# Drawing Partially Embedded and Simultaneously Planar Graphs

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**Abstract.** We investigate the problem of constructing planar drawings with few bends for two related problems, the *partially embedded graph* (PEG) problem—to extend a straight-line planar drawing of a subgraph to a planar drawing of the whole graph—and the *simultaneous planarity* (SEFE) problem—to find planar drawings of two graphs that coincide on shared vertices and edges. In both cases we show that if the required planar drawings exist, then there are planar drawings with a linear number of bends per edge and, in the case of simultaneous planarity, a constant number of crossings between every pair of edges. Our proofs provide efficient algorithms if the combinatorial embedding information about the drawing is given. Our result on partially embedded graph drawing generalizes a classic result of Pach and Wenger showing that any planar graph can be drawn with fixed locations for its vertices and with a linear number of bends per edge.

## 1 Introduction

In many practical applications we wish to draw a planar graph while satisfying some geometric or topological constraints. One natural situation is that we have a drawing of part of the graph and wish to extend it to a planar drawing of the whole graph. Pach and Wenger [20] considered a special case of this problem. They showed that any planar graph can be drawn with its vertices lying at pre-assigned points in the plane and with a linear number of bends per edge. In this case the pre-drawn subgraph has no edges.

If the pre-drawn subgraph $H$ has edges, a planar drawing of the whole graph $G$ extending the given drawing $\mathcal{H}$ of $H$ might not exist. Angelini et al. [1] gave a linear-time algorithm for the corresponding decision problem; the algorithm returns, for a positive answer, a planar embedding of $G$ that *extends* that of $\mathcal{H}$ (i.e., if we restrict the embedding of $G$ to the edges and vertices of $H$, we obtain the embedding corresponding to $\mathcal{H}$). If one does not care about maintaining the actual planar drawing of $H$ this is the end of the story, since standard methods can be used to find a straight-line planar drawing of $G$ in which the drawing of $H$ is topologically equivalent to the one of $\mathcal{H}$. In

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this paper we show how to draw $G$ while preserving the actual drawing $\mathcal{H}$ of $H$, so that each edge has a linear number of bends. This bound is worst-case optimal, as proved by Pach and Wenger [20] in the special case in which $H$ has no edges.

A result analogous to ours was claimed by Fowler et al. [10] for the special case in which $H$ has the same vertex set as $G$. Their algorithm draws the edges of $G$ one by one in a certain order, and they claim a linear number of bends per edge. However, we give an example where their algorithm produces exponentially many bends, confirming a claim of Schaefer [23] that greedy extensions can in general give many bends.

We also address the *simultaneous planarity* problem [4], also known as "simultaneous embedding with fixed edges (SEFE)". The SEFE problem is strongly related to the partially embedded graph problem and—in a sense we will make precise later—generalizes it. We are given two planar graphs $G_1$ and $G_2$ that share a *common subgraph* $G$ (i.e., $G$ is composed of those vertices and edges that belong to both $G_1$ and $G_2$). We wish to find a *simultaneously planar drawing*, i.e., a planar drawing of $G_1$ and a planar drawing of $G_2$ that coincide on $G$. Graphs $G_1$ and $G_2$ are *simultaneously planar* if they admit such a drawing. Both $G_1$ and $G_2$ may have *private* edges that are not part of $G$. In a simultaneous planar drawing the private edges of $G_1$ may cross the private edges of $G_2$. The simultaneous planarity problem arises in information visualization when we wish to display two relationships on two overlapping element sets.

The decision version of the simultaneous planarity problem is not known to be **NP**-complete, nor solvable in polynomial time, though it is **NP**-complete if more than two graphs are given [11]. However, there is a combinatorial characterization of simultaneous planarity, based on the concept of a "compatible embedding", due to Jünger and Schulz [16] (see below for details). Erten and Kobourov [8], who first introduced the problem, gave an efficient drawing algorithm for the special case where the two graphs share vertices but no edges. In this case, a simultaneous planar drawing always exists, and they construct a drawing in which each edge has at most three bends and therefore any two edges cross (when they legally can) at most 16 times. In this paper we show that if two graphs have a simultaneous planar drawing, then there is a drawing in which every edge has a linear number of bends and in which any two edges cross at most 24 times. Our result is algorithmic, assuming a compatible embedding is given.

More formally, our paper addresses the following two problems:

- **Planarity of a partially embedded graph (PEG).** Given a planar graph $G$ and a straight-line planar drawing $\mathcal{H}$ of a subgraph $H$ of $G$, find a planar drawing of $G$ that extends $\mathcal{H}$ (see [1,15]).
- **Simultaneous planarity (SEFE).** Given two planar graphs $G_1$ and $G_2$ that share a subgraph $G$, find planar drawings of $G_1$ and $G_2$ that are the same on the shared subgraph (see [4]).

We prove the following results:

**Theorem 1.** *Let $G$ be an $n$-vertex planar graph, let $H$ be a subgraph of $G$, and let $\mathcal{H}$ be a straight-line planar drawing of $H$. Suppose that $G$ has a planar embedding $\mathcal{E}$ that extends $\mathcal{H}$. Then we can construct a planar drawing of $G$ in $O(n^2)$-time which realizes $\mathcal{E}$, extends $\mathcal{H}$, and has at most $102|V(H)| + 12$ bends per edge.*

**Theorem 2.** *Let $G_1$ and $G_2$ be simultaneously planar graphs on a total of $n$ vertices with a shared subgraph $G$. Then there is a simultaneous planar drawing in which any edge of $G_1 - G$ and any edge of $G_2 - G$ intersect at most $24$ times, and one of the following properties holds:*

1. *each edge of $G$ is straight, and each private edge of $G_1$ and of $G_2$ has at most $72n$ bends; also, vertices, bends, and crossings lie on an $O(n^2) \times O(n^2)$ grid; or*
2. *each edge of $G_1$ is straight and each private edge of $G_2$ has at most $102|V(H)|+12$ bends per edge.*

*If we are given a compatible embedding of the two graphs, we can construct such drawings in $O(n^2)$ time.*

Theorem 1 generalizes Pach and Wenger's result, which corresponds to the special case in which the pre-drawn subgraph has no edges. Observe that Theorem 1 directly provides a weak form of Theorem 2: If $G_1$ and $G_2$ are simultaneously planar, then they admit a compatible embedding. We can hence take any straight-line planar drawing of $G_1$ realizing the embedding and extend the induced drawing of $G$ to a drawing of $G_2$. By Theorem 1, we obtain a simultaneous planar drawing where each edge of $G_1$ is straight and each private edge of $G_2$ has at most $102|V(H)| + 12$ bends per edge. Our stronger result of 24 crossings between any two edges is obtained by modifying the proof of Theorem 1, rather than applying that result directly.

We note that Grilli et al. [12] have a paper in this conference with a result similar to Theorem 2. They show, using different techniques, that two simultaneously planar graphs have a simultaneous planar drawing with at most 9 bends per edge, vastly better than our $72n$ bound. Our primary goal, however, was to reduce crossings rather than bends. We achieve $24$ crossings per pair of edges. They do not address the number of crossings, but the obvious bound from their result is $100$ crossings per pair of edges. We also achieve a polynomial-size grid, but the obvious way of forcing their drawing onto a polynomial-sized grid increases the number of bends per edge to $300n$.

## 1.1 Related Work

The decision version of simultaneous planarity generalizes partially embedded planarity: given an instance $(G, H, \mathcal{H})$ of the latter problem, we can augment $\mathcal{H}$ to a drawing of a 3-connected graph $G_1$ and let $G_2 = G$. Then $G_1$ and $G_2$ are simultaneously planar if and only if $G$ has a planar embedding extending $\mathcal{H}$. In the other direction, the algorithm [1] for testing planarity of partially embedded graphs solves the special case of the simultaneous planarity problem in which the embedding of the common graph $G$ is fixed (which happens, e.g., if $G$ or one of the two graphs is 3-connected).

Several optimization versions of partially embedded planarity and simultaneous planarity are **NP**-hard. Patrignani showed that testing whether there is a straight-line drawing of a planar graph $G$ extending a given drawing of a subgraph of $G$ is **NP**-complete [21], so bend minimization in partial embedding extensions is **NP**-complete; Patrignani's result holds even if a combinatorial embedding of $G$ is given.[1] Bend minimiza-

[1] Patrignani does not explicitly claim **NP**-completeness in the case in which the embedding of $G$ is fixed, but that can be concluded by checking his construction; only the variable gadget, pictured in his Figure 3, needs minor adjustments.

tion in simultaneous planar drawings is **NP**-hard, since it is **NP**-hard to decide whether there is a straight-line simultaneous drawing [9]. Crossing minimization in simultaneous planar drawings is also **NP**-hard, as follows from an **NP**-hardness result on *anchored planar drawings* by Cabello and Mohar [5] (see Section 4).

As mentioned above, the special cases of PEG and SEFE in which there are no edges in the pre-drawn subgraph and in the common subgraph have been already studied.

Concerning PEG, Pach and Wenger [20] proved the following result: given an $n$-vertex planar graph $G$ with fixed vertex locations, a planar drawing of $G$ in which each edge has at most $120n$ bends can be constructed in $O(n^2)$ time. They also proved that such a bound is tight in the worst case. A $3n+2$ upper bound improving upon the $120n$ upper bound of Pach and Wenger has been proved by Badent et al. [2].

Concerning SEFE, Erten and Kobourov [8] proved the following result: given two planar graphs $G_1$ and $G_2$ sharing some vertices and no edges with a total number of $n$ vertices, there is an $O(n)$-time algorithm to construct a simultaneous planar drawing of $G_1$ and $G_2$ on a grid of size $O(n^2) \times O(n^2)$, with at most 3 bends per edge, hence at most 16 crossings between any edge of $G_1$ and any edge of $G_2$. Building on Kaufmann and Wiese's drawing algorithm [17], the number of bends per edge and the number of crossings per pair of edges can be reduced to 2 and 9, respectively, at the expense of an exponential increase in the area of the simultaneous drawing.

Haeupler et al. [13] showed that if two simultaneously planar graphs $G_1$ and $G_2$ share a subgraph $G$ that is connected, then there is a simultaneous planar drawing in which any edge of $G_1 - G$ and any edge of $G_2 - G$ intersect at most once. Introducing vertices at crossing points yields a planar graph, and a straight-line drawing of that graph provides a simultaneous planar drawing with $O(n)$ bends per edge, $O(n)$ crossings per edge, and with vertices, bends, and crossings on an $O(n^2) \times O(n^2)$ grid. Our result generalizes this to the case where the common graph $G$ is not necessarily connected.

### 1.2 Graph Drawing Terminology

A *rotation system* for a graph is a cyclic ordering of the edges incident to each vertex. A rotation system of a connected graph determines its *facial walks*—the closed walks in which each edge $(u, v)$ is followed by the next edge $(v, w)$ in the cyclic order at $v$. The facial walks are the boundaries of the *faces* in an embedding of the graph. The *size* $|W|$ of a facial walk $W$ is the length of $W$ (edge repetitions are counted). A rotation system is *planar* if it corresponds to a planar drawing; a *planar embedding* of a connected graph consists of a planar rotation system together with a specified outer face.

These definitions do not handle the situation in which the graph is not connected. Following Jünger and Schulz [16], we define a *topological embedding* of a (possibly non-connected) graph as follows: We specify a planar embedding for each connected component. This determines a set of inner faces. For each connected component we specify a "containing" face, which may be an inner face of some other component or the unique outer face. Furthermore, we forbid cycles of containment—in other words, if a connected component is contained in an inner face, which is contained in a component, etc., then this chain of containments must lead eventually to the unique outer face.

A *facial boundary* in a topological embedding of a graph is the collection of facial walks along the (not necessarily connected) boundary of a face. Each face (unless it is the outer face) has a distinguished facial walk we call the *outer* facial walk separating the remaining *inner* facial walks from the outer face of the embedding. The *size* of a facial boundary is the sum of the sizes of the facial walks part of the facial boundary.

A *compatible embedding* of two planar graphs $G_1$ and $G_2$ consists of topological embeddings of $G_1$ and $G_2$ such that the common subgraph $G$ inherits the same topological embedding from $G_1$ as from $G_2$ (where a subgraph inherits a topological embedding in a straightforward way; in particular, if we remove an edge that disconnects the graph, the face containment is determined by the edge that was removed). Jünger and Schulz [16] proved that $G_1$ and $G_2$ are simultaneously planar if and only if they have a compatible embedding. For that proof, they construct a simultaneous planar drawing of $G_1$ and $G_2$ by extending a drawing of $G$ (thus proving a form of our Theorem 1). However, their method does not yield any bounds on the number of bends or crossings.

## 2 Partially Embedded Graphs

In this section we prove Theorem 1. We will construct a planar drawing of $G$ that extends $\mathcal{H}$, assuming that we are given a planar embedding of $G$ that extends $\mathcal{H}$. It suffices to prove the result for a single face $F$ of $\mathcal{H}$ and the connected components of $G$ that lie inside or on the boundary of $F$ and are connected to $H$.

Pach and Wenger [20] proved their upper bound on the number of bends needed to draw a graph with fixed vertex locations by drawing a tree with leaves at the fixed vertex locations, and "routing" all the edges close to the tree, sometimes crossing the tree but never crossing each other. We will adapt their method to our setting.

One important difference is that we have to deal with fixed facial boundaries instead of fixed vertex locations. The solution is natural: We contract each facial boundary $W_i$ of $F$ to a single vertex $v_i$, fix vertex $v_i$ inside $F$ near $W_i$, and then apply the Pach-Wenger method to draw the contracted graph on the fixed vertex locations $v_i$. This must be done while keeping the drawing inside $F$. We keep the drawing at a small distance from the boundary of $F$, inside a polygonal region $F'$ that is an "inner approximation" of $F$. Inside $F'$ we draw a tree $T$ with its leaves $v_i$ at the fixed vertex locations, suitably bounding the size of $T$ in order to get our bound on the number of bends. We then route the edges of the contracted graph close to $T$ as in Pach-Wenger. Finally, to get back our uncontracted graph, we route the edges incident to $v_i$ to their true endpoint on the facial boundary $W_i$—these routes use the empty buffer zone between $F$ and $F'$.

We now fill in further details. We use $n_A$ and $m_A$ for the number of vertices and edges in subgraph $A$. Let $W_i$, with $1 \leq i \leq b$, be the boundary walks of $F$.

We now introduce the concept of inner $\varepsilon$-approximations. The *Hausdorff distance* $d_H(A, B)$ of two sets (in a space with metric $d$) is defined as:[2]

$$\max \{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}.$$

Intuitively, the Hausdorff distance measures how far a point in one set can be from the other set. Sets $A$ and $B$ are *$\varepsilon$-close* if $d_H(A, B) < \varepsilon$. Then $A$ is an *inner $\varepsilon$-approximation of* $B$ if they are $\varepsilon$-close and there is a $\delta > 0$ so that all the points $\delta$-close to $A$ are a subset of $B$. The next lemma deals with inner $\varepsilon$-approximations of $F$.

[2] The underlying metric $d$ can be Euclidean or some other appropriate metric.

**Lemma 1.** *Let $k$ be the size of the boundary of $F$. For any $\varepsilon > 0$ we can efficiently construct an inner $\varepsilon$-approximation $F'$ of $F$ whose boundary has size $3k$ (see Figure 1).*

We prove Lemma 1 using Lemma 2 in which, for every sufficiently small $\varepsilon > 0$ we construct a closed polygonal arc $P_\varepsilon$ that is $\varepsilon$-close to the facial walk, does not have too many bends, and so that the simple polygon bounded by $P_{\varepsilon'}$ lies in the interior of the simple polygon bounded by $P_\varepsilon$ for all $0 < \varepsilon' < \varepsilon$ (in particular, any two polygonal arcs are disjoint). There are various ways to achieve this. Pach and Wenger [20] use the Minkowski sum of the facial walk (in their case the facial walk of a tree) and a square diamond centered at 0. We use a slightly different construction, because it seems easier (both computationally and conceptually) and it gives a slightly better bound on the number of bends (which is what we are most interested in); namely for the facial walk of an $n$-vertex tree, Pach and Wenger construct a polygonal arc with $4n - 2$ vertices, while our polygonal arcs have $2n - 2$ vertices. Our construction does have one disadvantage: the resulting drawings will get rather tight for sharp (acute or obtuse) angles (the Minkowski-sum construction has the same problem for highly obtuse angles only).

**Lemma 2.** *Let $W$ be a facial walk in a face $F$ of a drawing of a graph $G$ in the plane. We can efficiently construct a disjoint family of polygonal arcs $P_\varepsilon$ so that $P_\varepsilon$ is $\varepsilon$-close to $W$ and each $P_\varepsilon$ has at most* $\max\{3, |W|\}$ *vertices.*

*Proof.* Let $e, v, f$ be a *corner* of $W$, that is, two consecutive edges $e$, $f$ and their shared vertex $v$. At $v$ erect the angle bisector of $e$ and $f$ of length $\varepsilon$ (inside $F$), and let $v'$ be the endpoint of the bisector different from $v$. For computational reasons, it may be better to use the $\ell_1$-norm at this point (the Euclidean norm will lead to square root expressions in the coordinates). If $(v_i)_{i=1}^k$ is the sequence of vertices along $W$, with $k = |W|$, then $(v_i')_{i=1}^k$ defines a closed polygonal arc. If $\varepsilon$ is sufficiently small, namely less than half the distance between any vertex of $W$ and a non-adjacent edge on $W$, the arc is free of self-crossings, and therefore bounds a simple polygon with $|W|$ vertices. There are two special cases in which this argument does not work: if the boundary walk is a boundary walk on an isolated vertex or an isolated edge. In both of these cases, we can approximate $W$ using a triangular shape. □

Lemma 2 allows us to replace a facial boundary with a *simple polygon with holes*, that is, a collection of closed polygonal arcs that bound a face which is very close to the original boundary, has bounded complexity, and can be constructed efficiently. This leads to a proof of Lemma 1. Namely, approximate each facial walk of the facial boundary with an $\varepsilon$-close polygonal arc lying in $F$. The union of those arcs is a simple polygon with holes as long as $\varepsilon$ is less than half the distance between any two non-adjacent vertices or edges. The upper bound of $3k$ will generally be a large overestimate, but allows for the possibility that all the inner walks are walks on isolated vertices.

We now return to the proof of Theorem 1. After constructing an inner $\varepsilon$-approximation $F'$ of $F$ by using Lemma 1, the next step is to construct tree $T$. Triangulate $F'$ using at most $m_{F'} + 2(b-2)$ triangles[3] and use a result of Bern and Gilbert [3] to construct a

[3] Every $n$-vertex polygon with $b$ boundary components can be triangulated by inserting edges in $O(n \log n)$ time. The number of resulting triangles is $n + 2(b-2)$ (see [19, Lemma 5.1]).

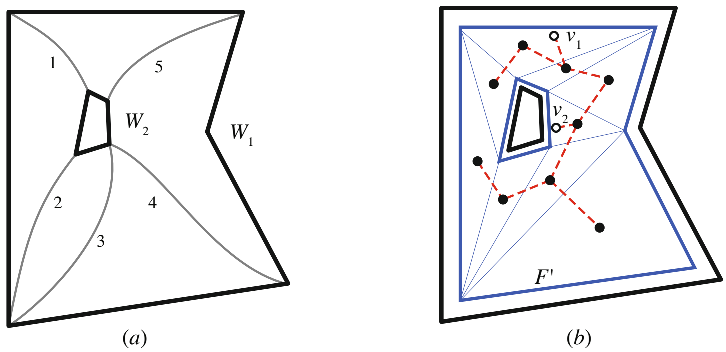


**Fig. 1.** A face $F$ with outer and inner and boundary walks $W_1$ and $W_2$. (*a*) The 5 edges of $G - H$. (*b*) The inner approximation $F'$ (heavy blue lines), a triangulation of it (fine lines), and the dual spanning tree (dashed red) with extra vertices $v_1$ and $v_2$ close to $W_1$ and $W_2$, respectively.

straight-line drawing of the dual of the triangulation. Bern and Gilbert place a vertex at the *incenter* of each triangle (where the angle bisectors of the triangle meet) and prove that the straight-line edge joining two vertices in adjacent triangles lies within the union of the two triangles. Now take a spanning tree $T$ of the dual. For each boundary walk $W_i$, we augment $T$ with a new leaf $v_i$ close to $W_i$ and inside $F'$. This adds $b$ vertices to $T$, so the number of vertices of $T$ is now $n_T = m_{F'} + 3b - 4$.

Let $G_F$ be the embedded multi-graph obtained by restricting $G$ to vertices and edges lying inside or on the boundary of $F$ and by contracting each boundary walk $W_i$ of $F$ to a single vertex $v_i$. We can now use the following result (extending ideas of Pach and Wenger) to embed $G_F$ close to $T$.

**Lemma 3.** *Let $G$ be a multi-graph with a given planar embedding and fixed locations for a subset $U \subseteq V(G)$ of its vertices. Suppose we are given a straight-line drawing of a tree $T$ whose leaves include all the vertices in $U$ at their fixed locations. Then for every $\varepsilon > 0$ there is a planar poly-line drawing of $G$ that is $\varepsilon$-close to $T$, that realizes the given embedding, where the vertices in $U$ are at their fixed locations, and where each edge has at most $12n_T$ bends. Moreover, each edge of $G$ comes close to each vertex in $U$ at most six times (where coming close means entering and leaving an $\varepsilon$-neighborhood of the vertex or terminating at the vertex).*

The proof of Lemma 3 is long and involved, hence we defer it to the end of the section, and we first proceed with the reminder of the proof of Theorem 1.

We use Lemma 3 to embed $G_F$ along $T$ so that vertices $v_i$ are drawn at their fixed locations. Each edge of $G_F$ has at most $12n_T$ bends.

We now want to connect edges in $G_F$ to the boundary components they belong to. We will use the buffer between $F'$ and $F$ to do this. In fact, we need to split the buffer zone into two, so we apply Lemma 1 a second time to obtain an inner $\varepsilon/2$-approximation $F''$ of $F$, so that $F' \subseteq F'' \subseteq F$. See Figure 2. The size of the boundary of $F''$ is at most $3m_F$ (just like $F'$). Now for each walk $W_i$ we extend the edges ending at $v_i$ to their endpoint on $W_i$. Since we maintained the cyclic order of $G_F$-edges at $v_i$, we can simply

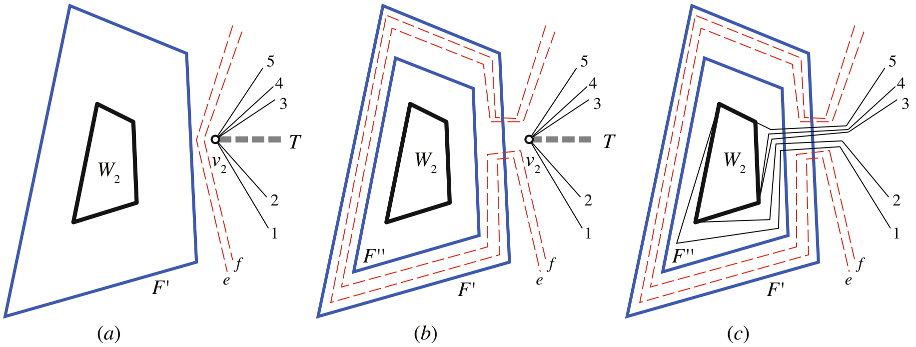


**Fig. 2.** A close-up of the situation near inner boundary walk $W_2$. (*a*) After drawing $G_F$ around the tree $T$ (heavy dashed line), edges $1, \ldots, 5$ are incident to $v_2$ in the correct cyclic order, but two other edges $e$ and $f$ pass by between $v_2$ and $F'$. (*b*) We add a second approximation $F''$ and route the edges $e$ and $f$ (in dashed red) around $W_2$ in the buffer zone between $F''$ and $F'$. (*c*) We route the edges incident to $W_2$ in the buffer zone between $F$ and $F''$.

route these edges around $W_i$ using approximations to $W_i$ via Lemma 1, and we can do so in $F - F''$. This adds at most $m_{W_i} + 2$ bends to an edge with endpoint on $W_i$; the two additional bends are needed to separate edges at $v_i$, and turn to connect to $W_i$. There is one difficulty: there are edges of $G_F$ that pass by $v_i$, separating it from the segment of $F'$ close to $v_i$ (which is our gate to $W_i$). To remedy this difficulty, we first route all of these edges around the whole obstacle $W_i$ in the $F'' - F'$ part of the buffer, which adds $m_{W_i} + 2$ bends to an edge every time it passes $v_i$. Now we are free to route the $G_F$-edges incident to $v_i$ to their endpoints along $W_i$. Since an edge can pass by and/or terminate at a vertex at most six times, the total number of additional bends in each edge caused by going around $W_i$ is $6(m_{W_i} + 2) \leq 6(m_{F'} + 2) \leq 18m_F + 12$. Since each $G_F$ edge started with $12n_T$ bends, each $G_F$ edge now has at most $12n_T + 18m_F + 12$ bends. Using $m_F \leq m_H \leq 3n_H$, and $n_T \leq m_{F'} + 3b - 4 \leq 3m_F + 3b - 4 \leq 4n_H$ we conclude that each edge has at most $48n_H + 54n_H + 12 = 102n_H + 12$ bends.

Let us now analyze the running time of the algorithm. Most of the steps in the construction can be performed in linear time. Building the triangulation takes time $O(n_H \log n_H)$. The overall running time is thus bounded by the size of the resulting drawing which contains a linear number of edges each with a linear number of bends, yielding the quadratic running time.

We conclude the section by proving Lemma 3. Pach and Wenger's [20] algorithm to draw a planar graph $G$ with vertices at fixed locations has three ingredients: $(i)$ they show how to assume that $G$ is Hamiltonian, $(ii)$ they show how to draw the Hamiltonian cycle of $G$, and $(iii)$, they show how to draw the remaining edges of $G$. In order to prove Lemma 3, we will follow their structure closely. We will use their result $(i)$ directly:

**Lemma 4 (Pach, Wenger [20]).** *Given a planar graph G we can in linear time construct a Hamiltonian graph $G'$ with $|E(G')| \leq 5|E(G)| - 10$ by adding and subdividing edges of G (each edge is subdivided by at most two new vertices).*

We will use a slightly stronger version of Lemma 4 in which $G$ is allowed to be a mulitgraph. Pach and Wenger's proof of Lemma 4 works for this case.

For part $(ii)$ Pach and Wenger show that a Hamiltonian cycle can be drawn at fixed vertex locations $\varepsilon$-close to a star connecting all the vertices. For our application, we replace their star with a straight-line drawing of a tree $T$ whose leaves are the vertices $v_i$. Independently of our result, the generalization of part $(ii)$ to trees has essentially been shown by Chan et al. [6]. Since their goal was the minimization of the edge lengths, they did not give an estimate on the number of bends. We now show how to draw the Hamiltonian cycle. We will later show how to draw the remaining edges.

**Lemma 5.** *Let $C$ be a cycle with fixed vertex locations, and suppose we are given a straight-line planar drawing of a tree $T$, in which the vertices of $C$ are leaves of $T$ at their fixed locations. Then for every $\varepsilon > 0$ there is a planar poly-line drawing of $C$ with at most $2|E(T)| - 1$ bends per edge and $\varepsilon$-close to $T$.*

*Proof.* Let $p_1, \ldots, p_n$ be the vertices of $C$ in their order along the cycle. We build a planar poly-line drawing of $C$ as follows. Let $\Theta_i$ be an $i\varepsilon/n$-approximation of $T$ for $1 \leq i < n$ (which we can construct using Lemma 2). We start at $p_1$. Suppose we have already built the poly-line drawing of $p_1, \ldots, p_i$ and we want to add $p_i p_{i+1}$. Let $Q_i$ be the unique path in $T$ connecting $p_i$ to $p_{i+1}$. Create $\Theta'_i$ from $\Theta_i$ by keeping only the vertices of $\Theta_i$ close to (approximating) vertices in $T_i := \bigcup_{j \leq i} Q_j$. This removes parts of the walk along $\Theta_i$ which we patch up as follows: suppose $v$ is an interior vertex of $T_i$, and $v$ is incident to $e$ which does not lie on $T_i$. Then $v$ is approximated by two vertices $v_1$ and $v_2$ which lie on bisectors formed by $e$ with neighboring edges. Now $v_1$ and $v_2$ belong to $\Theta'_i$, but the path along $\Theta_i$ between them got removed (since $e$ does not belong to $T_i$). We add $v_1 v_2$ to $\Theta'_i$ to connect them. Note that $v_1 v_2$ does not pass through $v$ since $v$ is incident to at least three edges ($e$ and two edges of $T_i$), and it does not cross any edges of any $\Theta'_j$ with $j < i$, since $T_i$ is monotone: if $e \notin E(\Theta_i)$, then $e \notin E(\Theta_j)$ for $j < i$. See Figure 3 for an illustration. Now both $p_i$ and $p_{i+1}$ correspond to unique vertices on $\Theta'_i$ (since they are leaves), so we can pick a facial walk $v_1, \ldots, v_k$ on $\Theta'_i$ which connects $p_i$ to $p_{i+1}$ and which avoids passing by $p_1$. We now add line segments $p_i v_2$, $v_2 v_3$, …, $v_{k-2} v_{k-1}$, $v_{k-1} p_{i+1}$ to the poly-line drawing of $C$. We treat the final edge $p_n p_1$ similarly, except that we move along $\Theta'_{n-1}$ back to $p_1$ in the last step, which we can do, since none of the intermediate paths passed by $p_1$. Each edge of $C$ is replaced by a polygonal arc with at most $2|E(T)| - 1$ bends. □

As mentioned earlier, the following lemma is close to a result by Chan et al. [6], except for the claim about the number of bends, and the rotation system (which we require for our main result).

**Lemma 6.** *Let $G$ be a Hamiltonian multi-graph with a given planar embedding and fixed vertex locations. Suppose we are given a straight-line drawing of a tree $T$ whose leaves include all the vertices of $G$ at their fixed locations. Then for every $\varepsilon > 0$ there is a planar poly-line drawing of $G$ that is $\varepsilon$-close to $T$, that realizes the given embedding, where the vertices of $G$ are at their fixed locations, where each edge has at most $4|E(T)| - 1$ bends, and where each edge comes close to any leaf of $T$ at most twice.*

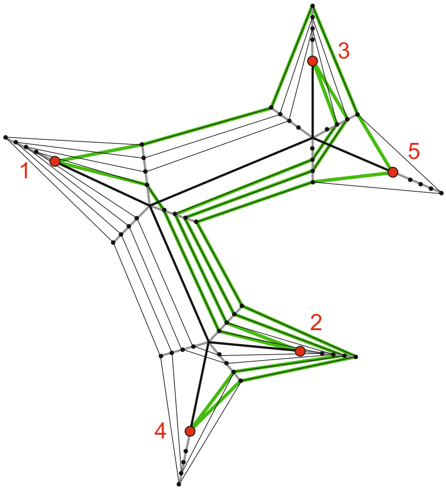


**Fig. 3.** The underlying tree $T$ is in black (thick edges), angle bisectors in gray; the $\Theta'_i$ are drawn as thin black edges; to reduce clutter, we are not showing the remaining edges of $\Theta_i$; the drawing of $C$ is indicated by the green line.

The obvious idea—routing edges along the Hamiltonian cycle $C$—only gives a quadratic bound on the number of bends, since each edge would follow the path of a linear number of edges of $C$, and each edge of $C$ has a linear number of bends. Pach and Wenger came up with an ingenious way to construct auxiliary curves with few bends based on the level curves $\Theta'_i$ which carry the cycle $C$ in the proof of Lemma 5.

*Proof.* Let $C$ be the Hamiltonian cycle of $G$ and let $G_1$ and $G_2$ be the two outerplanar graphs composed of $C$ and, respectively, of the edges of $G$ outside and inside $C$. Using Lemma 5 we find a planar poly-line drawing of $C$ on $V(G)$. We need to show how to draw $G_1$ and $G_2$ respecting the planar embeddings induced by the given embedding of $G$. Let $n = |V(G)|$ and $m_i = |E(G_i)|$. We only describe how to draw $G_1$, since $G_2$ can be handled analogously. Let $\Delta_{i,k}$, $1 \leq k \leq m_1$ be a $k\varepsilon/(nm_1)$-approximation of $\Theta'_i$ constructed using Lemma 2. For a fixed $i$, each $\Delta_{i,k}$ crosses $C$ twice: when $C$ moves from $p_i$ to $\Theta'_{i+1}$, and when it finally moves back from $\Theta'_n$ to $p_1$. As in Pach and Wenger, we can then split $\Delta_{i,k}$ at the crossings and connect their free ends to $p_1$ and $p_i$, resulting (for each $k$) in two curves $\Delta'_{i,k}$ and $\Delta''_{i,k}$ connecting $p_1$ to $p_i$, where $\Delta'_{i,k}$ lies outside $C$ (these are the curves we use for $G_1$) and $\Delta''_{i,k}$ inside $C$ (these are the curves we use for $G_2$). Each such curve has at most $2|E(T)| - 1$ bends. As in the proof of Pach and Wenger, we can create edges $p_ip_j \in E(G_1)$ by concatenating $\Delta'_{i,k}$ with $\Delta'_{j,k}$. Since we chose $m_1$ such approximations, we can do this for each edge in $G_1$. There are two problems remaining: edges $p_ip_j$ now all pass through $p_1$ and they could potentially cross (rather than just touch) there. Pach and Wenger show that any two edges touch, so the drawing can be modified close to $p_1$ so as to separate all edges $p_ip_j$ from each other. This introduces at most one more bend per edge, so that the resulting edges have

$2(2|E(T)|-1)+1 = 4|E(T)|-1$ bends. Finally, note that each edge $p_i p_j$ comes close to each leaf of $T$ (including $p_1$) at most twice, once for $\Delta'_{i,k}$ and once for $\Delta'_{j,k}$. □

Now we are ready to finish the proof of Lemma 3. We show how to apply Lemma 6 in case $G$ is not Hamiltonian, and not all its vertices are assigned fixed locations.

By Lemma 4, we can construct a graph $G'$ with a Hamiltonian cycle $C$ by subdividing each edge of $G$ at most twice, and by adding some edges, where $G'$ has a planar embedding extending the embedding of $G$. Traverse $C$: whenever we encounter an edge of $C$ with at least one endpoint not in $U$, contract that edge. This yields a new Hamiltonian graph $G''$ with $V(G'') = U$ and a planar embedding induced by the planar embedding of $G'$. Use Lemma 6 to embed $G''$ at the fixed vertex locations, and $\varepsilon$-close to $T$, so that each edge of $G''$ has at most $4|E(T)| - 1$ bends. Each vertex $u \in U$ of $G''$ corresponds to a set of vertices $V_u \subseteq V(G')$ which was contracted to $u$, so the subgraph $G'_u$ of $G'$ induced by $V_u$ is connected. Since we embedded $G''$ with the induced planar embedding of $G'$, we can now do some surgery to turn $u$ back into $G'_u$.

To this end, we define a graph $G^+_u$, which consists of $G'_u$, of a cycle $C_u$ containing $G'_u$ in its interior, and of some further edges. Each vertex of $C_u$ corresponds to an edge of $G'$ "incident to" $G'_u$, i.e., with an end-vertex in $V_u$ and with an end-vertex not in $V_u$. Vertices appear in $C_u$ in the same order as the corresponding edges incident to $G'_u$ leave $G'_u$ (this order also corresponds to the cyclic order of the edges incident to $v$ in $G''$); each vertex of $C_u$ corresponding to an edge $e$ of $G'$ is connected to the end-vertex of $e$ in $V_u$. Finally, $G^+_u$ contains further edges that triangulate its internal faces.

Now consider a small disk $\delta$ around $u$. We erase the part of the drawing of $G''$ inside $\delta$. We construct a straight-line convex drawing of $G^+_u$ in which each vertex of $C_u$ is mapped to the point in which the corresponding edge crosses the boundary of $\delta$. This drawing always exists (and can be constructed efficiently), given that $G^+_u$ is 2-connected and internally-triangulated. Removing the edges that triangulate the internal faces of $G^+_u$ completes the reintroduction of $G'_u$.

Overall, we added one bend to an edge with exactly one endpoint in $V_u$. Since an edge can have endpoints in at most two $V_u$, this process adds at most two bends per edge, so every edge has at most $4|E(T)|+1$ bends. Since each edge of $G$ was subdivided at most twice to obtain $G'$, each edge of $G$ has at most $3(4|E(T)|+1)+2 = 12|E(T)|+5 < 12|V(T)|$ bends. Each edge of $G'$ comes close to each leaf of $T$ at most twice, so each edge of $G$ comes close to each vertex of $U$ at most six times. This concludes the proof of Lemma 3.

## 3 Extending Partial Straight-Line Planar Drawings Greedily

Let $G$ be an $n$-vertex plane graph, let $H$ be a spanning subgraph of $G$, let $\mathcal{H}$ be a straight-line planar drawing of $H$, and let $\sigma = [e_1, \dots, e_m]$ be an ordering of the edges in $G \setminus H$. A drawing $\Gamma$ of $G$ *greedily extends* $\mathcal{H}$ *with respect to* $\sigma$ if it is obtained by drawing edges $e_1, \dots, e_m$ in this order, so that $e_i$ is drawn as a polygonal curve that respects the embedding of $G$ and with the minimum number of bends, for $i = 1, \dots, m$.

Fowler *et al.* claimed in [10] that, for every ordering $\sigma$ of the edges in $G \setminus H$ such that the edges between distinct connected components of $H$ precede edges between vertices

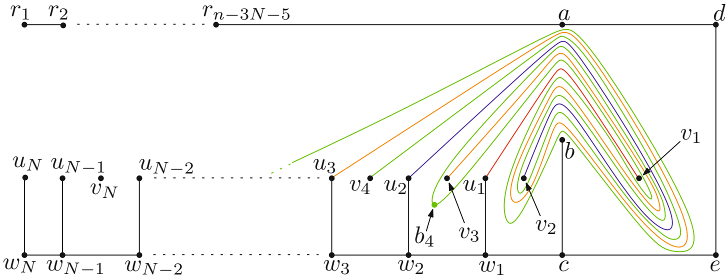


**Fig. 4.** A drawing $\Gamma$ of $G$ that greedily extends $\mathcal{H}$ with respect to $\sigma$. Drawing $\mathcal{H}$ consists of the black circles. The first edges $n - N - 1$ edges in $\sigma$ are (black) straight-line segments. The last $N$ edges $(u_i, v_i)$ are (colored) polygonal lines whose bends have been made smooth to improve the readability. Only four of the latter edges are shown.

in the same connected component of $H$, there exists a drawing $\Gamma$ of $G$ greedily extending $\mathcal{H}$ with respect to $\sigma$ where each edge has $O(n)$ bends. However, in the following we confirm a claim of Schaefer [23] stating that greedy extensions do not, in general, lead to drawings with a polynomial number of bends.

**Theorem 3.** *For every $n$, there exists an $n$-vertex plane graph $G$, a planar drawing $\mathcal{H}$ of the spanning empty subgraph $H$ of $G$, and an order $\sigma$ of the edges in $G$ such that any drawing of $G$ that greedily extends $\mathcal{H}$ with respect to $\sigma$ has edges with $2^{\Omega(n)}$ bends.*

*Proof.* We adapt an example by Kratochvíl and Matoušek [18]. Refer to Fig. 4. Let $N = \lfloor \frac{n}{3} \rfloor - 6$, for any integer $n$. Graph $H$ consists of $n$ isolated vertices; namely vertices $u_1, \dots, u_N, v_1, \dots, v_N, w_1, \dots, w_N, a, b, c, d, e, r_1, \dots, r_{n-3N-5}$. The first $n - N - 1$ edges in $\sigma$ are $(u_i, w_i)$ for $i = 1, \dots, N$, $(w_i, w_{i+1})$ for $i = 1, \dots, N-1$, $(r_i, r_{i+1})$ for $i = 1, \dots, n - 3N - 6$, $(c, w_1)$, $(b, c)$, $(c, e)$, $(e, d)$, $(a, d)$, and $(a, r_{n-3N-5})$. All these edges are straight-line segments in any drawing $\Gamma$ of $G$ that greedily extends $\mathcal{H}$ with respect to $\sigma$. The last $N$ edges in $\sigma$ are $(u_1, v_1), \dots, (u_N, v_N)$ in this order.

Consider any drawing $\Gamma$ of $G$ that greedily extends $\mathcal{H}$ with respect to $\sigma$. We claim that edge $(u_i, v_i)$ has $2^{i-1}$ bends in $\Gamma$. In fact, it suffices to prove that $(u_i, v_i)$ has $2^{i-1}$ intersections with the straight-line segment $\overline{ab}$ in $\Gamma$. Indeed, $(u_1, v_1)$ has exactly one intersection with $\overline{ab}$ in $\Gamma$. Inductively assume that $(u_i, v_i)$ has $2^{i-1}$ intersections with $\overline{ab}$ in $\Gamma$; we prove that $(u_{i+1}, v_{i+1})$ has $2^i$ intersections with $\overline{ab}$ in $\Gamma$. This proof is accomplished by citing Kratochvíl and Matoušek [18] almost *verbatim*. Since $(u_{i+1}, v_{i+1})$ does not cross $(u_i, v_i)$, it has a bend $b_{i+1}$ around $v_i$, i.e., inside the square defined by $u_{i-2}$, $w_{i-2}$, $w_{i-1}$, and $u_{i-1}$. Thus the polygonal curve representing $(u_{i+1}, v_{i+1})$ in $\Gamma$ consists of two parts – one from $u_{i+1}$ to $b_{i+1}$, the other from $b_{i+1}$ to $v_{i+1}$. Both of these parts may be used as an edge joining $u_i$ and $v_i$ – after contracting $u_{i+1}$ and $v_{i+1}$ into $u_i$, and $b_{i+1}$ into $v_i$. Hence, by induction, each of these two parts has $2^{i-1}$ intersections with $\overline{ab}$, and the whole edge $(u_{i+1}, v_{i+1})$ has $2^i$ intersections with $\overline{ab}$.

Hence, in any drawing $\Gamma$ of $G$ that greedily extends $\mathcal{H}$ with respect to $\sigma$, one edge has $2^{N-1} = 2^{\lfloor \frac{n}{3} \rfloor - 7} \in 2^{\Omega(n)}$ bends, which concludes the proof.

Note that the graph $G$ in the proof of Theorem 3 is a tree, thus all of its edges connect vertices in distinct connected components of $H$. □

## 4 Simultaneous Planarity

Before turning to our algorithm for drawing simultaneously planar graphs, we justify our claim that minimizing the number of crossings in a simultaneous planar drawing is **NP**-hard. This result follows from Cabello and Mohar's proof of **NP**-hardness for the *anchored planarity* problem [5, Theorem 2.1], but a more direct proof of a slightly stronger result is possible by reduction from the **NP**-complete crossing number problem. We briefly explain the reduction. Given a graph $G$ with $m$ edges, subdivide each edge $2m$ times. Let $G_1$ consist of all the edges incident to the original vertices of $G$ together with every other edge along the paths connecting the original vertices. Let $G_2$ consist of the remaining edges. Note that $G_1$ and $G_2$ do not share any edges. It can be easily seen that the crossing number of $G$ equals the smallest number of crossings between edges of $G_1$ and edges of $G_2$ in a simultaneous drawing of $G_1$ and $G_2$.[4] We now turn to the proof of Theorem 2.

*Proof (of Theorem 2).* We show how to find in $O(n^2)$ time a simultaneous planar drawing $\Gamma$ such that any private edge of $G_1$ and any private edge of $G_2$ intersect at most 24 times, such that every edge of $G_1$ is straight, and such that every private edge of $G_2$ has at most $102|V(H)| + 12$ bends. In order to construct a simultaneous planar drawing $\Gamma'$ on an $O(n^2) \times O(n^2)$ grid such that any private edge of $G_1$ and any private edge of $G_2$ intersect at most 24 times, such that each edge of $G$ is straight, and such that every private edge has at most $72n$ bends, it suffices to introduce dummy vertces at the $O(n^2)$ crossing points in $\Gamma$, and then to construct a straight-line drawing of the resulting planar graph on a small grid. In particular, the number of bends per edge in $\Gamma'$ is at most $72n$, since each edge in $\Gamma$ crosses less than $3n$ edges, each at most 24 times.

We start by constructing any straight-line planar drawing $\Gamma_1$ of $G_1$. We now construct a drawing $\Gamma_2$ of $G_2$ by exploiting an approach analogous to the one of the proof of Theorem 1. Drawing $\Gamma_1$ induces a straight-line planar drawing $\Gamma$ of $G$. Thus, in order to determine $\Gamma_2$, it remains to describe how to draw the private edges of $G_2$. We will accomplish this independently for each face $F$ of $G$.

We construct a triangulation $\Sigma$ of $F$ by using all the vertices and edges of $G_1$ that lie inside $F$, as well as some extra edges. Next, we execute the same algorithm as for the proof of Theorem 2. Namely, we construct a straight-line drawing of the dual $D$ of $\Sigma$ and we take a spanning tree $T$ of $D$. For each boundary walk $W_i$ of $F$, we augment $T$ with a leaf $v_i$ close to $W_i$ and inside $F'$, where $F'$ is an inner $\varepsilon$-approximation of $F$. Let $G_2^F$ be the embedded multi-graph obtained by restricting $G_2$ to the vertices and edges inside or on the boundary of $F$, and by contracting each boundary walk $W_i$ of $F$ to a single vertex $v_i$. We use Lemma 3 to construct a planar poly-line drawing of $G_2^F$ that realizes the given embedding, that is $\varepsilon$-close to $T$, and in which vertices $v_i$ maintain their fixed locations. Finally, we reconnect edges in $G_2^F$ to the boundary components they belong to. In order to do this, we first "wrap" the edges of $G_2^F$ passing by a vertex

[4] Using the fact that crossing number is hard for cubic graphs [14], we can even show that minimizing the number of crossings in a simultaneous drawing of two graphs one of which is the disjoint union of paths of length at most two and the other is a matching is **NP**-hard. This is in some sense sharp, since the union of two matchings is always planar.

$v_i$ around $W_i$, and we then extend the edges of $G_2^F$ incident to $v_i$ to their endpoint on $W_i$, by routing them around $W_i$.

By construction every edge of $G_1$ is straight. By Theorem 1 every private edge of $G_2$ has at most $102|V(H)| + 12$ bends. Also, the algorithmic steps are the same as for the proof of Theorem 1, hence the algorithm runs in $O(n^2)$ time. It remains to prove that any private edge of $G_1$ and any private edge of $G_2$ intersect at most 24 times.

Consider any private edge $e$ of $G_2$ and any private edge $e'$ of $G_1$. Recall that $e'$ is an edge of $\Sigma$. Denote by $W_i$ and $W_j$ the boundary walks the end-vertices of $e'$ belong to. Edge $e$ intersects $e'$ in two situations: when passing by $v_i$ or $v_j$ and when passing by the point $p_T$ in which the edge of $D$ dual to $e'$ crosses $e'$. We prove that each of these two types of intersections happens at most 12 times.

For the first type of intersections, we have by Lemma 3 that edge $e$ passes by each of $v_i$ or $v_j$ at most 6 times, hence at most 12 times in total. For the second type of intersections, we have by Lemma 4 that edge $e$ is subdivided into at most three edges $e_1$, $e_2$, and $e_3$ in order to turn $G_2^F$ into a Hamiltonian graph. For each $j = 1, 2, 3$, $e_j$ either belongs to the Hamiltonian cycle of the subdivided $G_2^F$ or not. In the former case, $e_j$ is drawn as part of an $i\varepsilon/n$-approximation $\Theta_i$ of $T$, as in the proof of Lemma 5, hence it crosses $e'$ at most twice. In the latter case, $e_j$ is composed of two parts, denoted by $\Delta'_{p,k}$ and $\Delta'_{q,k}$, or by $\Delta''_{p,k}$ and $\Delta''_{q,k}$ in the proof of Lemma 6. Each of $\Delta'_{p,k}$, $\Delta'_{q,k}$, $\Delta''_{p,k}$ and $\Delta''_{q,k}$ is part of a $k\varepsilon/(nm_1)$-approximation of $\Theta'_i$, which is part of $\Theta_i$. Hence, each of $\Delta'_{p,k}$, $\Delta'_{q,k}$, $\Delta''_{p,k}$ and $\Delta''_{q,k}$ crosses $e'$ at most twice; thus $e_j$ crosses $e'$ at most four times, and $e$ crosses $e'$ close to $p_T$ at most 12 times. □

## 5 Conclusions and Open Problems

We proved that if a graph has a planar drawing extending a straight-line planar drawing of a subgraph then there is such a drawing with at most $102n + O(1)$ bends per edge. This is asymptotically tight, but can the constant 102 be reduced? Our second result is that any two simultaneously planar graphs have a simultaneous planar drawing with at most 24 crossings per pair of edges and a linear number of bends per edge with a drawing on a polynomial-sized grid. The only lower bound on the number of crossings between two edges in a simultaneous planar drawing is 2 (see [7] or the figure in the margin for the entry "simultaneous crossing number" in [22]). There is a large gap between 2 and 24. Can two edges be forced to cross more than twice in a simultaneous planar drawing? Grilli et al. [12] showed that two simultaneously planar graphs have a drawing with at most 9 bends per edge, though with a larger constant for the number of crossings and not on a grid. Is it possible to achieve the best of both results: 9 bends per edge, 24 crossings per pair of edges, and a nice grid?

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