱ) can be removed from A (see Fig. 2(d)). After repeating this reduction for all the con-edges in A, an equivalent instance A is eventually obtained in which Property 1 is satisfied. Observe that the described simplification can be easily performed in $O(|C|^2)$ time. Thus, we get the following:

Lemma 2. *Assume that the* PSSTTM *problem can be solved in $f(|A|)$ time for instances satisfying Property 1. Then the c-planarity of any embedded flat clustered graph C with at most two vertices per cluster on each face can be tested in $O(f(|A|) + |C|^2)$ time.*

Proof. Consider any embedded flat clustered graph C with at most two vertices per cluster on each face. Conditions (1) and (2) in Lemma 1 can be tested in $O(|C|)$ time (see [11]); hence, testing the c-planarity of C is equivalent to solve the PSSTTM problem for A. Finally, as described before the lemma, there exists an $O(|C|^2)$-time algorithm that modifies multigraph A so that it satisfies Property 1. □

3 Algorithm Outline

In this section we give an outline of our algorithm for testing the existence of a planar set S of spanning trees for A. We assume that A satisfies Property 1.

Our algorithm repeatedly tries to detect certain substructures in A. When it does find one of such substructures, the algorithm either "simplifies" A or concludes that A does not admit any planar set of spanning trees. For example, if a cluster α exists such that $A[\alpha]$ is not connected, then the algorithm concludes that no planar set of spanning trees exists and terminates; as another example, if conflicting con-edges e_α and e_β for clusters α and β exist in A such that e_α is a bridge for $A[\alpha]$, then the algorithm determines that e_α has to be in S and that e_β can be assumed not to be in S.

If the algorithm determines that certain edges have to be in S or can be assumed not to be in S, these edges are contracted or removed, respectively. Given a set $A' \subseteq A$, the operation of *removing* A' from A consists of updating $A := A \setminus A'$. Given a set $A' \subseteq A$, the operation of *contracting* the edges in A' consists of identifying the end-vertices of each con-edge e in A' (all the con-edges different from e and incident to the end-vertices of e remain in A), and of updating $A := A \setminus A'$.

Edges are removed or contracted only when this does not alter the possibility of finding a planar set of spanning trees for A. Also, contractions are only applied to con-edges that cross no other con-edges; hence, after any contraction, graph K_A only changes for the removal of the isolated vertices corresponding to the contracted edges.

As a consequence of a removal or of a contraction operation, the number of edges in A decreases, that is, A is "simplified". After any simplification due to the detection of a certain substructure in A, the algorithm will run again all previous tests for the detection of the other substructures. In fact, it is possible that a certain substructure arises from performing a simplification on A (e.g., a bridge might be present in A after a set of edges has been removed from A). Since detecting each substructure that leads to a simplification in A can be performed in quadratic time, and since the initial size of A is linear in the size of C, the algorithm has a cubic running time.

If none of the four tests (called TEST 1–4) and none of the eight simplifications (called SIMPLIFICATION 1–8) described in Section 4 applies to A, then A is a *single-conflict* multigraph. That is, each con-edge in A crosses at most one con-edge in A. A linear-time algorithm for deciding the existence of a planar set of spanning trees in a single-conflict multigraph A is known [11]. Hence, our algorithm uses that algorithm [11] to conclude the test of the existence of a planar set of spanning trees in A.

4 Algorithm

To ease the reading and avoid text duplication, when introducing a new lemma we always assume, without making it explicit, that all the previously defined simplifications do not apply, and that all the previously defined tests fail. Also, we do not make explicit the removal and contraction operations that we perform, as they straight-forwardly follow from the statement of each lemma. We start with the following test.

Lemma 3 **(**TEST 1**).** *Let α be a cluster such that $A[\alpha]$ is disconnected. Then, there exists no planar set S of spanning trees for A.*

Proof. No set $S \subseteq A$ is such that $S[\alpha]$ induces a tree spanning the vertices in α. □

We continue with the following simplification.

Lemma 4 **(**SIMPLIFICATION 1**).** *Let e be a bridge of $A[\alpha]$. Then, for every planar set S of spanning trees for A, we have $e \in S$.*

Proof. Graph $A[\alpha] \setminus \{e\}$ is disconnected; hence, by Lemma 3, no planar set of spanning trees for A exists with $e \notin S$. □

The following lemma is used massively in the remainder of the paper.

Lemma 5. *Let $e_\alpha, e_\beta \in A$ be con-edges such that $e_\alpha \otimes e_\beta$. Let S be a planar set of spanning trees for A and suppose that $e_\alpha \notin S$. Then, $e_\beta \in S$.*

Proof sketch. If S contains neither e_α nor e_β, then the two paths in S connecting the end-vertices of e_α and connecting the end-vertices of e_β cross, a contradiction. □

The algorithm continues with the following test.

Lemma 6 **(**TEST 2**).** *If the conflict graph K_A is not bipartite, then there exists no planar set S of spanning trees for A.*

Proof sketch. If an odd cycle $\mathcal{C}$ exists in K_A, then by repeated applications of Lemma 5 and of the fact that S does not contain two conflicting edges, we get that any edge of $\mathcal{C}$ simultaneously should be in S and should not be in S, a contradiction. □

The contraction of con-edges chosen to be in S might lead to self-loops in A, a situation that is handled in the following.

Lemma 7 **(**SIMPLIFICATION 2**).** *Let $e \in A$ be a self-loop. Then, for every planar set S of spanning trees for A, we have $e \notin S$.*

Proof. Since a tree does not contain any self-loop, the lemma follows. □

Con-edges that do not cross any other con-edge can be safely chosen to be in S.

Lemma 8 **(**SIMPLIFICATION 3**).** *Let e be any con-edge in A that does not have a conflict with any other con-edge in A. Then, there exists a planar set S of spanning trees for A if and only if there exists a planar set S' of spanning trees for A such that $e \in S'$.*

Proof sketch. Let S be any planar set of spanning trees for A. If $e \notin S$, then $S \cup \{e\}$ contains a cycle $\mathcal{C}$ of con-edges for the same cluster. Let e' be any edge of $\mathcal{C}$ different from e. Then, $S' = S \cup \{e\} \setminus \{e'\}$ is a planar set of spanning trees for A. □

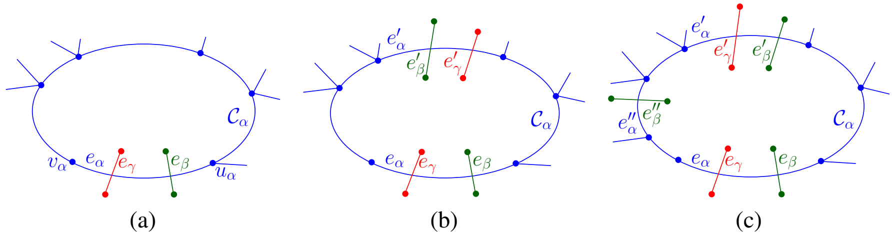

Fig. 3. The setting for (a) Lemma 9, (b) Lemma 10, and (c) Lemma 11

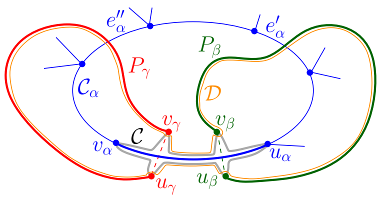

Fig. 4. Illustration for the proof of Lemma 9

In the next three lemmata we deal with the following setting. Assume that there exist con-edges $e_\alpha, e_\beta, e_\gamma \in A$ for distinct clusters α, β, and γ, respectively, such that $e_\alpha \otimes e_\beta$ and $e_\alpha \otimes e_\gamma$. Since TEST 2 fails on A, e_β does not cross e_γ. Let $\mathcal{C}_\alpha$ be any of the two facial cycles of $A[\alpha]$ incident to e_α, where a facial cycle of $A[\alpha]$ is a simple cycle all of whose edges appear on the boundary of a single face of $A[\alpha]$. Assume w.l.o.g. that e_α is crossed first by e_β and then by e_γ when $\mathcal{C}_\alpha$ is traversed clockwise. See Fig. 3(a).

The next lemma presents a condition in which we can delete e_α from A.

Lemma 9 (SIMPLIFICATION 4). *Suppose that there exists no con-edge of $\mathcal{C}_\alpha$ different from e_α that has a conflict with both a con-edge for β and a con-edge for γ. Then, for every planar set S of spanning trees for A, we have $e_\alpha \notin S$.*

Proof sketch. See Fig. 4. Let u_α and v_α (u_β and v_β, u_γ and v_γ) be the end-vertices of e_α (resp. of e_β, of e_γ). By Property 1, a closed simple curve $\mathcal{C}$ can be drawn through u_α, u_β, u_γ, v_α, v_γ, and v_β, with e_α, e_β, and e_γ in its interior and every other con-edge for α, β, and γ in its exterior. If $e_\alpha \in S$, then $e_\beta, e_\gamma \notin S$. Then, the path P_β in S connecting u_β and v_β and the path P_γ in S connecting u_γ and v_γ cross $\mathcal{C}_\alpha$ at different edges e'_α and e''_α. Hence, the end-vertices of e'_α are on different sides of the cycle $\mathcal{D}$ composed of P_β, of P_γ, and of the paths in $\mathcal{C}$ between u_β and u_γ and between v_β and v_γ. However, no con-edge for α in S crosses $\mathcal{D}$, hence S does not connect α. □

The next two lemmata state conditions in which no planar set of spanning trees for A exists. Their statements are illustrated in Figs. 3(b) and 3(c), respectively; further, they can be proved with arguments similar to the ones in the proof of Lemma 9.

Lemma 10 (TEST 3). *Suppose that there exist con-edges $e'_\alpha, e'_\beta, e'_\gamma \in A$ for clusters α, β, and γ, respectively, such that $e'_\alpha \neq e_\alpha$, e'_α belongs to $\mathcal{C}_\alpha$, and $e'_\alpha \otimes e'_\beta$ as well*

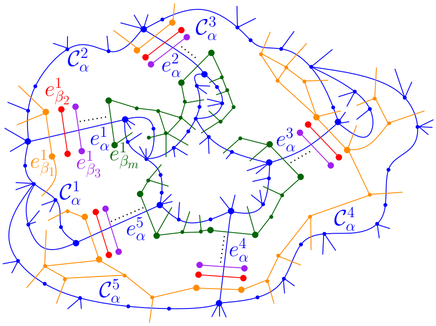

Fig. 5. A partial representation of the α-donut for e_α

as $e'_\alpha \otimes e'_\gamma$. Assume that e'_α is crossed first by e'_β and then by e'_γ when $\mathcal{C}_\alpha$ is traversed clockwise. Then, no planar set of spanning trees for A exists.

Lemma 11 (TEST 4)**.** *Suppose that con-edges $e'_\alpha, e''_\alpha \in A$ for α exist in $\mathcal{C}_\alpha$, and such that e_α, e''_α, and e'_α occur in this order along $\mathcal{C}_\alpha$, when clockwise traversing $\mathcal{C}_\alpha$. Suppose also that there exist con-edges $e'_\beta, e''_\beta \in A$ for β and $e'_\gamma \in A$ for γ such that $e'_\alpha \otimes e'_\beta$, $e'_\alpha \otimes e'_\gamma$, and $e''_\alpha \otimes e''_\beta$. Then, no planar set of spanning trees for A exists.*

If SIMPLIFICATIONS 1–4 do not apply to A and TESTS 1–4 fail on A, then the con-edges for a cluster α that are crossed by con-edges for (at least) two other clusters have a nice structure, that we call *α-donut* (see Fig. 5). Consider a con-edge $e_\alpha \in A$ for α crossing con-edges $e_{\beta_1}, \dots, e_{\beta_m}$ for clusters $\beta_1, \dots, \beta_m$, with $m \geq 2$. An α-donut for e_α consists of a sequence $e_\alpha^1, \dots, e_\alpha^k, e_\alpha^{k+1}$ of con-edges for α with $k \geq 2$, called *spokes* of the α-donut, of a sequence $\mathcal{C}_\alpha^1, \dots, \mathcal{C}_\alpha^k, \mathcal{C}_\alpha^{k+1} = \mathcal{C}_\alpha^1$ of facial cycles in $A[\alpha]$, and of sequences $e_{\beta_j}^1, \dots, e_{\beta_j}^k$ of con-edges for β_j, for each $1 \leq j \leq m$, such that the following hold for every $1 \leq i \leq k$: (a) $e_\alpha = e_\alpha^1$; (b) $e_\alpha^i \otimes e_{\beta_j}^i$, for every $1 \leq j \leq m$; (c) $\mathcal{C}_\alpha^i$ and $\mathcal{C}_\alpha^{i+1}$ share edge e_α^i; (d) edge e_α^i is crossed by $e_{\beta_1}^i, \dots, e_{\beta_m}^i$ in this order when $\mathcal{C}_\alpha^i$ is traversed clockwise; (e) all the con-edges of $\mathcal{C}_\alpha^{i+1}$ encountered when clockwise traversing $\mathcal{C}_\alpha^{i+1}$ from e_α^i to e_α^{i+1} do not cross any con-edge for $\beta_2, \dots, \beta_m$; and (f) all the con-edges of $\mathcal{C}_\alpha^{i+1}$ encountered when clockwise traversing $\mathcal{C}_\alpha^{i+1}$ from e_α^{i+1} to e_α^i do not cross any con-edge for $\beta_1, \dots, \beta_{m-1}$. We have the following.

Lemma 12. *For every con-edge $e_\alpha \in A$ for α, there exists an α-donut for e_α.*

Proof sketch. Let $e_\alpha^1 = e_\alpha$ and let $\mathcal{C}_\alpha^1$ and $\mathcal{C}_\alpha^2$ be the facial cycles incident to e_α^1. Since SIMPLIFICATION 1 does not apply to A, $\mathcal{C}_\alpha^1 \neq \mathcal{C}_\alpha^2$. Let $e_{\beta_1}^1, \dots, e_{\beta_m}^1$ be the con-edges for $\beta_1, \dots, \beta_m$, respectively, ordered as they cross e_α^1 when clockwise traversing $\mathcal{C}_\alpha^1$. Since SIMPLIFICATION 4 does not apply to A and TESTS 3-4 fail on A, a con-edge e_α^2 exists in $\mathcal{C}_\alpha^2$ that is crossed by con-edges $e_{\beta_1}^2, \dots, e_{\beta_m}^2$ for $\beta_1, \dots, \beta_m$, respectively, in this order when clockwise traversing $\mathcal{C}_\alpha^2$; further, all the con-edges of $\mathcal{C}_\alpha^2$ encountered when clockwise traversing $\mathcal{C}_\alpha^2$ from e_α^1 to e_α^2 (from e_α^2 to e_α^1) do not cross any con-edges for $\beta_2, \dots, \beta_m$ (resp. for $\beta_1, \dots, \beta_{m-1}$). This argument is repeated for $i = 3, \dots, k$ to determine a facial cycle $\mathcal{C}_\alpha^i$ containing e_α^{i-1} and to determine edges $e_\alpha^i, e_{\beta_1}^i, \dots, e_{\beta_m}^i$.

Eventually, facial cycle $\mathcal{C}_\alpha^{k+1} = \mathcal{C}_\alpha^1$ of $A[\alpha]$ is considered in which the two con-edges that are crossed by con-edges for all of $\beta_1, \dots, \beta_m$ are e_α^k and e_α^1. □

The α-donut for e_α can be computed efficiently. Further, we have the following lemma, whose proof is similar to the one of Lemma 9.

Lemma 13. *Let $e_\alpha^1, \dots, e_\alpha^k$ be the spokes of the α-donut for e_α. Then, if a planar set S of spanning trees for A exists, it contains exactly one of $e_\alpha^1, \dots, e_\alpha^k$.*

Consider a con-edge e for a cluster α. The *conflicting structure* $M(e)$ of e is a sequence of sets $H_0(e), L_1(e), H_1(e), L_2(e), H_2(e), \dots$ of con-edges which correspond to the layers of a BFS traversal starting at e of the connected component of K_A containing e. That is: $H_0(e) = \{e\}$; then, for $i \geq 1$, $L_i(e)$ is the set of con-edges that cross con-edges in $H_{i-1}(e)$ and that are not in $L_{i-1}(e)$, and $H_i(e)$ is the set of con-edges that cross con-edges in $L_i(e)$ and that are not in $H_{i-1}(e)$.

We now study the conflicting structures of the spokes $e_\alpha^1, \dots, e_\alpha^k$ of the α-donut of a con-edge e_α for α. No two edges in a set $H_i(e_\alpha)$ or in a set $L_i(e_\alpha)$ have a conflict, as otherwise TEST 2 would succeed. Also, by Lemma 5, any planar set S of spanning trees for A contains either all the edges in $\bigcup_i H_i(e_\alpha)$ or all the edges in $\bigcup_i L_i(e_\alpha)$.

Assume that e_α has a conflict with at least two con-edges for other clusters. For any $1 \leq i \leq k$, we say that e_α^i and e_α^{i+1} have *isomorphic conflicting structures* if they belong to isomorphic connected components of K_A and if the vertices of these components that are in correspondence under the isomorphism represent con-edges for the same cluster. Formally, e_α^i and e_α^{i+1} have isomorphic conflicting structures if there exists a bijective mapping δ from the edges in $M(e_\alpha^i)$ to the edges in $M(e_\alpha^{i+1})$ such that: (1) e is a con-edge for a cluster ϱ if and only if $\delta(e)$ is a con-edge for ϱ, for every $e \in M(e_\alpha^i)$; (2) $e \in H_j(e_\alpha^i)$ if and only if $\delta(e) \in H_j(e_\alpha^{i+1})$, for every $e \in M(e_\alpha^i)$; (3) $e \in L_j(e_\alpha^i)$ if and only if $\delta(e) \in L_j(e_\alpha^{i+1})$, for every $e \in M(e_\alpha^i)$; and (4) $e \otimes f$ if and only if $\delta(e) \otimes \delta(f)$, for every $e, f \in M(e_\alpha^i)$. Observe that the isomorphism of two conflicting structures can be tested efficiently.

We will prove in the following four lemmata that by examining the conflicting structures for the spokes of the α-donut for e_α, a decision on whether some spoke is or is not in S can be taken without loss of generality. We start with the following:

Lemma 14 (SIMPLIFICATION 5)**.** *Suppose that spokes e_α^i and e_α^{i+1} have isomorphic conflicting structures. Then, there exists a planar set S of spanning trees for A if and only if there exists a planar set S' of spanning trees for A such that $e_\alpha^i \notin S'$.*

Proof sketch. Suppose that a planar set S of spanning trees for A exists with $e_\alpha^i \in S$. By Lemma 5, $\bigcup_j H_j(e_\alpha^i) \subseteq S$ and $S \cap \bigcup_j L_j(e_\alpha^i) = \emptyset$. By Lemma 13, $e_\alpha^{i+1} \notin S$, hence $\bigcup_j L_j(e_\alpha^{i+1}) \subseteq S$ and $S \cap \bigcup_j H_j(e_\alpha^{i+1}) = \emptyset$. Let S' be the set of con-edges obtained from S by removing $\bigcup_j H_j(e_\alpha^i)$ and $\bigcup_j L_j(e_\alpha^{i+1})$ and by adding $\bigcup_j L_j(e_\alpha^i)$ and $\bigcup_j H_j(e_\alpha^{i+1})$. The lemma follows from the claim that S' is a planar set of spanning trees for A. The proof for this claim consists of two parts. In the first one, it is shown that no two con-edges in S' cross, by exploiting the absence of crossings in S and the properties of $M(e_\alpha^i)$ and $M(e_\alpha^{i+1})$. In the second one, it is shown that, for each cluster μ, the graph induced by the con-edges in $S'[\mu]$ is a tree that spans the vertices in μ; this proof uses topological arguments to establish that the only con-edge for μ in $S' \setminus S$ has its end-vertices in different connected components of the graph obtained from $S[\mu]$ by removing the only con-edge for μ in $S \setminus S'$. □

Next, we study non-isomorphic spokes. Let e^i_α be a spoke of the α-donut for e_α. Assume that $L_1(e^i_\alpha)$ contains a con-edge e^i_β for a cluster β, and that $H_1(e^i_\alpha)$ contains a con-edge e^i_γ for a cluster γ, where $e^i_\alpha \otimes e^i_\beta$ and $e^i_\beta \otimes e^i_\gamma$. By Property 1, since e^i_γ and e^i_α belong to the same connected component of K_A and do not cross (as otherwise TEST 2 would succeed), it follows that e^i_γ does not cross any con-edge for α, hence it lies in one of the two faces f^i_α and f^{i+1}_α of $A[\alpha]$ that e^i_α shares with spokes e^{i-1}_α and e^{i+1}_α, respectively. Assume w.l.o.g. that e^i_γ lies in f^{i+1}_α. By Lemma 12, $L_1(e^{i+1}_\alpha)$ contains a con-edge e^{i+1}_β for β, where $e^{i+1}_\alpha \otimes e^{i+1}_\beta$.

The next two lemmata discuss the case in which $M(e^{i+1}_\alpha)$ contains a con-edge for γ that has a conflict with e^{i+1}_β and the case in which it does not. We start with the latter.

Lemma 15 (SIMPLIFICATION 6)**.** *Suppose that no con-edge e^{i+1}_γ for γ exists such that $e^{i+1}_\gamma \otimes e^{i+1}_\beta$, and that a planar set S of spanning trees for A exists. Then, $e^i_\alpha \in S$.*

Proof sketch. If a planar set S of spanning trees for A exists with $e^i_\alpha \notin S$, then by Lemma 5 we have $e^i_\beta \in S$ and $e^i_\gamma \notin S$. Then, the paths P^i_α and P^i_γ connecting the end-vertices of e^i_α and e^i_γ, together with a closed simple curve $\mathcal{C}$ surrounding e^i_α, e^i_β, and e^i_γ form a closed curve $\mathcal{D}$ that contains vertices in β on both sides. However, $\mathcal{D}$ cannot be crossed by any con-edge for β in S, thus S does not connect β. □

Lemma 16 (SIMPLIFICATION 7). *Suppose that a con-edge e^{i+1}_γ for γ exists with $e^{i+1}_\gamma \otimes e^{i+1}_\beta$. If a planar set S of spanning trees for A exists, then either $e^i_\alpha \in S$ or $e^{i+1}_\alpha \in S$.*

Proof sketch. By Lemma 13, at most one out of e^i_α and e^{i+1}_α belongs to S. To prove that at least one out of e^i_α and e^{i+1}_α belongs to S, by Lemma 5 it suffices to prove that at most one out of e^i_β and e^{i+1}_β belongs to S. This is accomplished again by Lemma 13 and by proving that e^{i+1}_β is a spoke of the β-donut for e^i_β. □

Observe that Simplification 7 can be applied in the case in which the α-donut for e_α has at least three spokes. Namely, in that case, by Lemmata 13 and 16 all the spokes different from e^i_α and e^{i+1}_α can be removed from A.

Next, assume that there exists an α-donut with exactly two spokes e^1_α and e^2_α. Consider the smallest $j \geq 1$ such that one of the following holds:
(1) there exist con-edges $e_\mu \in L_j(e^a_\alpha)$ and $e_\nu \in H_{j-1}(e^a_\alpha)$ for clusters μ and ν, resp., such that $e_\mu \otimes e_\nu$, and there exists no con-edge $g_\mu \in L_j(e^b_\alpha)$ for μ such that $g_\mu \otimes g_\nu$ with g_ν con-edge for ν in $H_{j-1}(e^b_\alpha)$, for some $a, b \in \{1, 2\}$ with $a \neq b$; or
(2) there exist con-edges $e_\mu \in H_j(e^a_\alpha)$ and $e_\nu \in L_j(e^a_\alpha)$ for clusters μ and ν, resp., such that $e_\mu \otimes e_\nu$, and there exists no con-edge $g_\mu \in H_j(e^b_\alpha)$ for μ such that $g_\mu \otimes g_\nu$ with g_ν con-edge for ν in $L_j(e^b_\alpha)$, for some $a, b \in \{1, 2\}$ with $a \neq b$. We have the following.

Lemma 17 (SIMPLIFICATION 8). *Assume that a planar set S of spanning trees for A exists. Then, $e_\mu \in S$.*

Proof sketch. The proof uses topological arguments to establish the following claim: If $e_\mu \notin S$, then the end-vertices of e_μ are on diffent sides of a cycle composed of con-edges for ν that cannot be crossed by con-edges for μ in S, hence S does not connect μ, a contradiction. □

We now prove that our simplifications form a "complete set".

Lemma 18. *Suppose that* SIMPLIFICATIONS 1–8 *do not apply to* A *and that* TESTS 1–4 *fail on* A*. Then, every con-edge in* A *crosses exactly one con-edge in* A*.*

Proof sketch. Since SIMPLIFICATIONS 2-3 do not apply to A, every con-edge crosses at least one con-edge. Suppose, for a contradiction, that there exists a con-edge for a cluster α that has a conflict with at least two con-edges. Since SIMPLIFICATIONS 1–4 do not apply to A and TESTS 1–4 fail on A, by Lemma 12 there exists an α-donut with spokes $e^1_\alpha, \ldots, e^k_\alpha$. If the conflicting structures of e^1_α and e^2_α are isomorphic, then SIMPLIFICATION 5 applies to A. Otherwise, if $k \geq 3$ and the conflicting structures of e^1_α and e^2_α are isomorphic (not isomorphic) when restricted to sets $H_0(e^j_\alpha)$, $L_1(e^j_\alpha)$, and $H_1(e^j_\alpha)$, then SIMPLIFICATION 7 (resp. SIMPLIFICATION 6) applies to A. If $k = 2$ and the conflicting structures of e^1_α and e^2_α are not isomorphic, then SIMPLIFICATION 8 applies to A. This provides a contradiction. □

A linear-time algorithm to determine whether a planar set S of spanning trees exists for a single-conflict graph is known [11]. We thus finally get:

Theorem 1. *There exists an* $O(|C|^3)$*-time algorithm to test the c-planarity of an embedded flat clustered graph* C *with at most two vertices per cluster on each face.*

Proof. Multigraph A can be easily constructed in $O(|C|^2)$ time, so that A has $O(|C|)$ vertices and edges and satisfies Property 1. By Lemma 2, it suffices to show how to solve the PSSTTM problem for A in $O(|C|^3)$ time. Algorithm 1 correctly determines whether a planar set S of spanning trees for A exists, by Lemmata 3–18. It can be easily tested in $O(|A|^2)$ time whether the pre-conditions of each of SIMPLIFICATIONS 1–8 and TESTS 1–4 are satisfied; also, the actual simplifications, that is, removing and contracting edges in A, can be performed in $O(|A|)$ time. Furthermore, the algorithm in [11] runs in $O(|A|)$ time. Since the number of performed tests and simplifications is in $O(|A|)$, the total running time is in $O(|A|^3)$, and hence in $O(|C|^3)$. □

5 Conclusions

We presented a polynomial-time algorithm for testing c-planarity of embedded flat clustered graphs with at most two vertices per cluster on each face. An interesting extension of our results would be to devise an FPT algorithm to test c-planarity of embedded flat clustered graphs, where the parameter is the maximum number k of vertices of the same cluster on any face. Several key lemmata (e.g. Lemmata 5 and 6) do not apply if $k > 2$, hence even an algorithm with running time $n^{O(f(k))}$ seems to be an elusive goal.

References

1. Angelini, P., Di Battista, G., Frati, F., Jelínek, V., Kratochvíl, J., Patrignani, M., Rutter, I.: Testing planarity of partially embedded graphs. In: SODA 2010, pp. 202–221. ACM (2010)

2. Angelini, P., Frati, F., Patrignani, M.: Splitting clusters to get C-planarity. In: Eppstein, D., Gansner, E.R. (eds.) GD 2009. LNCS, vol. 5849, pp. 57–68. Springer, Heidelberg (2010)
3. Bertolazzi, P., Di Battista, G., Liotta, G., Mannino, C.: Upward drawings of triconnected digraphs. Algorithmica 12(6), 476–497 (1994)
4. Chimani, M., Di Battista, G., Frati, F., Klein, K.: Advances on testing c-planarity of embedded flat clustered graphs. CoRR, abs/1408.2595 (2014)
5. Chimani, M., Gutwenger, C., Jansen, M., Klein, K., Mutzel, P.: Computing maximum c-planar subgraphs. In: Tollis, I.G., Patrignani, M. (eds.) GD 2008. LNCS, vol. 5417, pp. 114–120. Springer, Heidelberg (2009)
6. Chimani, M., Klein, K.: Shrinking the search space for clustered planarity. In: Didimo, W., Patrignani, M. (eds.) GD 2012. LNCS, vol. 7704, pp. 90–101. Springer, Heidelberg (2013)
7. Cornelsen, S., Wagner, D.: Completely connected clustered graphs. J. Discrete Algorithms 4(2), 313–323 (2006)
8. Cortese, P.F., Di Battista, G., Frati, F., Patrignani, M., Pizzonia, M.: C-planarity of c-connected clustered graphs. J. Graph Algorithms Appl. 12(2), 225–262 (2008)
9. Cortese, P.F., Di Battista, G., Patrignani, M., Pizzonia, M.: Clustering cycles into cycles of clusters. J. Graph Alg. Appl. 9(3), 391–413 (2005)
10. Dahlhaus, E.: A linear time algorithm to recognize clustered planar graphs and its parallelization. In: Lucchesi, C.L., Moura, A.V. (eds.) LATIN 1998. LNCS, vol. 1380, pp. 239–248. Springer, Heidelberg (1998)
11. Di Battista, G., Frati, F.: Efficient c-planarity testing for embedded flat clustered graphs with small faces. J. Graph Alg. Appl. 13(3), 349–378 (2009)
12. Didimo, W., Giordano, F., Liotta, G.: Overlapping cluster planarity. J. Graph Algorithms Appl. 12(3), 267–291 (2008)
13. Feng, Q.W., Cohen, R.F., Eades, P.: Planarity for clustered graphs. In: Moore, W., Luk, W. (eds.) FPL 1995. LNCS, vol. 975, pp. 213–226. Springer, Heidelberg (1995)
14. Garg, A., Tamassia, R.: On the computational complexity of upward and rectilinear planarity testing. SIAM Journal on Computing 31(2), 601–625 (2001)
15. Goodrich, M.T., Lueker, G.S., Sun, J.Z.: C-planarity of extrovert clustered graphs. In: Healy, P., Nikolov, N.S. (eds.) GD 2005. LNCS, vol. 3843, pp. 211–222. Springer, Heidelberg (2006)
16. Gutwenger, C., Jünger, M., Leipert, S., Mutzel, P., Percan, M., Weiskircher, R.: Advances in c-planarity testing of clustered graphs. In: Goodrich, M.T., Kobourov, S.G. (eds.) GD 2002. LNCS, vol. 2528, pp. 220–235. Springer, Heidelberg (2002)
17. Jelínek, V., Jelínková, E., Kratochvíl, J., Lidický, B.: Clustered planarity: Embedded clustered graphs with two-component clusters. In: Tollis, I.G., Patrignani, M. (eds.) GD 2008. LNCS, vol. 5417, pp. 121–132. Springer, Heidelberg (2009)
18. Jelínek, V., Suchý, O., Tesař, M., Vyskočil, T.: Clustered planarity: Clusters with few outgoing edges. In: Tollis, I.G., Patrignani, M. (eds.) GD 2008. LNCS, vol. 5417, pp. 102–113. Springer, Heidelberg (2009)
19. Jelínková, E., Kára, J., Kratochvíl, J., Pergel, M., Suchý, O., Vyskocil, T.: Clustered planarity: Small clusters in cycles and Eulerian graphs. J. Graph Alg. Appl. 13(3), 379–422 (2009)
20. Kratochvíl, J., Lubiw, A., Nesetril, J.: Noncrossing subgraphs in topological layouts. SIAM J. Discrete Math. 4(2), 223–244 (1991)
21. Schaefer, M.: Toward a theory of planarity: Hanani-Tutte and planarity variants. J. Graph Algorithms Appl. 17(4), 367–440 (2013)
22. Schaeffer, S.E.: Graph clustering. Computer Science Review 1(1), 27–64 (2007)
23. Tamassia, R.: On embedding a graph in the grid with the minimum number of bends. SIAM J. Comput. 16(3), 421–444 (1987)