Drawing Planar Graphs with Reduced Height

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Abstract. A straight-line (respectively, polyline) drawing Γ of a planar graph G on a set L_k of k parallel lines is a planar drawing that maps each vertex of G to a distinct point on L_k and each edge of G to a straight line segment (respectively, a polygonal chain with the bends on L_k) between its endpoints. The height of Γ is k, i.e., the number of lines used in the drawing. In this paper we compute new upper bounds on the height of polyline drawings of planar graphs using planar separators. Specifically, we show that every n-vertex planar graph with maximum degree Δ, having a simple cycle separator of size λ, admits a polyline drawing with height $4n/9+O(\lambda\Delta)$, where the previously best known bound was $2n/3$. Since $\lambda \in O(\sqrt{n})$, this implies the existence of a drawing of height at most $4n/9 + o(n)$ for any planar triangulation with $\Delta \in o(\sqrt{n})$. For n-vertex planar 3-trees, we compute straight-line drawings with height $4n/9 + O(1)$, which improves the previously best known upper bound of $n/2$. All these results can be viewed as an initial step towards compact drawings of planar triangulations via choosing a suitable embedding of the input graph.

1 Introduction

A *polyline drawing* of a planar graph G is a planar drawing of G such that each vertex of G is mapped to a distinct point in the Euclidean plane, and each edge is mapped to a polygonal chain between its endpoints. Let $L_k = \{l_1, l_2, \ldots, l_k\}$ be a set of k horizontal lines such that for each $i \leq k$, line l_i passes through the point $(0, i)$. A polyline drawing of G is called a *polyline drawing on* L_k if the vertices and bends of the drawing lie on the lines of L_k. The *height* of such a drawing is k, i.e., the number of parallel horizontal lines used by the drawing. Such a drawing is also referred to as a *k-layer drawing* in the literature [13,18]. Let Γ be a polyline drawing of G. We call Γ a *t-bend polyline drawing* if each of its edges has at most t bends. Thus a 0-bend polyline drawing is also known as a *straight-line drawing.* Drawing planar graphs on a small integer grid is an active research area in graph drawing [7,16], which is motivated by the need of compact layout of VLSI circuits and visualization of software architecture. Since simultaneously optimizing the

* Work of the author is supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).

** Work of the author is supported in part by a University of Manitoba Graduate Fellowship.

C. Duncan and A. Symvonis (Eds.): GD 2014, LNCS 8871, pp. 392–403, 2014.

width and height of the drawing is very challenging, researchers have also focused their attention on optimizing one dimension of the drawing [6,11,13,17], while the other dimension is unbounded. In this paper we develop new techniques that can produce drawings with small height. We distinguish between the terms 'plane' and 'planar'. A *plane graph* is a planar graph with a fixed combinatorial embedding and a specified outer face. While drawing a planar graph, we allow the output to represent any planar embedding of the graph. On the other hand, while drawing a plane graph, the output is further constrained to respect the input embedding.

State-of-the-art algorithms that compute straight-line drawings of n-vertex plane graphs on an $(O(n) \times 2n/3)$-size grid imply an upper bound of $2n/3$ on the height of straight-line drawings [5,6]. This bound is tight for plane graphs, i.e., there exist n-vertex plane graphs such as plane nested triangles graphs and some plane 3-trees that require a height of $2n/3$ in any of their straight-line drawings. Recall that an n-vertex *nested triangles graph* is a plane graph formed by a sequence of $n/3$ vertex disjoint cycles, $\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_{n/3}$, where for each $i \in \{2, \ldots, n/3\}$, cycle $\mathtt{C}_i$ contains the cycles $\mathtt{C}_1, \ldots, \mathtt{C}_{i-1}$ in its interior, and a set of edges that connect each vertex of $\mathtt{C}_i$ to a distinct vertex in $\mathtt{C}_{i-1}$. Besides, a *plane* 3*-tree* is a triangulated plane graph that can be constructed by starting with a triangle, and then repeatedly adding a vertex to some inner face of the current graph and triangulating that face.

The $2n/3$ upper bound on the height is also the currently best known bound for polyline drawings, even for planar graphs, i.e., when we are allowed to choose a suitable embedding for the output drawing. Frati and Patrignani [10] showed that in the variable embedding setting, an n-vertex nested triangles graph can be drawn with height at most $n/3 + O(1)$, which is significantly smaller than the lower bound of $2n/3$ in the fixed embedding setting. Similarly, Hossain et al. [11] showed that an *universal set* of $n/2$ horizontal lines can support all n-vertex planar 3-trees, i.e., every planar 3-tree admits a drawing with height at most $n/2$. They also showed that $4n/9$ lines suffice for some subclasses of planar 3-trees, and asked whether $4n/9$ is indeed an upper bound for planar 3-trees.

In the context of optimization, Dujmović et al.[9] gave fixed-parameter-tractable (FPT) algorithms, parameterized by pathwidth, to decide whether a planar graph admits a straight-line drawing on k horizontal lines. Drawings with minimum number of parallel lines have been achieved for trees [13]. Recently, Biedl [2] gave an algorithm to approximate the height of straight-line drawings of 2-connected outerplanar graphs within a factor of 4.

Contributions. In this paper we show that every n-vertex planar graph with maximum degree Δ, having a simple cycle separator of size λ, admits a drawing with height $4n/9+O(\lambda\Delta)$, which is better than the previously best known bound of $2n/3$ for any $\lambda\Delta \in o(n)$. This result is an outcome of a new application of the planar separator theorem [8]. Although the technique is simple, it has the potential to be a powerful tool while computing compact drawings for planar triangulations in the variable embedding setting. If the input graphs are restricted to planar 3-trees, then we can improve the upper bound to $4n/9 + O(1)$, which settles the question of Hossain et al. [11]. Furthermore, the drawing we construct in this case is a straight-line drawing.

2 Preliminary Definitions and Results

Let G be an n-vertex triangulated plane graph. A simple cycle C in G is called a *cycle separator* if the interior and the exterior of C each contains at most $2n/3$ vertices. Let v_1, v_n and v_2 be the outer vertices of G in clockwise order on the outer face. Let $\sigma = (v_1, v_2, ..., v_n)$ be an ordering of all vertices of G. By G_k, $2 \leq k \leq n$, we denote the subgraph of G induced by $v_1 \cup v_2 \cup ... \cup v_k$. For each G_k, the notation P_k denotes the path (while walking clockwise) on the outer face of G_k that starts at v_1 and ends at v_2. We call σ a *canonical ordering* of G with respect to the outer edge (v_1, v_2) if for each k, $3 \leq k \leq n$, the following conditions are satisfied [7]:

(a) G_k is 2-connected and internally triangulated.
(b) If $k \leq n$, then v_k is an outer vertex of G_k and the neighbors of v_k in G_{k-1} are consecutive on P_{k-1}.

Let P_k, for some $k \in \{3, 4, \ldots, n\}$, be the path $w_1(= v_1), \ldots, w_l, v_k(= w_{l+1})$, $w_r, \ldots, w_t(= v_2)$. The edges (w_l, v_k) and (v_k, w_r) are the *l-edge* and *r-edge* of v_k, respectively. The other edges incident to v_k in G_k are called the *m-edges.* For example, in Figure 1(c), the edges $(v_6, v_1), (v_6, v_4)$, and (v_5, v_6) are the l-, r- and m-edges of v_6, respectively. Let E_m be the set of all m-edges in G. Then the graph T_{v_n} induced by the edges in E_m is a tree with root v_n. Similarly, the graph T_{v_1} induced by all l-edges except (v_1, v_n) is a tree rooted at v_1 (Figure 1(b)), and the graph T_{v_2} induced by all r-edges except (v_2, v_n) is a tree rooted at v_2. These three trees form the *Schnyder realizer* [16] of G.

Lemma 1 (Bonichon et al. [4]). *The total number of leaves in all the trees in any Schnyder realizer of an n-vertex triangulation is at most* $2n - 5$.

Let G be a planar graph and let Γ be a straight-line drawing on k parallel lines. By $l(v)$, where v is a vertex of G, we denote the horizontal line in Γ that passes through v. We now have the following lemma that bounds the height of a straight-line drawing in terms of the number of leaves in a Schnyder tree. The

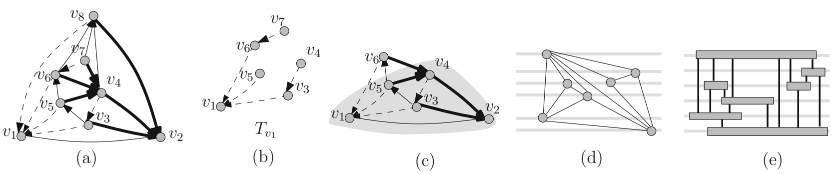

Fig. 1. (a) A plane triangulation G with a canonical ordering. The associated realizer, where the l-, r- and m- edges are shown in dashed, bold-solid, and thin-solid lines, respectively. (b) T_{v_1}. (c) Neighbors of v_6 in G_6. (d)–(e) Illustrating Lemma 3.

lemma can be derived from the known straight-line [5] and polyline drawing algorithms [4]. We omit the proof due to space constraints.

Lemma 2. *Let G be an n-vertex plane triangulation and let v_1, v_n, v_2 be the outer vertices of G in clockwise order on the outer face. Assume that T_{v_n} has at most p leaves. Then for any placement of v_n on line l_1 or l_{p+2}, there exists a straight-line drawing Γ of G on L_{p+2} such that v_2 and v_1 lie on lines l_{p+2} and l_1, respectively.*

Chrobak and Nakano [6] showed that every planar graph admits a straight-line drawing with height $2n/3$. We now observe some properties of Chrobak and Nakano's algorithm [6]. Let G be a plane triangulation with n vertices and let x, y be two user prescribed outer vertices of G in clockwise order. Let Γ be the drawing of G produced by the Algorithm of Chrobak and Nakano [6]. Then Γ has the following properties:

(CN_1) Γ is a drawing on L_q, where $q \leq 2n/3$.
(CN_2) For the vertices x and y, we have $l(x) = l_1$ and $l(y) = l_q$ in Γ. The remaining outer vertex z lies on either l_1 or l_q.

Note that the user cannot choose the placement of z, i.e., the algorithm may produce a drawing where $l(x) = l_1, l(y) = l_q$ and $l(z) = l_1$, however, this does not imply that there exists another drawing where $l(x) = l_1, l(y) = l_q$ and $l(z) = l_q$. We end this section with the following lemma.

Lemma 3. *Let G be a plane graph and let Γ be a straight-line drawing of G on k horizontal lines, but the lines are not necessarily equally spaced. Then there exists a drawing Γ' of G on a set of k horizontal lines that are equally spaced. Furthermore, for every $i \in \{1, 2, \dots, k\}$, the left to right order of the vertices on the ith line in Γ coincides with that of Γ'.*

Proof (Outline). One can construct Γ' by first transforming Γ into a 'flat-visibility representation' on equally spaced horizontal lines, as shown in Figures 1(d)–(e), and then transforming this representation again into a straight-line drawing [1,3]. □

In the following sections we describe our drawing algorithms. Note that for simplicity we often omit the floor and ceiling functions while defining different parameters of the algorithms. One can describe a more careful computation using proper floor and ceiling functions, but that does not affect the asymptotic results discussed in this paper.

3 Drawing Triangulations with Small Height

Let $G = (V, E)$ be an n-vertex planar triangulation and let Γ be a planar drawing of G on the Euclidean plane. Let $C = (V_c, E_c)$ be a simple cycle separator of G of size λ. Let $G_i = (V_i, E_i)$ be the graph induced by the vertices that lie inside C and on the boundary of C. Similarly, let $G_o = (V_o, E_o)$ be the graph induced by the vertices that lie outside C and on the boundary of C. Specifically, $V = V_i \cup V_o$, $E = E_i \cup E_o$, $V_i \cap V_o = V_c$, and $E_i \cap E_o = E_c$. We now compute a polyline drawing of G.

3.1 Drawing Technique

If any edge $(a,b) \in E_c$ lies on the outer face of Γ, then we will draw G respecting the combinatorial embedding determined by Γ. Otherwise, there exists an edge $(a,b) \in E_c$ such that the face a,b,c with $c \in V_o$ does not lie interior to C. We redefine Γ as the embedding of G obtained by choosing a,b,c as the outer face, as illustrated in Figures 2(a)–(b).

Drawing G_i. Assume that $x = 4n/9 + 2\lambda/3 + 3$. Construct a plane graph G_i' by taking a copy of G_i from Γ, and then adding a vertex z to the outer face of G_i along with the edges (z,w), for all $w \in V_c$. Figure 2(c) illustrates G_i'. Since G_i has at most $(2n+3\lambda)/3$ vertices, we now use the algorithm of Chrobak and Nakano [6] to compute a drawing Γ_i of G_i' on L_x, where a,b lie on l_1, l_x and z lies on either l_1 or l_x. Assume without loss of generality that z is in the right half-plane of the line through a,b.

Drawing G_o. Take a copy of G_o from Γ. Let u be any vertex in G_o. Then by $d_o(u)$ we denote the degree of vertex u in G_o. Let the cycle C be $a(=w_1), w_2, \ldots, b(=w_\lambda)$. For each vertex $w_i \in V_c$, where $1 \leq i \leq \lambda$ and $w_{\lambda+1} = w_1$, if $d_o(w_i) > 3$, then replace (w_i, w_{i+1}) with a path $w_i, w_i^1, w_i^2, \ldots, w_i^{d_o(w_i)-3}, w_{i+1}$ of $d_o(w_i) - 3$ division vertices. Let $u_1, u_2, \ldots, u_{d_o(w_i)-2}$ be the neighbors of w_i in clockwise order outside of C. Then delete the edges from w_i to these neighbors, and add the edges $(w_i, u_1), (w_i^1, u_2), \ldots, (w_i^{d_o(w_i)-3}, u_{d_o(w_i)-2})$. Replace the edge $(w_1, w_\lambda^{d_o(w_\lambda)-3})$ by a path $w_1, w', w'', w_\lambda^{d_o(w_\lambda)-3}$, and redefine a and b such that $a = w''$ and $b = w'$. Let the resulting graph be H and let the newly constructed cycle be C'. Figure 2(d) illustrates H.

If z lies on l_1 in Γ_i, then we add the edges (a,w) to H, for each vertex w on C'. Otherwise, we add the edges (b,w) to H. Finally, we add a vertex z' on the outer face and triangulate H such that (a,b) remains an outer edge. Let the resulting graph be G_o'. Figure 2(e) illustrates G_o'. Observe that the number of vertices in G_o' is at most $2n/3 + \lambda\Delta + 3$. Hence we can use the algorithm of Chrobak and Nakano [6] to compute a drawing Γ_o of G_o' on L_y, where $y = (4n + 6\lambda\Delta + 18)/9$, such that a,b lie on l_1, l_y, respectively, and the segment ab is vertical. Assume without loss of generality that all the vertices of G_o' are in the right half-plane of the line through a,b.

Merging G_i and G_o. Without loss of generality assume that $l(z) = l_x$ in Γ_i, and recall that in this case b is adjacent to all the vertices on C' in Γ_o. Let ℓ_o be a vertical line to the right of segment ab in Γ_o such that all the other vertices of Γ_o are in the right half-plane of ℓ_o. Furthermore, ℓ_o must be close enough such that all the intersection points with the edges incident to b lie in between the horizontal line through b and the next horizontal line. For each intersection point, we insert a division vertex at that point and create a horizontal line through that vertex. We then delete vertex b from Γ_o, but not the division vertices. Figures 2(i)–(j) illustrate this scenario. By Lemma 3, we can modify Γ_o such that the horizontal lines are equally spaced. Since C' contains at most $\lambda\Delta$ vertices, Γ_o is a drawing on at most $y + \lambda\Delta$ horizontal lines. Similarly, we modify Γ_i, as follows.

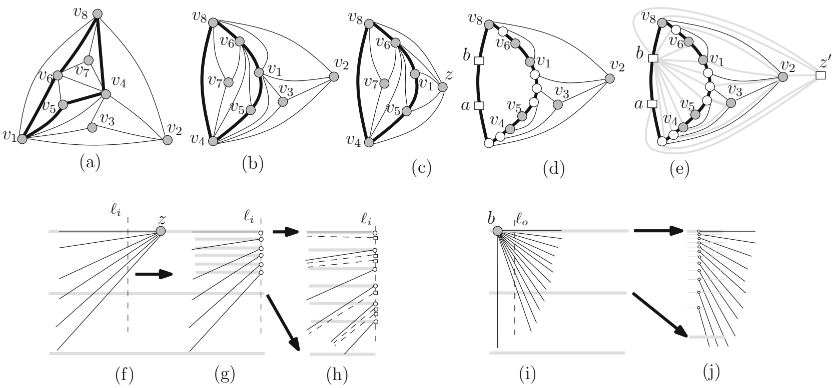

Fig. 2. (a) A plane triangulation G, where C is shown in bold. (b) Γ, where $v_4 = a, v_8 = b$ and $v_2 = c$. (c) G'_i. (d) H, where the division vertices are shown in white. (e) G'_o, where the edges added to H are shown in gray. (f)–(j) Drawing G.

Let ℓ_i be a vertical line to the left of z in Γ_i such that all the other vertices of Γ_i are in the left half-plane of ℓ_i. Furthermore, ℓ_i must be close enough such that all the intersection points with the edges incident to z lie in between two consecutive parallel lines. For each intersection point, we insert a division vertex at that point and create a horizontal line through that vertex. Let $v_1, v_2, \ldots, v_\lambda$ be the division vertices on ℓ_i in the order of decreasing y-coordinate, where for each $i \in \{1, 2, \ldots, \lambda\}$, v_i is incident to the vertex w_i on C. Delete vertex z, but not the division vertices. For each vertex w_i, if $d_o(w_i) > 3$, then we place a set of division vertices $v_i^1, v_i^2, \ldots, v_i^{d_o(w_i)-3}$ below v_i and above the horizontal line closest to v_i. Besides, these new division vertices must be sufficiently close to v_i such that drawing of the edges (w_i, v_i^j), where $1 \leq j \leq d_o(w_i) - 3$, do not create any edge crossing. Figures 2(f)–(h) illustrate this scenario. Finally, by Lemma 3, we can modify Γ_i such that the horizontal lines are equally spaced. Note that Γ_i is a drawing on at most $x + \lambda\Delta$ horizontal lines.

Since the division vertices in Γ_i and Γ_o take a set of consecutive horizontal lines from their respective topmost lines, it is straightforward to merge these two drawings on a set of $\lambda\Delta + \max\{x, y\} = 4n/9 + O(\lambda\Delta)$ horizontal lines. We delete the edges on C', and consider all vertices of C' as division vertices. Since the division vertices correspond to the bends, each edge may contain at most six bends (two bends to enter Γ_o from Γ_i, two bends on C', and two bends to return to Γ_i from Γ_o). Since there are at most $\lambda\Delta$ edges that may have bends, the number of bends is at most $6\lambda\Delta$ in total. However, via 'flat-visibility representation' (similar to the proof of Lemma 3) one can reduce the number of bends to enter and exit Γ_o by one, we omit the details due to space constraints. Hence the number of bends reduces to $4\lambda\Delta$. The following theorem summarizes the result of this section.

Theorem 1. *Let G be an n-vertex planar graph with maximum degree Δ. If G contains a simple cycle separator of size λ, then G admits a 4-bend polyline drawing with height $4n/9 + O(\lambda\Delta)$ and at most $4\lambda\Delta$ bends in total.*

Since every planar triangulation has a simple cycle separator of size $O(\sqrt{n})$ [8], we obtain the following corollary.

Corollary 1. *Every n-vertex planar triangulation with maximum degree $o(\sqrt{n})$ admits a polyline drawing with height at most $4n/9 + o(n)$.*

Pach and Tóth [15] showed that polyline drawings can be transformed into straight-line drawings while preserving the height if the polyline drawing is monotone, i.e., if every edge in the polyline drawing is drawn as a y-monotone curve. Unfortunately, our algorithm does not necessarily produce monotone drawings.

4 Drawing Planar 3-Trees with Small Height

In this section we examine straight-line drawings of planar 3-trees.

4.1 Technical Background

Let G be an n-vertex planar 3-tree and let Γ be a straight-line drawing of G. Then Γ can be constructed by starting with a triangle, which corresponds to the outer face of Γ, and then iteratively inserting the other vertices into the inner faces and triangulating the resulting graph. Let a, b, c be the outer vertices of Γ in clockwise order. If $n > 3$, then Γ has a unique vertex p that is incident to all the outer vertices. This vertex p is called the representative vertex of G.

For any cycle i, j, k in G, let G_{ijk} be the subgraph induced by the vertices i, j, k and the vertices lying inside the cycle. Let G^*_{ijk} be the number of vertices in G_{ijk}. The following two lemmas describe some known results.

Lemma 4 (Mondal et al. [14]). *Let G be a plane 3-tree and let i, j, k be a cycle of three vertices in G. Then G_{ijk} is a plane 3-tree.*

Lemma 5 (Hossain et al. [11]). *Let G be an n-vertex plane 3-tree with the outer vertices a, b, c in clockwise order. Let D be a drawing of the outer cycle a, b, c on L_n, where the vertices lie on l_1, l_k and l_i with $k \leq n$ and $i \in \{l_1, l_2, l_n, l_{n-1}\}$. Then G admits a straight-line drawing Γ on L_k, where the outer cycle of Γ coincides with D.*

Let G be a plane 3-tree and let a, b, c be the outer vertices of G. Assume that G has a drawing Γ on L_k, where a, b lie on lines l_1, l_k, respectively, and c lies on line l_i, where $1 \leq i \leq k$. Then the following properties hold for Γ [11].

Reshape. Let p, q and r be three different points on lines l_1, l_k and l_i, respectively. Then G has a drawing Γ' on L_k such that the outer face of Γ' coincides with triangle pqr (e.g., Figures 3(a)–(b)).

Stretch. For any integer $t \geq k$, G admits a drawing Γ' on L_t such that a, b, c lie on l_1, l_t, l_i, respectively (e.g., Figure 3(c)).

For any triangulation H with the outer vertices a, b, c, let $T_{a,H}, T_{b,H}, T_{c,H}$ be the Schnyder trees rooted at a, b, c, respectively. By $\texttt{leaf}(T)$ we denote the number of leaves in T. The following lemma establishes a sufficient condition for a plane 3-tree G to have a straight-line drawing with height at most $4(n+3)/9+4$.

Lemma 6. *Let G be an n-vertex plane 3-tree with outer vertices a, b, c in clockwise order. Let $w_1, \ldots, w_k (= p), w_{k+1} (= q), \ldots, w_t (= c)$ be the maximal path P such that each vertex on P is adjacent to both a and b. Assume that $n' = n + 3$, and $x = 4n'/9$. If $G^*_{apq} \leq (n' + 2)/3$, $G^*_{bpq} \leq G^*_{abp} \leq n'/2$ and $\max_{\forall i>k+1}\{G^*_{aw_iw_{i-1}}, G^*_{bw_iw_{i-1}}\} \leq 4n'/9$, then G admits a drawing with height at most $4n'/9 + 4$.*

Proof. Let H be the subgraph of G induced by the vertices $\{a, b\} \cup \{w_k, \ldots, w_t\}$. The idea of the proof is first to construct a drawing of H on L_{x+4}, and then to extend it to the required drawing using Lemmas 2–5. We distinguish two cases depending on whether $\texttt{leaf}(T_{p,G_{abp}}) \leq x$ or not.

Case 1 ($\texttt{leaf}(T_{p,G_{abp}}) \leq x$). Since $G^*_{bqp} \leq n'/2$, by Lemma 1, one of the trees in the Schnyder realizer of G_{bqp} has at most $n'/3 \leq x$ leaves. We now draw G_{abq} considering the following scenarios.

- **Case 1A ($\texttt{leaf}(T_{p,G_{bqp}}) \leq x$).** By Lemma 2 and the Stretch condition, G_{abp} admits a drawing Γ_{abp} on L_{x+2} such that the vertices a, b, p lie on l_1, l_{x+2}, l_{x+2}, respectively. Similarly, G_{bqp} admits a drawing Γ_{bpq} on L_{x+2} such that the vertices q, b, p lie on l_1, l_{x+2}, l_{x+2}, respectively, as shown in Figure 3(d). By the Stretch property, Γ_{abp} can be extended to a drawing Γ'_{abp} on L_{x+3}, where a, b, p lie on l_1, l_{x+3}, l_{x+2}, respectively. Similarly, Γ_{bqp} can be extended to a drawing Γ'_{bqp} on L_{x+3}, where q, b, p lie on l_1, l_{x+3}, l_{x+2}, respectively. Since $G^*_{apq} \leq (n'+2)/3$, by Lemma 5 and the Stretch condition, G_{apq} admits a drawing Γ_{apq} on $L_{(n'+2)/3}$. Finally, by the Stretch property Γ_{apq} can be extended to a drawing Γ'_{apq} on L_{x+2} such that a, p, q lie on l_1, l_{x+2}, l_1, respectively, and by the Reshape property we can merge these drawings to obtain a drawing of G_{abq} on L_{x+3}. Figure 3(e) depicts an illustration.
- **Case 1B ($\texttt{leaf}(T_{b,G_{bqp}}) \leq x$).** By Lemma 2 and the Stretch condition, G_{abp} admits a drawing Γ_{abp} on L_{x+2} such that the vertices a, b, p lie on l_1, l_{x+2}, l_1,

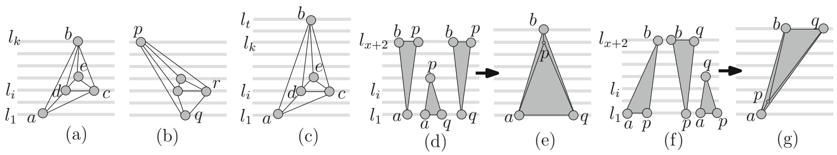

Fig. 3. (a)–(b) Illustrating Reshape. (c) Illustrating Stretch. (d)–(e) Illustration for Case 1A. (d)–(e) Illustration for Case 1B.

respectively. Similarly, G_{bqp} admits a drawing Γ_{bpq} on L_{x+2} such that the vertices p, b, q lie on l_1, l_{x+2}, l_{x+2}, respectively. By Lemma 5, G_{apq} admits a drawing Γ_{apq} on $L_{(n'+2)/3}$ such that a, p, q lie on l_1, l_1, l_{x+2}, respectively. Finally, by Stretch and Reshape we can merge these drawings to obtain a drawing of G_{abq} on L_{x+3}. Figures 3(f)–(g) show an illustration.

Case 1C ($\texttt{leaf}(T_{q,G_{bqp}}) \leq x$). The drawing of this case is similar to Case 1B. The only difference is that we use $T_{q,G_{bqp}}$ while drawing G_{bqp}.

Observe that each of the Cases 1A–1C produces a drawing of G_{abq} such that a, b lie on l_1, l_{x+3}, respectively, and q lies on either l_1 or l_{x+3}. We stretch it to a drawing on L_{x+4} such that a, b lies on l_1, l_{x+4}, respectively, and q lies on either l_2 or l_{x+3}. If q lies on l_2, then we draw the path $w_{k+1}, \ldots, w_t(= c)$ in a zigzag fashion, placing the vertices on l_2 and l_{x+3} alternatively such that each vertex is visible to both a and b. Similarly, if q lies on l_{x+3}, then we place the vertices $w_{k+1}, \ldots, w_t(= c)$ on l_{x+3} and l_2 alternatively, as shown in Figure 4(a). For each $i > k+1$, Lemma 4 ensures that the graphs $G_{aw_iw_{i-1}}$ and $G_{bw_iw_{i-1}}$ are plane 3-trees. Since $\max_{\forall i>k+1}\{G^*_{aw_iw_{i-1}}, G^*_{bw_iw_{i-1}}\} \leq x$, we can draw $G_{aw_iw_{i-1}}$ and $G_{bw_iw_{i-1}}$ using Lemma 5 inside their corresponding triangles.

Case 2 ($\texttt{leaf}(T_{p,G_{abp}}) > x$). Since $G^*_{abp} \leq n'/2$, by Lemma 1, $\texttt{leaf}(T_{a,G_{abp}}) + \texttt{leaf}(T_{b,G_{abp}}) \leq n' - \texttt{leaf}(T_{p,G_{abp}}) \leq 5n'/9$. Hence we draw G_{abq} considering the following scenarios.

Case 2A ($\texttt{leaf}(T_{a,G_{abp}}) \leq x$ and $\texttt{leaf}(T_{b,G_{abp}}) \leq x$). Since $G^*_{bqp} \leq n'/2$, by Lemma 1, one of the trees in the Schnyder realizer of G_{bqp} has at most $n'/3 \leq x$ leaves. If $\texttt{leaf}(T_{p,G_{bpq}}) \leq x$, then we draw G_{abq} on L_{x+3}, where a, b, p, q lie on $l_1, l_{x+3}, l_{x+2}, l_1$, respectively, as in Figure 4(b). Otherwise, either $\texttt{leaf}(T_{b,G_{bpq}}) \leq x$ or $\texttt{leaf}(T_{q,G_{bpq}}) \leq x$. In this case we draw G_{abq} on L_{x+3}, where a, b, p, q lie on $l_1, l_{x+3}, l_2, l_{x+3}$, respectively, as in Figure 4(c).

Case 2B ($\texttt{leaf}(T_{a,G_{abp}}) > x$ and $\texttt{leaf}(T_{b,G_{abp}}) \leq n'/9$). If $\texttt{leaf}(T_{p,G_{bpq}}) \leq n'/3$, then we first draw G_{bpq} using Lemma 2 such that b, p, q lie on $l_{n'/3+2}, l_{n'/3+2}, l_1$, respectively, and then use the Stretch condition to shift b to l_{x+3}. By Lemma 2 and the Stretch condition, there exists a drawing of G_{abp} on L_{x+3} with a, b, p lying on $l_1, l_{x+3}, l_{n'/3+2}$, respectively. Since $G^*_{apq} \leq (n'+2)/3$, we can draw G_{apq} using Lemma 5 inside triangle apq. Figure 4(d) illustrates the scenario after applying Stretch and Reshape.

If $\texttt{leaf}(T_{p,G_{bpq}}) > n'/3$, then by Lemma 1 either $\texttt{leaf}(T_{b,G_{bpq}}) \leq n'/3$ or $\texttt{leaf}(T_{q,G_{bpq}}) \leq n'/3$. Hence we can use Lemma 2 and the Stretch condition

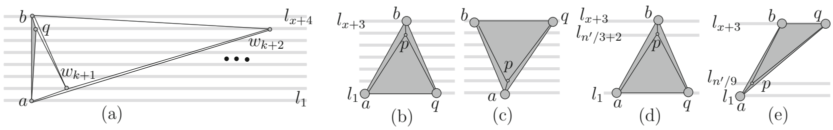

Fig. 4. (a) Illustrating Case 1. (b)–(c) Illustrating Case 2A. (d)–(e) Case 2B.

to draw G_{bpq} such that b, p, q lie on $l_{x+3}, l_{n'/9}, l_{x+3}$, respectively. On the other hand, we use Lemma 2 and the Stretch condition to draw G_{abp} such that a, b, p lie on $l_1, l_{x+3}, l_{n'/9}$, respectively. Since $G^*_{apq} \leq (n'+2)/3$, we can draw G_{apq} using Lemma 5 inside triangle apq. Figure 4(e) illustrates the scenario.

Case 2C ($\texttt{leaf}(T_{a,G_{abp}}) \leq n'/9$ and $\texttt{leaf}(T_{b,G_{abp}}) > x$). Since each of the edges among (a, b) and (b, p) spans at least $n'/9 + 2$ parallel lines in Case 2B, the drawing in this case is analogous to Case 2B. The only difference is that we use $T_{a,G_{abp}}$ while drawing G_{abp}.

Each of the Cases 2A–2C produces a drawing of G_{abq} such that a, b lies on l_1, l_{x+3}, respectively, and q lies on either l_1 or l_{x+3}. Hence we can extend these drawings to draw G as in Case 1. □

4.2 Drawing Algorithm

Decomposition. Let G be an n-vertex plane 3-tree with the outer vertices a, b, c and the representative vertex p. A tree spanning the inner vertices of G is called the *representative tree* T if it satisfies the following conditions [14]:

(a) If $n = 3$, then T is empty.
(b) If $n = 4$, then T consists of a single vertex.
(c) If $n > 4$, then the root p of T is the representative vertex of G and the subtrees rooted at the three clockwise ordered children p_1, p_2 and p_3 of p in T are the representative trees of G_{abp}, G_{bcp} and G_{cap}, respectively.

Recall that every n-vertex tree T' has a vertex v' such that the connected components of $T' \setminus v'$ are all of size at most $n/2$ [12]. Such a vertex v in T corresponds to a decomposition of G into four smaller plane 3-trees G_1, G_2, G_3, and G_4, as follows.

- The plane 3-tree G_i, where $1 \leq i \leq 3$, is determined by the representative tree rooted at the ith child of v, and thus contains at most $(n+3)/2$ vertices.
- The plane 3-tree G_4 is obtained by deleting v and the vertices from G that are descendent of v in T, and contains at most $(n+3)/2$ vertices.

Drawing Technique. Without loss generality assume that $G^*_3 \leq G^*_2 \leq G^*_1$. If G_1 is incident to the outer face of G, then let (a, b) be the corresponding outer edge. Otherwise, G_1 does not have any edge incident to the outer face of G. In this case there exists an inner face f in G that is incident to G_1, but does not belong to G_1. We choose f as the outer face of G, and now we have an edge (a, b) of G_1 that is incident to the outer face. Let $P{=}(w_1, \ldots, w_k(= p), w_{k+1}(= q), \ldots, w_t)$ be the maximal path in G such that each vertex on P is adjacent to both a and b, where $\{a, b, p\}, \{a, p, q\}, \{b, q, p\}$ are the outer vertices of G_1, G_2, G_3, respectively. Assume that $n' = n + 3$ and $x = 4n'/9$. We draw G on L_{x+4} by distinguishing two cases depending on whether $G^*_4 > x$ or not.

Case 1 ($G^*_4 > x$). Observe that $G^*_2 \leq G^*_1 \leq n'/2$ and since $G^*_3 + G^*_2 + G^*_1 \leq n' + 5 - G^*_4$, we have $G^*_3 \leq 5n'/27 + 5/3 \leq n'/3$ for sufficiently large values of n.

If $\max_{\forall i>k+1}\{G^*_{aw_iw_{i-1}}, G^*_{bw_iw_{i-1}}\} \leq x$ holds, then G admits a drawing on L_{x+4} by Lemma 6. We may thus assume that there exists some $j > q$ such that either $G^*_{aw_jw_{j-1}} > x$ or $G^*_{bw_jw_{j-1}} > x$. Hence $\max_{\forall i>k+1, i\neq j}\{G^*_{aw_iw_{i-1}}, G^*_{bw_iw_{i-1}}\} \leq n'/9$.

We first show that G_{abq} can be drawn on L_{x+3} in two ways: One drawing Γ_1 contains the vertices a, b, q on l_1, l_{x+3}, l_2, respectively, and the other drawing Γ_2 contains a, b, q on l_1, l_{x+3}, l_{x+2}, respectively. We then extend these drawings to obtain the required drawing of G. Consider the following scenarios depending on whether $G^*_1 \leq x$ or not.

- If $G^*_1 \leq x$, then $G^*_3 \leq G^*_2 \leq G^*_1 \leq x$. Here we draw the subgraph G' induced by the vertices a, b, p, q such that they lie on $l_1, l_{x+3}, l_{x+2}, l_2$, respectively. Since $G^*_3 \leq G^*_2 \leq G^*_1 \leq x$, by Lemma 5, G_1, G_2 and G_3 can be drawn inside their corresponding triangles, which corresponds to Γ_1. Similarly, we can find another drawing Γ_2 of G_{abq}, where the vertices a, b, p, q lie on $l_1, l_{x+3}, l_2, l_{x+2}$, respectively.
- If $G^*_1 > x$, then $G^*_3 \leq G^*_2 \leq n'/9$. We use Chrobak and Nakano's algorithm [6] and the Stretch condition to draw G_1 on L_{x+3} layers such that a, b lie on l_1, l_{x+3}, respectively, and p lies either on l_2 or $l_{n'/3+2}$. If $l(p) = l_2$ (i.e., Γ_2), then we place q on l_{x+2}. Otherwise, $l(p) = l_{n'/3+2}$ (i.e., Γ_1), and we place q on l_2. Since $G^*_3 \leq G^*_2 \leq n'/9$, for each of these two placements we can draw G_2 and G_3 using Lemma 5 inside their corresponding triangles.

We now show how to extend the drawing of G_{abq} to compute the drawing of G. Consider two scenarios depending on whether $G^*_{aw_jw_{j-1}} > x$ or $G^*_{bw_jw_{j-1}} > x$.

- Assume that $G^*_{aw_jw_{j-1}} > x$. Shift b to l_{x+4}, and draw the path $w_{k+1}, \ldots, w_{j-1}$ in a zigzag fashion, placing the vertices on l_2 and l_{x+3} alternatively, such that $l(w_{k+1}) \neq l(w_{k+2})$, and each vertex is visible to both a and b. Choose Γ_1 or Γ_2 such that the edge (a, w_{j-1}) spans at least $x+3$ lines. We now draw $G_{aw_jw_{j-1}}$ using Chrobak and Nakano's algorithm [6]. Since $x < G_{aw_jw_{j-1}} \leq n'/2$, we can draw $G_{aw_jw_{j-1}}$ on at most $n'/3$ parallel lines. By the Stretch and Reshape conditions, we merge this drawing with the current drawing such that w_j lies on either l_{x+3} or $l_{n'/9+2}$. Since $G^*_{bw_jw_{j-1}} \leq n'/9$, we can draw $G_{bw_jw_{j-1}}$ inside its corresponding triangle using Lemma 5. Since $\max_{\forall i>j}\{G^*_{aw_iw_{i-1}}, G^*_{bw_iw_{i-1}}\} \leq n'/9$, it is straightforward to extend the current drawing to a drawing of G on $x+4$ parallel lines by continuing the path $w_j, \ldots, w_t$ in the zigzag fashion.
- Assume that $G^*_{bw_jw_{j-1}} > x$. The drawing in this case is similar to the case when $G^*_{aw_jw_{j-1}} > x$. The only difference is that while drawing the path $w_{k+1}, \ldots, w_{j-1}$, we choose Γ_1 or Γ_2 such that the edge (b, w_{j-1}) spans at least $x+3$ lines.

Case 2 ($G^*_4 \leq x$). Observe that $G^*_2 \leq G^*_1 \leq n'/2$. Since $G^*_3 \leq G^*_2 \leq G^*_1$ and $G^*_3 + G^*_2 + G^*_1 = n + 5$, we have $G^*_3 \leq (n'+2)/3$. Hence G admits a drawing on L_{x+4} by Lemma 6.

The following theorem summarizes the result of this section.

Theorem 2. *Every n-vertex planar 3-tree admits a straight-line drawing with height $4(n+3)/9 + 4 = 4n/9 + O(1)$.*

Acknowledgement. We thank the anonymous reviewers for their detailed feedback to improve the presentation of the paper.

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